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Research Article

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Pentagonal quasigroups, their translatability and parastrophes

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Abstract: Any pentagonal quasigroup Q is proved to have the product $xy = \varphi(x) + y - \varphi(y)$, where $(Q, +)$ is an Abelian group, φ is its regular automorphism satisfying $\varphi^4 - \varphi^3 + \varphi^2 - \varphi + \varepsilon = 0$ and ε is the identity mapping. All Abelian groups of order $n < 100$ inducing pentagonal quasigroups are determined. The variety of commutative, idempotent, medial groupoids satisfying the pentagonal identity $(xy \cdot x)y \cdot x = y$ is proved to be the variety of commutative, pentagonal quasigroups, whose spectrum is $\{11^n : n = 0, 1, 2, \ldots\}$. We prove that the only translatable commutative pentagonal quasigroup is $xy = (6x + 6y)(\text{mod } 11)$. The parastrophes of a pentagonal quasigroup are classified according to well-known types of idempotent translatable quasigroups. The translatability of a pentagonal quasigroup induced by the group $\mathbb{Z}_n$ and its automorphism $\varphi(x) = ax$ is proved to determine the value of a and the range of values of n.

Keywords: quasigroup, pentagonal quasigroup, translatability, idempotent

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1 Introduction

A latin square $n \times n$ is *k-translatable* if it is obtained by the following rule: if the first row is $a_1, a_2, \ldots, a_n$, then the qth row is obtained from the $(q-1)$-st row by taking the last k entries in the $(q-1)$-st row and inserting them as the first k entries of the qth row and by taking the first $n-k$ entries of the $(q-1)$-st row and inserting them as the last $n-k$ entries of the qth row, where $q \in \{2, 3, \ldots, n\}$. An algebraic interpretation of translatable latin squares is translatable quasigroups.

Pentagonal quasigroups are medial idempotent quasigroups with a beautiful geometric interpretation. Any identity in the pentagonal quasigroup can be interpreted as a theorem of the Euclidean geometry which can be proved directly, but the theory of pentagonal quasigroups gives a better insight into the mutual relations of such theorems.

This paper was inspired by the work of Vidak in [1]. It is also a continuation of the ideas appearing in [2]. All results here follow from the main result, Theorem 2.1, which gives a new characterization of a pentagonal quasigroup $(Q, \cdot)$ in terms of a regular automorphism φ on an Abelian group $(Q, +)$, where $xy = \varphi(x) + y - \varphi(y)$, $\varphi^4 - \varphi^3 + \varphi^2 - \varphi + \varepsilon = 0$ and ε is the identity mapping on Q. We say then that $(Q, +)$ *induces* the pentagonal quasigroup $(Q, \cdot)$.

Notice that $xy = (\varepsilon - \varphi)(y) + x - (\varepsilon - \varphi)(x)$. The characterization of a pentagonal quasigroup $(Q, \cdot)$ given by Vidak in [1] is that $xy = \psi(y) + x - \psi(x)$ for some automorphism ψ on an Abelian group $(Q, +)$, where $\psi^4 - 3\psi^3 + 4\psi^2 - 2\psi + \varepsilon = 0$. Now since, when $\varphi^4 - \varphi^3 + \varphi^2 - \varphi + \varepsilon = 0$, $(\varepsilon - \varphi)^4 - 3(\varepsilon - \varphi)^3 + 4(\varepsilon - \varphi)^2 - 2(\varepsilon - \varphi) + \varepsilon = 0$, we can think of ψ as equal to $\varepsilon - \varphi$.

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In Theorem 3.3, we prove that a pentagonal quasigroup induced by the group $\mathbb{Z}_n$ has the form $xy = (ax + (1-a)y)(\text{mod } n)$, where $(a^4 - a^3 + a^2 - a + 1) = 0(\text{mod } n)$. Vidak's identity gives the second component, namely, $\psi(x) = (1-a)x(\text{mod } n)$.

As a consequence of our characterization, all Abelian groups of order $n < 100$ that induce pentagonal quasigroups are determined. Also, the variety of commutative, idempotent, medial groupoids satisfying the pentagonal identity $(xy \cdot x)y \cdot x = y$ is proved in Corollary 3.10 to be the variety of commutative, pentagonal quasigroups, whose spectrum is $\{11^n : n = 0, 1, 2, \ldots\}$. The form of commutative pentagonal quasigroups is determined in Proposition 3.9 and as a corollary we prove that the only translatable commutative pentagonal quasigroup is $xy = (6x + 6y)(\text{mod } 11)$. In Theorem 4.2, we prove that the translatability of a pentagonal quasigroup induced by the group $\mathbb{Z}_n$ and its automorphism $\varphi(x) = ax$ determines the value of a and all the possible values of n. This characterizes translatable latin squares isotopic to the Cayley table of the cyclic group of order n.

Using results from [3] in the last table we classify the parastrophes of pentagonal quasigroups in terms of well-known types of idempotent translatable quasigroups and indirectly latin squares conjugates with an idempotent translatable latin square of certain types.

2 Existence of pentagonal quasigroups

All considered quasigroups are finite and have form $Q = \{1, 2, \ldots, n\}$ with the natural ordering, which is always possible by renumeration of elements. For simplicity, instead of $(x + y) \equiv z(\text{mod } n)$ we write $[x + y]_n = [z]_n$. Also, in calculations modulo n we identify 0 with n.

According to [1] a quasigroup $(Q, \cdot)$ is called *pentagonal* if it satisfies the following three identities:

$$xx = x, \tag{1}$$

$$xy \cdot zu = xz \cdot yu, \tag{2}$$

$$(xy \cdot x)y \cdot x = y. \tag{3}$$

Let us recall that a mapping φ of a group $(Q, +)$ onto $(Q, +)$ is called *regular* if $\varphi(x) = x$ holds only for $x = 0$.

Below we present a full characterization of pentagonal quasigroups.

Theorem 2.1. *A groupoid $(Q, \cdot)$ is a pentagonal quasigroup if and only if on Q one can define an Abelian group $(Q, +)$ and its regular automorphism φ such that*

$$x \cdot y = \varphi(x) + (\varepsilon - \varphi)(y), \tag{4}$$

$$\varphi^4 - \varphi^3 + \varphi^2 - \varphi + \varepsilon = 0, \tag{5}$$

where ε is the identity automorphism.

Proof. By the Toyoda theorem (see, for example, [4]), any quasigroup $(Q, \cdot)$ satisfying (1) and (2) can be presented in the form (4), where $(Q, +)$ is an Abelian group and φ is its automorphism. Applying this fact to (3) and putting $y = 0$ we obtain (5). From (5) it follows that the automorphism φ is regular.

Conversely, a groupoid $(Q, \cdot)$ defined by (4), where φ is a regular automorphism of an Abelian group $(Q, +)$, is a quasigroup satisfying (1) and (2). Applying (5) to $z = x - y$ and using (4), after simple calculations, we obtain (3). □

This means that pentagonal quasigroups are isotopic to the group inducing them. Thus, pentagonal quasigroups are isotopic if and only if they are induced by isomorphic groups.

Example 2.2. Let $(\mathbb{C}, +)$ be the additive group of complex numbers. Then $\varphi(z) = ze^{\frac{\pi}{5}i}$ is a regular automorphism of $(\mathbb{C}, +)$ satisfying (5). Thus, by Theorem 2.1, the set of complex numbers with multiplication defined by (4) is an infinite pentagonal quasigroup.

As a consequence of the aforementioned theorem we obtain:

Corollary 2.3. *On a pentagonal quasigroup* $(Q, \cdot)$ *one can define an Abelian group* $(Q, +)$ *and its regular automorphism* φ *such that* (4) *holds and*

$$\varphi^5 + \varepsilon = 0 \quad \text{and} \quad \varphi \neq -\varepsilon \quad \text{or} \quad \varphi = -\varepsilon \quad \text{and} \quad \exp(Q, +) = 5,$$

where ε *is the identity automorphism.*

Corollary 2.4. *A finite Abelian group inducing a pentagonal quasigroup is the direct product of cyclic groups of order* 5 *or has a regular automorphism of order* 10.

