

Research Article

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Global structure of sign-changing solutions for discrete Dirichlet problems

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Abstract: Let $T > 1$ be an integer, $\mathbb{T} := [1, T]_{\mathbb{Z}} = \{1, 2, \dots, T\}$, $\hat{\mathbb{T}} := \{0, 1, \dots, T + 1\}$. In this article, we are concerned with the global structure of the set of sign-changing solutions of the discrete second-order boundary value problem

$$\begin{cases} \Delta^2 u(x-1) + \lambda h(x)f(u(x)) = 0, & x \in \mathbb{T}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$, $sf(s) > 0$ for all $s \neq 0$ and $h: \hat{\mathbb{T}} \rightarrow [0, +\infty)$. By using the directions of a bifurcation, we obtain existence and multiplicity of sign-changing solutions of the above problem for λ lying in various intervals in \mathbb{R} . Moreover, we point out that these solutions change their sign exactly $k - 1$ times, where $k \in \{1, 2, \dots, T\}$.

Keywords: sign-changing solutions, bifurcation, Sturm-type comparison theorem, boundary value problem, difference equation

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1 Introduction

In this article, we are concerned with the discrete second-order boundary value problem

$$\begin{cases} \Delta^2 u(x-1) + \lambda h(x)f(u(x)) = 0, & x \in \mathbb{T}, \\ u(0) = u(T+1) = 0, \end{cases} \quad (1.1)$$

where $\mathbb{T} := [1, T]_{\mathbb{Z}} = \{1, 2, \dots, T\}$, $T > 1$ is an integer, $\hat{\mathbb{T}} := \{0, 1, \dots, T + 1\}$, $\lambda > 0$ is a parameter and h, f satisfy the following assumptions.

(A1) $h: \hat{\mathbb{T}} \rightarrow [0, +\infty)$ with $0 < h_* \leq h(t) \leq h^*$ on $\hat{\mathbb{T}}$ for some $h_*, h^* \in (0, \infty)$.

In assumption (A2) and throughout the article, we use the following well-known conceptions of a generalized zero and a simple generalized zero at $t \in \mathbb{T}$.

Definition 1.1. Suppose that a function $y: \hat{\mathbb{T}} \rightarrow \mathbb{R}$. If $y(t_0) = 0$, then t_0 is a zero of y . If $y(t_0) = 0$ or $y(t_0)y(t_0 + 1) < 0$ for some $t_0 \in \{1, \dots, T - 1\}$, then y has a generalized zero at $t_0 \in \mathbb{T}$.

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Definition 1.2. A function $y: \hat{\mathbb{T}} \rightarrow \mathbb{R}$ is said to have a simple generalized zero at $t_0 \in \mathbb{T}$ provided that one of the following conditions is satisfied:

- (i) $y(t_0) = 0$, $y(t_0 - 1)y(t_0 + 1) < 0$;
- (ii) $y(t_0)y(t_0 + 1) < 0$.

It is well-known that the eigenvalue problem

$$\begin{cases} \Delta^2 u(x-1) + \mu h(x)u(x) = 0, & x \in \mathbb{T}, \\ u(0) = u(T+1) = 0 \end{cases} \quad (1.2)$$

has a finite sequence of simple eigenvalues

$$0 < \mu_1 < \mu_2 < \dots < \mu_T,$$

and the eigenfunction ϕ_k corresponding to μ_k has exactly $k - 1$ simple generalized zeros in \mathbb{T} (see [1], and for other spectral results of related problems we refer to [2]). Let

$$0 = t_0 < t_1 < \dots < t_{k-1} < t_k = T + 1$$

be the generalized zeros of ϕ_k .

In the special case $h(x) \equiv 1$, the m th eigenvalue and eigenfunction corresponding to it are characterized by

$$\eta_m = 2 - 2 \cos \frac{m\pi}{T+1}, \quad \psi_m(t) = \sin\left(\frac{m}{T+1}\pi t\right),$$

respectively. Let

$$0 = \tau_0 < \tau_1 < \dots < \tau_{m-1} < \tau_m = T + 1$$

be the generalized zeros of $\sin\left(\frac{m}{T+1}\pi t\right)$ in $\hat{\mathbb{T}}$. Then,

$$\tau_{j+1} - \tau_j \in \left\{ t \in \mathbb{Z} \left| \frac{T+2-m}{m} \leq t \leq \frac{T+m}{m} \right. \right\}, \quad j \in \{0, 1, \dots, m-1\}.$$

We assume that

(A2) there exist two constants $m_1, m_2 \in \{1, 2, \dots, T\}$, such that

$$\eta_{m_1} \leq \mu_k h(t) \max_{s \in (0, \infty)} \frac{f(s)}{s} \leq \eta_{m_2}, \quad t \in \hat{\mathbb{T}} \quad (1.3)$$

for some $k \in \mathbb{N}$.

