DE GRUYTER Open Mathematics 2020; 18: 671–714

Review Article

Yongge Tian*

Miscellaneous equalities for idempotent matrices with applications

https://doi.org/10.1515/math-2020-0147
received August 8, 2019; accepted March 20, 2020

Abstract: This article brings together miscellaneous formulas and facts on matrix expressions that are composed by idempotent matrices in one place with cogent introduction and references for further study. The author will present the basic mathematical ideas and methodologies of the matrix analytic theory in a readable, up-to-date, and comprehensive manner, including constructions of various algebraic matrix identities composed by the conventional operations of idempotent matrices, and uses of the block matrix method in the derivation of closed-form formulas for calculating the ranks of matrix expressions that are composed by idempotent matrices. The author also determines the maximum and minimum ranks of some matrix pencils composed by the products of matrices and their generalized inverses and uses the ranks to characterize algebraic performance of the matrix pencils.

Keywords: idempotent matrix, block matrix, generalized inverse, equality, inequality, range, rank

MSC 2010: 15A03, 15A09, 15A27, 47A05

1 Introduction

Throughout this article, let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ complex matrices; A^T, A^*, $r(A)$, $\mathscr{R}(A)$, and $\mathscr{N}(A)$ denote the transpose, the conjugate transpose, the rank, the range (column space), and the kernel (null space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denote the identity matrix of order m; and $[A, B]$ denote a row block matrix consisting of A and B. The author next introduces the definition and notation of generalized inverses of matrix. The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the four Penrose equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA. \tag{1.1}$$

A matrix X is called a $\{i, \ldots, j\}$-generalized inverse of A, denoted by $A^{(i,\ldots,j)}$, if it satisfies the above ith, …, jth equations. The collection of all $\{i, \ldots, j\}$-generalized inverses of A is denoted by $\{A^{(i,\ldots,j)}\}$. In particular, the generalized inverses $A^\dagger$, $A^{(1,3,4)}$, $A^{(1,2,4)}$, $A^{(1,2,3)}$, $A^{(1,4)}$, $A^{(1,3)}$, $A^{(1,2)}$, and $A^{(1)}$ of A are called the eight commonly used generalized inverses of A, which are widely used in dealing with singular matrices in matrix and applications. Furthermore, let $P_A = AA^\dagger$, $E_A = I_m - AA^\dagger$, and $F_A = I_n - A^\dagger A$ stand for the three orthogonal projectors induced by A. In particular, a matrix X is called a $\{1\}$-inverse of A, denoted by A^-, if it satisfies $AXA = A$; the collection of all A^- is denoted by $\{A^-\}$. The Drazin inverse of a square matrix M, denoted by $X = M^D$, is defined to be the unique solution X to the three matrix equations $M^tXM = M^t$, $XMX = X$, and $MX = XM$, where t is the index of M, i.e., the smallest nonnegative integer t, such that $r(M^t) = r(M^{t+1})$. When $t = 1$, X is also called the group inverse of M and is denoted by $M^\#$; see [6,10,25] for more issues on generalized inverses of matrices.

*** Corresponding author: Yongge Tian,** College of Business and Economics, Shanghai Business School, Shanghai, China, e-mail: yongge.tian@gmail.com

Recall that a square matrix A is said to be idempotent if $A^2 = A$. Examples of idempotent matrices of the order m are the identity matrix I_m, the null matrix 0, and the matrix $(1/m)_{m\times m}$ among others. An idempotent matrix is often called an oblique projector whose null space is oblique to its range, in contrast to an orthogonal projector, whose null space is orthogonal to its range. Idempotent elements can be defined accordingly in various general algebraic structures, which are also fundamental objects and important tools in the investigation of the algebraic structures. As is known to all, idempotent matrices have strikingly simple and interesting properties. Especially, the two numbers that satisfy the real idempotent equation $x^2 = x$ are nothing but 0 and 1. Thus, idempotent matrices are the simplest and funniest objects in algebra in comparison with other issues.

The purpose of this study is to provide a good coverage of idempotent matrix mathematics, from the development of the basic language to the collection of miscellaneous formulas and facts on matrix expressions that are composed by idempotent matrices and their conventional algebraic operations in one place with cogent introduction and references for further study. The remainder of this article is organized as follows. In Section 2, the author revisits some simple and well-known formulas for calculating the ranks of matrices in the literature and introduces the block matrix method (BMM) and its applications in establishing expansion formulas for calculating the ranks of block matrices, products of matrices, generalized inverses, etc. In Section 3, the author presents miscellaneous known and novel algebraic matrix identities composed by two and three idempotent matrices and discusses the applications of these identities in determining inverses and generalized inverses of matrix products. In Section 4, the author establishes a series of closed-form formulas for calculating the ranks of matrix expressions composed by idempotent matrices and uses the rank formulas to characterize a variety of fundamental properties of the matrix expressions. In Section 5, the author derives the upper and lower bounds for the ranks of some matrix pencils that involve the products of matrices and their generalized inverses. Some remarks and open problems on idempotent matrices and related issues are addressed in Section 6.

2 How to establish matrix rank equalities by the BMM

The rank of a matrix is a quite basic concept in linear algebra, which can be defined by different manners and can be calculated directly by transforming the matrix to certain row and/or column echelon forms. One of the most important applications of the rank of matrix is to describe singularity and nonsingularity of the matrix, as well as the dimension of the row or column space of the matrix. Thus, people would always be of interest in establishing various nontrivial analytical formulas for calculating the ranks of matrices under various assumptions. In fact, people have established a quite large number of equalities and inequalities for the ranks of matrices and widely used them in matrix analysis and applications. The author starts with a simple and best-known fact in linear algebra: the rank equalities $r(A) = r(PA) = r(AQ) = r(PAQ)$ always hold provided P and Q are two nonsingular matrices. Based on these fundamental formulas, people can deduce numerous concrete rank formulas from different choices of A, P, and Q using the BMM and elementary block matrix operations (EBMOs). The rank of a matrix is closely connected with other issues of the matrix, such as nullity, singularity, nonsingularity, and number of singular values. One of the well-known connections is concerned with the nullity of matrix and its rank, which claims that $A = 0 \Leftrightarrow r(A) = 0$. Also note the rule $A = B \Leftrightarrow A - B = 0$ for any two matrices A and B of the same sizes, through which it is possible to transform the equality preserving the equivalence. Thus, we have the rule $A = B \Leftrightarrow A - B = 0 \Leftrightarrow r(A - B) = 0$. Furthermore, assume that S_1 and S_2 are two sets consisting of matrices of the same size. Then, from the above discussion, the following theoretical results on the relationship between the two matrix sets are obtained:

$$\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset \Leftrightarrow \min_{A\in\mathcal{S}_1, B\in\mathcal{S}_2} r(A - B) = 0,$$

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \Leftrightarrow \max_{A\in\mathcal{S}_1} \min_{B\in\mathcal{S}_2} r(A - B) = 0.$$

These results show that the rank of matrix has such an attractive feature that can be used to characterize matrix equalities and matrix set inclusions under general situations. To speak precisely, if certain closed-form formulas for calculating the rank of the difference $A - B$ are obtained, we can utilize the formulas to directly characterize the relationships between two matrices A and B as well as the relationships between two given matrix sets. Since the seminal work by Marsaglia and Styan in 1974 [19], this matrix rank method (MRM) has been identified to be a strong and available technique in the study of various complicated matrix expressions that involve inverses and generalized inverses of matrices. Perhaps, no methods in linear algebra and matrix theory, as described above, are more elementary and straightforward than the MRM in characterizing the equalities of matrices and properties of the matrix expressions.

It has a long history in linear algebra to establish equalities and inequalities for the ranks of matrices from theoretical and applied points of view. Especially, there is a major route to derive several simple expansion formulas for calculating the ranks of matrix expressions by means of constructing various specific block matrices. Here, the author presents several well-known simple and interesting examples:

$$r(I_m - A^2) = r(I_m + A) + r(I_m - A) - m, \tag{2.1}$$

$$r(A \pm A^2) = r(A) + r(I_m \pm A) - m, \tag{2.2}$$

$$r(A \pm A^3) = r(A) + r(I_m \pm A^2) - m, \tag{2.3}$$

$$r(A^2 \pm A^3) = r(A^2) + r(I_m \pm A) - m, \tag{2.4}$$

$$r(A^k \pm A^{k+1}) = r(A^k) + r(I_m \pm A) - m, \quad k \geq 3, \tag{2.5}$$

$$r[A(I_m \pm A)^2] = r(A) + r[(I_m \pm A)^2] - m, \tag{2.6}$$

$$r(A \pm ABA) = r(A) + r(I_n \pm BA) - n = r(A) + r(I_m \pm AB) - m, \tag{2.7}$$

$$r(A - AXBYA) = r(B - BYAXB) + r(A) - r(B), \tag{2.8}$$

where A, B, X, and Y are the matrices of appropriate sizes. These rank equalities seem quite neat in form and are easy to understand. On the other hand, they can be used as matrix analytic tools to establish various complicated equalities and inequalities for the ranks of matrices and to characterize various fundamental properties of matrices, such as nullity, singularity, nonsingularity, idempotency, tripotency, and involution. A strong method of establishing these rank equalities is to construct a series of block matrices with the given matrices as follows:

$$\begin{bmatrix} I_m & I_m + A \\ I_m - A & 0 \end{bmatrix}, \quad \begin{bmatrix} I_m & I_m \pm A \\ A & 0 \end{bmatrix}, \quad \begin{bmatrix} I_m & I_m \pm A^2 \\ A & 0 \end{bmatrix}, \quad \begin{bmatrix} I_m & I_m \pm A \\ A^2 & 0 \end{bmatrix}, \quad \begin{bmatrix} I_m & I_m \pm A \\ A^k & 0 \end{bmatrix}, \tag{2.9}$$

$$\begin{bmatrix} I_m & (I_m \pm A)^2 \\ A & 0 \end{bmatrix}, \quad \begin{bmatrix} I_n & I_n \pm BA \\ A & 0 \end{bmatrix}, \quad \begin{bmatrix} I_m & A \\ I_m \pm AB & 0 \end{bmatrix}, \quad \begin{bmatrix} A & AXB \\ BYA & B \end{bmatrix}, \tag{2.10}$$

and then to do some routine calculations of the ranks of the block matrices. Apparently, some of (2.1)–(2.8) occur in various textbooks in linear algebra and matrix theory [2,19,28,39]. The author hopes the above discussion gives the reader a familiar view of analytical expansion formulas for calculating the ranks of matrices, as well as the constructive use of the BMM that permits us to discover and prove these formulas.

In 1974, Marsaglia and Styan systematically approached in their study [19] a series of fundamental problems on the ranks of matrices and their generalized inverses and established a wide range of equalities and inequalities for the ranks of partitioned matrices and sums and products of matrices, and presented many practical applications of the rank equalities and inequalities in matrix theory and other fields. Here, the author presents several fundamental equalities and facts about the ranks of partitioned matrices and generalized inverses of the matrices in [19], which will be used in the sequel.

Lemma 2.1. [19] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, *and* $D \in \mathbb{C}^{l\times k}$. *Then,*

(a) *the following rank equalities*

$$r[A, B] = r(A) + r(B - AA^-B) = r(B) + r(A - BB^-A), \tag{2.11}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C - CA^-A) = r(C) + r(A - AC^-C), \tag{2.12}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[(I_m - BB^-)A(I_n - C^-C)], \tag{2.13}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & B - AA^-B \\ C - CA^-A & D - CA^-B \end{bmatrix} \tag{2.14}$$

hold for all A^-, B^-, and C^-, and the following rank inequalities

$$\max\{r(A) + r(B) - r(AA^-B), \quad r(A) + r(B) - r(BB^-A)\} \leq r[A, B] \leq r(A) + r(B), \tag{2.15}$$

$$\max\{r(A) + r(C) - r(AC^-C), \quad r(A) + r(C) - r(CA^-A)\} \leq r\begin{bmatrix} A \\ C \end{bmatrix} \leq r(A) + r(C) \tag{2.16}$$

hold for all A^-, B^-, and C^-.

(b) *If $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$, then*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D - CA^\dagger B). \tag{2.17}$$

(c) $r[A, B] = r(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A) \Leftrightarrow AA^\dagger B = B \Leftrightarrow E_A B = 0.$

(d) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*) \Leftrightarrow CA^\dagger A = C \Leftrightarrow CF_A = 0.$

(e) $r[A, B] = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\} \Leftrightarrow \mathscr{R}[(E_A B)^*] = \mathscr{R}(B^*) \Leftrightarrow \mathcal{R}[(E_B A)^*] = \mathscr{R}(A^*).$

(f) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C) \Leftrightarrow \mathscr{R}(A^*) \cap \mathscr{R}(C^*) = \{0\} \Leftrightarrow \mathscr{R}(CF_A) = \mathscr{R}(C) \Leftrightarrow \mathscr{R}(AF_C) = \mathscr{R}(A).$

(g) $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) \Rightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A), \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*),$ *and* $CA^\dagger B = D.$

Lemma 2.2. [19] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Then,*

$$r(AB) = r(A) + r(B) - n + r[(I_n - BB^-)(I_n - A^-A)], \tag{2.18}$$

$$r(ABC) = r(AB) + r(BC) - r(B) + r[(I_n - BC(BC)^-)B(I_p - (AB)^-AB)] \tag{2.19}$$

hold for all A^-, B^-, $(AB)^-$, and $(BC)^-$.

It can be deduced from (2.18) and (2.19) that

$$\min\{r(A), r(B)\} \geq r(AB) \geq \max\{0, r(A) + r(B) - n\}, \tag{2.20}$$

and

$$r(ABC) \geq \max\{0, \quad r(AB) + r(BC) - r(B)\} \geq \max\{0, \quad r(A) + r(B) + r(C) - n - p\}, \tag{2.21}$$

$$r(ABC) \leq \min\{r(AB), \quad r(BC)\} \leq \min\{r(A), r(B), r(C)\}, \tag{2.22}$$

which encompass the two famous Sylvester's law $r(AB) \geq r(A) + r(B), - n$ and Frobenius inequality $r(ABC) \geq r(AB) + r(BC) - r(B)$, [18].

Lemma 2.3. [19] *Let $A, B \in \mathbb{C}^{m\times n}$ and denote $P = \begin{bmatrix} A \\ B \end{bmatrix}$ and $Q = [A, B]$. Then,*

$$r(A + B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B) + r\left((I_{2m} - PP^-)\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}(I_{2n} - Q^-Q)\right) \tag{2.23}$$

holds for all P^- and Q^-. In particular, the following rank inequalities hold

$$r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B) \le r(A+B) \le \min\left\{ r\begin{bmatrix} A \\ B \end{bmatrix}, r[A, B] \right\} \le \min\{m, n, r(A) + r(B)\}; \quad (2.24)$$

the following equivalent facts hold

$$r(A+B) = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\} \text{ and } \mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\} \text{ for } A, B \in \mathbb{C}^{m\times n}. \quad (2.25)$$

Two new equalities for the rank of the partitioned matrix $[A, B]$ and their consequences are given below.

Lemma 2.4. [35] *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$ and denote $P_A = AA^\dagger$ and $P_B = BB^\dagger$. Then, we have the following results:*

(a) *The range equality below holds*

$$\mathscr{R}(A) \cap \mathscr{R}(B) = \mathscr{R}(P_A P_B) \cap \mathscr{R}(P_B P_A). \quad (2.26)$$

(b) *The rank equalities below hold*

$$r[A, B] = r(A) + r(B) - \dim[\mathscr{R}(P_A P_B) \cap \mathscr{R}(P_B P_A)], \quad (2.27)$$

$$r[A, B] = r(A) + r(B) - r(P_A P_B) - r(P_B P_A) + r[P_A P_B, P_B P_A]. \quad (2.28)$$

(c) $r[A, B] = r(A) + r(B) \Leftrightarrow r[P_A P_B, P_B P_A] = r(P_A P_B) + r(P_B P_A) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\} \Leftrightarrow \mathscr{R}(P_A P_B) \cap \mathscr{R}(P_B P_A) = \{0\}$.

(d) $r[A, B] = r(A) + r(B) - r(P_A P_B) \Leftrightarrow r[P_A P_B, P_B P_A] = r(P_A P_B) = r(P_B P_A) \Leftrightarrow \mathscr{R}(P_A P_B) = \mathscr{R}(P_B P_A) \Leftrightarrow P_A P_B = P_B P_A$.

(e) $r[A, B] = r[P_A P_B, P_B P_A] \Leftrightarrow r(A^*B) = r(A) = r(B)$.

It can be imagined that one can derive many more nontrivial expansion formulas for calculating the ranks of matrix expressions through use of the BMM. In other words, it seems more natural to consider various specified block matrices and to extend (2.1)–(2.10) to general forms under various assumptions. The principal issue dealt with here is to what extent these rank formulas generalize to cases with multiple matrices. In the remaining part of this section, the author shows how to construct three block matrices from the general solutions of some fundamental linear matrix equations and derive three rank equalities associated with the solutions.