The converse statement is not true. The automorphism $\varphi(x) = [4x]_{25}$ of the group $\mathbb{Z}_{25}$ is regular and satisfies the aforementioned condition, but $\mathbb{Z}_{25}$ with the multiplication $x \cdot y = [4x + 22y]_{25}$ is not a pentagonal quasigroup.

The following lemma is obvious.

Lemma 2.5. *The direct product of pentagonal quasigroups is also a pentagonal quasigroup.*

Corollary 2.6. *For every* t *there is a pentagonal quasigroup of order* 5^t.

Proof. For $t = 0$ it is trivial quasigroup. For $t = 1$ it is induced by the additive group $\mathbb{Z}_5$ and has the form $x \cdot y = [4x + 2y]_5$. For $t > 1$ it is the direct product of t copies of the last quasigroup. □

Proposition 2.7. *If finite Abelian groups* G_1 *and* G_2 *have relatively prime orders, then any pentagonal quasigroup induced by the group* $G_1 \times G_2$ *is the direct product of pentagonal quasigroups induced by groups* G_1 *and* G_2.

Proof. If G_1 and G_2 have relatively prime orders, then, according to Lemma 2.1 in [5], $\mathrm{Aut}(G_1 \times G_2) \cong \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$. So, each automorphism φ of $G_1 \times G_2$ can be treated as an automorphism of the form $\varphi = (\varphi_1, \varphi_2)$, where φ_1, φ_2 are automorphisms of G_1 and G_2, respectively. Obviously, φ is regular if and only if φ_1 and φ_2 are regular. Moreover, φ satisfies (5) if and only if φ_1 and φ_2 satisfy (5). Thus, a pentagonal quasigroup induced by $G_1 \times G_2$ is the direct product of pentagonal quasigroups induced by G_1 and G_2. □

To determine Abelian groups that induce pentagonal quasigroups, we will need the following theorem proved in [5].

Theorem 2.8. *The Abelian group* $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_m}}$ *has*

$$|\mathrm{Aut}(G)| = \prod_{k=1}^{m}\left(p^{d_k} - p^{k-1}\right)\prod_{j=1}^{m}\left(p^{\alpha_j}\right)^{m-d_j}\prod_{i=1}^{m}\left(p^{\alpha_i - 1}\right)^{m+1-c_i},$$

where $d_k = \max\{l : \alpha_l = \alpha_k\}$ *and* $c_k = \min\{l : \alpha_l = \alpha_k\}$.

3 Construction of pentagonal quasigroups

We start with the characterization of pentagonal quasigroups induced by $\mathbb{Z}_n$.

Theorem 3.1. *A groupoid* $(Q, \cdot)$ *of order* $n > 2$ *is a pentagonal quasigroup induced by the group* $\mathbb{Z}_n$ *if and only if there exist* $1 < a < n$ *such that* $(a, n) = (a - 1, n) = 1$, $x \cdot y = [ax + (1 - a)y]_n$ *and*

$$[a^4 - a^3 + a^2 - a + 1]_n = 0. \tag{6}$$

Proof. Automorphisms of the group $\mathbb{Z}_n$ have the form $\varphi(x) = ax$, where $(a, n) = 1$. Since $\varepsilon - \varphi$ also is an automorphism, $(a - 1, n) = 1$. Moreover, the equation $(a - 1)x = 0 (\text{mod } n)$ has $d = (a - 1, n)$ solutions (cf. [6]). So, $(a - 1, n) = 1$ means that the automorphism $\varphi(x) = ax$ is regular. Theorem 2.1 completes the proof. □

Theorem 3.2. *If a regular automorphism φ of an Abelian group $(Q, +)$ satisfies* (5), *then $(Q, *)$, $(Q, \circ)$ and $(Q, \diamondsuit)$ with the operations*

$$x * y = \varphi^2(y - x) + y, \quad x \circ y = \varphi^3(x - y) + y, \quad x \diamond y = \varphi^4(y - x) + y$$

are pentagonal quasigroups.

Proof. If φ and $(Q, +)$ are as in the assumption, then, by Theorem 2.1, $(Q, \cdot)$ with the operation $x \cdot y = \varphi(x - y) + y$ is a pentagonal quasigroup. From Vidak's results presented in [1] it follows that also $(Q, *)$, $(Q, \circ)$ and $(Q, \diamond)$, where $x * y = y \cdot (yx \cdot x)x$, $x \circ y = (yx \cdot y)x$ and $x \diamond y = (xy \cdot x)y$, are pentagonal quasigroups. Applying (4) and (5) to these operations we obtain our thesis. □

Theorem 3.3. *A pentagonal quasigroup induced by the group $\mathbb{Z}_n$ has one of the following forms:*

$$\begin{aligned} x \cdot y &= [ax + (1 - a)y]_n, \\ x \cdot y &= [-a^2x + (1 + a^2)y]_n, \\ x \cdot y &= [a^3x + (1 - a^3)y]_n, \\ x \cdot y &= [-a^4x + (1 + a^4)y]_n, \end{aligned}$$

where $1 < a < n - 1$ satisfy (6) *and $(a, n) = (a - 1, n) = 1$.*

When $a = n - 1$ there is only one pentagonal quasigroup. It is induced by $\mathbb{Z}_5$ and has the form $x \cdot y = [4x + 2y]_5$.

Proof. Equation (6) has no more than four solutions, so $\mathbb{Z}_n$ induces no more than four pentagonal quasigroups. Theorems 3.1 and 3.2 complete the proof for $1 < a < n - 1$. The case $a = n - 1$ is obvious. □

Note that for $[a + 1]_n \neq 0$ the equation (6) implies $[a^5 + 1]_n = 0$. Since it is also valid for $a = -1$ in a pentagonal quasigroup with $x \cdot y = [ax + (1 - a)y]_n$ we have

$$[a^5]_n = n - 1. \tag{7}$$

Proposition 3.4. *Let $(Q, \cdot)$ be a pentagonal quasigroup induced by the group $\mathbb{Z}_n$, where $n > 5$. If $m|n$, then $(Q, \cdot)$ contains a pentagonal subquasigroup of order m.*

Proof. If $m|n$, then the group $\mathbb{Z}_n$ contains a subgroup isomorphic to $\mathbb{Z}_m$. Let $x \cdot y = [ax + (1 - a)y]_n$. Since $1 < a < n - 1$, $(a, n) = (a - 1, n) = 1$, also $(a, m) = (a - 1, m) = 1$ and $[a^4 - a^3 + a^2 - a + 1]_m = 0$. Let $a' = [a]_m$. Then, as it is not difficult to see, $\mathbb{Z}_m$ with the multiplication $x \cdot y = [a'x + (1 - a')y]_m$ is a pentagonal quasigroup. □

Proposition 3.5. *If an Abelian group G inducing a pentagonal quasigroup has an element of order $k > 1$, then the number of such elements is greater than 3.*

Proof. An automorphism preserves the order of elements of G. So, if only one $x \in G$ has order $k > 1$, then $\varphi(x) = x$, which contradicts to the assumption on φ. If only two elements $x \neq y$ have order $k > 1$, then $\varphi(x) = y$ and $\varphi^2(x) = x$. Using (5) we get $3x = 2y$ and $3y = 2x$. Therefore, $2x = 3y = y + 3x$, which implies that $x = -y$. But $k(2x) = 2(kx) = 0$. Also $2x \neq 0$ or else $x = -x = y$, a contradiction. Thus, $2x$ has order k. Then $2x = x$ or $2x = y$. The first case is impossible. In the second $2x = y = 3y$ implies $2y = 0$, so $y = -y = x$, a contradiction. Therefore, G has at least three elements of order k.