From (1.3) and the Sturm-type comparison theorem, see Aharonov, Bohner and Elias [3], there exists

$$c_* = \min \left\{ t \in \mathbb{Z} \left| \frac{T+2-m_2}{m_2} \leq t \leq \frac{T+m_2}{m_2} \right. \right\}, \text{ such that}$$

$$t_{j+1} - t_j \geq c_*, \quad j \in \{0, 1, \dots, k-1\}.$$

Let χ_1 be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} \Delta^2 w(x-1) + \chi w(x) = 0, & x \in [1, \hat{c}-1]_{\mathbb{Z}}, \\ w(0) = w(\hat{c}) = 0, \end{cases} \quad (1.4)$$

where $\hat{c} \in \mathbb{T}$ satisfies $\frac{c_*}{2} \leq \hat{c} \leq \frac{c_*+1}{2}$, and let w_1 be an eigenfunction corresponding to χ_1 .

Furthermore, we assume that

(A3) there exist $\alpha > 0$, $f_0 > 0$ and $f_1 > 0$ such that

$$\lim_{|s| \rightarrow 0} \frac{f(s) - f_0 s}{s|s|^\alpha} = -f_1;$$

(A4) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $sf(s) > 0$ for $s \neq 0$ and $f_\infty := \lim_{|s| \rightarrow \infty} \frac{f(s)}{s} = 0$;

(A5) there exist $s_0 > 0$ and $0 < \sigma < 1$ such that

$$\min_{|s| \in \left[s_0, \frac{1}{\sigma}s_0\right]} \frac{f(s)}{s} \geq \frac{f_0}{\mu_k h_*} \chi_1.$$

In order to state our main result, we first recall some standard notations to describe the properties of sign-changing solutions. Let $Y = \{u|u: \mathbb{T} \rightarrow \mathbb{R}\}$ with the norm $\|u\|_Y = \max_{t \in \mathbb{T}} |u(t)|$. Let $X = \{u: \hat{\mathbb{T}} \rightarrow \mathbb{R} | u(0) = u(T+1) = 0\}$ with the norm $\|u\| = \max_{t \in \hat{\mathbb{T}}} |u(t)|$. For $k \geq 1$, $v \in \{+, -\}$, let S_k^v denote the set of functions in X such that

- (i) u has exactly $k - 1$ simple generalized zeros in \mathbb{T} ;
- (ii) $vu(1) > 0$.

Define $S_k = S_k^- \cup S_k^+$. They are disjoint and open in X . Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$. A S_k^v solution of (1.1) is a function $u \in S_k^v$ satisfying (1.1).

Our main result is the following.

Theorem 1.3. *For each fixed integer $k \in \{1, 2, \dots, T\}$ and each fixed $v \in \{+, -\}$, let (A1)–(A5) hold. Then, there exist $\lambda_* \in (0, \mu_k/f_0)$ and $\lambda^* > \mu_k/f_0$, such that*

- (i) (1.1) has at least one S_k^v solution if $\lambda = \lambda_*$;
- (ii) (1.1) has at least two S_k^v solutions if $\lambda_* < \lambda \leq \mu_k/f_0$;
- (iii) (1.1) has at least three S_k^v solutions if $\mu_k/f_0 < \lambda < \lambda^*$;
- (iv) (1.1) has at least two S_k^v solutions if $\lambda = \lambda^*$;
- (v) (1.1) has at least one S_k^v solution if $\lambda > \lambda^*$.

Existence and multiplicity of positive solutions for discrete boundary value problems have been extensively studied by several authors. We refer to Agarwal et al. [4,5], Cheng and Yen [6], Rachunkova and Tisdell [7], Ma [8], Rodriguez [9] and references therein. However, the methods in [4–9] are analytic techniques and various fixed point theorems, which cannot be used to prove the existence of S_k^v solutions.