Theorem 2.5. *Let $M \in \mathbb{C}^{m\times m}$ be given, and assume that $X, Y \in \mathbb{C}^{m\times m}$ are solutions of the following three matrix equations:*

$$MX = X, \quad YM = Y, \quad MY = XM. \quad (2.29)$$

Then,

$$r(X - Y) = r\begin{bmatrix} X \\ Y \end{bmatrix} + r[X, Y] - r(X) - r(Y). \quad (2.30)$$

Proof. First construct a block matrix from X and Y as follows:

$$N = \begin{bmatrix} -X & 0 & X \\ 0 & Y & Y \\ X & Y & 0 \end{bmatrix}. \quad (2.31)$$

Then, it is easy to verify by EBMOs that

$$P_1 N Q_1 = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ I_m & -I_m & I_m \end{bmatrix} N \begin{bmatrix} I_m & 0 & I_m \\ 0 & I_m & -I_m \\ 0 & 0 & I_m \end{bmatrix} = \begin{bmatrix} -X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & X - Y \end{bmatrix}, \quad (2.32)$$

and from (2.29) that

$$P_2NQ_2 = \begin{bmatrix} I_m & 0 & M \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} N \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ 0 & -M & I_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & X \\ 0 & 0 & Y \\ X & Y & 0 \end{bmatrix}. \tag{2.33}$$

Note that the block matrices P_1, Q_1, P_2, and Q_2 in (2.32) and (2.33) are all nonsingular. Then, both (2.32) and (2.33) imply that the rank of N satisfies the following two equalities

$$r(N) = r(P_1NQ_1) = r\begin{bmatrix} -X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & X-Y \end{bmatrix} = r(X-Y) + r(X) + r(Y),$$

$$r(N) = r(P_2NQ_2) = r\begin{bmatrix} 0 & 0 & X \\ 0 & 0 & Y \\ X & Y & 0 \end{bmatrix} = r\begin{bmatrix} X \\ Y \end{bmatrix} + r[X, Y].$$

Combining these two expansion formulas leads to (2.30). □

Theorem 2.6. *Let $A, B \in \mathbb{C}^{m\times m}$ be given, and assume that $X, Y \in \mathbb{C}^{m\times m}$ are solutions of the following equations:*

$$AX = X, \quad YB = Y, \quad AY = XB. \tag{2.34}$$

Then,

$$r(X-Y) = r\begin{bmatrix} X \\ Y \end{bmatrix} + r[X, Y] - r(X) - r(Y). \tag{2.35}$$

Proof. Let N be as given in (2.31). Then, a routine deduction by EBMOs shows that

$$r(N) = r\begin{bmatrix} -X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & X-Y \end{bmatrix} = r(X-Y) + r(X) + r(Y),$$

$$r(N) = r\begin{bmatrix} 0 & AY & X \\ 0 & Y & Y \\ X & Y & 0 \end{bmatrix} = r\begin{bmatrix} 0 & XB & X \\ 0 & YB & Y \\ X & Y & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & X \\ 0 & 0 & Y \\ X & Y & 0 \end{bmatrix} = r\begin{bmatrix} X \\ Y \end{bmatrix} + r[X, Y]$$

hold under (2.34). Combining these two expansion formulas leads to (2.35). □

Theorem 2.7. *Let $A, B \in \mathbb{C}^{m\times m}$ be given, and assume that $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{m\times p}$ are solutions of the following equations:*

$$AX = X, BY = Y, \quad \mathscr{R}(X) \supseteq \mathscr{R}(AY), \quad \mathscr{R}(Y) \supseteq \mathscr{R}(BX). \tag{2.36}$$

Then,

$$r[AY, BX] = r[X, Y] + r(AY) + r(BX) - r(X) - r(Y). \tag{2.37}$$

Proof. Construct a block matrix from X, Y, AY, and BX as follows: $N = \begin{bmatrix} X & 0 & AY & 0 \\ 0 & Y & 0 & BX \\ X & Y & 0 & 0 \end{bmatrix}$. Then, a routine deduction by EBMOs shows that

$$r(N) = r\begin{bmatrix} X & 0 & 0 & 0 \\ 0 & Y & 0 & 0 \\ 0 & 0 & -AY & -BX \end{bmatrix} = r[AY, BX] + r(X) + r(Y),$$

$$r(N) = r\begin{bmatrix} 0 & -AY & AY & 0 \\ -BX & 0 & 0 & BX \\ X & Y & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & AY & 0 \\ 0 & 0 & 0 & BX \\ X & Y & 0 & 0 \end{bmatrix} = r[X, Y] + r(AY) + r(BX)$$

hold under (2.36). Combining these two rank equalities leads to (2.37). □

The preceding proofs of (2.30), (2.35), and (2.37) are elementary and straightforward by way of the BMM and EBMOs, which incontrovertibly seem to have some magic features that link the ranks of different matrices in a remarkable manner. Note that the matrix equations in (2.29), (2.34), and (2.36) are quite fundamental in matrix analysis and have been widely studied in theory and applications [20]. Under the assumptions of these equations, (2.30), (2.35), and (2.37) link the solutions of these matrix equations and their algebraic operations. In this situation, there is a very strong intrinsic mathematical motivation to establish concrete rank formulas from (2.30), (2.35), and (2.37) for various solutions of the matrix equations. As demonstrated in Section 4, the above BMM which we advocate can be used to deduce many interesting and delightful formulas and facts concerning the ranks of idempotent matrices and related issues.

Recall that the dimension of a finite-dimensional vector space (linear subspace) over the complex number field $\mathbb{C}$, denoted by $\dim(\cdot)$, is the number of independent vectors required to span the vector space (linear subspace). Tian [34] has recently shown that

$$(k-1)\dim(\mathcal{M}_1 + \cdots + \mathcal{M}_k) + \dim(\widehat{\mathcal{M}}_1 \cap \cdots \cap \widehat{\mathcal{M}}_k) = \dim(\widehat{\mathcal{M}}_1) + \cdots + \dim(\widehat{\mathcal{M}}_k) \tag{2.38}$$

holds for a family of linear subspaces $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ in a finite-dimensional vector space over $\mathbb{C}$, where $\widehat{\mathcal{M}}_i = \mathcal{M}_1 + \cdots + \mathcal{M}_{i-1} + \mathcal{M}_{i+1} + \cdots + \mathcal{M}_k$, $i = 1, 2, \ldots, k$, which is an extension of the best-known dimension formula:

$$\dim(\mathcal{M}_1 + \mathcal{M}_2) + \dim(\mathcal{M}_1 \cap \mathcal{M}_2) = \dim(\mathcal{M}_1) + \dim(\mathcal{M}_2). \tag{2.39}$$

Concerning the rank of the product of two matrices, we have the following results.

Lemma 2.8. *Let $A, B, P, Q \in \mathbb{C}^{m\times m}$, and assume that*

$$AB = BA, \quad PA + BQ = I_m, \quad r[A, B] = m. \tag{2.40}$$

Then,

$$r(AB) = r(BA) = r(A) + r(B) - m, \tag{2.41}$$

$$\mathscr{R}(AB) = \mathscr{R}(BA) = \mathscr{R}(A) \cap \mathscr{R}(B). \tag{2.42}$$

Proof. It can be deduced from (2.40) and EBMOs that

$$r\begin{bmatrix} I_m & B \\ A & 0 \end{bmatrix} = r\begin{bmatrix} I_m & 0 \\ 0 & -AB \end{bmatrix} = m + r(AB),$$

$$r\begin{bmatrix} I_m & B \\ A & 0 \end{bmatrix} = r\begin{bmatrix} I_m - PA - BQ & B \\ A & 0 \end{bmatrix} = r\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} = r(A) + r(B).$$

Combining these two rank equalities leads to (2.41). Also by (2.39) and (2.40),

$$r(AB) = r(BA) = r(A) + r(B) - r[A, B] = \dim[\mathscr{R}(A) \cap \mathscr{R}(B)], \mathscr{R}(AB) = \mathscr{R}(BA) \subseteq \mathscr{R}(A) \cap \mathscr{R}(B).$$

Combining these two equalities leads to (2.42). □

We can finally prove a group of fundamental range equalities associated with the rank equalities in (2.1)–(2.6).

Theorem 2.9. *Let $A \in \mathbb{C}^{m\times m}$ be given. Then, the following matrix range equalities hold*

$$\mathscr{R}(I_m - A^2) = \mathscr{R}(I_m + A) \cap \mathscr{R}(I_m - A), \tag{2.43}$$

$$\mathscr{R}(A \pm A^2) = \mathscr{R}(A) \cap \mathscr{R}(I_m \pm A), \tag{2.44}$$

$$\mathscr{R}(A \pm A^3) = \mathscr{R}(A) \cap \mathscr{R}(I_m \pm A^2) = \mathscr{R}(A) \cap \mathscr{R}(I_m + A) \cap \mathscr{R}(I_m - A), \tag{2.45}$$

$$\mathscr{R}(A^2 \pm A^3) = \mathscr{R}(A^2) \cap \mathscr{R}(I_m \pm A), \tag{2.46}$$

$$\mathscr{R}(A^k \pm A^{k+1}) = \mathscr{R}(A^k) \cap \mathscr{R}(I_m \pm A), \quad k \geq 3, \tag{2.47}$$

$$\mathscr{R}[A(I_m \pm A)^2] = \mathscr{R}(A) \cap \mathscr{R}[(I_m \pm A)^2]. \tag{2.48}$$

Proof. We readily see from the definition of the range of matrix that

$$\begin{aligned}
\mathscr{R}(I_m - A^2) &= \mathscr{R}[(I_m + A)(I_m - A)] \subseteq \mathscr{R}(I_m + A) \cap R(I_m - A),\\
\mathscr{R}(A \pm A^2) &= \mathscr{R}[A(I_m \pm A)] \subseteq \mathscr{R}(A) \cap \mathscr{R}(I_m \pm A),\\
\mathscr{R}(A \pm A^3) &= \mathscr{R}[A(I_m \pm A^2)] \subseteq \mathscr{R}(A) \cap \mathscr{R}(I_m \pm A^2),\\
\mathscr{R}(A \pm A^3) &= \mathscr{R}[A(I_m + A)(I_m - A)] \subseteq \mathscr{R}(A) \cap \mathscr{R}(I_m + A) \cap \mathscr{R}(I_m - A),\\
\mathscr{R}(A^2 \pm A^3) &= \mathscr{R}[A^2(I_m \pm A)] \subseteq \mathscr{R}(A^2) \cap \mathscr{R}(I_m \pm A),\\
\mathscr{R}(A^k \pm A^{k+1}) &= \mathscr{R}[A^k(I_m \pm A)] \subseteq \mathscr{R}(A^k) \cap \mathscr{R}(I_m \pm A), \quad k \geq 3,\\
\mathscr{R}[A(I_m \pm A)^2] &\subseteq \mathscr{R}(A) \cap \mathscr{R}[(I_m \pm A)^2].
\end{aligned}$$

Also by (2.39), we can rewrite (2.1)–(2.6) as

$$\begin{aligned}
\dim[\mathscr{R}(I_m - A^2)] &= r(I_m + A) + r(I_m - A) - r[I_m + A, I_m - A] = \dim[\mathscr{R}(I_m + A) \cap \mathscr{R}(I_m - A)],\\
\dim[\mathscr{R}(A \pm A^2)] &= r(A) + r(I_m \pm A) - r[A, I_m \pm A] = \dim[\mathscr{R}(A) \cap \mathscr{R}(I_m \pm A)],\\
\dim[\mathscr{R}(A \pm A^3)] &= r(A) + r(I_m \pm A^2) - r[A, I_m \pm A^2] = \dim[\mathscr{R}(A) \cap \mathscr{R}(I_m \pm A^2)],\\
\dim[\mathscr{R}(A^2 \pm A^3)] &= r(A^2) + r(I_m \pm A) - r[A^2, I_m \pm A] = \dim[\mathscr{R}(A^2) \cap \mathscr{R}(I_m \pm A)],\\
\dim[\mathscr{R}(A^k \pm A^{k+1})] &= r(A^k) + r(I_m \pm A) - r[A^k, I_m \pm A] = \dim[\mathscr{R}(A^k) \cap \mathscr{R}(I_m \pm A)], \quad k \geq 3,\\
\dim[\mathscr{R}(A(I_m \pm A)^2)] &= r(A) + r[(I_m \pm A)^2] - r[A, (I_m \pm A)^2] = \dim[\mathscr{R}(A) \cap \mathscr{R}[(I_m \pm A)^2].
\end{aligned}$$

Applying Lemma 2.8 to the above range and dimension results leads to (2.43)–(2.48). □

The preceding results show that it is always possible to establish nontrivial rank formulas by means of constructing certain specified block matrices and making two kinds of calculations of the ranks of the block matrices by EBMOs. The author will present more specified rank equalities and their consequences in Sections 4 and 5 using the BMM.

3 Miscellaneous algebraic identities associated with two and three idempotent matrices

In this section, the author gathers a wide range of known or novel algebraic identities that are composed by the conventional matrix operations of two and three idempotent matrices and gives a variety of meaningful and interesting consequences and applications of these matrix identities. In the following, the notation

$$\tilde{A} = I_m - A, \quad \tilde{B} = I_m - B, \quad M = A + B$$

is used for two square matrices of the same sizes. The author starts with a list of equivalent facts for a matrix to be idempotent, which were summarized by Trenkler [42,43] as follows:

Lemma 3.1. [42,43] *Let $A \in \mathbb{C}^{m\times m}$. Then, the following statements are equivalent:*
(a) *A is an idempotent.*
(b) *$I_m - A$ is an idempotent.*
(c) *A^* is an idempotent.*
(d) *PAQ is idempotent for any matrices $P \in \mathbb{C}^{n\times m}$ and $Q \in \mathbb{C}^{m\times n}$ of such that $QP = I_m$.*
(e) *$(I_m - 2A)^2 = I_m$.*
(f) *$A^2(I_m - A) = A(I_m - A)^2 = 0$.*

(g) $A = PQ$, *where* $P \in \mathbb{C}^{m\times r}$, $Q \in \mathbb{C}^{r\times m}$, *and* $QP = I_r$.
(h) *A admits the decomposition* $A = P\,\mathrm{diag}(I_r, 0)P^{-1}$.
(i) $A = P\begin{bmatrix} I_r & R \\ 0 & 0 \end{bmatrix}P^*$, *where* $PP^* = P^*P = I_m$ *and* $R \in \mathbb{C}^{r\times(m-r)}$.
(j) $\mathscr{R}(A) \subseteq \mathscr{R}(I_m - A)$.
(k) $\mathscr{R}(I_m - A) \subseteq \mathscr{N}(A)$.
(l) $\mathbb{C}^m = \mathscr{R}(A) \oplus \mathscr{R}(I_m - A)$.
(m) $\mathscr{R}(A) \cap \mathscr{R}(I_m - A) = \{0\}$.
(n) $r(A) = \mathrm{trace}(A)$ *and* $r(I_m - A) = m - \mathrm{trace}(A)$.
(o) $r(I_m - A) = m - r(A)$.

The above lemma gives a clearer picture that the idempotency of a square matrix can be characterized in terms of other conventional matrix operations. On the other hand, there are a lot of opportunities to encounter idempotent matrices in different branches of matrix theory and applications. Here, the author refers a few of the examples that display the practical appearance of idempotent matrices:

(I) Idempotent matrices occur widely in the theory of generalized inverses of matrices, such as the two products AA^- and A^-A are always idempotent matrices for any matrix A^- that satisfies $AA^-A = A$; both $AA^\dagger$ and $A^\dagger A$ are Hermitian idempotent matrices for the Moore-Penrose inverse $A^\dagger$ of A; and the products $B(AB)^\dagger A$, $BC(ABC)^\dagger A$, and $C(ABC)^\dagger AB$ are all idempotent.
(II) Idempotent matrices play an important role in the theory of linear statistical models, in particular, the idempotent matrix $X(X^TVX)^\dagger X^TV$ and its variations have extensively been used to solve least-squares and weighted least-squares estimation problems in the statistical inferences of various regression models [23,24,26,40,41].
(III) If $A^2 = I_m$, then $(I_m \pm A)/2$ are idempotent; if $A^2 = -I_m$, then $(I_m \pm iA)/2$ are idempotent; if $A^2 = -A$, then $-A$ is idempotent; if $A^3 = A$, then $(A \pm A^2)/2$ are idempotent.
(IV) Any square matrix A that satisfies a quadratic equation $A^2 + aA + bI_m = 0$ can be written as $[A + (a/2)I_m]^2 = (a^2/4 - b)I_m$. If $a^2/4 - b \neq 0$, then we can also construct an idempotent matrix from this equality. Through these transformations, various results on idempotent matrices can be extended to other types of quadratic matrices.
(V) Any singular square matrix over an arbitrary field can be written as a product of a finite number of idempotent matrices [4,9,13,14]; there exist certain conditions under which a square matrix can be written as the sums and differences of idempotent matrices [15,16].

These apparent facts show that idempotency is one of the fundamental and intrinsic properties associated with matrices and their operations and thus can be used to describe algebraic performances of matrices under various assumptions. Because the idempotency of a matrix can be used to simplify multiplication operations of matrices, people can formulate various algebraic identities that involve idempotent matrices and use these identities in the investigation of various specified matrix analysis problems.

Next, the author presents an integrated account of algebraic matrix identities that are composed by two and three idempotent matrices and gives a variety of consequences and applications of these identities in determining the nonsingularity and the standard inverses of the matrices involved. The following three theorems are fundamental for establishing different kinds of matrix identities for two given idempotent matrices of the same order and their algebraic operations.

Theorem 3.2. [5] *Given two idempotent matrices* $A, B \in \mathbb{C}^{m\times m}$, *and two scalars* α *and* β *with* $\alpha \neq 0$, $\beta \neq 0$, *and* $\alpha + \beta \neq 0$, *the following matrix equalities*

$$\alpha A + \beta B = \lambda(I_m + \mu_1 A)M(I_m + \mu_2 B) = \lambda(I_m + \mu_2 B)M(I_m + \mu_1 A) \tag{3.1}$$

hold, where $\lambda = 2\alpha\beta(\alpha+\beta)^{-1}$, $\mu_1 = (2\beta)^{-1}(\alpha-\beta)$, *and* $\mu_2 = (2\alpha)^{-1}(\beta-\alpha)$, *and the matrices* $I_m + \mu_1 A$ *and* $I_m + \mu_2 B$ *are nonsingular. In particular,* $\alpha A + \beta B$ *is nonsingular if and only if* M *is nonsingular, in which case, the following two reverse order laws hold*

$$(\alpha A+\beta B)^{-1} = \lambda^{-1}(I_m+\mu_1 A)^{-1}M^{-1}(I_m+\mu_2 B)^{-1} = \lambda^{-1}(I_m+\mu_2 B)^{-1}M^{-1}(I_m+\mu_1 A)^{-1}. \tag{3.2}$$

Proof. Equation (3.1) can be verified by multiplying the matrices on the right-hand sides of the equalities. (3.2) is a direct consequence of (3.1) by the well-known reverse order law $(XYZ)^{-1} = Z^{-1}Y^{-1}X^{-1}$ for the inverse of the product of any three nonsingular matrices X, Y, and Z of the same size. □

Theorem 3.3. *Given two idempotent matrices* $A, B \in \mathbb{C}^{m\times m}$, *and two scalars* α and β, *the following four groups of factorization equalities hold*

$$\alpha AB+\beta BA = (\alpha A+\beta B)(M-I_m) = (M-I_m)(\beta A+\alpha B), \tag{3.3}$$

$$\alpha ABA+\beta BAB = (\alpha A+\beta B)(M-I_m)^2 = (M-I_m)(\beta A+\alpha B)(M-I_m) = (M-I_m)^2(\alpha A+\beta B), \tag{3.4}$$

$$\alpha(AB)^k+\beta(BA)^k = (\alpha A+\beta B)(M-I_m)^{2k-1} = (M-I_m)^{2k-1}(\beta A+\alpha B), \tag{3.5}$$

$$\alpha(ABA)^k+\beta(BAB)^k = (\alpha A+\beta B)(M-I_m)^{2k} = (M-I_m)^{2k}(\alpha A+\beta B). \tag{3.6}$$

In particular, $\alpha AB+\beta BA$ *is nonsingular* $\Leftrightarrow$ $\alpha(AB)^k+\beta(BA)^k$ *is nonsingular* $\Leftrightarrow$ $\alpha ABA+\beta BAB$ *is nonsingular* $\Leftrightarrow$ $\alpha(AB)^k+\beta(BA)^k$ *is nonsingular* $\Leftrightarrow$ $\alpha(ABA)^k+\beta(BAB)^k$ *is nonsingular* $\Leftrightarrow$ $\alpha A+\beta B$ *and* $M-I_m$ *are nonsingular* $\Leftrightarrow$ $\beta A+\alpha B$ *and* $M-I_m$ *are nonsingular, in which cases, the following reverse order laws hold*

$$(\alpha AB+\beta BA)^{-1} = (M-I_m)^{-1}(\alpha A+\beta B)^{-1} = (\beta A+\alpha B)^{-1}(M-I_m)^{-1}, \tag{3.7}$$

$$\begin{aligned}(\alpha ABA+\beta BAB)^{-1} &= (M-I_m)^{-2}(\alpha A+\beta B)^{-1} = (M-I_m)^{-1}(\beta A+\alpha B)^{-1}(M-I_m)^{-1}\\ &= (\alpha A+\beta B)^{-1}(M-I_m)^{-2},\end{aligned} \tag{3.8}$$

$$[\alpha(AB)^k+\beta(BA)^k]^{-1} = (M-I_m)^{-(2k-1)}(\alpha A+\beta B)^{-1} = (\beta A+\alpha B)^{-1}(M-I_m)^{-(2k-1)}, \tag{3.9}$$

$$[\alpha(ABA)^k+\beta(BAB)^k]^{-1} = (M-I_m)^{-2k}(\alpha A+\beta B)^{-1} = (\alpha A+\beta B)^{-1}(M-I_m)^{-2k}. \tag{3.10}$$

Proof. Equations (3.3)–(3.6) can be verified by multiplying the matrices on the right-hand sides. Equations (3.7)–(3.10) are direct consequences of (3.3)–(3.6) by reverse order laws for inverses of products of nonsingular matrices. □

Obviously, substituting (3.1) into (3.3)–(3.6), and (3.2) into (3.7)–(3.10) will yield several groups of new factorization equalities and new reverse order laws.