If G has three distinct elements x, y, z of order $k > 1$, then $\varphi(x) = y$, $\varphi^2(x) = \varphi(y) = z$, $\varphi^3(x) = \varphi(z) = x$. Obviously, $\varphi(x) \neq -x$, because $\varphi(x) = -x$ implies $x = \varphi^2(x) = z$, which is impossible. Thus, by Corollary 2.3, $0 = \varphi^5(x) + x = z + x$ and $0 = \varphi(z) + \varphi(x) = x + y$. So, $x + z = x + y$, a contradiction. Hence, G has more than three elements of order $k > 1$. □

Corollary 3.6. *Abelian groups of order n, where*
(*i*) $2|n$ *and* $4\nmid n$ *or*
(*ii*) $3|n$ *and* $9\nmid n$ *or*
(*iii*) $4|n$ *and* $8\nmid n$,
do not induce pentagonal quasigroups.

Proof. In the first case, a group has one element of order 2; in the second – two elements of order 3; in third case – one or three elements of order 2. □

Theorem 3.7. *A finite pentagonal quasigroup has order* $5s$ or $5s + 1$.

Proof. Suppose that a pentagonal quasigroup $(Q, \cdot)$ is induced by the group $(Q, +)$, where $Q = \{0, e_2, e_3, \dots, e_n\}$. Each automorphism ψ of this group can be identified with a permutation φ of the set $\{e_2, e_3, \dots, e_n\}$. Each such permutation is a cycle or can be decomposed into disjoint cycles. Since, by Corollary 2.3, $\varphi^2 = \varepsilon$ or $\varphi^{10} = \varepsilon$, a permutation φ can be decomposed into disjoint cycles of the length 2, 5 or 10. If φ contains a cycle of the length 2, then for some $e_i \in Q$ we have $\varphi(e_i) = e_j \neq e_i$ and $\varphi^2(e_i) = e_i$. If $e_j \neq -e_i$, then by Corollary 2.3, $-e_i = \varphi^5(e_i) = \varphi(e_i)$, a contradiction. Thus, $e_j = -e_i$ and consequently $5e_i = 0$, by (5). So, in this case 5 is a divisor of n. Hence, if φ is decomposed into k cycles of the length 2, then $(Q, +)$ has the order $n = 2k + 1 = 5t$. Since t must be odd, we see that in this case $n = 10s + 5$.

If φ contains a cycle of the length 5, then for some $e_i \in Q$ we have $\varphi^5(e_i) = e_i$ and $\varphi(e_i) \neq -e_i$. This, by Corollary 2.3, implies $2e_i = 0$. Thus, 2 is a divisor of n and $n > 5$. Moreover, each element of this cycle has order 2. Therefore, in the case when φ is decomposed into disjoint cycles of the length 5, the group $(Q, +)$ has $5s + 1$ elements and all non-zero elements have order 2. So, $(Q, +)$ is the direct product of copies of $\mathbb{Z}_2$. Thus, $n = 2^k = 5t + 1$. So, t is odd and, as in the previous case, $n = 10s + 6$. If φ is decomposed into cycles of the length 10, then obviously $n = 10s + 1$.

Now, if φ is decomposed into cycles of the length 2 and 5, then 10 divides n. Thus, $n = 10s$. If φ is decomposed into $p > 0$ cycles of the length 2 and $q > 0$ cycles of the length 10, then $n = 2p + 10q + 1$ and 5 divides n. Hence, $n = 10s + 5$. If φ is decomposed into $p > 0$ cycles of the length 5 and $q > 0$ cycles of the length 10, then $n = 5p + 10q + 1$ and 2 divides n. Hence, $n = 10s + 6$. Finally, if φ is decomposed into $p > 0$ cycles of the length 2, $q > 0$ cycles of the length 5 and r cycles of the length 10, then $n = 2p + 5q + 10r + 1$ and 5 divides n. Hence, $n = 10s + 5$. □

Corollary 3.8. *The smallest pentagonal quasigroup is induced by the group* $\mathbb{Z}_5$ *and has the form* $x \cdot y = [4x + 2y]_5$.

Proof. Indeed, by Theorem 3.7, $\mathbb{Z}_5$ is the smallest group that can be used in the construction of a pentagonal quasigroup. In this group only $a = 4$ satisfies (6). Thus, the multiplication of this quasigroup is defined by $x \cdot y = [4x + 2y]_5$. □

Proposition 3.9. *A groupoid* $(Q, \cdot)$ *is a commutative pentagonal quasigroup if and only if there exists an Abelian group* $(Q, +)$ *of exponent 11 such that* $x \cdot y = 6x + 6y$ *for all* $x, y \in Q$.

Proof. By Theorem 2.1 for a commutative pentagonal quasigroup there exists an Abelian group $(Q, +)$ and its automorphism φ such that $\varphi = \varepsilon - \varphi$. Thus, $\varepsilon = 2\varphi$. This, by (5), gives $\varphi(\varphi^3 - \varphi^2 + \varphi + \varepsilon) = 0$. Therefore, $\varphi(\varphi^2 - \varphi + 3\varepsilon) = 0$, and consequently, $\varphi^2 + 5\varphi = 0$, so $\varphi + 5\varepsilon = 0$. Hence, $11\varphi = 0$. Thus, $11x = 0$ for each $x \in Q$. Moreover, from $11\varphi = 0$ we obtain $\varphi(x) = -10\varphi(x) = -5x = 6x$ and $(\varepsilon - \varphi)(x) = x - 6x = -5x = 6x$. So, $\exp(Q, +) = 11$ and $x \cdot y = 6x + 6y$ for all $x, y \in Q$.

The converse statement is obvious. □

Corollary 3.10. *The variety of commutative, idempotent, medial groupoids satisfying the pentagonal identity is the variety of commutative, pentagonal quasigroups, whose spectrum is* $\{11^n : n = 0, 1, 2, \dots\}$.

Proof. It follows from Proposition 3.9 that the spectrum of the variety of commutative pentagonal quasigroups is $\{11^n : n = 0, 1, 2, \dots\}$. So, we need to only prove that a commutative, idempotent, medial groupoid $(Q, \cdot)$ satisfying the pentagonal identity is a quasigroup. Let $a, b \in Q$. Then the pentagonal identity ensures that the equations $xa = b$ and $ax = b$ have a solution $x = (ab \cdot a)b$. Suppose that $za = b$. Then $z = (az \cdot a)z \cdot a = (ba \cdot z)a = (ab \cdot z) \cdot aa = (ab \cdot a)b = x$ and the solution is unique. □

4 Translatable pentagonal quasigroups

Recall a quasigroup $(Q, \cdot)$, with $Q = \{1, 2, \dots, n\}$ and $1 \leqslant k < n$, is *k-translatable* if its multiplication table is obtained by the following rule: if the first row of the multiplication table is $a_1, a_2, \dots, a_n$, then the q-th row is obtained from the $(q-1)$-st row by taking the last k entries in the $(q-1)$-st row and inserting them as the first k entries of the q-th row and by taking the first $n-k$ entries of the $(q-1)$-st row and inserting them as the last $n-k$ entries of the qth row, where $q \in \{2, 3, \dots, n\}$. The multiplication in a k-translatable quasigroup is given by the formula $i \cdot j = [i+1]_n \cdot [j+k]_n = a_{[k-ki+j]_n}$ (cf. [2,7] or [3]). Moreover, Lemma 9.1 in [7] shows that a quasigroup of the form $x \cdot y = [ax + by]_n$ is k translatable only for k such that $[a + kb]_n = 0$. Thus, a pentagonal quasigroup induced by $\mathbb{Z}_n$ can be k-translatable only for $k \in \{2, 3, \dots, n-2\}$.

Theorem 4.1. *Every pentagonal quasigroup induced by* $\mathbb{Z}_n$ *is* k*-translatable for some* $k > 1$ *such that* $(k, n) = 1$. *If it has the form* $x \cdot y = [ax + (1-a)y]_n$, *then is* k*-translatable for* $k = [1 - a^3 - a]_n$.