By using the global bifurcation theory [10,11], Davidson and Rynne [12] and Luo and Ma [13] studied second-order Dirichlet boundary value problems on measure chain. It is well-known that measure chain $\hat{\mathbb{T}}$ denotes an arbitrary closed subset of real numbers \mathbb{R} , and we can regard the equations in [12,13] as difference equations when $\hat{\mathbb{T}} = \mathbb{Z}$. They showed that unbounded continua of nontrivial solutions prescribed nodal properties emanated from the trivial branch at the eigenvalues of the linearization problems. However, they only showed that for each fixed integer $k \geq 1$ and $v \in \{+, -\}$, there exists at least one nodal solution.

Remark 1.4. Existence of \hat{c} in (1.4) can be guaranteed by (A2). For example, let $T = 100$, $m_2 = 5$. By using *Mathematica 9.0*, the distance between two consecutive generalized zeros of $\psi_{m_2}(t)$ was found to be 20, which means $c_* = 20$. Thus, it is easy to check that $\hat{c} = 10$ by the definition of \hat{c} .

The rest of the article is organized as follows. In Section 2, we show the existence of bifurcation from some eigenvalues for the corresponding problem according to the standard argument and the rightward direction of bifurcation. In Section 3, the change in direction of bifurcation is given. The final section is devoted to show an *a priori* bound of solutions for (1.1) and complete the proof of Theorem 1.3.

2 Rightward bifurcation

We study the global behavior from the trivial branch with the rightward direction of bifurcation under suitable assumptions on h and f .

Lemma 2.1. *Suppose that (λ, u) is a nontrivial solution of (1.1). Then, there exists $k_0 \in \{1, 2, \dots, T\}$, such that*

$$u \in S_{k_0}^\vee.$$

Proof. Suppose on the contrary that for every $k \in \{1, 2, \dots, T\}$,

$$u \notin S_k^\vee.$$

Then, there exists $t_0 \in \mathbb{T}$ such that

$$u(t_0) = 0, \quad u(t_0 - 1)u(t_0 + 1) \geq 0. \quad (2.1)$$

Since $u \not\equiv 0$ on $\hat{\mathbb{T}}$, we may assume that

$$|u(t_0 - 1)| + |u(t_0 + 1)| > 0. \quad (2.2)$$

On the other hand, it follows from equation (1.1) and $f(0) = 0$ that

$$-\Delta^2 u(t_0 - 1) = \lambda h(t_0) f(u(t_0)) = 0,$$

which implies that

$$u(t_0 + 1) - 2u(t_0) + u(t_0 - 1) = 0.$$

However, by (2.2) and the fact $u(t_0) = 0$, we get

$$u(t_0 + 1)u(t_0 - 1) < 0, \quad (2.3)$$

which contradicts (2.1). \square

Next following the similar arguments in Gao and Ma [14], with obvious changes, we may get the following.

Lemma 2.2. *Assume that (A1), (A3) and (A4) hold. Let (A2) be satisfied for some $k \in \mathbb{N}$. Then for each $v \in \{+, -\}$, there exists an unbounded subcontinuum C_k^v which is branching from $(\mu_k/f_0, 0)$ for (1.1). Moreover, if $(\lambda, u) \in C_k^v$, then u is a S_k^v solution of (1.1).*

Lemma 2.3. *Assume that (A1), (A3) and (A4) hold. Let (A2) be satisfied for some $k \in \mathbb{N}$. Let u be a S_k^v solution of (1.1). Then, there exists a constant $C > 0$ independent of u such that*

$$|\Delta u(x)| \leq \lambda C \|u\|, \quad x \in [0, T]_{\mathbb{Z}}. \quad (2.4)$$

Proof. According to discrete Rolle's theorem (see [1]), there exists $x_0 \in \mathbb{T}$, such that $\Delta u(x_0) = 0$ or $\Delta u(x_0 - 1)\Delta u(x_0) < 0$. Then by a direct computation, it is easy to see that

$$\Delta u(x) = \Delta u(x_0) + \lambda \sum_{s=x+1}^{x_0} h(s)f(u(s)), \quad x \in \mathbb{T}. \quad (2.5)$$