Theorem 3.4. [44] *Given two idempotent matrices* $A, B \in \mathbb{C}^{m\times m}$, *and two scalars* α *and* β *with* $\alpha \neq -1, 0$ *and* $\beta \neq -1, 0$, *the following two identities hold*

$$I_m+\alpha A+\beta B = (I_m+\alpha A)(I_m-\lambda AB)(I_m+\beta B) = (I_m+\beta B)(I_m-\lambda BA)(I_m+\alpha A), \tag{3.11}$$

where $\lambda = \alpha\beta(1+\alpha)^{-1}(1+\beta)^{-1}$, *and* $I_m+\alpha A$ *and* $I_m+\beta B$ *are nonsingular. In particular,* $I_m-\lambda AB$ *is nonsingular if and only if* $I_m+\alpha A+\beta B$ *is nonsingular, and the following two reverse order laws hold*

$$(I_m+\alpha A+\beta B)^{-1} = (I_m+\alpha A)^{-1}(I_m-\lambda BA)^{-1}(I_m+\beta B)^{-1} = (I_m+\beta B)^{-1}(I_m-\lambda AB)^{-1}(I_m+\alpha A)^{-1}. \tag{3.12}$$

Proof. Multiplying the matrices on the right-hand sides of (3.11) yields the two equalities. (3.12) follows directly from (3.11). □

Equations (3.1)–(3.12) show that there exist essential links among any two given idempotent matrices of the same size. There is no doubt that (3.1)–(3.12) can be used to describe algebraic performances of the matrix expressions on the left-hand sides of the equalities, such as the ranks, ranges, nullity, r-potency, nilpotency, nonsingularity, inverses, generalized inverses, and norms of these matrix expressions. Note, in particular, that the term $\alpha A + \beta B$ occurs commonly in (3.1)–(3.12). Thus, more matrix identities can be derived from the various possible combinations of (3.1)–(3.12) when various specific conditions are met. As immediate consequences, we derive from (3.3)–(3.6) the following several groups of results for different choices of α and β.

Theorem 3.5. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, and a positive integer k, we have the following results*:

(a) *The matrix identities below hold*

$$AB - BA = (A - B)(M - I_m) = -(M - I_m)(A - B), \tag{3.13}$$

$$ABA - BAB = (A - B)(M - I_m)^2 = (M - I_m)^2(A - B) = (A - B) - (A - B)^3, \tag{3.14}$$

$$(AB - BA)^k = (-1)^{k(k-1)/2}(A - B)^k(M - I_m)^k = (-1)^{k(k-1)/2}(M - I_m)^k(A - B)^k, \tag{3.15}$$

$$(ABA - BAB)^k = (A - B)^k(M - I_m)^{2k} = (M - I_m)^{2k}(A - B)^k, \tag{3.16}$$

$$(AB)^k - (BA)^k = (A - B)(M - I_m)^{2k-1} = -(M - I_m)^{2k-1}(A - B), \tag{3.17}$$

$$(ABA)^k - (BAB)^k = (A - B)(M - I_m)^{2k} = (M - I_m)^{2k}(A - B). \tag{3.18}$$

(b) *The matrix identities below hold*

$$AB + BA = M(M - I_m) = (M - I_m)M, \tag{3.19}$$

$$ABA + BAB = M(M - I_m)^2 = (M - I_m)^2M, \tag{3.20}$$

$$(AB + BA)^k = M^k(M - I_m)^k = (M - I_m)^kM^k, \tag{3.21}$$

$$(ABA + BAB)^k = M^k(M - I_m)^{2k} = (M - I_m)^{2k}M^k, \tag{3.22}$$

$$(AB)^k + (BA)^k = M(M - I_m)^{2k-1} = (M - I_m)^{2k-1}M, \tag{3.23}$$

$$(ABA)^k + (BAB)^k = M(M - I_m)^{2k} = (M - I_m)^{2k}M. \tag{3.24}$$

(c) *The matrix identities below hold*

$$\begin{aligned} &AB - BA + (AB)^2 - (BA)^2 + \cdots + (AB)^k - (BA)^k \\ &\quad = (A - B)[(M - I_m) + (M - I_m)^3 + \cdots + (M - I_m)^{2k-1}] \\ &\quad = [(M - I_m) + (M - I_m)^3 + \cdots + (M - I_m)^{2k-1}](B - A), \end{aligned} \tag{3.25}$$

$$\begin{aligned} &AB + BA + (AB)^2 + (BA)^2 + \cdots + (AB)^k + (BA)^k \\ &\quad = M[(M - I_m) + (M - I_m)^3 + \cdots + (M - I_m)^{2k-1}] \\ &\quad = [(M - I_m) + (M - I_m)^3 + \cdots + (M - I_m)^{2k-1}]M, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &ABA - BAB + (ABA)^2 - (BAB)^2 + \cdots + (ABA)^k - (BAB)^k \\ &\quad = (A - B)[(M - I_m)^2 + (M - I_m)^4 + \cdots + (M - I_m)^{2k}] \\ &\quad = [(M - I_m)^2 + (M - I_m)^4 + \cdots + (M - I_m)^{2k}](A - B), \end{aligned} \tag{3.27}$$

$$\begin{aligned} &ABA + BAB + (ABA)^2 + (BAB)^2 + \cdots + (ABA)^k + (BAB)^k \\ &\quad = M[(M - I_m)^2 + (M - I_m)^4 + \cdots + (M - I_m)^{2k}] \\ &\quad = [(M - I_m)^2 + (M - I_m)^4 + \cdots + (M - I_m)^{2k}]M. \end{aligned} \tag{3.28}$$

(d) *$AB - BA$ is nonsingular $\Leftrightarrow$ $ABA - BAB$ is nonsingular $\Leftrightarrow$ $(AB)^k - (BA)^k$ is nonsingular $\Leftrightarrow$ $(ABA)^k - (BAB)^k$ is nonsingular $\Leftrightarrow$ $A - B$ and $M - I_m$ are nonsingular $\Leftrightarrow$ $A - B$ and $I_m - (A - B)^2$ are nonsingular, in which case, the following matrix identities hold*

$$(AB - BA)^{-1} = (M - I_m)^{-1}(A - B)^{-1} = -(A - B)^{-1}(M - I_m)^{-1}, \tag{3.29}$$

$$(ABA - BAB)^{-1} = (A - B)^{-1}(M - I_m)^{-2} = (M - I_m)^{-2}(A - B)^{-1}, \tag{3.30}$$

$$[(AB - BA)^k]^{-1} = (-1)^{k(k-1)/2}(A - B)^{-k}(M - I_m)^{-k} = (-1)^{k(k-1)/2}(M - I_m)^{-k}(A - B)^{-k}, \tag{3.31}$$

$$[(ABA - BAB)^k]^{-1} = (A - B)^{-k}(M - I_m)^{-2k} = (M - I_m)^{-2k}(A - B)^{-k}, \tag{3.32}$$

$$[(AB)^k - (BA)^k]^{-1} = (M - I_m)^{-(2k-1)}(A - B)^{-1} = -(A - B)^{-1}(M - I_m)^{-(2k-1)}, \tag{3.33}$$

$$[(ABA)^k - (BAB)^k]^{-1} = (A - B)^{-1}(M - I_m)^{-2k} = (M - I_m)^{-2k}(A - B)^{-1}. \tag{3.34}$$

(e) *$AB + BA$ is nonsingular $\Leftrightarrow$ $ABA + BAB$ is nonsingular $\Leftrightarrow$ $(AB)^k + (BA)^k$ is nonsingular $\Leftrightarrow$ $(ABA)^k + (BAB)^k$ is nonsingular $\Leftrightarrow$ M and $M - I_m$ are nonsingular, in which case, the following matrix identities hold*

$$(AB + BA)^{-1} = M^{-1}(M - I_m)^{-1} = (M - I_m)^{-1}M^{-1}, \tag{3.35}$$

$$(ABA + BAB)^{-1} = M^{-1}[(M - I_m)^{-2} = (M - I_m)^{-2}M^{-1}, \tag{3.36}$$

$$[(AB + BA)^k]^{-1} = M^{-k}(M - I_m)^{-k} = (M - I_m)^{-k}M^{-k}, \tag{3.37}$$

$$[(ABA + BAB)^k]^{-1} = M^{-k}(M - I_m)^{-2k} = (M - I_m)^{-2k}M^{-k}, \tag{3.38}$$

$$[(AB)^k + (BA)^k]^{-1} = M^{-1}(M - I_m)^{-(2k-1)} = (M - I_m)^{-(2k-1)}M^{-1}, \tag{3.39}$$

$$[(ABA)^k + (BAB)^k]^{-1} = M^{-1}(M - I_m)^{-2k} = (M - I_m)^{-2k}M^{-1}. \tag{3.40}$$

(f) *The following matrix identities hold*

$$(AB - BA)^D = (A - B)^D(M - I_m)^D = -(M - I_m)^D(A - B)^D, \tag{3.41}$$

$$(AB + BA)^D = M^D(M - I_m)^D = (M - I_m)^D M^D, \tag{3.42}$$

$$[(AB)^k - (BA)^k]^D = (A - B)^D[(M - I_m)^D]^{2k-1} = -[(M - I_m)^D]^{2k-1}(A - B)^D, \tag{3.43}$$

$$[(AB)^k + (BA)^k]^D = M^D[(M - I_m)^D]^{2k-1} = [(M - I_m)^D]^{2k-1}M^D, \tag{3.44}$$

$$[(ABA)^k - (BAB)^k]^D = (A - B)^D[(M - I_m)^D]^{2k} = [(M - I_m)^D]^{2k}(A - B)^D, \tag{3.45}$$

$$[(ABA)^k + (BAB)^k]^D = M^D[(M - I_m)^D]^{2k} = [(M - I_m)^D]^{2k}M^D. \tag{3.46}$$

Proof. Equations (3.13)–(3.24) follow from (3.25)–(3.28) for $\alpha = 1$ and $\beta = \pm 1$. Equations (3.25)–(3.28) follow from (3.13)–(3.24). Results (d) and (e) follow from (3.13)–(3.24). It follows from [31, corollary 5] that

$$MN = \pm NM \Rightarrow (MN)^D = N^D M^D.$$

Applying this result to (3.13)–(3.24) yields Result (f). □

Theorem 3.6. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, the following matrix identities hold*

$$(AB\tilde{A})^2 = (A\tilde{B}\tilde{A})^2 = (\tilde{A}BA)^2 = (\tilde{A}\tilde{B}A)^2 = 0, \tag{3.47}$$

$$(BA\tilde{B})^2 = (B\tilde{A}\tilde{B})^2 = (\tilde{B}AB)^2 = (\tilde{B}\tilde{A}B)^2 = 0, \tag{3.48}$$

$$AB\tilde{A} + A\tilde{B}\tilde{A} + \tilde{A}BA + \tilde{A}\tilde{B}A = 0, \tag{3.49}$$

$$BA\tilde{B} + B\tilde{A}\tilde{B} + \tilde{B}AB + \tilde{B}\tilde{A}B = 0, \tag{3.50}$$

$$AB + A\tilde{B} + \tilde{A}B + \tilde{A}\tilde{B} = I_m, \tag{3.51}$$

$$BA + B\tilde{A} + \tilde{B}A + \tilde{B}\tilde{A} = I_m, \tag{3.52}$$

$$ABA + A\tilde{B}A + \tilde{A}B\tilde{A} + \tilde{A}\tilde{B}\tilde{A} = 2I_m, \tag{3.53}$$

$$BAB + B\tilde{A}B + \tilde{B}A\tilde{B} + \tilde{B}\tilde{A}\tilde{B} = 2I_m, \tag{3.54}$$

$$(2A - I_m)(A - B) + (M - I_m)(2B - I_m) = I_m, \tag{3.55}$$

$$(A - B)(2A - I_m) + (2B - I_m)(M - I_m) = I_m, \tag{3.56}$$

$$(B - A)(2B - I_m) + (2A - I_m)(M - I_m) = I_m, \tag{3.57}$$

$$(2B - I_m)(B - A) + (M - I_m)(2A - I_m) = I_m, \tag{3.58}$$

$$(A - B)^2 + (I_m - M)^2 = I_m, \tag{3.59}$$

$$(A - B)^4 + 2(AB - BA)^2 + (I_m - M)^4 = I_m, \tag{3.60}$$

and the following matrix identities hold

$$(A - B)^2 = 2M - M^2, \tag{3.61}$$

$$(A - B)^2 = 2(2I_m - M) - (2I_m - M)^2, \tag{3.62}$$

$$(I_m - M)^2 = 2(\tilde{A} + B) - (\tilde{A} + B)^2, \tag{3.63}$$

$$(I_m - M)^2 = 2(A + \tilde{B}) - (A + \tilde{B})^2, \tag{3.64}$$

$$(AB - BA)^2 = (A - B)^2 - (A - B)^4 = (I_m - M)^2 - (I_m - M)^4, \tag{3.65}$$

$$(2^{-1}I_m - M)^2 = 4^{-1}I_m + AB + BA, \tag{3.66}$$

$$(2^{-1}I_m + A - B)^2 = 4^{-1}I_m + 2A - AB - BA, \tag{3.67}$$

$$(2^{-1}I_m - A + B)^2 = 4^{-1}I_m + 2B - AB - BA, \tag{3.68}$$

$$(3/2I_m - M)^2 = 5/4I_m - 2M + AB + BA, \tag{3.69}$$

$$7/4I_m - AB - BA = (2^{-1}I_m + A - B)^2 + (2^{-1}I_m - A + B)^2 + (3/2I_m - M)^2. \tag{3.70}$$

Proof. Equations (3.47)–(3.50) follow directly from the facts $A + \tilde{A} = B + \tilde{B} = I_m$ and $A\tilde{A} = \tilde{A}A = B\tilde{B} = \tilde{B}B = 0$. Equations (3.51)–(3.58) are determined by expanding the left-hand sides of these equalities and the corresponding deductive calculations. Adding (3.55) and (3.58) yields (3.59), which was first given in [17]; see also [1,3,8,22,27]. Equation (3.60) follows from squaring both sides of (3.59) and (3.15). Equations (3.61)–(3.64) follow from various variations of (3.59) (or direct expansions). The correctness of (3.66)–(3.70) can be verified from direct expansions of both sides of the equalities. □

In addition, it is easy to prove the following identities for two idempotent matrices of the same size.

Theorem 3.7. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, and a positive integer k, we have the following results:*
(a) *The matrix identities below hold*

$$A - ABA = A(A - B)^2 = (A - B)^2A, \tag{3.71}$$

$$B - BAB = B(A - B)^2 = (A - B)^2B, \tag{3.72}$$

$$(A - ABA)^k = A(A - B)^{2k} = (A - B)^{2k}A, \tag{3.73}$$

$$(B - BAB)^k = B(A - B)^{2k} = (A - B)^{2k}B, \tag{3.74}$$

$$ABA = A(M - I_m)^2 = (M - I_m)^2A, \tag{3.75}$$

$$BAB = B(M - I_m)^2 = (M - I_m)^2B, \tag{3.76}$$

$$(ABA)^k = A(M - I_m)^{2k} = (M - I_m)^{2k}A, \tag{3.77}$$

$$(BAB)^k = B(M - I_m)^{2k} = (M - I_m)^{2k}B, \tag{3.78}$$

$$(BA)^2 = BA(M - I_m)^2 = B(M - I_m)^2A, \tag{3.79}$$

$$(AB)^2 = AB(M - I_m)^2 = A(M - I_m)^2B, \tag{3.80}$$

$$(AB)^k = A(M - I_m)^kB, \tag{3.81}$$

$$(BA)^k = B(M - I_m)^k A. \tag{3.82}$$

(b) *The matrix identities below hold*

$$(A - ABA)^D = A[(A - B)^D]^2 = [(A - B)^D]^2 A, \tag{3.83}$$

$$(B - BAB)^D = B[(A - B)^D]^2 = [(A - B)^D]^2 B, \tag{3.84}$$

$$(ABA)^D = A[(M - I_m)^D]^2 = [(M - I_m)^D]^2 A, \tag{3.85}$$

$$(BAB)^D = B[(M - I_m)^D]^2 = [(M - I_m)^D]^2 B. \tag{3.86}$$

In the remaining part of this section, the author constructs several general matrix identities composed by three idempotent matrices and their algebraic operations.