Proof. Indeed, by Theorem 3.3, $x \cdot y = [ax + (1-a)y]_n$ and $[a^4 - a^3 + a^2 - a + 1]_n = 0$. Thus, $[a + (-a^3 - a + 1)(1-a)]_n = 0$, which, by Lemma 9.1 from [7], means that this quasigroup is k-translatable. Since $k + nt = 1 - a(a^2+1) = -a^2(a^2+1)$ and $(a, n) = 1$, each prime divisor of k and n is a divisor of a, which is impossible. So, $(k, n) = 1$. □

Theorem 4.2. *A groupoid* $(Q, \cdot)$ *of order* n *is a* k*-translatable pentagonal quasigroup,* $k > 1$, *if and only if it is of the form* $x \cdot y = [ax + (1-a)y]_n$, *where*

$$n | m = k^4 - 2k^3 + 4k^2 - 3k + 1 \quad \text{and} \quad a = [-k^3 + k^2 - 3k + 1]_n. \tag{8}$$

Proof. Suppose that $(Q, \cdot)$ is a k-translatable pentagonal quasigroup of order n. By Theorem 4.2 of [3] and Lemma 9.1 of [7] it is of the form $x \cdot y = [ax + (1-a)y]_n$ and $[a + (1-a)k]_n = 0$, where $1 < a < n$, $(a, n) = (a-1, n) = 1$. Thus,

$$[a + k]_n = [ka]_n \quad \text{and} \quad k = [(k-1)a]_n. \tag{9}$$

By Theorem 4.1, $k = [1 - a^3 - a]_n$. Therefore, $[a^3]_n = [1 - a - k]_n$. So,

$$[ka^3]_n = [k - ka - k^2]_n = [-a - k^2]_n. \tag{10}$$

By (7), we also have $[a^5]_n = [-1]_n$. Thus,

$$[ka^3]_n \overset{(9)}{=} [(k-1)a^4]_n, \quad [ka^4]_n = [1-k]_n, \quad [ka]_n = [(k-1)a^2]_n, \quad [ka^2]_n = [(k-1)a^3]_n.$$

Therefore, using pentagonality and the aforementioned identities, we obtain

$$\begin{aligned}0 &= [(k-1)(a^4 - a^3 + a^2 - a + 1)]_n\\ &= [(k-1)a^4 - (k-1)a^3 + (k-1)a^2 - (k-1)a + (k-1)]_n\\ &= [ka^3 - a^2 - 1]_n \overset{(9)}{=} [-a - k^2 - a^2 - 1]_n.\end{aligned}$$

Hence, $[a^2]_n = [-a - k^2 - 1]_n$, and as a consequence

$$[a+k]_n = [ka]_n = [(k-1)a^2]_n = [(k-1)(-a-k^2-1)]_n = [-k^3 + k^2 - 2k + 1]_n,$$

which implies the second equation of (8).

The first equation follows from the fact that

$$0 = [a + k - ka]_n = [k^4 - 2k^3 - 4k^2 - 3k + 1]_n.$$

Conversely, let $(Q, \cdot)$ be a groupoid of order $n > 1$ with $x \cdot y = [ax + (1-a)y]_n$, where n and a are as in (8). Then $(a, n) = (a - 1, n) = 1$. Indeed, each a prime divisor p of a and n is a divisor of $m - a = k^2(k^2 - k + 3)$. If $p|k$, then, by (8), $p|1$, a contradiction. So, $p|(k^2 - k + 3)$ and $p\nmid n$, but then $p|k(k^2 - k + 3) = 1 - a$. This is also impossible. Hence, $(a, n) = 1$. Similarly, $(a - 1, n) = 1$. Thus, $1 < a < n$ and, as a consequence, $(Q, \cdot)$ is a quasigroup. Since $[a + k(1 - a)]_n = 0$, by Lemma 9.1 from [7], it is k-translatable. This implies (9).

Now, using (9) and (8), we obtain

$$[k^2a]_n = [k(ka)]_n = [k(k+a)]_n = [k^2 + k + a]_n \overset{(8)}{=} [-k^3 + 2k^2 - 2k + 1]_n \tag{11}$$

and

$$\begin{aligned}[k^2a^2]_n &= [(k^2a)a]_n \overset{(11)}{=} [-k^2(k+a) + 2k(k+a) - 2ka + a]_n\\ &= [-k^3 - k^2a + 2k^2 + a]_n \overset{(11)}{=} [-k^3 - k^2 - k - a + 2k^2 + a]_n\\ &= [-k^3 + k^2 - k]_n.\end{aligned}$$

That is,

$$[k^2a^2]_n = [-k^3 + k^2 - k]_n \quad \text{and} \quad [k^3a^2]_n = [-k^4 + k^3 - k^2]_n.$$

Then

$$\begin{aligned}[a^2]_n &\overset{(8)}{=} [-kk^2a + k^2a - 3ka + a]_n\\ &\overset{(10),(9)}{=} [(k^4 - 2k^3 + 2k^2 - k) + (-k^3 + 2k^2 - 2k + 1) + (-3k - 3a + a)]_n\\ &\overset{(8)}{=} [-k^3 - 3k - 2a]_n \overset{(8)}{=} [k^3 - 2k^2 + 3k - 2]_n.\end{aligned}$$

Consequently,

$$[ka^2]_n = [k^4 - 2k^3 + 3k^2 - 2k]_n \overset{(8)}{=} [-k^2 + k - 1]_n.$$

Now, using the aforementioned identities, we obtain

$$[a^3]_n = [-k^3a^2 + k^2a^2 - 3ka^2 + a^2]_n = [k^3 - k^2 + 2k]_n \overset{(8)}{=} [1 - a - k]_n.$$

Therefore, $[a^4]_n = [a - a^2 - ak]_n$ and

$$[a^4 - a^3 + a^2 - a + 1]_n = [1 - ak - a^3]_n = [a + k(1-a)]_n = 0,$$

which, by Theorem 3.3, shows that $(Q, \cdot)$ is a pentagonal quasigroup. □

Corollary 4.3. *For every $k > 1$ there exists at least one k-translatable pentagonal quasigroup.*

Proof. One k-translatable pentagonal quasigroup is defined by Theorem 4.2. In this quasigroup a and n are as in (8). If m is a divisor of n and $a = [b]_n$, then $a = [b]_m$. Thus, $(\mathbb{Z}_m, \cdot)$ with $x \cdot y = [a'x + (1 - a')y]_m$, $a' = [-k^3 + k^2 - 3k + 1]_m$, also is a k-translatable pentagonal quasigroup. □

According to Theorem 4.2 for $k = 2$, we have $n = m = 11$ and $a = 2$. So for $k = 2$ there is only one k-translatable pentagonal quasigroup induced by $\mathbb{Z}_n$. It has the form $x \cdot y = [2x + 10y]_{11}$. For $k = 3$, $m = 55$, $a = [29]_n$ and $n|m$, there are three k-translatable pentagonal quasigroups induced by $\mathbb{Z}_n$. They have the form: $x \cdot y = [29x + 27y]_{55}$, $x \cdot y = [7x + 5y]_{11}$ and $x \cdot y = [4x + 2y]_5$. Other calculations for $k \leqslant 20$ are presented as follows:

k	$x \cdot y$
2	$[2x + 10y]_{11}$
3	$[4x + 2y]_5$, $[7x + 5y]_{11}$, $[29x + 27y]_{55}$
4	$[122x + 60y]_{181}$
5	$[347x + 115]_{461}$
6	$[794x + 198y]_{991}$
7	$[27x + 5y]_{31}$, $[52x + 10y]_{61}$, $[1577x + 315]_{1891}$
8	$[190x + 472y]_{661}$, $[2834x + 472y]_{3305}$
9	$[8x + 4y]_{11}$, $[308x + 184y]_{491}$, $[4727x + 675y]_{5401}$
10	$[6x + 6y]_{11}$, $[593x + 169y]_{761}$, $[7442x + 930y]_{8371}$
11	$[29x + 3y]_{31}$, $[362x + 40]_{401}$, $[11189x + 1243y]_{12431}$
12	$[14x + 58y]_{71}$, $[138x + 114y]_{251}$, $[16202x + 1620y]_{17821}$
13	$[25x + 17y]_{41}$, $[24x + 32y]_{55}$, $[112x + 10y]_{121}$, $[189x + 17y]_{205}$,
	$[354x + 252y]_{605}$, $[189x + 2067y]_{2255}$, $[2895x + 2067y]_{4961}$, $[22739x + 2076]_{24805}$
14	$[472x + 2590y]_{3061}$, $[31082x + 2590y]_{33671}$
15	$[4x + 38y]_{41}$, $[79x + 1013y]_{1091}$, $[41537x + 3195y]_{44731}$
16	$[54434x + 3888y]_{58321}$
17	$[42x + 90y]_{131}$, $[465x + 107y]_{571}$, $[70127x + 4675y]_{74801}$
18	$[13350x + 5562y]_{18911}$, $[88994x + 5562y]_{94555}$
19	$[111437x + 6555y]_{117991}$
20	$[17x + 85y]_{101}$, $[70x + 62y]_{131}$, $[118x + 994y]_{1111}$, $[987x + 455y]_{1441}$,
	$[5572x + 7660y]_{13231}$, $[137882x + 7660y]_{145541}$

Let $\mathbb{Z}_{11}^*$ be the pentagonal quasigroup with the multiplication $x \cdot y = [6x + 6y]_{11}$. By the aforementioned result, a finite commutative pentagonal quasigroup is the direct product of m copies of $\mathbb{Z}_{11}^*$ but for $m > 1$, as it is shown below, they are not translatable.

Theorem 4.4. *$\mathbb{Z}_{11}^*$ is the only translatable commutative pentagonal quasigroup.*

Proof.

Let $(Q, \cdot)$ be a commutative pentagonal quasigroup. By definition, an infinite quasigroup cannot be translatable. So, $(Q, \cdot)$ must be finite. By Proposition 3.9 its order is $n = 11^m$.

If $m = 1$, then, by Proposition 3.9, the multiplication of $(Q, \cdot)$ has the form $x \cdot y = [6x + 6y]_{11}$. From the multiplication table of this quasigroup, it follows that it is k-translatable for $k = 10 = n - 1$. So, for $m = 1$, our theorem is valid.

Now let $m > 1$ and $(Q, \cdot)$ be $(n - 1)$-translatable. According to Lemma 2.7 in [2], we can assume that Q is ordered in the following way: $x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(n)}$, where $x^{(1)} = (1, 0, 0, \dots, 0)$. Then the multiplication table of $(Q, \cdot)$ has the following form:

$\cdot$	$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$\cdots\cdots$	$x^{(n)}$
$x^{(1)}$	$x^{(1)}\cdot x^{(1)}$	$x^{(1)}\cdot x^{(2)}$	$x^{(1)}\cdot x^{(3)}$	$\cdots\cdots$	$x^{(1)}\cdot x^{(n)}$
$x^{(2)}$	$x^{(2)}\cdot x^{(1)}$	$x^{(2)}\cdot x^{(2)}$	$x^{(2)}\cdot x^{(3)}$	$\cdots\cdots$	$x^{(2)}\cdot x^{(n)}$
$x^{(3)}$	$x^{(3)}\cdot x^{(1)}$	$x^{(3)}\cdot x^{(2)}$	$x^{(3)}\cdot x^{(3)}$	$\cdots\cdots$	$x^{(3)}\cdot x^{(n)}$
$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$
$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$

Since $(Q,\cdot)$ is $(n-1)$-translatable, $x^{(2)}\cdot x^{(t)} = x^{(1)}\cdot x^{(t+1)}$ for all $t \in \mathbb{Z}_n$.

Let $x^{(2)} = (a_1, a_2, \dots, a_m)$. We will prove by induction that

$$x^{(t+1)} = (ta_1 - (t-1), ta_2, ta_3, \dots, ta_m) \quad \forall t \in \mathbb{Z}_n.$$

The induction hypothesis is clearly true, by definition, for $t = 1$. Assume that the induction hypothesis is true for all $s \leqslant t$. Then

$$x^{(t)} = ((t-1)a_1 - (t-2), (t-1)a_2, (t-1)a_3, \dots, (t-1)a_m).$$

Suppose that $x^{(t+1)} = (z_1, z_2, z_3, \dots, z_m)$. Since $x^{(2)}\cdot x^{(t)} = x^{(1)}\cdot x^{(t+1)}$, we have $6x^{(2)} + 6x^{(t)} = 6x^{(1)} + 6x^{(t+1)}$. The last expression means that

$$(6a_1 + 6(t-1)a_1 - 6(t-2), 6ta_2, 6ta_3, \dots, 6ta_m) = (6 + 6z_1^{(t)}, 6z_2^{(t)}, 6z_3^{(t)}, \dots, 6z_m^{(t)}).$$

Hence, $6z_1^{(t)} = 6(ta_1 - (t-1))$, which implies $z_1^{(t)} = ta_1 - (t-1)$. Also $z_s^{(t)} = ta_s$ for all $s = 2, 3, \dots, m$. So, $x^{(t+1)} = (ta_1 - (t-1), ta_2, ta_3, \dots, ta_m)$, as required.

Now, $x^{(12)} = (11a_1 - 10, 11a_2, 11a_3, \dots, 11a_m) = (-10, 0, 0, \dots, 0) = x^{(1)}$, a contradiction because all $x^{(1)}$, $x^{(2)}, x^{(3)}, \dots, x^{(n)}$ are different. So, for $m > 1$ a quasigroup $(Q,\cdot)$ cannot be $(n-1)$-translatable. □

Suppose that $(G,\cdot)$ is a commutative pentagonal quasigroup and a, b are two distinct elements of G. Then it is straighforward to prove that a and b generate the subquasigroup

$$\langle a, b\rangle = \{a, b, ab, aba, bab, aba\cdot a, aba\cdot b, bab\cdot a, bab\cdot b, (aba\cdot a)b, (bab\cdot b)a\}$$

and that $\langle a, b\rangle$ is isomorphic to $\mathbb{Z}_{11}^*$. Then we take $c \notin \langle a, b\rangle$, if c exists.

Lemma 4.5. $\langle a, b\rangle \cap \langle b, c\rangle = \{b\}$.

Proof. From the multiplication table of $\mathbb{Z}_{11}^*$ we see that any two distinct elements generate $\mathbb{Z}_{11}^*$. Hence, $\langle a, b\rangle \cap \langle b, c\rangle$ cannot contain b and another element of $\langle a, b\rangle \cap \langle b, c\rangle$, or else $c \in \langle a, b\rangle = \langle b, c\rangle$, a contradiction. □

Theorem 4.6. *$H = \langle a, b\rangle\langle b, c\rangle$ is a commutative pentagonal subquasigroup of $(G,\cdot)$ isomorphic to $\mathbb{Z}_{11}^* \times \mathbb{Z}_{11}^*$.*

Proof. Since $(G,\cdot)$ is medial, $(\langle a, b\rangle\langle b, c\rangle)(\langle a, b\rangle\langle b, c\rangle) \subseteq \langle a, b\rangle\langle b, c\rangle$. Note that $\langle a, b\rangle \subseteq \langle a, b\rangle b \subseteq \langle a, b\rangle\langle b, c\rangle$ and $\langle b, c\rangle \subseteq b\langle b, c\rangle \subseteq \langle a, b\rangle\langle b, c\rangle$. Hence, the commutative pentagonal quasigroup $H = \langle a, b\rangle\langle b, c\rangle \supseteq \langle a, b\rangle \cup \{c\}$ has more than 11 elements and less than or equal to 121 elements. Therefore, as we have already seen, H has 121 elements and is isomorphic to $\mathbb{Z}_{11}^* \times \mathbb{Z}_{11}^*$. This completes the proof. □