Noting that (A3) and (A4) imply that

$$|f(s)| \leq f^*|s|, \quad s \in \mathbb{R} \quad (2.6)$$

for some $f^* > 0$, it follows from (2.5) that

(i) if $\Delta u(x_0) = 0$, then

$$|\Delta u(x)| = \lambda \sum_{s=x+1}^{x_0} |h(s)f(u(s))| \leq \lambda f^* \|u\| \sum_{s=x+1}^{x_0} h(s) \leq \lambda f^* \sum_{s=0}^{T+1} h(s) \|u\|;$$

(ii) if $\Delta u(x_0 - 1)\Delta u(x_0) < 0$, then

$$|\Delta u(x)| = \left| \Delta u(x_0) + \lambda \sum_{s=x+1}^{x_0} h(s)f(u(s)) \right| \leq \left(2 + \lambda f^* \sum_{s=0}^{T+1} h(s) \right) \|u\|. \quad \square$$

Lemma 2.4. Assume that (A1), (A3) and (A4) hold. Let (A2) be satisfied for some $k \in \mathbb{N}$. Let $\{(\lambda_n, u_n)\}$ be a sequence of S_k^v solutions to (1.1), which satisfies $\|u_n\| \rightarrow 0$ and $\lambda_n \rightarrow \mu_k/f_0$. Let $\phi_k(x)$ be the k th eigenfunction of (1.2), which satisfies $\|\phi_k\| = 1$. Then, there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/\|u_n\|$ converges uniformly to ϕ_k on \mathbb{T} .

Proof. Set $v_n := u_n/\|u_n\|$. Then, it is easy to see that $\|v_n\| = 1$. It follows from Lemma 2.3 that $\|\Delta v_n\|$ is bounded, so there is a subsequence of v_n uniformly convergent to a limit v . Furthermore, there exists a subsequence of it such that $\Delta v_n(0)$ converges to some constant c . We again denote by v_n the subsequence. We note that $v \in Y$, $v(0) = v(T+1) = 0$ and $\|v\| = 1$. Rewriting equation (1.1) with $(\lambda, u) = (\lambda_n, u_n)$, we obtain

$$\Delta u_n(x) = \Delta u_n(0) - \lambda_n \sum_{t=1}^x h(t)f(u_n(t)). \quad (2.7)$$

Dividing both sides of (2.7) by $\|u_n\|$, we get

$$\Delta v_n(x) = \Delta v_n(0) - \lambda_n \sum_{t=1}^x h(t) \frac{f(u_n(t))}{u_n(t)} v_n(t) := w_n(x). \quad (2.8)$$

Noting that $u_n(x) \rightarrow 0$ for all $x \in \hat{\mathbb{T}}$, so for each fixed $t \in \hat{\mathbb{T}}$, one has $\frac{f(u_n(t))}{u_n(t)} \rightarrow f_0$. It follows from (2.6) that $w_n(x)$ converges to

$$w(x) := c - \mu_k \sum_{t=1}^x h(t)v(t) \quad (2.9)$$

for each fixed $x \in \hat{\mathbb{T}}$. Therefore, by recalling (2.8), one has

$$v_n(x) = \sum_{s=1}^{x-1} w_n(s).$$

The fact coupled with (2.9) yields that $v_n(x)$ converges to

$$v(x) = \sum_{s=1}^{x-1} \left(c - \mu_k \sum_{t=1}^s h(t)v(t) \right),$$

which implies that v is a nontrivial solution of (1.2) with $\lambda = \mu_k$, and hence $v \equiv \phi_k$. \square

Lemma 2.5. Assume that (A1), (A3) and (A4) hold. Let (A2) be satisfied for some $k \in \mathbb{N}$. Let C_k^v be as in Lemma 2.2. Then there exists $\delta > 0$ such that for $(\lambda, u) \in C_k^v$ and $|\lambda - \mu_k/f_0| + \|u\| \leq \delta$, one has $\lambda > \mu_k/f_0$.