Theorem 3.8. *Given three idempotent matrices $A, B, C \in \mathbb{C}^{m\times m}$ with $S = A + B + C$, three scalars α, β, γ, and a positive integer k, we have the following results*:

(a) *The matrix identities below hold*

$$\alpha(AB + AC) + \beta(BA + BC) + \gamma(CA + CB) = (\alpha A + \beta B + \gamma C)(S - I_m), \tag{3.87}$$

$$\alpha(BA + CA) + \beta(AB + CB) + \gamma(AC + BC) = (S - I_m)(\alpha A + \beta B + \gamma C), \tag{3.88}$$

$$\begin{aligned}(\alpha + \beta)(AB + BA) + (\alpha + \gamma)(AC + CA) + (\beta + \gamma)(BC + CB)\\ = (\alpha A + \beta B + \gamma C)(S - I_m) + (S - I_m)(\alpha A + \beta B + \gamma C),\end{aligned} \tag{3.89}$$

$$(\alpha - \beta)(AB - BA) + (\alpha - \gamma)(AC - CA) + (\beta - \gamma)(BC - CB) = (\alpha A + \beta B + \gamma C)S - S(\alpha A + \beta B + \gamma C), \tag{3.90}$$

$$\alpha(B + C)A(B + C) + \beta(A + C)B(A + C) + \gamma(A + B)C(A + B) = (S - I_m)(\alpha A + \beta B + \gamma C)(S - I_m). \tag{3.91}$$

(b) *The matrix identities below hold*

$$(A + B)^2 + (A + C)^2 + (B + C)^2 = S(I_m + S), \tag{3.92}$$

$$(A - B)^2 + (A - C)^2 + (B - C)^2 = S(3I_m - S) = 9/4I_m - (S - 3/2I_m)^2, \tag{3.93}$$

$$AB + BA + AC + CA + BC + CB = S(S - I_m) = (S - 2^{-1}I_m)^2 - 4^{-1}I_m, \tag{3.94}$$

$$(AB + BA + AC + CA + BC + CB)^k = S^k(S - I_m)^k. \tag{3.95}$$

(c) *The equivalent facts below hold*

$$\begin{aligned}&\alpha(AB + AC) + \beta(BA + BC) + \gamma(CA + CB) = 0 \Leftrightarrow (\alpha A + \beta B + \gamma C)(S - I_m) = 0,\\ &\alpha(BA + CA) + \beta(AB + CB) + \gamma(AC + BC) = 0 \Leftrightarrow (S - I_m)(\alpha A + \beta B + \gamma C) = 0,\\ &(\alpha + \beta)(AB + BA) + (\alpha + \gamma)(AC + CA) + (\beta + \gamma)(BC + CB) = 0\\ &\quad\Leftrightarrow (\alpha A + \beta B + \gamma C)(S - I_m) + (S - I_m)(\alpha A + \beta B + \gamma C) = 0,\\ &(\alpha - \beta)(AB - BA) + (\alpha - \gamma)(AC - CA) + (\beta - \gamma)(BC - CB) = 0\\ &\quad\Leftrightarrow (\alpha A + \beta B + \gamma C)S = S(\alpha A + \beta B + \gamma C),\\ &\alpha(B + C)A(B + C) + \beta(A + C)B(A + C) + \gamma(A + B)C(A + B) = 0\\ &\quad\Leftrightarrow (S - I_m)(\alpha A + \beta B + \gamma C)(S - I_m) = 0.\end{aligned}$$

(d) *The equivalent facts below hold*

$$\begin{aligned}&(A + B)^2 + (A + C)^2 + (B + C)^2 = 0 \Leftrightarrow S^2 + S = 0,\\ &(A + B)^2 + (A + C)^2 + (B + C)^2 = I_m \Leftrightarrow S^2 + S = I_m,\\ &(A - B)^2 + (A - C)^2 + (B - C)^2 = 0 \Leftrightarrow (2S - 3I_m)^2 = 9I_m,\\ &(A - B)^2 + (A - C)^2 + (B - C)^2 = 9/8I_m \Leftrightarrow (2S - 3I_m)^2 = 9I_m,\\ &(A - B)^2 + (A - C)^2 + (B - C)^2 = 3I_m \Leftrightarrow (2S - 3I_m)^2 = -3I_m,\\ &(A - B)^2 + (A - C)^2 + (B - C)^2 = 9/4I_m \Leftrightarrow (2S - 3I_m)^2 = 0,\\ &AB + BA + AC + CA + BC + CB = kI_m \Leftrightarrow (I_m - 2S)^2 = (4k + 1)I_m, \quad k = 0, 1,\ldots,6.\end{aligned}$$

(e) $\alpha(AB + AC) + \beta(BA + BC) + \gamma(CA + CB)$ *is nonsingular* $\Leftrightarrow$ $\alpha(BA + CA) + \beta(AB + CB) + \gamma(AC + BC)$ *is nonsingular* $\Leftrightarrow$ $\alpha(B + C)A(B + C) + \beta(A + C)B(A + C) + \gamma(A + B)C(A + B)$ $\Leftrightarrow$ *both* $\alpha A + \beta B + \gamma C$ and $S - I_m$ *are nonsingular, in which cases, the following equalities hold*

$$[\alpha(AB + AC) + \beta(BA + BC) + \gamma(CA + CB)]^{-1} = (S - I_m)^{-1}(\alpha A + \beta B + \gamma C)^{-1},$$
$$[\alpha(BA + CA) + \beta(AB + CB) + \gamma(AC + BC)]^{-1} = (\alpha A + \beta B + \gamma C)^{-1}(S - I_m)^{-1},$$
$$[\alpha(B + C)A(B + C) + \beta(A + C)B(A + C) + \gamma(A + B)C(A + B)]^{-1} = (S - I_m)^{-1}(\alpha A + \beta B + \gamma C)^{-1}(S - I_m)^{-1}.$$

(f) $(A + B)^2 + (A + C)^2 + (B + C)^2$ *is nonsingular if and only if* S *and* $I_m + S$ *are nonsingular, in which case*, $[(A + B)^2 + (A + C)^2 + (B + C)^2]^{-1} = S^{-1}(I_m + S)^{-1}$.

(g) $(A - B)^2 + (A - C)^2 + (B - C)^2$ *is nonsingular if and only if* S *and* $3I_m - S$ *are nonsingular, in which case*, $[(A - B)^2 + (A - C)^2 + (B - C)^2]^{-1} = S^{-1}(3I_m - S)^{-1}$.

(h) $AB + BA + AC + CA + BC + CB$ *is nonsingular if and only if* S *and* $I_m - S$ *are nonsingular, in which case*, $(AB + BA + AC + CA + BC + CB)^{-1} = S^{-1}(I_m - S)^{-1}$.

Proof. Multiplying the matrices on the right-hand sides of (3.87), (3.88), and (3.91) yields the three equalities. The sum and difference of (3.87) and (3.88) result in (3.89) and (3.90), respectively. Equations (3.92)–(3.95) follow from direct expansions. Results (c)–(h) follow from (3.87)–(3.95). □

Apparently, the algebraic matrix identities in Theorem 3.8 can be extended without much effort to a family of idempotent matrices of the same size. This fact demonstrates that all idempotent matrices are undoubtedly linked to each other through a variety of nontrivial matrix equalities, which in turn can be utilized to describe algebraic performance of matrix expressions that involve idempotent matrices.

4 Miscellaneous rank and range formulas for idempotent matrices

People have posed and approached a large number of problems on idempotent matrices, one of which is to establish various exact formulas for calculating the ranks of matrix expressions that are composed by idempotent matrices. With the background material we have accumulated in the previous sections, it is possible to establish a wide range of analytical formulas for calculating the ranks of matrix expressions composed by idempotent matrices and obtain many meaningful consequences, including the characterizations of relationships among given idempotent matrices and their operations and the derivation of matrix range equalities.

The author begins with a group of fundamental rank equalities for two idempotent matrices and their consequences that arise from Lemma 2.1.

Theorem 4.1. *Given two idempotent matrices* $A, B \in \mathbb{C}^{m\times m}$, *we have the following results*:

(a) *The rank equalities below hold*

$$r[A, B] = r(A) + r(\tilde{A}B) = r(B) + r(\tilde{B}A), \tag{4.1}$$

$$r[\tilde{A}, B] = r(AB) - r(A) + m = r(B) + r(\tilde{B}\tilde{A}), \tag{4.2}$$

$$r[A, \tilde{B}] = r(BA) - r(B) + m = r(A) + r(\tilde{A}\tilde{B}), \tag{4.3}$$

$$r[\tilde{A}, \tilde{B}] = r\begin{bmatrix} A \\ B \end{bmatrix} - r(A) - r(B) + m, \tag{4.4}$$

$$r\begin{bmatrix} A \\ B \end{bmatrix} = r(A) + r(B\tilde{A}) = r(B) + r(A\tilde{B}), \tag{4.5}$$

$$r\begin{bmatrix} \tilde{A} \\ B \end{bmatrix} = r(BA) - r(A) + m = r(B) + r(\tilde{A}\tilde{B}), \tag{4.6}$$

$$r\begin{bmatrix} A \\ \tilde{B} \end{bmatrix} = r(AB) - r(B) + m = r(A) + r(\tilde{B}\tilde{A}), \tag{4.7}$$

$$r\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} = r[A, B] - r(A) - r(B) + m, \tag{4.8}$$

and the rank inequalities below hold

$$r[A, B] \geq \max\{r(A) + r(B) - r(AB), r(A) + r(B) - r(BA)\}, \tag{4.9}$$

$$r[\tilde{A}, B] \geq \max\{r(\tilde{A}) + r(B) - r(\tilde{A}B), r(\tilde{A}) + r(B) - r(B\tilde{A})\}, \tag{4.10}$$

$$r[A, \tilde{B}] \geq \max\{r(A) + r(\tilde{B}) - r(A\tilde{B}), r(A) + r(\tilde{B}) - r(\tilde{B}A)\}, \tag{4.11}$$

$$r[\tilde{A}, \tilde{B}] \geq \max\{r(\tilde{A}) + r(\tilde{B}) - r(\tilde{A}\tilde{B}), r(\tilde{A}) + r(\tilde{B}) - r(\tilde{B}\tilde{A})\}. \tag{4.12}$$

(b) $r[A, B] = r(A) \Leftrightarrow \mathscr{R}(A) \supseteq \mathscr{R}(B) \Leftrightarrow AB = B$.

(c) $r\begin{bmatrix} A \\ B \end{bmatrix} = r(A) \Leftrightarrow \mathscr{R}(A^*) \supseteq \mathscr{R}(B^*) \Leftrightarrow BA = B$.

(d) $r[\tilde{A}, B] = m \Leftrightarrow r(AB) = r(A)$.

(e) $r[A, \tilde{B}] = m \Leftrightarrow r(BA) = r(B)$.

(f) $r[\tilde{A}, \tilde{B}] = m \Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} = r(A) + r(B)$.

(g) $r\begin{bmatrix} \tilde{A} \\ B \end{bmatrix} = m \Leftrightarrow r(BA) = r(A)$.

(h) $r\begin{bmatrix} A \\ \tilde{B} \end{bmatrix} = m \Leftrightarrow r(AB) = r(B)$.

(i) $r\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} = m \Leftrightarrow r[A, B] = r(A) + r(B)$.

Proof. Choosing $A^- = A$ in (2.11) and (2.12) results in (4.1) and (4.5). Replacing A and B with $\tilde{A}$ and $\tilde{B}$ in (4.1) and (4.5), respectively, results in (4.2)–(4.4), and (4.6)–(4.8). It follows from the well-known rank inequality $r(A - B) \geq r(A) - r(B)$ that $r(\tilde{A}B) \geq r(A) - r(AB)$ and $r(\tilde{B}A) \geq r(A) - r(BA)$. Substituting these two inequalities into (4.1) yields (4.9). Replacing A and B with $\tilde{A}$ and $\tilde{B}$ in (4.9), respectively, results in (4.10)–(4.12). Results (b)–(i) are consequences of (4.1)–(4.8). □

Theorem 4.2. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, we have the following results*:

(a) *The rank equalities below hold*

$$r(AB) = r(A) + r(B) - m + r(\tilde{B}\tilde{A}), \tag{4.13}$$

$$r(BA) = r(A) + r(B) - m + r(\tilde{A}\tilde{B}), \tag{4.14}$$

$$r(AB) + r(\tilde{A}\tilde{B}) = r(BA) + r(\tilde{B}\tilde{A}). \tag{4.15}$$

(b) *If $BA = 0$, then $r(AB) = r(I_m - M) + r(A) + r(B) - m = r(I_m - M) - r(\tilde{A}\tilde{B})$.*

(c) *If $AB = 0$, then $r(BA) = r(I_m - M) + r(A) + r(B) - m = r(I_m - M) - r(\tilde{B}\tilde{A})$.*

(d) *If $AB = BA = 0$, then $r(I_m - M) = m - r(A) - r(B)$.*

(e) $r(AB) = r(A) + r(B) - m \Leftrightarrow \tilde{B}\tilde{A} = 0 \Leftrightarrow \mathscr{N}(A) \subseteq \mathscr{R}(B) \Leftrightarrow \mathscr{R}(A^*) \supseteq \mathscr{N}(B^*)$.

(f) $r(AB) = r(BA) \Leftrightarrow r(\tilde{A}\tilde{B}) = r(\tilde{B}\tilde{A})$.

Proof. Equations (4.13) and (4.14) follow from (2.18). Substituting (4.13) into (4.14) yields (4.15). Results (b)–(f) are direct consequences of (4.13)–(4.15). □

Theorem 4.3. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, we have the following results:*
(a) *The rank equalities below hold*

$$r(\tilde{A}BA) = r(\tilde{A}\tilde{B}A) = r[A, B] + r(BA) - r(A) - r(B) = r(BA) - \dim[\mathscr{R}(A) \cap \mathscr{R}(B)], \tag{4.16}$$

$$r(AB\tilde{A}) = r(A\tilde{B}\tilde{A}) = r\begin{bmatrix} A \\ B \end{bmatrix} + r(AB) - r(A) - r(B) = r(AB) - \dim[\mathscr{R}(A^*) \cap \mathscr{R}(B^*)], \tag{4.17}$$

$$r(\tilde{B}AB) = r(\tilde{B}\tilde{A}B) = r[A, B] + r(AB) - r(A) - r(B) = r(AB) - \dim[\mathscr{R}(A) \cap \mathscr{R}(B)], \tag{4.18}$$

$$r(BA\tilde{B}) = r(B\tilde{A}\tilde{B}) = r\begin{bmatrix} A \\ B \end{bmatrix} + r(BA) - r(A) - r(B) = r(BA) - \dim[\mathscr{R}(A^*) \cap \mathscr{R}(B^*)]. \tag{4.19}$$

(b) *The equivalent facts below hold*

$$\begin{aligned} \tilde{A}BA = 0 &\Leftrightarrow \tilde{A}\tilde{B}A = 0 \Leftrightarrow \mathscr{R}(BA) \subseteq \mathscr{R}(A) \Leftrightarrow \mathscr{R}(\tilde{B}A) \subseteq \mathscr{R}(A) \\ &\Leftrightarrow \mathscr{R}(BA) = \mathscr{R}(A) \cap \mathscr{R}(B) \Leftrightarrow \mathscr{R}(\tilde{B}A) = \mathscr{R}(A) \cap \mathscr{R}(\tilde{B}) \\ &\Leftrightarrow r[A, B] = r(A) + r(B) - r(BA) \Leftrightarrow r[A, \tilde{B}] = r(A) + r(\tilde{B}) - r(\tilde{B}A), \end{aligned} \tag{4.20}$$

$$\begin{aligned} AB\tilde{A} = 0 &\Leftrightarrow A\tilde{B}\tilde{A} = 0 \Leftrightarrow \mathscr{R}[(AB)^*] \subseteq \mathscr{R}(A^*) \Leftrightarrow \mathscr{R}[(A\tilde{B})^*] \subseteq \mathscr{R}(A^*) \\ &\Leftrightarrow \mathscr{R}[(AB)^*] = \mathscr{R}(A^*) \cap \mathscr{R}(B^*) \Leftrightarrow \mathscr{R}[(A\tilde{B})^*] = \mathscr{R}(A^*) \cap \mathscr{R}(\tilde{B}^*) \\ &\Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} = r(A) + r(B) - r(AB) \Leftrightarrow r\begin{bmatrix} A \\ \tilde{B} \end{bmatrix} = r(A) + r(\tilde{B}) - r(A\tilde{B}), \end{aligned} \tag{4.21}$$

$$\begin{aligned} \tilde{B}AB = 0 &\Leftrightarrow \tilde{B}\tilde{A}B = 0 \Leftrightarrow \mathscr{R}(AB) \subseteq \mathscr{R}(B) \Leftrightarrow \mathscr{R}(AB) \subseteq \mathscr{R}(B) \\ &\Leftrightarrow \mathscr{R}(AB) = \mathscr{R}(A) \cap \mathscr{R}(B) \Leftrightarrow \mathscr{R}(\tilde{A}B) = \mathscr{R}(\tilde{A}) \cap \mathscr{R}(B) \\ &\Leftrightarrow r[A, B] = r(A) + r(B) - r(AB) \Leftrightarrow r[\tilde{A}, B] = r(\tilde{A}) + r(B) - r(\tilde{A}B), \end{aligned} \tag{4.22}$$

$$\begin{aligned} BA\tilde{B} = 0 &\Leftrightarrow B\tilde{A}\tilde{B} = 0 \Leftrightarrow \mathscr{R}[(BA)^*] \subseteq \mathscr{R}(B^*) \Leftrightarrow \mathscr{R}[(B\tilde{A})^*] \subseteq \mathscr{R}(B^*) \\ &\Leftrightarrow \mathscr{R}[(BA)^*] = \mathscr{R}(A^*) \cap \mathscr{R}(B^*) \Leftrightarrow \mathscr{R}[(B\tilde{A})^*] = \mathscr{R}(\tilde{A}^*) \cap \mathscr{R}(B^*) \\ &\Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} = r(A) + r(B) - r(BA) \Leftrightarrow r\begin{bmatrix} \tilde{A} \\ B \end{bmatrix} = r(\tilde{A}) + r(B) - r(B\tilde{A}). \end{aligned} \tag{4.23}$$

(c) *The equivalent facts below hold*

$$\mathscr{R}[(\tilde{A}BA)^*] = \mathscr{R}[(BA)^*] \Leftrightarrow \mathscr{R}[(\tilde{B}AB)^*] = \mathscr{R}[(AB)^*] \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}, \tag{4.24}$$

$$\mathscr{R}(AB\tilde{A}) = \mathscr{R}(AB) \Leftrightarrow \mathscr{R}(BA\tilde{B}) = \mathscr{R}(BA) \Leftrightarrow \mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\}. \tag{4.25}$$

Proof. Note from Lemma 3.1(m) that $\mathscr{R}(\tilde{B}A) \cap \mathscr{R}(BA) = \{0\}$. Thus, we obtain from (2.11), Lemma 2.1(e), and EBMOs that

$$\begin{aligned} r(\tilde{A}BA) &= r[A, BA] - r(A) = r[\tilde{B}A, BA] - r(A) = r(\tilde{B}A) + r(BA) - r(A) \\ &= r[A, B] + r(BA) - r(A) - r(B) = r(BA) - \dim[\mathscr{R}(A) \cap \mathscr{R}(B)] \quad (\text{by } 2.39)), \end{aligned}$$

establishing (4.16). Equations (4.17)–(4.19) can be established by a similar approach. Setting both sides of (4.16)–(4.19) equal to zero leads to the equivalent facts in (4.20)–(4.23). Setting both sides of (4.17)–(4.19) equal to $r(AB)$ or $r(BA)$ and combining the rank equalities obtained with the following obvious facts $\mathscr{R}[(\tilde{A}BA)^*] \subseteq \mathscr{R}[(BA)^*]$, $\mathscr{R}[(\tilde{B}AB)^*] \subseteq \mathscr{R}[(AB)^*]$, $\mathscr{R}(AB\tilde{A}) \subseteq \mathscr{R}(AB)$, and $\mathscr{R}(BA\tilde{B}) \subseteq \mathscr{R}(BA)$ lead to the equivalent facts in (4.24) and (4.25). □

Next, the author gives some groups of known equalities and inequalities for the ranks of $A \pm B$ and $AB \pm BA$ and their consequences with easy proofs.