Corollary 4.7. *$\mathbb{Z}_{11}^* \times \mathbb{Z}_{11}^*$ is generated by three distinct elements.*

Corollary 4.8. *If $xy = zw \in \langle a, b\rangle\langle b, c\rangle$, then $x = z$ and $y = w$.*

Corollary 4.9. *If $d \notin \langle a, b\rangle\langle b, c\rangle$, then $(\langle a, b\rangle\langle b, c\rangle)\langle b, d\rangle$ is a commutative pentagonal quasigroup of order 11^3 and is isomorphic to $\mathbb{Z}_{11}^* \times \mathbb{Z}_{11}^* \times \mathbb{Z}_{11}^*$.*

Corollary 4.10. *$\mathbb{Z}_{11}^{*} \times \mathbb{Z}_{11}^{*} \times \mathbb{Z}_{11}^{*}$ is generated by four distinct elements.*

Corollary 4.11. *The direct product of n copies of $\mathbb{Z}_{11}^{*}$ is generated by $n + 1$ distinct elements.*

5 Groups inducing pentagonal quasigroups

Pentagonal quasigroups are very large. Using Theorem 4.2 we can determine all pentagonal quasigroups induced by $\mathbb{Z}_n$. Below we present several such quasigroups. For $a = 3$ there is only one such quasigroup. It is induced by the group $\mathbb{Z}_{61}$. Its multiplication is defined by $x \cdot y = [3x - 2y]_{61} = [3x + 59y]_{61}$. This quasigroup is 32-translatable. For $a = 4$ there are three such quasigroups. They are induced by $\mathbb{Z}_5$, $\mathbb{Z}_{41}$, $\mathbb{Z}_{205}$ and are 3-, 15-, 138-translatable, respectively.

a	2	3	4	5	6	7	8	9
n	11	61	5, 41, 205	521	11, 101, 1111	11, 191, 2101	11, 331, 3641	1181, 5905
k	2	32	3, 15, 138	392	10, 82, 890	3, 33, 1752	9, 143, 3122	444, 5168

a	10	11	12	13	14	15	16
n	9091	13421	19141	2411	71, 101, 355, 505, 7171, 35855	31, 1531, 47461	61681
k	8082	12080	17402	202	83, 71, 83, 273, 4414, 33098	21, 1204, 44072	57570

a	17	18	19	20
n	71, 101, 781, 1111, 7171, 78881	9041, 99451	55, 2251, 11255, 24761, 123805	152381
k	41, 19, 538, 625, 2242, 73952	3192, 93602	53, 2127, 4378, 17884, 116928	144362

a	21	22	23	24
n	185641	224071	31, 41, 211, 1271, 6541, 8651, 268181	55, 5791, 28955, 63701, 318505
k	176360	213402	25, 28, 48, 521, 893, 5113, 255992	13, 2267, 15108, 49854, 304658

The aforementioned table shows that from groups $\mathbb{Z}_n$ for $n < 24$ only groups $\mathbb{Z}_5$ and $\mathbb{Z}_{11}$ determine pentagonal quasigroups. To determine other groups of order $n < 100$ inducing pentagonal quasigroups observe that from Corollary 2.3 and Theorem 3.7 it follows that an Abelian group inducing a pentagonal quasigroup is the direct product of several copies of the group $\mathbb{Z}_5$ or has a regular automorphism $\varphi \neq \varepsilon$ of order 10. Observe that from Proposition 2.7, Corollary 3.6, Theorem 3.7 and the above table the possible values of $n < 100$ are $n \in \{5, 11, 16, 25, 31, 40, 41, 45, 55, 56, 61, 71, 80, 81\}$.

For $n = 5$ we have one pentagonal quasigroup, and for $n = 11$ there are four such quasigroups (see the aforementioned table). For $n = 16$ we have five Abelian groups of order 16: $\mathbb{Z}_{16}$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_2^4$. From the aforementioned table it follows that the group $\mathbb{Z}_{16}$ does not induce any pentagonal quasigroup. Groups $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ do not have automorphisms of order 10 (Theorem 2.8), so they cannot be considered as a group inducing pentagonal quasigroups. The group $\mathbb{Z}_2^4$ can be treated as a vector space V over $\mathbb{Z}_2$. Then, by Corollary 2.3, automorphisms φ interesting for us are linear endomorphisms of V for which $\lambda = -1$ is an eigenvalue of φ^5. From these endomorphisms we select those satisfying (5). There is 1,344 such endomorphisms, so the group $\mathbb{Z}_2^4$ induces 1,344 pentagonal quasigroups.

The group $\mathbb{Z}_{25}$ has four elements of order 5, namely, 5, 10, 15 and 20. Thus, $\varphi(5) \in \{10, 15, 20\}$. Therefore, φ restricted to the set $\{5, 10, 15, 20\}$ has the form $\varphi(x) = ax$, where $a \in \{2, 3, 4\}$, but such φ does not satisfy (5). Hence, $\mathbb{Z}_{25}$ does not induce a pentagonal quasigroup. The group $\mathbb{Z}_5 \times \mathbb{Z}_5$ induces 24 pentagonal quasigroups. These quasigroups are induced by matrices

$$A \in \left\{ \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \right\}$$

and $-A^2, A^3, -A^4$.

Pentagonal quasigroups of order 31 are induced by the group $\mathbb{Z}_{31}$. They are determined by an automorphism $\varphi(x) = ax$, where $a \in \{15, 23, 27, 29\}$ (see table below).

From Abelian groups of order 40 the groups $\mathbb{Z}_{40}$, $\mathbb{Z}_2 \times \mathbb{Z}_{20}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{10}$ have one or three elements of order 2, so they cannot induce pentagonal quasigroups. In the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ only elements $(0, 0, 2)$, $(0, 0, 4)$, $(0, 0, 6)$, $(0, 0, 8)$ have order 5. Thus, $\varphi(0, 0, 2) \in \{(0, 0, 4), (0, 0, 6), (0, 0, 8)\}$. But then $\varphi^5(0, 0, 2) + (0, 0, 2) \neq (0, 0, 0)$, a contradiction. Therefore, there are no pentagonal quasigroups of order 40.

Pentagonal quasigroups of order 41 can be calculated by solution of the equation (6) or (7). The solutions are $a = 4, 23, 25, 31$. So there are four such quasigroups.

For $n = 45$ there are two Abelian groups: $\mathbb{Z}_{45}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{15}$. The first group has two elements of order 3, so by Proposition 3.5 it cannot induce pentagonal quasigroups. The second group has four elements of order 5. The smallest is $(0, 3)$. Thus, $\varphi(0, 3) = (0, 3a)$ for $a = 2, 3, 4$. But then $\varphi^5(0, 3) + (0, 3) \neq (0, 0)$. Thus, pentagonal quasigroups of order 45 do not exist.

For $n = 55$ there exists only one Abelian group: $\mathbb{Z}_{55}$. Its automorphisms have form $\varphi(x) = ax$, where $(a, 55) = 1$. The automorphisms inducing pentagonal quasigroups should satisfy (7). It is easily to see, that for $k = 0, 1, 2, \ldots, 9$ the last digit of k^5 is k. So, for $a = mk$ the last digit of a^5 is also k. Since $a^5 + 1$ must be divided by 5, $a = m4$ or $a = m9$. The aforementioned table shows that the smallest possible value of a is 19. Because 44 is divided by 11, $44^5 + 1$ cannot be divided by 11. Thus, 44 should be omitted. Also, $54 = (-1)(\mathrm{mod}\ 55)$ should be omitted. By direct calculation we can see that from other $a < 54$ acceptable are 24, 29 and 39. Hence, there are four pentagonal quasigroups of order 55. They are isomorphic to the direct product of pentagonal quasigroups induced by $\mathbb{Z}_5$ and $\mathbb{Z}_{11}$.