Proof. Suppose on the contrary that there exists a sequence $\{(\lambda_n, u_n)\}$ such that $(\lambda_n, u_n) \in C_k^v$, which satisfies $\lambda_n \rightarrow \mu_k/f_0$, $\|u_n\| \rightarrow 0$ and $\lambda_n \leq \mu_k/f_0$. According to Lemma 2.4, there exists a subsequence of $\{u_n\}$, for convenience denote it by $\{u_n\}$, such that $\frac{u_n}{\|u_n\|}$ converges uniformly to ϕ_k on $\hat{\mathbb{T}}$, where $\phi_k(x) \in S_k^v$ is the k th eigenfunction of (1.2) with $\|\phi_k\| = 1$. Multiplying the equation of (1.1) with $(\lambda, u) = (\lambda_n, u_n)$ by u_n and by a direct computation, one has

$$\lambda_n \sum_{x=0}^T h(x) f(u_n(x)) u_n(x) = \sum_{x=0}^T |\Delta u_n(x)|^2,$$

and accordingly

$$\lambda_n \sum_{x=0}^T h(x) \frac{f(u_n(x))}{\|u_n\|} \frac{u_n(x)}{\|u_n\|} = \sum_{x=0}^T \frac{|\Delta u_n(x)|^2}{\|u_n\|^2}. \quad (2.10)$$

It follows from Lemma 2.4 that after taking a subsequence and relabeling if necessary, $\frac{u_n}{\|u_n\|}$ converges to ϕ_k in Y .

$$\sum_{x=0}^T |\Delta \phi_k(x)|^2 = \mu_k \sum_{x=0}^T h(x) |\phi_k(x)|^2,$$

then coupled with (2.10), one has

$$\lambda_n \sum_{x=0}^T h(x) f(u_n(x)) u_n(x) = \mu_k \sum_{x=0}^T h(x) |u_n(x)|^2 - \zeta(n) \|u_n\|^2$$

with a function $\zeta: \mathbb{N} \rightarrow \mathbb{R}$ satisfying

$$\lim_{n \rightarrow \infty} \zeta(n) = 0.$$

That is,

$$\sum_{x=0}^{T+1} h(x) \frac{f(u_n(x)) - f_0 u_n(x)}{|u_n(x)|^\alpha u_n(x)} \left| \frac{u_n(x)}{\|u_n\|} \right|^{2+\alpha} = \frac{1}{\lambda_n \|u_n\|^\alpha} \left[(\mu_k - f_0 \lambda_n) \sum_{x=0}^{T+1} h(x) \left| \frac{u_n(x)}{\|u_n\|} \right|^2 - \zeta(n) \right].$$

From condition (A3), we have

$$\sum_{x=0}^{T+1} h(x) \frac{f(u_n(x)) - f_0 u_n(x)}{|u_n(x)|^\alpha u_n(x)} \left| \frac{u_n(x)}{\|u_n\|} \right|^{2+\alpha} \rightarrow -f_1 \sum_{x=0}^{T+1} h(x) |\phi_k(x)|^{2+\alpha} < 0,$$

and

$$\sum_{x=0}^{T+1} h(x) \left| \frac{u(x)}{\|u_n\|} \right|^2 \rightarrow \sum_{x=0}^{T+1} h(x) |\phi_k(x)|^2 > 0.$$

This contradicts $\lambda_n \leq \mu_k / f_0$. □

3 Direction turn of bifurcation

In this section, we show that the connected components grow to the left at some point under (A5) condition.

Lemma 3.1. Assume that (A2) holds for some $k \in \mathbb{N}$. Let u be a S_k^v solution of (1.1). Then there exist two generalized zeros α_u and β_u , such that

$$\beta_u - \alpha_u \geq c_*,$$

$$|u| > 0, \quad x \in [\alpha_u + 1, \beta_u - 1]_{\mathbb{Z}},$$

$$\|u\| = |u(t_0)| \quad \text{for some } t_0 \in [\alpha_u, \beta_u]_{\mathbb{Z}}.$$

Moreover,

$$\sigma \|u\| \leq |u(x)| \leq \|u\|, \quad x \in \left\{ t \in \mathbb{Z} : \frac{3\alpha_u + \beta_u}{4} \leq t \leq \frac{\alpha_u + 3\beta_u}{4} \right\} := J_u.$$

where $\sigma = \min \left\{ \frac{\min J_u}{\beta_u - \alpha_u}, \frac{\beta_u - \max J_u}{\beta_u - \alpha_u} \right\}$.

Proof. It is an immediate consequence of the fact that u is concave down in $\hat{\mathbb{T}}$. \square

Lemma 3.2. Assume that (A1) and (A5) hold. Let σ be as in Lemma 3.1 and u be a S_k^v solution of (1.1) with $\|u\| = \frac{1}{\sigma} S_0$. Then $\lambda < \mu_k/f_0$.