Theorem 4.4. [36] *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, we have the following results:*
(a) *The rank equalities below hold*

$$r(A - B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B) = r\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} + r[\tilde{A}, \tilde{B}] + r(\tilde{A}) + r(\tilde{B}), \tag{4.26}$$

$$r(A - B) = r(A\tilde{B}) + r(\tilde{A}B) = r(\tilde{B}A) + r(B\tilde{A}). \tag{4.27}$$

(b) *If $AB = 0$, then $r(A - B) = r(\tilde{B}A) + r(B\tilde{A}) = r(A) + r(B)$.*
(c) *If $BA = 0$, then $r(A - B) = r(A\tilde{B}) + r(\tilde{A}B) = r(A) + r(B)$.*
(d) *$A = B \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B)$ and $\mathscr{R}(A^*) = \mathscr{R}(B^*) \Leftrightarrow \mathscr{N}(A) = \mathscr{N}(B)$ and $\mathscr{N}(A^*) = \mathscr{N}(B^*)$.*
(e) *$r(A - B) = m \Leftrightarrow r(\tilde{A}B) = r(B\tilde{A}) = r(\tilde{A}) = r(B) \Leftrightarrow r(A\tilde{B}) = r(\tilde{B}A) = r(A) = r(\tilde{B})$.*
(f) *$A + B = I_m \Leftrightarrow m + r(AB) + r(BA) = r(A) + r(B) \Leftrightarrow \mathscr{R}(\tilde{A}) = \mathscr{R}(B)$ and $\mathscr{R}(\tilde{A}^*) = \mathscr{R}(B^*) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(\tilde{B})$ and $\mathscr{R}(A^*) = \mathscr{R}(\tilde{B}^*)$.*
(g) *$r(A - B) = r(A) - r(B) \Leftrightarrow ABA = B \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$.*
(h) *$r(A - B) = m \Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} = r[A, B] = r(A) + r(B) = m \Leftrightarrow \mathscr{R}(A) \oplus \mathscr{R}(B) = \mathscr{R}(A^*) \oplus \mathscr{R}(B^*) = \mathbb{C}^m \Leftrightarrow \mathscr{N}(A) \oplus \mathscr{N}(B) = \mathscr{N}(A^*) \oplus \mathscr{N}(B^*) = \mathbb{C}^m$.*

Proof. Setting $X = A$ and $Y = B$, and $X = \tilde{A}$ and $Y = \tilde{B}$, respectively, in Theorem 2.6 leads to the two equalities in (4.26). Applying (2.11) to (4.26) leads to (4.27). Results (b)–(h) are direct consequences of (4.26) and (4.27). □

Theorem 4.5. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$ and a scalar $\lambda \neq 0$, we have the following results:*
(a) *The rank equalities below hold*

$$r\begin{bmatrix} \lambda I_m + A & AB \\ BA & \lambda I_m + B \end{bmatrix} = m + r(\lambda I_m + M), \tag{4.28}$$

$$r\begin{bmatrix} \lambda I_m + AB & A \\ B & \lambda I_m + BA \end{bmatrix} = m + r(\lambda I_m + AB + BA). \tag{4.29}$$

(b) $r\begin{bmatrix} \lambda I_m + A & AB \\ BA & \lambda I_m + B \end{bmatrix} = 2m \Leftrightarrow r(I_m + M) = m.$
(c) *If $AB = 0$ or $BA = 0$, then $r(\lambda I_m + M) = m$.*
(d) $r\begin{bmatrix} \lambda I_m + AB & A \\ B & \lambda I_m + BA \end{bmatrix} = 2m \Leftrightarrow r(\lambda I_m + AB + BA) = m.$
(e) $r\begin{bmatrix} \lambda I_m + AB & A \\ B & \lambda I_m + BA \end{bmatrix} = m \Leftrightarrow \lambda I_m + AB + BA = 0.$

Proof. Applying (2.7) to the two pairs of products $\lambda^{-1}\begin{bmatrix} A \\ B \end{bmatrix}[A, B]$ and $\lambda^{-1}[A, B]\begin{bmatrix} A \\ B \end{bmatrix}$, $\lambda^{-1}\begin{bmatrix} A \\ B \end{bmatrix}[B, A]$ and $\lambda^{-1}[B, A]\begin{bmatrix} A \\ B \end{bmatrix}$ leads to (4.28) and (4.29). Results (b)–(e) are direct consequences of (4.28) and (4.29). □

Theorem 4.6. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, we have the following results:*
(a) *The rank equalities below hold*

$$r(I_m - M) = r\begin{bmatrix} \tilde{A} & AB \\ BA & \tilde{B} \end{bmatrix} - m = r\begin{bmatrix} A & \tilde{A}\tilde{B} \\ \tilde{B}\tilde{A} & B \end{bmatrix} - m, \tag{4.30}$$

$$r(I_m - M) = r(AB) + r(BA) - r(A) - r(B) + m, \tag{4.31}$$

$$r(I_m - M) = r(\tilde{A}\tilde{B}) + r(\tilde{B}\tilde{A}) + r(A) + r(B) - m. \tag{4.32}$$

(b) $A + B = I_m \Leftrightarrow r\begin{bmatrix} \tilde{A} & AB \\ BA & \tilde{B} \end{bmatrix} = m \Leftrightarrow r\begin{bmatrix} A & \tilde{A}\tilde{B} \\ \tilde{B}\tilde{A} & B \end{bmatrix} = m \Leftrightarrow r(AB) + r(BA) = r(A) + r(B) - m \Leftrightarrow r(A) + r(B) + r(\tilde{A}\tilde{B}) + r(\tilde{A}\tilde{B}) = m \Leftrightarrow AB = BA = 0$ *and* $r(A) + r(B) = m \Leftrightarrow \mathscr{R}(A) = \mathscr{N}(B)$ *and* $\mathscr{R}(A^*) = \mathscr{N}(B^*) \Leftrightarrow \mathscr{N}(A) = \mathscr{R}(B)$ *and* $\mathscr{N}(A^*) = \mathscr{R}(B^*)$.

(c) $r(I_m - M) = m \Leftrightarrow r\begin{bmatrix} \tilde{A} & AB \\ BA & \tilde{B} \end{bmatrix} = 2m \Leftrightarrow r\begin{bmatrix} A & \tilde{A}\tilde{B} \\ \tilde{B}\tilde{A} & B \end{bmatrix} = 2m \Leftrightarrow r(AB) = r(BA) = r(A) = r(B) \Leftrightarrow \mathscr{R}(A) \oplus \mathscr{N}(B) = \mathscr{R}(A^*) \oplus \mathscr{N}(B^*) = \mathbb{C}^m \Leftrightarrow \mathscr{N}(A) \oplus \mathscr{R}(B) = \mathscr{N}(A^*) \oplus \mathscr{R}(B^*) = \mathbb{C}^m$.

Proof. Setting $\lambda = -1$ in (4.28) leads to the first equality in (4.30); replacing A and B with $\tilde{A}$ and $\tilde{B}$ in the first equality in (4.28) leads to the second equality in (4.30). Replacing A and B with $\tilde{A}$ and $\tilde{B}$ in (4.26) and (4.27) and simplifying by (4.26) and (4.27) lead to (4.31) and (4.32). Results (b) and (c) are direct consequences of (4.30)–(4.32). □

Next, the author presents some known closed-form formulas in [32,33,36,38] for calculating the ranks of $A + B$ and gives their proofs by the BMM.

Theorem 4.7. *Given two idempotent matrices* $A, B \in \mathbb{C}^{m\times m}$, *and two scalars* α and β *with* $\alpha \neq 0$, $\beta \neq 0$, *and* $\alpha + \beta \neq 0$, *we have the following results*:

(a) $r(\alpha A + \beta B) = r(A + B)$ *holds.*

(b) *The rank equalities below hold*

$$r(A + B) = r\begin{bmatrix} A & B \\ B & 0 \end{bmatrix} - r(B) = r\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} - r(A), \tag{4.33}$$

$$r(A + B) = r[A\tilde{B}, B] = r[B\tilde{A}, A] = r\begin{bmatrix} \tilde{B}A \\ B \end{bmatrix} = r\begin{bmatrix} \tilde{A}B \\ A \end{bmatrix}, \tag{4.34}$$

$$r(A + B) = r(\tilde{B}A\tilde{B}) + r(B) = r(\tilde{A}B\tilde{A}) + r(A), \tag{4.35}$$

$$r(A + B) = r\begin{bmatrix} A & B & 0 \\ B & 0 & A \end{bmatrix} - r[A, B] = r\begin{bmatrix} A & B \\ B & 0 \\ 0 & A \end{bmatrix} - r\begin{bmatrix} A \\ B \end{bmatrix}, \tag{4.36}$$

$$r(A + B) = r\begin{bmatrix} \tilde{B}A \\ \tilde{A}B \end{bmatrix} + \dim[\mathscr{R}(A) \cap \mathscr{R}(B)] = r[A\tilde{B}, B\tilde{A}] + \dim[\mathscr{R}(A^*) \cap \mathscr{R}(B^*)], \tag{4.37}$$

$$r(A + B) = r(A) + r(B) - m + r\left(\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}[\tilde{A}, \tilde{B}]\right), \tag{4.38}$$

$$r(A + B) = r\begin{bmatrix} I_m + 2A & I_m \\ I_m & I_m + 2B \end{bmatrix} - m. \tag{4.39}$$

(c) *If* $AB = BA$, *then* $r(A + B) = r[A, B] = r\begin{bmatrix} A \\ B \end{bmatrix}$.

(d) *If* $AB = 0$ *or* $BA = 0$, *then* $r(A + B) = r(A) + r(B)$.

(e) $r(A + B) = r(A) + r(B) \Leftrightarrow r\begin{bmatrix} A & B \\ B & 0 \end{bmatrix} = r(A) + 2r(B) \Leftrightarrow r\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} = r(A) + 2r(B) \Leftrightarrow r(\tilde{B}A\tilde{B}) = r(A)$

$\Leftrightarrow r(\tilde{A}B\tilde{A}) = r(B) \Leftrightarrow r\left(\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}[\tilde{A}, \tilde{B}]\right) = m \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$ and $\mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\}$.

(f) $A + B = 0 \Leftrightarrow A = B = 0$.

(g) $r(A+B) = m \Leftrightarrow r(\tilde{A}B\tilde{A}) = r(\tilde{A}) \Leftrightarrow r(\tilde{B}A\tilde{B}) = r(\tilde{B}) \Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} = m$ *and* $\mathscr{R}\begin{bmatrix} A \\ B \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} B \\ 0 \end{bmatrix} = \{0\} \Leftrightarrow r[A, B] = m$ *and* $\mathscr{R}\begin{bmatrix} A^* \\ B^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} B^* \\ 0 \end{bmatrix} = \{0\} \Leftrightarrow r\begin{bmatrix} B \\ A \end{bmatrix} = m$ *and* $\mathscr{R}\begin{bmatrix} B \\ A \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} A \\ 0 \end{bmatrix} = \{0\} \Leftrightarrow r[A, B] = m$ *and* $\mathscr{R}\begin{bmatrix} B^* \\ A^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} A^* \\ 0 \end{bmatrix} = \{0\} \Leftrightarrow r\left(\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}[\tilde{A}, \tilde{B}]\right) + r(A) + r(B) = 2m \Leftrightarrow r\begin{bmatrix} I_m + 2A & I_m \\ I_m & I_m + 2B \end{bmatrix} = 2m.$

(h) $r(A+B) \geq \max\{r(A), r(B), r(A-B)\}$. *In particular, if* $r(A-B) = m$, *then* $r(A+B) = m$.

Proof. Result (a) follows directly from (3.1). It can be deduced by EBMOs that

$$r\begin{bmatrix} A & 0 & A \\ 0 & B & B \\ A & B & 0 \end{bmatrix} = r\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -A-B \end{bmatrix} = r(A) + r(B) + r(A+B).$$

On the other hand, we obtain from $A^2 = A$, $B^2 = B$, and EBMOs that

$$r\begin{bmatrix} A & 0 & A \\ 0 & B & B \\ A & B & 0 \end{bmatrix} = r\begin{bmatrix} A & 0 & A \\ -BA & 0 & B \\ A & B & 0 \end{bmatrix} = r\begin{bmatrix} 2A & 0 & A \\ 0 & 0 & B \\ A & B & 0 \end{bmatrix} = r\begin{bmatrix} 2A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & -\frac{1}{2}A \end{bmatrix} = r\begin{bmatrix} A & B \\ B & 0 \end{bmatrix} + r(A).$$

Combining these two rank equalities yields (4.33). Applying (2.11)–(2.13) to the two block matrices in (4.33) yields the equalities in (4.34) and (4.35), respectively. By (2.14) and EBMOs,

$$r\begin{bmatrix} A & B & 0 \\ B & 0 & A \end{bmatrix} = r\begin{bmatrix} \tilde{B}A \\ \tilde{A}B \end{bmatrix} + r(A) + r(B),$$

$$\begin{aligned} r\begin{bmatrix} A & B & 0 \\ B & 0 & A \end{bmatrix} &= r\begin{bmatrix} A & B-AB & 0 \\ B-BA & -B & A \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & B-AB & 0 \\ B-BA & -B & A \end{bmatrix} \\ &= r(A) + r\begin{bmatrix} 0 & B-AB & 0 \\ B-BA & -AB & A \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & B-AB & 0 \\ B-BA & 0 & A \end{bmatrix} \\ &= r(A) + r(B-AB) + r[B-BA, A] = [A, B] + r\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} - r(A) \\ &= r[A, B] + r(A+B) \quad \text{(by (4.33))} \\ &= r(A+B) + r(A) + r(B) - \dim[\mathscr{R}(A) \cap \mathscr{R}(B)], \end{aligned}$$

$$r\begin{bmatrix} A & B \\ B & 0 \\ 0 & A \end{bmatrix} = r[A\tilde{B}, B\tilde{A}] + r(A) + r(B),$$

$$r\begin{bmatrix} A & B \\ B & 0 \\ 0 & A \end{bmatrix} = r\begin{bmatrix} A \\ B \end{bmatrix} + r(A+B) = r(A+B) + r(A) + r(B) - \dim[\mathscr{R}(A^*) \cap \mathscr{R}(B^*)].$$

Combining these equalities leads to (4.36) and (4.37). By (2.13) and EBMOs,

$$\begin{aligned} r\left(\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}[\tilde{A}, \tilde{B}]\right) &= r\begin{bmatrix} \begin{bmatrix} I_m & I_m \\ I_m & I_m \end{bmatrix} & \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} - 2r\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = r\begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & 0 & -A & B \\ 0 & -A & -A & 0 \\ 0 & B & 0 & 0 \end{bmatrix} - 2r\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ &= r\begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & -A & 0 \\ 0 & B & 0 & 0 \end{bmatrix} - 2r(A) - 2r(B) = m + r\begin{bmatrix} A & B \\ B & 0 \end{bmatrix} - r(A) - 2r(B) \\ &= r(A+B) - r(A) - r(B) + m \quad \text{(by (4.33))}, \end{aligned}$$

and

$$r\begin{bmatrix} \begin{bmatrix} I_m & I_m \\ I_m & I_m \end{bmatrix} & \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = r\begin{bmatrix} \begin{bmatrix} I_m/2 & I_m/2 \\ I_m/2 & I_m/2 \end{bmatrix} & \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$
$$= r\begin{bmatrix} I_m/2 + A & I_m/2 \\ I_m/2 & I_m/2 + B \end{bmatrix} + r(A) + r(B) \quad (\text{by } (4.33)),$$
$$= r\begin{bmatrix} I_m + 2A & I_m \\ I_m & I_m + 2B \end{bmatrix} + r(A) + r(B),$$

establishing (4.38) and (4.39). Results (b)–(f) are direct consequences of (4.33)–(4.39).

By (4.35), the two inequalities $r(A + B) \geq r(A)$ and $r(A + B) \geq r(B)$ hold, and by (2.24) and (4.26), the inequality $r(A + B) \geq r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] - r(A) - r(B) = r(A - B)$ holds. Combining these inequalities leads to Result (g). □

Theorem 4.8. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, we have the following results*:

(a) *The rank equalities below hold*

$$r(I_m + A - B) = r(BAB) - r(B) + m, \tag{4.40}$$
$$r(I_m - A + B) = r(ABA) - r(A) + m, \tag{4.41}$$
$$\begin{aligned} r(2I_m - M) &= r(I_m - AB) = r(I_m - BA) = r(I_m - ABA) = r(I_m - BAB) \\ &= r(B\tilde{A}B) - r(B) + m = r(A\tilde{B}A) - r(A) + m, \end{aligned} \tag{4.42}$$
$$r(2I_m - M) = r\begin{bmatrix} A\tilde{B} \\ B\tilde{A} \end{bmatrix} + \dim[\mathcal{N}(A) \cap \mathcal{N}(B)] = r[\tilde{A}B, \tilde{B}A] + \dim[\mathcal{N}(A^*) \cap \mathcal{N}(B^*)], \tag{4.43}$$
$$r(2I_m - M) = r\left(\begin{bmatrix} A \\ B \end{bmatrix}[A, B]\right) - r(A) - r(B) + m, \tag{4.44}$$
$$r(2I_m - M) = r\left(\begin{bmatrix} A \\ B \end{bmatrix}[A, B]\right) + r\left(\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}[\tilde{A}, \tilde{B}]\right) - r(M), \tag{4.45}$$
$$r(2I_m - M) = r\begin{bmatrix} 3I_m - 2A & I_m \\ I_m & 3I_m - 2B \end{bmatrix} - m, \tag{4.46}$$
$$r(A + ABA) = r(I_m + BAB) + r(A) - m, \tag{4.47}$$
$$r(B + BAB) = r(I_m + ABA) + r(B) - m. \tag{4.48}$$

(b) $ABA = 0 \Leftrightarrow r(I_m - A + B) = m - r(A)$.

(c) $BAB = 0 \Leftrightarrow r(I_m + A - B) = m - r(B)$.

(d) $r(ABA) = r(A) \Leftrightarrow r(I_m - A + B) = m$.

(e) $r(BAB) = r(B) \Leftrightarrow r(I_m + A - B) = m$.

(f) $ABA = A \Leftrightarrow r(2I_m - M) = m - r(A) \Leftrightarrow r(I_m - BAB) = m - r(A)$.

(g) $BAB = B \Leftrightarrow r(2I_m - M) = m - r(B) \Leftrightarrow r(I_m - ABA) = m - r(B)$.

(h) $ABA = A$ and $BAB = B \Leftrightarrow r(2I_m - M) = m - r(A) = m - r(B) \Leftrightarrow r(I_m - BAB) = r(I_m - ABA) = m - r(A) = m - r(B)$.

(i) $A + B = 2I_m \Leftrightarrow A = B = I_m$.

(j) $r(2I_m - M) = m \Leftrightarrow r(I_m - AB) = m \Leftrightarrow r(I_m - BA) = m \Leftrightarrow r(I_m - ABA) = m \Leftrightarrow r(I_m - BAB) = m \Leftrightarrow r(A\tilde{B}A) = r(A) \Leftrightarrow r(B\tilde{A}B) = r(B) \Leftrightarrow r\begin{bmatrix} A\tilde{B} \\ B\tilde{A} \end{bmatrix} + \dim[\mathcal{N}(A) \cap \mathcal{N}(B)] = m$

$$\Leftrightarrow r[\tilde{A}B, \tilde{B}A] + \dim[\mathscr{N}(A^*) \cap \mathscr{N}(B^*)] = m \Leftrightarrow r\left(\begin{bmatrix} A \\ B \end{bmatrix}[A, B]\right) = r(A) + r(B)$$

$$\Leftrightarrow r\left(\begin{bmatrix} A \\ B \end{bmatrix}[A, B]\right) + r\left(\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}[\tilde{A}, \tilde{B}]\right) = m + r(M) \Leftrightarrow r\begin{bmatrix} 3I_m - 2A & I_m \\ I_m & 3I_m - 2B \end{bmatrix} = 2m.$$

(k) $r(A + B) = r\left(\begin{bmatrix} A \\ B \end{bmatrix}[A, B]\right) \Leftrightarrow r(2I_m - M) = r\left(\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}[\tilde{A}, \tilde{B}]\right)$.

(l) $r(I_m + ABA) = m \Leftrightarrow r(B + BAB) = r(B)$.

(m) $r(I_m + BAB) = m \Leftrightarrow r(A + ABA) = r(A)$.

Proof. By (4.33) and (2.13),

$$r(A + \tilde{B}) = r\begin{bmatrix} A & \tilde{B} \\ \tilde{B} & 0 \end{bmatrix} - r(\tilde{B}) = r(BAB) - r(B) + m,$$

as required in (4.40). Equation (4.41) can be established similarly. Next, by (4.33) and (2.13),

$$r(2I_m - M) = r(\tilde{A} + \tilde{B}) = r\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & 0 \end{bmatrix} - r(\tilde{B}) = r(B\tilde{A}B) - r(B) + m;$$

and by (2.14) and EBMO,

$$r\begin{bmatrix} I_m & AB \\ BA & B \end{bmatrix} = r\begin{bmatrix} I_m & 0 \\ 0 & B\tilde{A}B \end{bmatrix} = r\begin{bmatrix} I_m - ABA & 0 \\ 0 & B \end{bmatrix} = m + r(B\tilde{A}B) = r(I_m - ABA) + r(B).$$

Combining these equalities leads to the equalities in (4.42). Applying (4.37) to $\tilde{A} + \tilde{B}$ leads to (4.43). Applying (4.38) to $\tilde{A} + \tilde{B}$ leads to (4.44). Substituting (4.38) to (4.44) leads to (4.45). Replacing A and B with $\tilde{A}$ and $\tilde{B}$, respectively, in (4.39) leads to (4.46). By (2.17) and EBMO,

$$r\begin{bmatrix} I_m & AB \\ BA & -B \end{bmatrix} = r\begin{bmatrix} I_m & 0 \\ 0 & -B - BAB \end{bmatrix} = r\begin{bmatrix} I_m + ABA & 0 \\ 0 & -B \end{bmatrix} = m + r(B + BAB) = r(I_m + ABA) + r(B),$$

establishing (4.47). Equation (4.48) can be established similarly. Results (b)–(m) are direct consequences of (4.40)–(4.48). □

More rank formulas for two idempotent matrices and miscellaneous consequences are given below.