From Abelian groups of order 56 groups $\mathbb{Z}_{56}$, $\mathbb{Z}_2 \times \mathbb{Z}_{28}$, $\mathbb{Z}_4 \times \mathbb{Z}_{14}$ have one or three elements of order 2. Thus, they cannot induce pentagonal quasigroups. The group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}$ has six elements of order 7. In the same manner as in the case of groups of order 45, we can prove that this group cannot induce pentagonal quasigroups.

Pentagonal quasigroups of prime orders 61 and 71 can be calculated in the same way as for $n = 41$. Results are presented in the table below.

An Abelian group G of order 80 can be decomposed into the direct product of two groups H and $\mathbb{Z}_5$, where H is a group of order 16. From groups of order 16 only $\mathbb{Z}_2^4$ induces pentagonal quasigroups. So, by Proposition 2.7, from groups of order 81 only $\mathbb{Z}_2^4 \times \mathbb{Z}_5$ induces pentagonal quasigroups. We have 1,344 such quasigroups.

The group $\mathbb{Z}_{81}$ has only two elements of order 2, so, by Proposition 3.5, this group cannot be inducing group for a pentagonal quasigroup. Theorem 2.8 shows that from other Abelian groups of order 81 only the group $\mathbb{Z}_3^4$ can have an automorphism of order 10. Using a computer software we calculate 303,264 of such automorphisms satisfying (5). So, $\mathbb{Z}_3^4$ induces 303,264 pentagonal quasigroups.

In this way, we have proved the following:

Theorem 5.1. *The groups of order $n < 100$ that induce pentagonal quasigroups are* $\mathbb{Z}_5$, $\mathbb{Z}_{11}$, $\mathbb{Z}_2^4$, $\mathbb{Z}_5^2$, $\mathbb{Z}_{31}$, $\mathbb{Z}_{41}$, $\mathbb{Z}_{55}$, $\mathbb{Z}_{61}$, $\mathbb{Z}_{71}$, $\mathbb{Z}_2^4 \times \mathbb{Z}_5$ *and* $\mathbb{Z}_3^4$.

For $n < 100$ pentagonal quasigroups induced by $\mathbb{Z}_n$ are as follows:

$n = 5$	$4x + 2y$			
$n = 11$	$2x + 10y$	$6x + 6y$	$7x + 5y$	$8x + 4y$
$n = 31$	$15x + 17y$	$23x + 9y$	$27x + 5y$	$29x + 3y$
$n = 41$	$4x + 38y$	$23x + 19y$	$25x + 17y$	$31x + 11y$
$n = 55$	$19x + 37y$	$24x + 32y$	$29x + 27y$	$39x + 17y$
$n = 61$	$3x + 59y$	$27x + 35y$	$41x + 21y$	$52x + 10y$
$n = 71$	$14x + 58y$	$17x + 55y$	$46x + 26y$	$66x + 6y$

6 Parastrophes of pentagonal quasigroups

Each quasigroup $(Q, \cdot)$ determines five new quasigroups $Q_i = (Q, \circ_i)$ with the operations $\circ_i$ defined as follows:

$$\begin{aligned} x \circ_1 y &= z \leftrightarrow x \cdot z = y, \\ x \circ_2 y &= z \leftrightarrow z \cdot y = x, \\ x \circ_3 y &= z \leftrightarrow z \cdot x = y, \\ x \circ_4 y &= z \leftrightarrow y \cdot z = x, \\ x \circ_5 y &= z \leftrightarrow y \cdot x = z. \end{aligned}$$

Such defined (not necessarily distinct) quasigroups are called *parastrophes* or *conjugates* of Q.

Parastrophes of each quasigroup can be divided into separate classes containing isotopic parastrophes. The number of such classes is always 1, 2, 3 or 6 (cf. [8]). In some cases (described in [9]), parastrophes of a given quasigroup Q are pairwise equal. Parastrophes do not save properties of the initial quasigroup. Parastrophes of an idempotent quasigroup are idempotent quasigroups, but parastrophes of a pentagonal quasigroup are not pentagonal quasigroups, in general.

Let $(Q, \cdot)$ be a pentagonal quasigroup induced by the group $\mathbb{Z}_n$. Then $x \cdot y = [ax + (1 - a)y]_n$ and $[a^4 - a^3 + a^2 - a + 1]_n = 0$. Such quasigroup is k-translatable for $k = [1 - a^3 - a]_n$. Since $[a(1 - a + a^2 - a^3)]_n = 1 = [(1 - a)(a^3 + a)]_n$, from Theorems 5.1 and 5.3 in [3] we obtain the following characterization of parastrophes of pentagonal quasigroups.

Proposition 6.1. *If $(Q, \cdot)$ is a pentagonal quasigroup with multiplication $x \cdot y = [ax + (1 - a)y]_n$, then its parastrophe*

$$\begin{aligned} &x \circ_1 y = [(1 - a^3 - a)x + (a^3 + a)y]_n && \text{is } k\text{-translatable for } k = a, \\ &x \circ_2 y = [-a^4 x + (a^4 + 1)y]_n && \text{is } k\text{-translatable for } k = [a^3 + a]_n, \\ &x \circ_3 y = [(a^4 + 1)x + (-a^4)y]_n && \text{is } k\text{-translatable for } k = [1 - a]_n, \\ &x \circ_4 y = [(a^3 + a)x + (1 - a - a^3)y]_n && \text{is } k\text{-translatable for } k = [-a^4]_n, \\ &x \circ_5 y = [(1 - a)x + (a)y]_n && \text{is } k\text{-translatable for } k = [a^4 + 1]_n. \end{aligned}$$

Using Proposition 6.1 we can show for which values of a and n parastrophes of a pentagonal quasigroup with the multiplication $x \cdot y = [ax + (1 - a)y]_n$ are pentagonal, quadratical ($xy \cdot x = zx \cdot yz$), hexagonal ($x \cdot yx = y$), GS-quasigroups ($x(xy \cdot z) \cdot z = y$), ARO-quasigroups ($xy \cdot y = yx \cdot x$), Stein quasigroups ($x \cdot xy = yx$), right modular ($xy \cdot z = zy \cdot x$) and C3 quasigroups ($(xy \cdot y)y = x$).

We start with the lemma that is a consequence of our results proved in [3].

Lemma 6.2. *Let $(Q, \cdot)$ be a quasigroup of the form $x \cdot y = [ax + (1 - a)y]_n$. Then*
$[2a^2 - 2a + 1]_n = 0$ *if it is quadratical* (Theorem 4.8 *in* [10]),
$[a^2 - a + 1]_n = 0$ *if it is hexagonal,*
$[a^2 - a - 1]_n = 0$ *if it is a GS-quasigroup,*

$[2a^2]_n = 1$ *if it is an ARO-quasigroup,*
$[a^2 - 3a + 1]_n = 0$ *if it is a Stein quasigroup,*
$[a^2 + a - 1]_n = 0$ *if it is right modular,*
$[a^3]_n = 1$ *if it is a C3 quasigroup.*

Using the aforementioned characterization and the fact that a quasigroup of the form $x \cdot y = [ax + (1 - a)y]_n$ is k-translatable if and only if $[a + (1 - a)k]_n = 0$ (cf. [2,7] or [3]) we obtain:

Lemma 6.3. *A quasigroup of the form* $x \cdot y = [ax + (1 - a)y]_n$ *is*
$[1 - 2a]_n$*-translatable if and only if it is quadratical,*
$[1 - a]_n$*-translatable if and only if it is hexagonal,*
$[a + 1]_n$*-translatable if and only if it is a GS-quasigroup,*
$[-2a - 1]_n$*-translatable if and only if if it is an ARO-quasigroup,*
$[a - 1]_n$*-translatable if and only if it is a Stein quasigroup,*
$[-1 - a]_n$*-translatable if and only if it is right modular.*
A C3 quasigroup is k-translatable for k such that $[(1 - a^2)k]_n = 1$.