Proof. Let u be a S_k^v solution of (1.1). It follows from Lemma 3.1 that

$$\sigma \|u\| \leq |u(x)| \leq \|u\|, \quad x \in J_u.$$

We note that u is a solution of

$$\Delta^2 u(x) + \lambda h(x) \frac{f(u(x))}{u(x)} u(x) = 0, \quad x \in J_u.$$

Suppose on the contrary that $\lambda \geq \mu_k/f_0$. Then for $x \in J_u$, we have from (A5) that

$$\lambda h(x) \frac{f(u(x))}{u(x)} \geq \frac{\mu_k}{f_0} h_* \frac{f_0}{\mu_k h_*} \chi_1 = \chi_1.$$

Choose $b > 0$, such that $[b, b + \hat{c}]_{\mathbb{Z}} \subset J_u$. Set

$$y(x) = w_1(x - b), \quad x \in [b, b + \hat{c}]_{\mathbb{Z}},$$

then

$$\begin{cases} \Delta^2 y(x - 1) + \chi_1 y(x) = 0, & x \in [a, b]_{\mathbb{Z}}, \\ y(b) = y(b + \hat{c}) = 0. \end{cases}$$

Combine this with a generalized version of the Sturm-type comparison theorem [3]. We deduce that u has at least one generalized zero on J_u . This contradicts the fact that $u(x) > 0$ on J_u . \square

4 Second turn and proof of Theorem 1.3

The main ingredient of this section is an *a priori* estimate and finally we shall give a proof of Theorem 1.3.

Lemma 4.1. Assume that (A3) and (A4) hold. Let (λ, u) be a S_k^v solution of (1.1). Then, there exists $\lambda_* > 0$ such that $\lambda \geq \lambda_*$.

Proof. From Lemma 2.3, there exists a constant $C > 0$, which is independent of u such that (2.4) holds. Let $\|u\| = u(x_0)$, then it follows from (2.4) that

$$\|u\| = |u(x_0)| = \sum_{s=0}^{x_0-1} \Delta u(s) \leq \sum_{s=0}^{x_0-1} \lambda C \|u\| \leq \lambda C T \|u\|, \quad (4.1)$$

that is, $\lambda \geq (CT)^{-1}$. \square

Lemma 4.2. Assume that (A1) and (A4) hold. Let (A2) be satisfied for some $k \in \mathbb{N}$. Let u be an S_k^v solution of (1.1). Then, there exists a constant $C > 0$ independent of u such that for $\underline{f}(s) = \min_{0 \leq t \leq s} f(t)/t$, one has $\lambda \underline{f}(\|u\|) \leq C$.

Proof. Let

$$0 = t_0 < t_1 < \cdots < t_k = T + 1$$

be the generalized zeros of u . Then, by (A2) we may get

$$t_{j+1} - t_j \geq c_*, \quad j \in \{0, 1, \dots, k-1\}.$$

We may assume that

$$\|u\| = u(x_0), \quad x_0 \in [t_l, t_{l+1}]_{\mathbb{Z}}, \quad (4.2)$$

or

$$\|u\| = -u(x_0), \quad x_0 \in [t_l, t_{l+1}]_{\mathbb{Z}}. \quad (4.3)$$

We only deal with case (4.2) since case (4.3) can be treated by the similar way.

Let

$$J_l = \left\{ t \in \mathbb{Z} : \frac{3t_l + t_{l+1}}{4} \leq t \leq \frac{t_l + 3t_{l+1}}{4} \right\}.$$

Let $\hat{t} \in \mathbb{T}$ satisfy $\frac{t_{l+1} - t_l}{2} \leq \hat{t} \leq \frac{t_{l+1} - t_l + 1}{2}$ and