Theorem 4.9. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, we have the following results*:

$$r[M, I_m + M] = r[M, A + \tilde{B}] = r[M, \tilde{A} + B] = r[M, I_m - M] = m, \tag{4.49}$$

$$r[A - B, I_m + M] = r[A - B, A + \tilde{B}] = r[A - B, \tilde{A} + B] = r[A - B, I_m - M] = m, \tag{4.50}$$

$$r[M^2, (I_m + M)^2] = r[M^2, (A + \tilde{B})^2] = r[M^2, (\tilde{A} + B)^2] = r[M^2, (I_m - M)^2] = m, \tag{4.51}$$

$$r[(A - B)^2, (A + \tilde{B})^2] = r[(A - B)^2, (\tilde{A} + B)^2] = r[(A - B)^2, (I_m - M)^2] = m. \tag{4.52}$$

Proof. Follows from verifications by EBMOs. □

Theorem 4.10. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, we have the following results*:

(a) *The rank formulas below hold*

$$r(AB - BA) = r(A - B) + r(I_m - M) - m, \tag{4.53}$$

$$r(I_m + M^2) = r(iI_m + M) + r(iI_m - M) - m, \tag{4.54}$$

$$r(I_m - M^2) = r(I_m + M) + r(I_m - M) - m, \tag{4.55}$$

$$r(M+M^2) = r(2M+AB+BA) = r(M) + r(I_m+M) - m, \tag{4.56}$$

$$r(AB+BA) = r(M-M^2) = r(M) + r(I_m-M) - m, \tag{4.57}$$

$$r(M+M^3) = r(M) + r(iI_m+M) + r(iI_m-M) - 2m, \tag{4.58}$$

$$r(M-M^3) = r(M) + r(I_m+M) + r(I_m-M) - 2m, \tag{4.59}$$

$$r[I_m+(A-B)^2] = r(iI_m+A-B) + r(iI_m-A+B) - m, \tag{4.60}$$

$$r[(I_m-M)^2] = r[I_m-(A-B)^2] = r(A+\tilde{B}) + r(\tilde{A}+B) - m, \tag{4.61}$$

$$r[(A-B)+(A-B)^2] = r(2A-AB-BA) = r(A-B) + r(A+\tilde{B}) - m, \tag{4.62}$$

$$r[(A-B)-(A-B)^2] = r(2B-AB-BA) = r(A-B) + r(\tilde{A}+B) - m, \tag{4.63}$$

$$r[(A-B)+(A-B)^3] = r(A-B) + r(iI_m+A-B) + r(iI_m-A+B) - 2m, \tag{4.64}$$

$$\begin{aligned} r(ABA-BAB) &= r[(A-B)-(A-B)^3] = r(A-B) + r[(I_m-M)^2] - m \\ &= r(A-B) + r(A+\tilde{B}) + r(\tilde{A}+B) - 2m, \end{aligned} \tag{4.65}$$

$$r[I_m+(\tilde{A}+B)^2] = r[(i-1)I_m+A-B] + r[(i+1)I_m+A-B] - m, \tag{4.66}$$

$$r[I_m-(\tilde{A}+B)^2] = r(A-B) + r(2I_m-A+B) - m, \tag{4.67}$$

$$r[(\tilde{A}+B)+(\tilde{A}+B)^2] = r(\tilde{A}+B) + r(2I_m-A+B) - m, \tag{4.68}$$

$$r[(\tilde{A}+B)-(\tilde{A}+B)^2] = r(A-B) + r(\tilde{A}+B) - m, \tag{4.69}$$

$$r[(\tilde{A}+B)+(\tilde{A}+B)^3] = r(\tilde{A}+B) + r[(i-1)I_m+A-B] + r[(i+1)I_m+A-B] - 2m, \tag{4.70}$$

$$r[(\tilde{A}+B)-(\tilde{A}+B)^3] = r(A-B) + r(\tilde{A}+B) + r(2I_m-A+B) - 2m, \tag{4.71}$$

$$r[I_m+(I_m-M)^2] = r[(i-1)I_m+M] + r[(i+1)I_m+M] - m, \tag{4.72}$$

$$\begin{aligned} r[(A-B)^2] &= r(M-AB-BA) = r[I_m-(I_m-M)^2] = r[(M/2)^2-M/2] \\ &= r(M) + r(2I_m-M) - m, \end{aligned} \tag{4.73}$$

$$r[(I_m-M)+(I_m-M)^2] = r(I_m-M) + r(2I_m-M) - m, \tag{4.74}$$

$$r[(I_m-M)+(I_m-M)^3] = r(I_m-M) + r[(i-1)I_m+M] + r[(i+1)I_m+M] - 2m, \tag{4.75}$$

$$r[(I_m-M)-(I_m-M)^3] = r(M) + r(I_m-M) + r(2I_m-M) - 2m, \tag{4.76}$$

$$\begin{aligned} r(ABA+BAB) &= r[M(I_m-M)^2] = r[(I_m-M)^2M] = r(M) + r[(I_m-M)^2] - m \\ &= r(M) + r(A+\tilde{B}) + r(\tilde{A}+B) - 2m, \end{aligned} \tag{4.77}$$

$$\begin{aligned} r\,\mathrm{d}[(AB-BA)^2] &= r[I_m-(A-B)^4-(I_m-M)^4] = r[(A-B)^2-(A-B)^4] \\ &= r[(I_m-M)^2-(I_m-M)^4] = r[(A-B)^2] + r[(I_m-M)^2] - m \\ &= r(M) + r(A+\tilde{B}) + r(\tilde{A}+B) + r(2I_m-M) - 3m, \end{aligned} \tag{4.78}$$

$$r(I_m-AB-BA) = r[(\sqrt{5}-1)/2I_m+M] + r[(\sqrt{5}+1)/2I_m-M] - m, \tag{4.79}$$

$$r(2I_m-AB-BA) = r(I_m+M) + r(2I_m-M) - m; \tag{4.80}$$

(b) *The range formulas below hold*

$$\mathscr{R}(AB-BA) = \mathscr{R}(A-B) \cap (I_m-M), \tag{4.81}$$

$$\mathscr{R}(I_m+M^2) = \mathscr{R}(iI_m+M) \cap \mathscr{R}(iI_m-M), \tag{4.82}$$

$$\mathscr{R}(I_m-M^2) = \mathscr{R}(I_m+M) \cap \mathscr{R}(I_m-M), \tag{4.83}$$

$$\mathscr{R}(M+M^2) = \mathscr{R}(2M+AB+BA) = \mathscr{R}(M) \cap (I_m+M), \tag{4.84}$$

$$\mathscr{R}(AB + BA) = \mathscr{R}(M - M^2) = \mathscr{R}(M) \cap \mathscr{R}(I_m - M), \tag{4.85}$$

$$\mathscr{R}(M + M^3) = \mathscr{R}(M) \cap \mathscr{R}(iI_m + M) \cap \mathscr{R}(iI_m - M), \tag{4.86}$$

$$\mathscr{R}(M - M^3) = \mathscr{R}(A + B) \cap \mathscr{R}(I_m + M) \cap \mathscr{R}(I_m - M), \tag{4.87}$$

$$\mathscr{R}[I_m + (A - B)^2] = \mathscr{R}(iI_m + A - B) \cap \mathscr{R}(iI_m - A + B), \tag{4.88}$$

$$\mathscr{R}[(I_m - M)^2] = \mathscr{R}[I_m - (A - B)^2] = \mathscr{R}(A + \tilde{B}) \cap \mathscr{R}(\tilde{A} + B), \tag{4.89}$$

$$\mathscr{R}[(A - B) + (A - B)^2] = \mathscr{R}(A - B) \cap \mathscr{R}(A + \tilde{B}), \tag{4.90}$$

$$\mathscr{R}[(A - B) - (A - B)^2] = \mathscr{R}(A - B) \cap \mathscr{R}(\tilde{A} + B), \tag{4.91}$$

$$\mathscr{R}[(A - B) + (A - B)^3] = \mathscr{R}(A - B) \cap \mathscr{R}(iI_m + A - B) \cap \mathscr{R}(iI_m - A + B), \tag{4.92}$$

$$\mathscr{R}(ABA - BAB) = \mathscr{R}[(A - B) - (A - B)^3] = \mathscr{R}(A - B) \cap \mathscr{R}(A + \tilde{B}) \cap \mathscr{R}(\tilde{A} + B), \tag{4.93}$$

$$\mathscr{R}[I_m + (\tilde{A} + B)^2] = \mathscr{R}[(i - 1)I_m + A - B] \cap \mathscr{R}[(i + 1)I_m + A - B], \tag{4.94}$$

$$\mathscr{R}[I_m - (\tilde{A} + B)^2] = \mathscr{R}(A - B) \cap \mathscr{R}(2I_m - A + B), \tag{4.95}$$

$$\mathscr{R}[(\tilde{A} + B) + (\tilde{A} + B)^2] = \mathscr{R}(\tilde{A} + B) \cap \mathscr{R}(2I_m - A + B), \tag{4.96}$$

$$\mathscr{R}[(\tilde{A} + B) - (\tilde{A} + B)^2] = \mathscr{R}(A - B) \cap \mathscr{R}(\tilde{A} + B), \tag{4.97}$$

$$\mathscr{R}[(\tilde{A} + B) + (\tilde{A} + B)^3] = \mathscr{R}(\tilde{A} + B) \cap \mathscr{R}[(i - 1)I_m + A - B] \cap \mathscr{R}[(i + 1)I_m + A - B], \tag{4.98}$$

$$\mathscr{R}[(\tilde{A} + B) - (\tilde{A} + B)^3] = \mathscr{R}(A - B) \cap \mathscr{R}(\tilde{A} + B) \cap \mathscr{R}(2I_m - A + B), \tag{4.99}$$

$$\mathscr{R}[I_m + (I_m - M)^2] = \mathscr{R}[(i - 1)I_m + M] \cap \mathscr{R}[(i + 1)I_m + M], \tag{4.100}$$

$$\mathscr{R}[(A - B)^2] = \mathscr{R}[(M/2)^2 - M/2] = \mathscr{R}(M) \cap \mathscr{R}(2I_m - M), \tag{4.101}$$

$$\mathscr{R}[(I_m - M) + (I_m - M)^2] = \mathscr{R}(I_m - M) \cap \mathscr{R}(2I_m - M), \tag{4.102}$$

$$\mathscr{R}[(I_m - M) + (I_m - M)^3] = \mathscr{R}(I_m - M) \cap \mathscr{R}[(i - 1)I_m + M] \cap \mathscr{R}[(i + 1)I_m + M], \tag{4.103}$$

$$\mathscr{R}[(I_m - M) - (I_m - M)^3] = \mathscr{R}(M) \cap \mathscr{R}(I_m - M) \cap \mathscr{R}(2I_m - M), \tag{4.104}$$

$$\mathscr{R}(ABA + BAB) = \mathscr{R}(M) \cap \mathscr{R}[(I_m - M)^2] = \mathscr{R}(M) \cap \mathscr{R}(A + \tilde{B}) \cap \mathscr{R}(\tilde{A} + B), \tag{4.105}$$

$$\begin{aligned}\mathscr{R}(AB - BA)^2] &= \mathscr{R}[(A - B)^2] \cap \mathscr{R}[(I_m - M)^2] \\ &= \mathscr{R}(M) \cap \mathscr{R}(A + \tilde{B}) \cap \mathscr{R}(\tilde{A} + B) \cap \mathscr{R}(2I_m - M),\end{aligned} \tag{4.106}$$

$$\mathscr{R}(I_m - AB - BA) = \mathscr{R}[(\sqrt{5} - 1)/2I_m + M] \cap \mathscr{R}[(\sqrt{5} + 1)/2I_m - M], \tag{4.107}$$

$$\mathscr{R}(2I_m - AB - BA) = \mathscr{R}(I_m + M) \cap \mathscr{R}(2I_m - M); \tag{4.108}$$

(c) *The equivalent facts below hold*

⟨1⟩ $r(A - B) = m \Leftrightarrow r(M - AB - BA) = m \Leftrightarrow r[I_m - (I_m - M)^2] = m$
$\Leftrightarrow r[(M/2)^2 - M/2] = m \Leftrightarrow r(M) = r(2I_m - M) = m$;

⟨2⟩ $r(I_m - M) = m \Leftrightarrow r(A + \tilde{B}) = r(\tilde{A} + B) = m$;

⟨3⟩ $r(AB - BA) = m \Leftrightarrow r[(A - B)^4 - (A - B)^2] = m \Leftrightarrow r[(I_m - M)^4 - (I_m - M)^2] = m$
$\Leftrightarrow r(A - B) = r(I_m - M) = m \Leftrightarrow r(M) = r(A + \tilde{B}) = r(\tilde{A} + B) = r(2I_m - M) = m$;

⟨4⟩ $r(I_m + M^2) = m \Leftrightarrow r(iI_m + M) = r(iI_m - M) = m$;

⟨5⟩ $r(I_m - M^2) = m \Leftrightarrow r(I_m + M) = r(I_m - M) = m$;

⟨6⟩ $r(M + M^2) = m \Leftrightarrow r(M) = r(I_m + M) = m$;

⟨7⟩ $r(AB + BA) = m \Leftrightarrow r(M - M^2) = m \Leftrightarrow r[(I_m - M) - (I_m - M)^2] = m \Leftrightarrow r(M) = r(I_m - M) = m$;

⟨8⟩ $r(M + M^3) = m \Leftrightarrow r(M) = r(iI_m + M) = r(iI_m - M) = m$;

⟨9⟩ $r(M - M^3) = m \Leftrightarrow r(M) = r(I_m + M) = r(I_m - M) = m$;

⟨10⟩ $r[I_m + (A - B)^2] = m \Leftrightarrow r(iI_m + A - B) = r(iI_m - A + B) = m$;

⟨11⟩ $r\,[I_m - (A-B)^2] = m \Leftrightarrow r\,(I_m - M) = m \Leftrightarrow r\,(A+\tilde{B}) = r\,(\tilde{A}+B) = m$;
⟨12⟩ $r\,[(A-B) + (A-B)^2] = m \Leftrightarrow r\,(A-B) = r\,(A+\tilde{B}) = m$;
⟨13⟩ $r\,[(A-B) - (A-B)^2] = m \Leftrightarrow r\,(A-B) = r\,(\tilde{A}+B) = m$;
⟨14⟩ $r\,[(A-B) + (A-B)^3] = m \Leftrightarrow r\,(A-B) = r\,(iI_m + A - B) = r\,(iI_m - A + B) = m$;
⟨15⟩ $r\,[(A-B) - (A-B)^3] = m \Leftrightarrow r\,(A-B) = r\,(A+\tilde{B}) = r\,(\tilde{A}+B) = m$;
⟨16⟩ $r\,(ABA - BAB) = m \Leftrightarrow r\,[(A-B) - (A-B)^3] = m \Leftrightarrow r\,(A-B) + r\,[(I_m - M)^2] = m$
$\Leftrightarrow r\,(A-B) + r\,(A+\tilde{B}) + r\,(\tilde{A}+B) = 2m$;
⟨17⟩ $r\,[I_m + (\tilde{A}+B)^2] = m \Leftrightarrow r\,[(i-1)I_m + A - B] + r\,[(i+1)I_m + A - B] = m$;
⟨18⟩ $r\,[I_m - (\tilde{A}+B)^2] = m \Leftrightarrow r\,(A-B) = r\,(2I_m - A + B) = m$;
⟨19⟩ $r\,[(\tilde{A}+B) + (\tilde{A}+B)^2] = m \Leftrightarrow r\,(\tilde{A}+B) = r\,(2I_m - A + B) = m$;
⟨20⟩ $r\,[(\tilde{A}+B) - (\tilde{A}+B)^2] = m \Leftrightarrow r\,(A-B) = r\,(\tilde{A}+B) = m$;
⟨21⟩ $r\,[(\tilde{A}+B) + (\tilde{A}+B)^3] = m \Leftrightarrow r\,(\tilde{A}+B) = r\,[(i-1)I_m + A - B] = r\,[(i+1)I_m + A - B] = m$;
⟨22⟩ $r\,[(\tilde{A}+B) - (\tilde{A}+B)^3] = m \Leftrightarrow r\,(A-B) = r\,(\tilde{A}+B) = r\,(2I_m - A + B) = m$;
⟨23⟩ $r\,[I_m + (I_m - M)^2] = m \Leftrightarrow r\,[(i-1)I_m + M] = r\,[(i+1)I_m + M] = m$;
⟨24⟩ $r\,[I_m - (I_m - M)^2] = m \Leftrightarrow r\,(A-B) = m \Leftrightarrow r\,(M) = r\,(2I_m - M) = m$;
⟨25⟩ $r\,[(I_m - M) + (I_m - M)^2] = m \Leftrightarrow r\,(I_m - M) = r\,(2I_m - M) = m$;
⟨26⟩ $r\,[(I_m - M) + (I_m - M)^3] = m \Leftrightarrow r\,(I_m - M) = r\,[(i-1)I_m + M] = r\,[(i+1)I_m + M] = m$;
⟨27⟩ $r\,[(I_m - M) - (I_m - M)^3] = m \Leftrightarrow r\,(M) = r\,(I_m - M) = r\,(2I_m - M) = m$;
⟨28⟩ $r\,(ABA + BAB) = m \Leftrightarrow r\,(M) = r\,(A+\tilde{B}) = r\,(\tilde{A}+B) = m$;
⟨29⟩ $r\,(I_m - AB - BA) = m \Leftrightarrow r\,[(\sqrt{5}-1)/2I_m + M] = r\,[(\sqrt{5}+1)/2I_m - M] = m$;
⟨30⟩ $r\,(2I_m - AB - BA) = m \Leftrightarrow r\,(I_m + M) = r\,(2I_m - M) = m$;