Using these two lemmas we can determine properties of parastrophes of pentagonal quasigroups induced by $\mathbb{Z}_n$. We start with Q_1.

- Suppose that Q_1 is pentagonal. Then $a = [1 - (1 - a^3 - a)^3 - (1 - a^3 - a)]_n$, from translatability, and $[(1 - a^3 - a)^2]_n = [-a^2 - a - 1]_n$, from (6). Then we have $[(1 - a^3 - a)^3]_n = [(-a^2 - a - 1)(1 - a^3 - a)]_n = [a^4 + 2a^3 - 2]_n \overset{(6)}{=} [3a^3 - a^2 + a - 3]_n$. Therefore, $a = [1 - (1 - a^3 - a)^3 - (1 - a^3 - a)]_n = [-2a^3 + a^2 + 3]_n$, whence, multiplying by a^2, we obtain $[a^4 - a^3]_n = [-3a^2 - 2]_n$. This, by (6), shows that $[2a^2 + a + 1]_n = 0$. Multiplying this equation by a^3 and applying (6) we get $[a^4 + a^3]_n = 2$. Adding this equation to $[a^4 - a^3]_n = [-3a^2 - 2]_n$ we obtain $[2a^4]_n = [-3a^2]_n$. Thus, $[2a^2]_n = [-3]_n$ and consequently, $[a^4 - a^3]_n = [-a^2 - 2a^2 - 2]_n = [-a^2 + 1]_n$. Hence, $[a^4 - a^3 + a^2]_n = 1$, which by (6) implies $a = 2$ and $n = 11$.
- Suppose that Q_1 is quadratical. Then, $a = [1 - 2(1 - a^3 - a)]_n$ by Lemmas 6.2 and 6.3. Hence, $[2a^3]_n = [1 - a]_n$. Also $0 = [2(1 - a^3 - a)^2 - 2(1 - a^3 - a) + 1]_n = [2a^4 - 2a - 1]_n$. So, $[2a + 1]_n = [2a^3 a]_n = [(1 - a)a]_n = [a - a^2]_n$. Consequently, $[a^2]_n = [-a - 1]_n$ and $0 = [2a^4 - 2a^3 + 2a^2 - 2a + 2]_n = [(2a + 1) - (1 - a) + 2(-a - 1) - 2a + 2]_n = [-a]_n$, a contradiction. So, Q_1 cannot be quadratical.
- Q_1 is never hexagonal. Indeed, Q_1 is a-translatable and $[a^3 + a]_n$-translatable as a hexagonal quasigroup. Hence, $a = [a^3 + a]_n$, which implies $[a^3]_n = 0$. Thus, $0 = [a^5]_n = [-1]_n$, a contradiction.
- If Q_1 is a GS-quasigroup, then $a = [(1 - a^3 - a) + 1]_n$. Hence, $[a^3]_n = [2 - 2a]_n$, $[a^4]_n = [2a - 2a^2]_n$, $[-1]_n = [a^5]_n = [2a^2 - 2a^3]_n = [2a^2 + 4a - 4]_n$, $[2a^2]_n = [3 - 4a]_n$, $[a^4]_n = [6a - 3]_n$. Then $0 = [2a^4 - 2a^3 + 2a^2 - 2a + 2]_n = [10a - 5]_n$. So, $[5a]_n = [10a^2]_n = [15 - 20a]_n$, i.e., $[25a]_n = [15]_n$. Thus, $[5a]_n = 5$ and $5 = [10a]_n = [5a + 5a]_n = [10]_n$. Therefore, $n = 5$ and $x \cdot y = [4x + 2y]_5$.
- If Q_1 is an ARO-quasigroup, then $[2(1 - a^3 - a)^2]_n = 1$, so $[2a^2]_n = [-2a - 3]_n$. Also $a = [-2(1 - a^3 - a) - 1]_n$. Thus, $[2a^3]_n = [3 - a]_n$, $[2a^4]_n = [3a - a^2]_n$ and $0 = [2a^4 - 2a^3 + 2a^2 - 2a + 2]_n = [-a^2 - 4]_n$, i.e., $[a^2]_n = [-4]_n$. Hence, $[-8]_n = [2a^2]_n = [-2a - 3]_n$ which gives $[2a]_n = 5$. So, $[-16]_n = [4a^2]_n = [25]_n$. Therefore, $n = 41$, $a = 23$.
- If Q_1 is a Stein quasigroup, then $a = [(1 - a^3 - a) -]_n$. So, $[a^3]_n = [-2a]_n$, $[a^3]_n = [-2a]_n$, $[a^4]_n = [-2a^2]_n$, $[-1]_n = [a^5]_n = [-2a^3]_n$, $a = [2a^4]_n$, $[a^2]_n = [-2]_n$, $a = [2a^4]_n = [8]_n$, $[a^3]_n = [-4a^4]_n = [-16]_n$. Thus, by (6), we obtain $[11]_n = 0$. Hence, $n = 11$ and $a = 8$.
- If Q_1 is right modular, then $a = [-1 - (1 - a^3 - a)]_n$. Hence, $[a^3]_n = 2$, $[a^4]_n = [2a]_n$, $[-1]_n = [a^5]_n = [2a^2]_n$, $a = [-2a^3]_n = [-4]_n$. This by (6) implies $n = 11$, $a = 7$.
- If Q_1 is a $C3$ quasigroup, then $1 = [(1 - (1 - a^3 - a)^2)a]_n$. Hence, $[a^3 + a^2 + 2a - 1]_n = 0$, which, by (6), gives $[a^4 + 2a^2 + a]_n = 0$. So, $[-1 + 2a^3 + a^2]_n = 0$. Comparing this equation with $[a^3 + a^2 + 2a - 1]_n = 0$ we obtain $[a^3]_n = [2a]_n$. So, $[-1]_n = [2a^3]_n = [4a]_n$ and $1 = [2a^3 + a^2]_n = [-1 + a^2]_n$. Thus, $[a^2]_n = 2$ and $[-a]_n = [4a^2]_n = 8$. Therefore, $2 = [a^2]_n = [64]_n$. Consequently, $n = 31$, $a = 23$.

In other cases the proof is very similar, so we omit it.

The result of calculations is presented in the table below. In this table, the intersection of the ARO-row with the Q_3-column means that for a pentagonal quasigroup Q its parastrophe Q_3 is an ARO-quasigroup only in the case when $x \cdot y = [14x + 58y]_{71}$.

	Q	Q_1	Q_2	Q_3	Q_4	Q_5
pentaq.		$[2x + 10y]_{11}$	always	never	$[6x + 6y]_{11}$	$[6x + 6y]_{11}$
quadrat.	$[4x + 2y]_5$	never	$[4x + 2y]_5$	$[4x + 2y]_5$	never	$[4x + 2y]_5$
hexag.	never	never	never	never	never	never
GS	$[8x + 4y]_{11}$	$[4x + 2y]_5$	$[7x + 5y]_{11}$	$[7x + 5y]_{11}$	$[4x + 2y]_5$	$[8x + 4y]_{11}$
ARO	$[27x + 5y]_{31}$	$[23x + 19y]_{41}$	$[23x + 9y]_{31}$	$[14x + 58y]_{71}$	$[25x + 17y]_{41}$	$[66x + 6y]_{71}$
Stein	$[4x + 2y]_5$	$[8x + 4y]_{11}$	never	$[8x + 4y]_{11}$	$[7x + 5y]_{11}$	$[7x + 5y]_{11}$
r. mod.	$[7x + 5y]_{11}$	$[7x + 5y]_{11}$	never	$[4x + 2y]_5$	$[8x + 4y]_{11}$	$[4x + 2y]_5$
C3	never	$[23x + 9y]_{31}$	never	$[23x + 9y]_{31}$	$[27x + 5y]_{31}$	$[27x + 5y]_{31}$

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