$$J_1 := [\min J_l, \hat{t}], \quad J_2 := [\hat{t}, \max J_l].$$

If $u(t_l) = u(t_{l+1}) = 0$, then it follows from (4.2) that

$$\begin{aligned} \|u\| &\geq |u(\hat{t})| = \lambda \sum_{s=t_l}^{t_{l+1}} [G(\hat{t}, s)h(s)f(u(s))] \\ &= \lambda \|u\| \sum_{s=t_l}^{t_{l+1}} \left[G(\hat{t}, s)h(s) \frac{f(u(s))}{\|u\|} \right] \\ &\geq \lambda \|u\| \sum_{s \in J_l} \left[G(\hat{t}, s)h(s) \min_{\sigma\|u\| \leq r \leq \|u\|} \frac{f(r)u(s)}{r} \frac{1}{\|u\|} \right] \\ &\geq \sigma \lambda \|u\| \sum_{s \in J_l} [G(\hat{t}, s)h(s)\underline{f}(\|u\|)] \\ &\geq \sigma \lambda \|u\| \underline{f}(\|u\|) h_*(t_{l+1} - t_l) \min_{s \in J_l} \{G(\hat{t}, s) : s \in J_l\}, \end{aligned}$$

where

$$G(\hat{t}, s) = \frac{1}{T+1} \begin{cases} (T+1-\hat{t})s, & 0 \leq s \leq \hat{t} \leq T+1, \\ (T+1-s)\hat{t}, & 0 \leq \hat{t} \leq s \leq T+1. \end{cases}$$

If $u(t_l) = A \neq 0$, $u(t_{l+1}) = B \neq 0$, then it follows from (4.2) again that

$$\begin{aligned} \|u\| &\geq |u(\hat{t})| = \left| \tilde{u}(\hat{t}) + \lambda \sum_{s=t_l}^{t_{l+1}} [G(\hat{t}, s)h(s)f(u(s))] \right| \\ &\geq \lambda \left| \sum_{s=t_l}^{t_{l+1}} [G(\hat{t}, s)h(s)f(u(s))] \right| - |\tilde{u}(\hat{t})| \\ &\geq \lambda \left| \sum_{s=t_l}^{t_{l+1}} [G(\hat{t}, s)h(s)f(u(s))] \right| - \|u\|, \end{aligned}$$

where $\tilde{u}(\hat{t})$ is the solution of $\Delta^2 \tilde{u}(\hat{t} - 1) = 0$ with boundary value condition $\tilde{u}(t_l) = A$, $\tilde{u}(t_{l+1}) = B$. Then by the similar argument as above, we complete the proof. \square

Lemma 4.3. Assume that (A1) and (A3) hold, (A2) be satisfied for some $k \in \mathbb{N}$. Let J be a compact interval in $(0, \infty)$. Then, there exists a constant $M_J > 0$ such that for all $\lambda \in J$ and given $v \in \{+, -\}$, one has all possible S_k^v solutions u of (1.1) satisfy $\|u\| \leq M_J$.

Proof. Suppose on the contrary that there exists a sequence $\{u_n\}$ of S_k^ν solutions of (1.1) with $\{\lambda_n\} \subset J \triangleq [a, b]$ and $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Then by the same argument used in the proof of Lemma 4.2, with obvious changes, we may get

$$\|u_n\| = u_n(x_0), \quad x_0 \in [t_m, t_{m+1}]_{\mathbb{Z}}.$$

Let

$$\rho \in \left(0, \frac{1}{bQ}\right),$$

where $Q = \sum_{s=1}^T G(t, s)h(s)$. Then by (A3), there exists $u_\rho > 0$ such that $u > u_\rho$ implies $f(u) < \rho u$.

Let $m_\rho \triangleq \max_{u \in [0, u_\rho]} f(u)$ and let $A_n \triangleq \{t \in [t_m, t_{m+1}]_{\mathbb{Z}} : u_n(t) \leq u_\rho\}$ and $B_n \triangleq \{t \in [t_m, t_{m+1}]_{\mathbb{Z}} : u_n(t) > u_\rho\}$.

Then we have,

$$\begin{aligned} u_n(\delta_n) &= \tilde{u}(\delta_n) + \sum_{s=t_m+1}^{t_{m+1}} \lambda_n G(\delta_n, s)h(s)f(u_n(s)) \\ &= \tilde{u}(\delta_n) + \sum_{s \in A_n} \lambda_n G(\delta_n, s)h(s)f(u_n(s)) + \sum_{s \in B_n} \lambda_n G(\delta_n, s)h(s)f(u_n(s)) \end{aligned}$$

for $0 \leq s \leq \delta_n$. Thus,

$$\frac{1}{\lambda_n} \leq \frac{\tilde{u}(\delta_n)}{\lambda_n \|u_n\|} + \frac{m_\rho Q}{\|u_n\|} + \sum_{s \in B_n} \frac{G(\delta_n, s)h(s)f(u_n(s))}{\|u_n\|}.$$