(d) *The equivalent facts below hold*
⟨1⟩ $(A-B)^2 = 0 \Leftrightarrow (I_m - M)^2 = I_m \Leftrightarrow (M/2)^2 = M/2 \Leftrightarrow (I_m - M/2)^2 = I_m - M/2 \Leftrightarrow r\,(M) + r(2I_m - M) = m$
$\Leftrightarrow \mathscr{R}(M) \cap \mathscr{R}(2I_m - M) = \{0\}$;
⟨2⟩ $(A-B)^2 = 2^{-1}I_m \Leftrightarrow (I_m - M)^2 = 2^{-1}I_m$;
⟨3⟩ $(A-B)^2 = I_m \Leftrightarrow (I_m - M)^2 = 0 \Leftrightarrow r\,(A+\tilde{B}) + r\,(I_m + B - A) = m$;
⟨4⟩ $(A-B)^4 = 0 \Leftrightarrow M^4 = 4M^3 - 4M^2 \Leftrightarrow 2(AB - BA)^2 + (A + B - I_m)^4 = I_m \Leftrightarrow (A-B)^2 = (AB - BA)^2$;
⟨5⟩ $(A-B)^2 = (I_m - M)^2 \Leftrightarrow (A-B)^4 = (I_m - M)^4$;
⟨6⟩ $(I_m - M)^2 = 0 \Leftrightarrow 2(A+\tilde{B}) = (A+\tilde{B})^2 \Leftrightarrow 2(\tilde{A}+B) = (\tilde{A}+B)^2$;
⟨7⟩ $(I_m - M)^4 = 0 \Leftrightarrow (A-B)^4 + 2(AB - BA)^2 = I_m \Leftrightarrow (I_m - M)^2 = (AB - BA)^2$;
⟨8⟩ $AB = BA \Leftrightarrow A(A \pm B) = (A \pm B)A \Leftrightarrow B(A \pm B) = (A \pm B)B \Leftrightarrow (A-B)(A+B) = (A+B)(A-B)$
$\Leftrightarrow r\,(A-B) + r\,(I_m - M) = m \Leftrightarrow \mathscr{R}(A-B) \cap (I_m - M) = \{0\}$;
⟨9⟩ $(A+B)^2 = I_m \Leftrightarrow r\,(I_m + M) + r\,(I_m - M) = m$;
⟨10⟩ $(A+B)^2 = -I_m \Leftrightarrow r\,(iI_m + M) + r\,(iI_m - M) = m$;
⟨11⟩ $AB + BA = 0 \Leftrightarrow M^2 = M \Leftrightarrow (I_m - M)^2 = I_m - M \Leftrightarrow (M - 2^{-1}I_m)^2 = 4^{-1}I_m \Leftrightarrow r\,(M) + r\,(I_m - M) = m$
$\Leftrightarrow \mathscr{R}(M) \cap \mathscr{R}(I_m - M) = \{0\}$;
⟨12⟩ $M^2 = -M \Leftrightarrow r\,(M) + r\,(I_m + M) = m$;
⟨13⟩ $M^3 = M \Leftrightarrow r\,(M) + r\,(I_m + M) + r\,(I_m - M) = 2m$;
⟨14⟩ $M^3 = -M \Leftrightarrow r\,(M) + r\,(iI_m + M) + r\,(iI_m - M) = 2m$;
⟨15⟩ $(A-B)^2 = I_m \Leftrightarrow r\,[(I_m - M)^2] = r\,(A+\tilde{B}) + r\,(\tilde{A}+B) = m$;
⟨16⟩ $(A-B)^2 = -I_m \Leftrightarrow r\,(iI_m + A - B) + r\,(iI_m - A + B) = m$;
⟨17⟩ $(A-B)^2 = A - B \Leftrightarrow r\,(A-B) + r\,(\tilde{A}+B) = m$;
⟨18⟩ $(A-B)^2 = -(A-B) \Leftrightarrow r\,(A-B) + r\,(A+\tilde{B}) = m$;
⟨19⟩ $ABA = BAB \Leftrightarrow (A-B)^3 = A - B \Leftrightarrow r\,(A-B) + r\,(A+\tilde{B}) + r\,(\tilde{A}+B) = 2m$
$\Leftrightarrow \mathscr{R}(A-B) \cap \mathscr{R}(A+\tilde{B}) \cap \mathscr{R}(\tilde{A}+B) = \{0\}$;
⟨20⟩ $(A-B)^3 = -(A-B) \Leftrightarrow r\,(A-B) + r\,(iI_m + A - B) + r\,(iI_m - A + B) = 2m$;

(e) *The equivalent facts below hold*

⟨1⟩ $(\tilde{A} + B)^2 = -I_m \Leftrightarrow r[(i-1)I_m + A - B] + r[(i+1)I_m + A - B] = m$;

⟨2⟩ $(\tilde{A} + B)^2 = I_m \Leftrightarrow r(A - B) + r(2I_m - A + B) = m$;

⟨3⟩ $(\tilde{A} + B)^2 = -(\tilde{A} + B) \Leftrightarrow r(\tilde{A} + B) + r(2I_m - A + B) = m$;

⟨4⟩ $(\tilde{A} + B)^2 = (\tilde{A} + B) \Leftrightarrow r(A - B) + r(\tilde{A} + B) = m$;

⟨5⟩ $(\tilde{A} + B)^3 = -(\tilde{A} + B) \Leftrightarrow r(\tilde{A} + B) + r[(i-1)I_m + A - B] + r[(i+1)I_m + A - B] = 2m$;

⟨6⟩ $(\tilde{A} + B)^3 = \tilde{A} + B \Leftrightarrow r(A - B) + r(\tilde{A} + B) + r(2I_m - A + B) = 2m$;

⟨7⟩ $(I_m - M)^2 = -I_m \Leftrightarrow r[(i-1)I_m + M] + r[(i-1)I_m - M] = m$;

⟨8⟩ $(I_m - M)^2 = -(I_m - M) \Leftrightarrow r(I_m - M) + r(2I_m - M) = m$;

⟨9⟩ $(I_m - M)^3 = -(I_m - M) \Leftrightarrow r(I_m - M) + r[(i-1)I_m + M] + r[(i+1)I_m + M] = 2m$;

⟨10⟩ $(I_m - M)^3 = I_m - M \Leftrightarrow r(M) + r(I_m - M) + r(2I_m - M) = 2m$;

⟨11⟩ $ABA + BAB = 0 \Leftrightarrow r(M) + r(A + \tilde{B}) + r(\tilde{A} + B) = 2m \Leftrightarrow \mathscr{R}(A + B) \cap \mathscr{R}[(I_m - M)^2] = \{0\}$
$\Leftrightarrow \mathscr{R}(A + B) \cap \mathscr{R}(A + \tilde{B}) \cap \mathscr{R}(\tilde{A} + B) = \{0\}$;

⟨12⟩ $(AB - BA)^2 = 0 \Leftrightarrow (A - B)^4 = (A - B)^2 \Leftrightarrow (I_m - M)^4 = (I_m - M)^2 (A - B)^4 + (I_m - M)^4 = I_m$
$\Leftrightarrow r(M) + r(A + \tilde{B}) + r(\tilde{A} + B) + r(2I_m - M) = 3m \Leftrightarrow \mathscr{R}[(A - B)^2] \cap \mathscr{R}[(I_m - M)^2] = \{0\}$
$\Leftrightarrow \mathscr{R}(M) \cap \mathscr{R}(A + \tilde{B}) \cap \mathscr{R}(\tilde{A} + B) \cap \mathscr{R}(2I_m - M) = \{0\}$;

⟨13⟩ $AB + BA = -2I_m \Leftrightarrow (M - 2^{-1}I_m)^2 = -\frac{7}{4}I_m$;

⟨14⟩ $AB + BA = -I_m \Leftrightarrow (M - 2^{-1}I_m)^2 = -\frac{3}{4}I_m$;

⟨15⟩ $AB + BA = -4^{-1}I_m \Leftrightarrow (M - 2^{-1}I_m)^2 = 0$;

⟨16⟩ $AB + BA = \frac{3}{4}I_m \Leftrightarrow (M - 2^{-1}I_m)^2 = I_m$;

⟨17⟩ $AB + BA = I_m \Leftrightarrow (M - 2^{-1}I_m)^2 = \frac{5}{4}I_m \Leftrightarrow r[(\sqrt{5} - 1)/2I_m + M] + r[(\sqrt{5} + 1)/2I_m - M] = m$;

⟨18⟩ $AB + BA = 2I_m \Leftrightarrow (2^{-1}I_m - M)^2 = \frac{9}{4}I_m \Leftrightarrow r(I_m + M) + r(2I_m - M) = m$.

Proof. It can be deduced from (3.13), (3.59), and EBMOs that

$$r\begin{bmatrix} I_m & I_m - M \\ A - B & 0 \end{bmatrix} = r\begin{bmatrix} I_m & 0 \\ 0 & (A - B)(I_m - M) \end{bmatrix} = r(AB - BA) + m,$$

$$r\begin{bmatrix} I_m & I_m - M \\ A - B & 0 \end{bmatrix} = r\begin{bmatrix} I_m - (A - B)^2 - (I_m - M)^2 & I_m - M \\ A - B & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} 0 & I_m - M \\ A - B & 0 \end{bmatrix} = r(A - B) + r(I_m - M).$$

Combining these rank equalities leads to (4.53). Equations (4.54)–(4.60), (4.62)–(4.64), (4.66)–(4.72) and (4.74)–(4.76) follow from applying (2.1)–(2.3) to the left-hand sides of the equalities. Equations (4.61) and (4.73) follow from applying (2.1) to (3.59). Equation (4.65) follows from applying (2.1) and (4.61) to (3.14). Equation (4.77) follows from applying (2.6) and (3.55) to (3.20). By (3.13), (3.59), and block elementary operations of matrices,

$$r\begin{bmatrix} I_m & (I_m - M)^2 \\ (A - B)^2 & 0 \end{bmatrix} = r\begin{bmatrix} I_m & 0 \\ 0 & (A - B)^2 (I_m - M)^2 \end{bmatrix} = r(AB - BA)^2 + m,$$

$$r\begin{bmatrix} I_m & (I_m - M)^2 \\ (A - B)^2 & 0 \end{bmatrix} = r\begin{bmatrix} I_m - (A - B)^2 - (I_m - M)^2 & (I_m - M)^2 \\ (A - B)^2 & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} 0 & (I_m - M)^2 \\ (A - B)^2 & 0 \end{bmatrix} = r[(A - B)^2] + r[(I_m - M)^2].$$

Combining these rank equalities with (3.60), (3.65), (4.61), and (4.73) leads to (4.78). Equations (4.79)–(4.84) follow from applying (2.1) to (3.66)–(3.70). Results (b) follow from applying (2.43)–(2.48) and (4.49)–(4.52) to (4.53)–(4.84). Results (c) and (d) are direct consequences of (4.53)–(4.84). □

Theorem 4.11. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, $\alpha \neq -1, 0$ and $\beta \neq -1, 0$, we have the following result*:

$$r(I_m - \lambda AB) = r(I_m - \lambda BA) = r(I_m + \alpha A + \beta B), \tag{4.109}$$

where $\lambda = \alpha\beta(1+\alpha)^{-1}(1+\beta)^{-1}$. In particular,

$$r(I_m + AB) = r(I_m + BA) = r(\sqrt{2}I_m + A - B) = r(\sqrt{2}I_m - A + B). \tag{4.110}$$

Proof. Equation (4.109) follows from (3.11). Equation (4.110) follows from (4.109) by setting $\alpha = -\beta = \pm 1/\sqrt{2}$. □

Theorem 4.12. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, we have the following results*:
(a) *The rank equalities below hold*

$$r[AB - (AB)^2] = r(I_m - AB) + r(AB) - m = r(2I_m - M) + r(AB) - m, \tag{4.111}$$

$$\begin{aligned} r[AB - (AB)^3] &= r(I_m + AB) + r(I_m - AB) + r(AB) - 2m \\ &= r(\sqrt{2}I_m + A - B) + r(2I_m - M) + r(AB) - 2m \\ &= r(\sqrt{2}I_m - A + B) + r(2I_m - M) + r(AB) - 2m, \end{aligned} \tag{4.112}$$

$$r[ABA - (ABA)^2] = r(ABA) + r(I_m - ABA) - m = r(2I_m - M) + r(I_m - A + B) + r(A) - 2m, \tag{4.113}$$

$$\begin{aligned} r[ABA - (ABA)^3] &= r(I_m + ABA) + r(I_m - ABA) + r(ABA) - 2m \\ &= r(I_m + ABA) + r(2I_m - M) + r(I_m - A + B) + r(A) - 2m. \end{aligned} \tag{4.114}$$

(b) $(AB)^2 = AB \Leftrightarrow r(I_m - AB) = m - r(AB) \Leftrightarrow r(2I_m - M) = m - r(AB)$.
(c) $(AB)^3 = AB \Leftrightarrow r(I_m + AB) + r(I_m - AB) + r(AB) = 2m \Leftrightarrow r(\sqrt{2}I_m + A - B) + r(2I_m - M) + r(AB) = 2m$.
(d) $(ABA)^2 = ABA \Leftrightarrow r(ABA) + r(I_m - ABA) = m \Leftrightarrow r(2I_m - M) + r(I_m - A + B) + r(A) = 2m$.

Proof. Applying (2.1) and (4.42) to $AB - (AB)^2$ yields (4.111). Applying (2.3), (4.42), and (4.111) to $AB - (AB)^3$ yields (4.112). Applying (2.1), (4.41), and (4.42) to $ABA - (ABA)^2$ yields (4.113). Applying (2.3) and (4.113) to $ABA - (ABA)^3$ yields (4.114). Results (b) and (c) are direct consequences of (4.111)–(4.114). □

We now go back to the cases of the k-powers $(AB)^k$, $(AB)^k$, $(AB)^kA$, and $(BA)^kB$ for two idempotent matrices A and B. From Theorems 2.5–2.7, we can easily obtain a group of nice rank formulas as follows.

Theorem 4.13. *Given two idempotent matrices $A, B \in \mathbb{C}^{m\times m}$, and an integer $k \geq 1$, we have the following results*:
(a) *The rank equalities below hold*

$$r[(AB)^k, (BA)^k] = r[A, B] + r[(AB)^k] + r[(BA)^k] - r(A) - r(B), \tag{4.115}$$

$$r[(AB)^kA, (BA)^kB] = r[A, B] + r[(AB)^kA] + r[(BA)^kB] - r(A) - r(B), \tag{4.116}$$

$$r\begin{bmatrix} (AB)^k \\ (BA)^k \end{bmatrix} = r\begin{bmatrix} A \\ B \end{bmatrix} + r[(AB)^k] + r[(BA)^k] - r(A) - r(B), \tag{4.117}$$

$$r\begin{bmatrix} (AB)^kA \\ (BA)^kB \end{bmatrix} = r\begin{bmatrix} A \\ B \end{bmatrix} + r[(AB)^kA] + r[(BA)^kB] - r(A) - r(B), \tag{4.118}$$

$$r[(AB)^k - (BA)^k] = r\begin{bmatrix} (AB)^k \\ (BA)^k \end{bmatrix} + r[(AB)^k, (BA)^k] - r[(AB)^k] - r[(BA)^k], \tag{4.119}$$

$$r[(AB)^k - (BA)^k] = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A, B] + r[(AB)^k] + r[(BA)^k] - 2r(A) - 2r(B), \tag{4.120}$$

$$r[(AB)^kA - (BA)^kB] = r\begin{bmatrix} (AB)^kA \\ (BA)^kB \end{bmatrix} + r[(AB)^kA, (BA)^kB] - r[(AB)^kA] - r[(BA)^kB], \tag{4.121}$$

$$r[(AB)^kA-(BA)^kB]=r\begin{bmatrix}A\\B\end{bmatrix}+r[A,B]+r[(AB)^kA]+r[(BA)^kB]-2r(A)-2r(B), \tag{4.122}$$

$$r[(AB)^k+(BA)^k]=r\begin{bmatrix}(AB)^k & (BA)^k\\(BA)^k & 0\end{bmatrix}-r[(BA)^k]=r\begin{bmatrix}(BA)^k & (AB)^k\\(AB)^k & 0\end{bmatrix}-r[(AB)^k], \tag{4.123}$$

$$r[(AB)^kA+(BA)^kB]=r\begin{bmatrix}(AB)^kA & (BA)^kB\\(BA)^kB & 0\end{bmatrix}-r[(BA)^kB]=r\begin{bmatrix}(BA)^kB & (AB)^kA\\(AB)^kA & 0\end{bmatrix}-r[(AB)^kA]. \tag{4.124}$$

(b) $r[(AB)^k,(BA)^k]=r[(AB)^k]+r[(BA)^k]\Leftrightarrow r[A,B]=r(A)+r(B)$
$\Leftrightarrow \mathscr{R}[(AB)^k]\cap\mathscr{R}[(BA)^k]=\{0\}\Leftrightarrow\mathscr{R}(A)\cap\mathscr{R}(B)=\{0\}$.

(c) $r[(AB)^k,(BA)^k]=r[A,B]\Leftrightarrow\mathscr{R}[(AB)^k]=\mathscr{R}(A)$ *and* $\mathscr{R}[(BA)^k]=\mathscr{R}(B)$.

(d) $(AB)^k=(BA)^k\Leftrightarrow\mathscr{R}[(AB)^k]=\mathscr{R}[(BA)^k]$ *and* $\mathscr{R}[(A^*B^*)^k]=\mathscr{R}[(B^*A^*)^k]$
$\Leftrightarrow r[A,B]=r(A)+r(B)-r[(AB)^k]$ *and* $r\begin{bmatrix}A\\B\end{bmatrix}=r(A)+r(B)-r[(BA)^k]$.

(e) $r[(AB)^kA,(BA)^kB]=r[(AB)^kA]+r[(BA)^kB]\Leftrightarrow r[A,B]=r(A)+r(B)$
$\Leftrightarrow\mathscr{R}[(AB)^kA]\cap\mathscr{R}[(BA)^kB]=\{0\}\Leftrightarrow\mathscr{R}(A)\cap\mathscr{R}(B)=\{0\}$.

(f) $r[(AB)^kA,(BA)^kB]=r[A,B]\Leftrightarrow\mathscr{R}[(AB)^kA]=\mathscr{R}(A)$ *and* $\mathscr{R}[(BA)^kB]=\mathscr{R}(B)$.

(g) $(AB)^kA=(BA)^kB\Leftrightarrow\mathscr{R}[(AB)^kA]=\mathscr{R}[(BA)^kB]$ *and* $\mathscr{R}[(A^*B^*)^kA^*]=\mathscr{R}[(B^*A^*)^kB^*]$
$\Leftrightarrow r[A,B]=r(A)+r(B)-r[(AB)^kA]$ *and* $r\begin{bmatrix}A\\B\end{bmatrix}=r(A)+r(B)-r[(BA)^kB]$.

(h) $r[(AB)^k+(BA)^k]\geq\max\{r[(AB)^k],r[(BA)^k],r[(AB)^k-(BA)^k]\}$.
In particular, if $r[(AB)^k-(BA)^k]=m$, *then* $r[(AB)^k+(BA)^k]=m$.

(i) $r[(AB)^kA+(BA)^kB]\geq\max\{r[(AB)^kA],r[(BA)^kB],r[(AB)^kA-(BA)^kB]\}$.
In particular, if $r[(AB)^kA-(BA)^kB]=m$, *then* $r[(AB)^kA+(BA)^kB]=m$.