On B_n , $u_n(s) > u_\rho$ implies $\frac{f(u_n(s))}{\|u_n\|} \leq \frac{f(u_n(s))}{u_n(s)} \leq \rho$. Accordingly,

$$\frac{1}{\lambda_n} \leq \frac{\tilde{u}(\delta_n)}{\lambda_n \|u_n\|} + \frac{m_\rho Q}{\|u_n\|} + \rho Q.$$

Since $0 < a < \lambda_n \leq b$ for all n , we have $\frac{1}{b} \leq \frac{1}{\lambda_n} \leq \frac{1}{a}$ for all n and

$$\frac{1}{b} < \frac{\tilde{u}(\delta_n)}{a \|u_n\|} + \frac{m_\rho Q}{\|u_n\|} + \rho Q.$$

According to the fact $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$, we get

$$\frac{1}{b} < \rho Q < \frac{1}{b}.$$

This contradiction completes the proof. \square

Lemma 4.4. Assume that (A1), (A3) and (A4) hold. Let (A2) hold for some $k \in \mathbb{N}$. Then there exists $\{(\lambda_n, u_n)\}$ such that $(\lambda_n, u_n) \in C_k^\nu$, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ and $\|u_n\| \rightarrow \infty$.

Proof. We only deal with the case $\nu = +$.

It follows from Lemma 2.2 that C_k^ν is unbounded, so there exists a sequence $\{(\lambda_n, u_n)\}$ of solutions of (1.1) such that $\{(\lambda_n, u_n)\} \subset \mathbb{R}^+ \times S_k^+$ and $|\lambda_n| + \|u_n\| \rightarrow \infty$. Moreover, Lemma 4.1 implies that $\lambda_n > 0$. Suppose on the contrary that there exists a bounded subsequence $\{(\lambda_{n_k}, u_{n_k})\}$. Then from Lemma 4.3, $\|u_{n_k}\|$ is bounded, which contradicts the fact that $|\lambda_n| + \|u_n\| \rightarrow \infty$. So, $\lambda_n \rightarrow \infty$. Then, Lemma 4.2 implies that $f(\|u_n\|) \rightarrow 0$. It deduces from (A4) that $\|u_n\| \rightarrow \infty$. \square

Proof of Theorem 1.3. Let C_k^ν be as in Lemma 2.2. In order to simplify the proof, we only deal with the case C_k^+ .

It follows from Lemma 2.5 that C_k^+ is branching from $(\mu_k/f_0, 0)$ and goes rightward. Let $\{(\lambda_n, u_n)\}$ be as in Lemma 4.4. Then, there exists $(\lambda_0, u_0) \in C_k^+$ such that $\|u_0\| = \frac{1}{\sigma} s_0$. By Lemma 3.2, one has $\lambda_0 < \mu_k/f_0$.

Coupled with Lemmas 2.5, 3.2 and 4.3, C_k^+ passes through some points $(\mu_k/f_0, v_1)$ and $(\mu_k/f_0, v_2)$ with

$$\|v_1\| < \frac{1}{\sigma}s_0 < \|v_2\|.$$

By Lemmas 2.5, 3.2 and 4.3 again, there exist $\underline{\lambda}$ and $\bar{\lambda}$, which satisfy $0 < \underline{\lambda} < \mu_k/f_0 < \bar{\lambda}$ and

(i) if $\lambda \in (\mu_k/f_0, \bar{\lambda}]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in C_k^+$ and

$$\|u\| < \|v\| < \frac{1}{\sigma}s_0;$$

(ii) if $\lambda \in [\underline{\lambda}, \mu_k/f_0]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in C_k^+$ and

$$\|u\| < \frac{1}{\sigma}s_0 < \|v\|.$$

Define

$$\lambda^* = \sup\{\bar{\lambda}: \bar{\lambda} \text{ satisfies (i)}\}, \lambda_* = \inf\{\underline{\lambda}: \underline{\lambda} \text{ satisfies (ii)}\}.$$

Then by the standard argument, (1.1) has a S_k^+ solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively. Lemma 3.2 implies that there exists w such that for each $\lambda > \mu_k/f_0$, one has $(\lambda, w) \in C_k^v$ and $\|w\| > \frac{1}{\sigma}s_0$. This completes the proof. \square

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