Proof. Let $X=(AB)^{k-1}A$ and $Y=(BA)^{k-1}B$ as well as $X=(AB)^k$ and $Y=(BA)^k$. Then, they satisfy (2.36). In such cases, (2.37) becomes (4.115) and (4.116), respectively. Equations (4.117) and (4.118) are established by taking transpose of (4.115) and (4.116), respectively. Let $M=A$, $X=(AB)^k$, and $Y=(BA)^k$. Then, they satisfy (2.34), thus (2.35) becomes (4.121). Let $X=(AB)^kA$ and $Y=(BA)^kB$. Then, they satisfy (2.34), thus (2.35) becomes (4.121). Substituting (4.115)–(4.118) into (4.119) and (4.121) yields (4.120) and (4.122), respectively. We next can deduce from $A^2=A$, $B^2=B$, $B(AB)^k=(BA)^kB$, and EBMOs that

$$r\begin{bmatrix}(AB)^k & 0 & (AB)^k\\0 & (BA)^k & (BA)^k\\(AB)^k & (BA)^k & 0\end{bmatrix}=r\begin{bmatrix}(AB)^k & 0 & 0\\0 & (BA)^k & 0\\0 & 0 & -(AB)^k-(BA)^k\end{bmatrix}=r[(AB)^k]+r[(BA)^k]+r[(AB)^k+(BA)^k],$$

$$r\begin{bmatrix}(AB)^k & 0 & (AB)^k\\0 & (BA)^k & (BA)^k\\(AB)^k & (BA)^k & 0\end{bmatrix}=r\begin{bmatrix}(AB)^k & 0 & (AB)^k\\-B(AB)^k & 0 & (BA)^k\\(AB)^k & (BA)^k & 0\end{bmatrix}=r\begin{bmatrix}2(AB)^k & 0 & (AB)^k\\0 & 0 & (BA)^k\\(AB)^k & (BA)^k & 0\end{bmatrix}$$
$$=r\begin{bmatrix}2(AB)^k & 0 & 0\\0 & 0 & (BA)^k\\0 & (BA)^k & -\frac{1}{2}(AB)\end{bmatrix}=r\begin{bmatrix}(AB)^k & (BA)^k\\(BA)^k & 0\end{bmatrix}+r[(AB)^k].$$

Combining these two rank equalities leads to (4.123). Further by $A^2=A$, $B^2=B$, $B(AB)^k=(BA)^kB$, and EBMOs,

$$r\begin{bmatrix}(AB)^kA & 0 & (AB)^kA\\0 & (BA)^kB & (BA)^kB\\(AB)^kA & (BA)^kB & 0\end{bmatrix}=r\begin{bmatrix}(AB)^kA & 0 & 0\\0 & (BA)^kB & 0\\0 & 0 & -(AB)^kA-(BA)^kB\end{bmatrix}$$
$$=r[(AB)^kA]+r[(BA)^kB]+r[(AB)^kA+(BA)^kB],$$

$$r\begin{bmatrix} (AB)^kA & 0 & (AB)^kA \\ 0 & (BA)^kB & (BA)^kB \\ (AB)^kA & (BA)^kB & 0 \end{bmatrix} = r\begin{bmatrix} (AB)^kA & 0 & (AB)^kA \\ -B(AB)^k & A0 & (BA)^kB \\ (AB)^kA & (BA)^kB & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} 2(AB)^kA & 0 & (AB)^kA \\ 0 & 0 & (BA)^kB \\ (AB)^kA & (BA)^kB & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} 2(AB)^kA & 0 & 0 \\ 0 & 0 & (BA)^kA \\ 0 & (BA)^kB & -\frac{1}{2}(AB)^kA \end{bmatrix}$$
$$= r\begin{bmatrix} (AB)^kA & (BA)^kB \\ (BA)^kB & 0 \end{bmatrix} + r[(AB)^kA].$$

Combining these two rank equalities leads to (4.124). Results (b)–(g) follow directly from (4.115), (4.116), and (4.119)–(4.122). By (4.123), the two inequalities $r[(AB)^k + (BA)^k] \geq r[(AB)^k]$ and $r[(AB)^k + (BA)^k] \geq r[(BA)^k]$ hold, and by (2.24) and (4.119), the inequality $r[(AB)^kA + (BA)^kB] \geq r\begin{bmatrix} (AB)^k \\ (BA)^k \end{bmatrix} + r[(AB)^k, (BA)^k] - r[(AB)^k] - r[(BA)^k] = r[(AB)^kA - (BA)^kB]$ holds. Combining these inequalities leads to Result (h). Result (i) can be established by a similar way. □

Some results in Theorem 4.13 were established in the literature [33,36–38,45,46].

We next derive from Theorem 3.8 some rank and range formulas for expressions composed by three idempotent matrices and their consequences.

Theorem 4.14. *Given three idempotent matrices $A, B, C \in \mathbb{C}^{m\times m}$ and $S = A + B + C$, we have the following results*:

(a) *The following rank formulas hold*

$$r[(A+B)^2 + (A+C)^2 + (B+C)^2] = r(S+S^2) = r(S) + r(I_m + S) - m, \tag{4.125}$$

$$r[(A-B)^2 + (A-C)^2 + (B-C)^2] = r(3S-S^2) = r(S) + r(3I_m - S) - m, \tag{4.126}$$

$$\begin{gathered} r(kI_m - AB - BA - AC - CA - BC - CB) = r[(\sqrt{4k+1}+1)/2I_m - S] \\ +r[(\sqrt{4k+1}-1)/2I_m + S] - m \end{gathered} \tag{4.127}$$

for $k = 0, 1, \ldots, 6$.

(b) *The following range equalities*

$$\mathscr{R}[(A+B)^2 + (A+C)^2 + (B+C)^2] = \mathscr{R}(S+S^2) = \mathscr{R}(S) \cap \mathscr{R}(I_m + S), \tag{4.128}$$

$$\mathscr{R}[(A-B)^2 + (A-C)^2 + (B-C)^2] = \mathscr{R}(3S-S^2) = \mathscr{R}(S) \cap \mathscr{R}(3I_m - S), \tag{4.129}$$

$$\begin{gathered} \mathscr{R}(kI_m - AB - BA - AC - CA - BC - CB) \\ = \mathscr{R}[(\sqrt{4k+1}+1)/2I_m - S] \cap \mathscr{R}[(\sqrt{4k+1}-1)/2I_m + S] \end{gathered} \tag{4.130}$$

hold for $k = 0, 1, \ldots, 6$.

(c) $(A+B)^2 + (A+C)^2 + (B+C)^2 = 0 \Leftrightarrow S + S^2 = 0 \Leftrightarrow r(S) + r(I_m + S) = m$.

(d) $(A-B)^2 + (A-C)^2 + (B-C)^2 = 0 \Leftrightarrow (S/3)^2 = S/3 \Leftrightarrow r(S) + r(3I_m - S) = m$.

(e) $AB + BA + AC + CA + BC + CB = kI_m \Leftrightarrow r[(\sqrt{4k+1}+1)/2I_m - S] + r[(\sqrt{4k+1}-1)/2I_m + S] = 0$, $k = 0, 1, \ldots, 6$.

Proof. Applying (2.1) to (3.92)–(3.94) leads to (4.125)–(4.127). Results (b)–(e) follow from (4.125)–(4.127). □

Furthermore, it is desired to deduce various expansion formulas for calculating the ranks of matrix expressions composed by a family of idempotent matrices. Next, the author presents some general rank equalities associated with a family of idempotent matrices.

Theorem 4.15. *Given a family of idempotent matrices $A_1, A_2,\dots, A_k \in \mathbb{C}^{m\times m}$, and denote $S = A_1 + A_2 +\cdots+ A_k$, $A = \operatorname{diag}(A_1, A_2,\dots, A_k)$, and $\tilde{A}_i = I_m - A_i$, $i = 1, 2,\dots, k$, we have the following results:*

(a) *The rank equalities below hold*

$$r\left(\begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix}[A_1, \dots, A_k]\right) = r\begin{bmatrix} (k+1)I_m - kA_1 & \cdots & I_m \\ \vdots & \ddots & \vdots \\ I_m & \cdots & (k+1)I_m - kA_k \end{bmatrix} + r(A) - km, \tag{4.131}$$

$$r\left(\begin{bmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_k \end{bmatrix}[\tilde{A}_1, \dots, \tilde{A}_k]\right) = r\begin{bmatrix} I_m + kA_1 & \cdots & I_m \\ \vdots & \ddots & \vdots \\ I_m & \cdots & I_m + kA_k \end{bmatrix} - r(A), \tag{4.132}$$

$$r\left(A - \begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix}[A_1, \dots, A_k]\right) = r(A) + r(I_m - S) - m, \tag{4.133}$$

$$r\left((I_{km} - A) - \begin{bmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_k \end{bmatrix}[\tilde{A}_1, \dots, \tilde{A}_k]\right) = r[(k-1)I_m - S)] - r(A) + (k-1)m. \tag{4.134}$$

(b) $r\left(\begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix}[A_1, \dots, A_k]\right) = m \Leftrightarrow r\begin{bmatrix} (k+1)I_m - kA_1 & \cdots & I_m \\ \vdots & \ddots & \vdots \\ I_m & \cdots & (k+1)I_m - kA_k \end{bmatrix} = (k+1)m - r(A).$

(c) $r\left(\begin{bmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_k \end{bmatrix}[\tilde{A}_1, \dots, \tilde{A}_k]\right) = m \Leftrightarrow r\begin{bmatrix} I_m + kA_1 & \cdots & I_m \\ \vdots & \ddots & \vdots \\ I_m & \cdots & I_m + kA_k \end{bmatrix} = m + r(A).$

(d) $A = \begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix}[A_1, \dots, A_k] \Leftrightarrow A_iA_j = 0, \ \ i \neq j, \ \ i, j = i = 1, 2, \dots, k \Leftrightarrow S^2 = S$ and $r(S) = r(A)$

$\Leftrightarrow r(I_m - S) = m - r(A).$

Proof. Let $J = [I_m,\dots, I_m]$. Then, $(J^TJ/k)^2 = J^TJ/k$. By (2.13) and (4.33),

$$r\left(\begin{bmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_k \end{bmatrix}[\tilde{A}_1, \dots, \tilde{A}_k]\right) = r\begin{bmatrix} J^TJ & A \\ A & 0 \end{bmatrix} - 2r(A) = r\begin{bmatrix} J^TJ/k & A \\ A & 0 \end{bmatrix} - 2r(A)$$
$$= r(J^TJ/k + A) + r(A) = r(J^TJ + kA) + r(A),$$

establishing (4.132), this proof was first given in [39]. Replacing $\tilde{A}_i$ with A_i in (4.132) leads to (4.131). By $JAJ^T = S$,

$$r\begin{bmatrix} -I_m & 0 & I_m \\ 0 & A & AJ^T \\ I_m & JA & 0 \end{bmatrix} = r\begin{bmatrix} -I_m & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_m - S \end{bmatrix} = r\begin{bmatrix} 0 & 0 & I_m \\ 0 & A - AJ^TJA & 0 \\ I_m & 0 & 0 \end{bmatrix}$$
$$= r(A) + r(I_m - S) + m = r(A - AJ^TJA) + 2m,$$

establishing (4.133). Replacing A_i with $\tilde{A}_i$ in (4.133) leads to (4.134). Results (b)–(d) follow from (4.131)–(4.133). □

Theorem 4.16. *Given a family of idempotent matrices $A_1, A_2, \ldots, A_k \in \mathbb{C}^{m\times m}$, we have the following results:*

(a) *The rank equality below holds*

$$r[A_1\hat{A}_1, A_2\hat{A}_2, \ldots, A_k\hat{A}_k] = r(A_1\hat{A}_1) + r(A_2\hat{A}_2) + \cdots + r(A_k\hat{A}_k) + r(\tilde{A}) - r(A_1) - r(A_2) - \cdots - r(A_k), \tag{4.135}$$

where $\tilde{A} = [A_1, A_2, \ldots, A_k]$ and $\hat{A}_i = [A_1, \ldots, A_{i-1}, 0, A_{i+1}, \ldots, A_k]$.

(b) $r[A_1\hat{A}_1, A_2\hat{A}_2, \ldots, A_k\hat{A}_k] = r(A_1\hat{A}_1) + r(A_2\hat{A}_2) + \cdots + r(A_k\hat{A}_k) \Leftrightarrow r(\tilde{A}) = r(A_1) + r(A_2) + \cdots + r(A_k)$.

(c) $r[A_1\hat{A}_1, A_2\hat{A}_2, \ldots, A_k\hat{A}_k] = r(\tilde{A}) \Leftrightarrow \mathscr{R}(A_i\hat{A}_i) = \mathscr{R}(A_i), \ \ i = 1, 2, \ldots, k$.

(d) $r[A_1\hat{A}_1, A_2\hat{A}_2, \ldots, A_k\hat{A}_k] = m \Leftrightarrow r(\tilde{A}) = m$ *and* $\mathscr{R}(A_i\hat{A}_i) = \mathscr{R}(A_i), \ \ i = 1, 2, \ldots, k$.

(e) *If* $A_1\hat{A}_1 = A_2\hat{A}_2 = \ldots = A_k\hat{A}_k = 0$, *then* $r(\tilde{A}) = r(A_1) + r(A_2) + \cdots + r(A_k)$.

(f) $r(\tilde{A}) \geq r(A_1) + r(A_2) + \cdots + r(A_k) - r(A_1\hat{A}_1) - r(A_2\hat{A}_2) - \cdots - r(A_k\hat{A}_k)$ *holds.*

Proof. A much trickier block matrix composed by the family of idempotent matrices is given by

$$X = \begin{bmatrix} A_1 & 0 & \cdots & 0 & A_1\hat{A}_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & 0 & A_2\hat{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k & 0 & 0 & \cdots & A_k\hat{A}_k \\ A_1 & A_2 & \cdots & A_k & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It is easy to verify that the above construction leads to the following two rank equalities:

$$\begin{aligned} r(X) &= r\begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -A_1\hat{A}_1 & -A_2\hat{A}_2 & \cdots & -A_k\hat{A}_k \end{bmatrix} \\ &= r(A_1) + r(A_2) + \cdots + r(A_k) + r[A_1\hat{A}_1, A_2\hat{A}_2, \ldots, A_k\hat{A}_k], \end{aligned}$$

$$\begin{aligned} r(X) &= r\begin{bmatrix} 0 & -A_1A_2 & \cdots & -A_1A_k & A_1\hat{A}_1 & 0 & \cdots & 0 \\ -A_2A_1 & 0 & \cdots & -A_2A_k & 0 & A_2\hat{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_kA_1 & -A_kA_2 & \cdots & 0 & 0 & 0 & \cdots & A_k\hat{A}_k \\ A_1 & A_2 & \cdots & A_k & 0 & 0 & \cdots & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & \cdots & 0 & A_1\hat{A}_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & A_2\hat{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & A_k\hat{A}_k \\ A_1 & A_2 & \cdots & A_k & 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= r(A_1\hat{A}_1) + r(A_2\hat{A}_2) + \cdots + r(A_k\hat{A}_k) + r(A). \end{aligned}$$

Combining these two equalities leads to (4.135). Results (b)–(f) follow directly from (4.135). □

It is easy to see that (4.135) for $k = 2$ and (4.115) for $k = 1$ are the same. Choosing $k = 3$ in (4.135), this allows us to obtain the following result.

Corollary 4.17. *Given three idempotent matrices $A, B, C \in \mathbb{C}^{m\times m}$, we have the following results*:

(a) *The rank equality below holds*

$$r[A, B, C] = r(A) + r(B) + r(C) - r[AB, AC] - r[BA, BC] - r[CA, CB] + r[AB, AC, BA, BC, CA, CB].$$

(b) *If $AB = BA$, $AC = CA$, and $BC = CB$, then*

$$r[A, B, C] = r(A) + r(B) + r(C) - r[AB, AC] - r[BA, BC] - r[CA, CB] + r[AB, AC, BC].$$

(c) $r[AB, AC, BA, BC, CA, CB] = r[AB, AC] + r[BA, BC] + r[CA, CB] \Leftrightarrow r[A, B, C] = r(A) + r(B) + r(C)$.

(d) $r[AB, AC, BA, BC, CA, CB] = r[A, B, C] \Leftrightarrow \mathscr{R}[AB, AC] = \mathscr{R}(A), \ \ \mathscr{R}[BA, BC] = \mathscr{R}(B)$, *and* $\mathscr{R}[CA, CB] = \mathscr{R}(C)$.

(e) $r[AB, AC, BA, BC, CA, CB] = m \Leftrightarrow r[A, B, C] = m, \ \ \mathscr{R}[AB, AC] = \mathscr{R}(A), \ \ \mathscr{R}[BA, BC] = \mathscr{R}(B)$, *and* $\mathscr{R}[CA, CB] = \mathscr{R}(C)$.

(f) *If* $AB = BA = AC = CA = BC = CB = 0$, *then* $r[A, B, C] = r(A) + r(B) + r(C)$.
(g) $r[A, B, C] \geq r(A) + r(B) + r(C) - r[AB, AC] - r[AB, BC] - r[AC, BC]$ *holds.*

Finally, we deduce from Theorem 4.13(a) the following result on the ranks of a partitioned matrix and generalized inverses of its submatrices.

Corollary 4.18. *Given* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{m\times p}$, *the following rank equality*

$$r\left([A, B] - [A, B]\begin{bmatrix} A^- \\ B^- \end{bmatrix}[A, B]\right) = r(AA^-B) + r(BB^-A) + r[A, B] - r(A) - r(B) \tag{4.136}$$

holds for all A^- *and* B^-. *In particular, the following results hold.*
(a) *The maximum and minimum ranks of* (4.136) *with respect to* A^- *and* B^- *are given by*

$$\max_{A^-,B^-} r\left([A, B] - [A, B]\begin{bmatrix} A^- \\ B^- \end{bmatrix}[A, B]\right) = r[A, B] - |r(A) - r(B)|, \tag{4.137}$$

$$\min_{A^-,B^-} r\left([A, B] - [A, B]\begin{bmatrix} A^- \\ B^- \end{bmatrix}[A, B]\right) = r(A) + r(B) - r[A, B]. \tag{4.138}$$

(b) $\{[A, B]^-\} \cap \left\{\begin{bmatrix} A^- \\ B^- \end{bmatrix}\right\} \neq \emptyset \Leftrightarrow r[A, B] = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$.

(c) $\{[A, B]^-\} \supseteq \left\{\begin{bmatrix} A^- \\ B^- \end{bmatrix}\right\} \Leftrightarrow r[A, B] = |r(A) - r(B)| \Leftrightarrow A = 0$ *or* $B = 0$.

Proof. By definition, $\begin{bmatrix} A^- \\ B^- \end{bmatrix}$ is a generalized inverse of $[A, B]$ if and only if $[A, B]\begin{bmatrix} A^- \\ B^- \end{bmatrix}[A, B] = [A, B]$. On the other hand, it is easy to verify that

$$[A, B] - [A, B]\begin{bmatrix} A^- \\ B^- \end{bmatrix}[A, B] = [A, B] - [(AA^- + BB^-)A, (AA^- + BB^-)B] = -[BB^-A, AA^-B] \tag{4.139}$$

holds for all A^- and B^-. Note that both AA^- and BB^- are idempotent. Applying (4.115) to (4.139), we obtain

$$\begin{aligned} r[BB^-A, AA^-B] &= r[AA^-BB^-, BB^-AA^-] \\ &= r(AA^-BB^-) + r(BB^-AA^-) + r[AA^-, BB^-] - r(AA^-) - r(BB^-) \\ &= r(AA^-B) + r(BB^-A) + r[A, B] - r(A) - r(B), \end{aligned} \tag{4.140}$$

as required for (4.136). Applying the following two known rank formulas

$$\max_{A^-} r(D - CA^-B) = \min\left\{r[C, D], r\begin{bmatrix} B \\ D \end{bmatrix}, r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A)\right\},$$

$$\min_{A^-} r(D - CA^-B) = r(A) + r[C, D] + r\begin{bmatrix} B \\ D \end{bmatrix} + r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix}$$

in [30] to AA^-B and BB^-A gives

$$\max_{A^-} r(AA^-B) = \max_{B^-} r(BB^-A) = \min\{r(A), r(B)\},$$
$$\min_{A^-} r(AA^-B) = \min_{B^-} r(BB^-A) = r(A) + r(B) - r[A, B].$$

Substituting these two results into (4.140) yields

$$\begin{aligned} \max_{A^-,B^-} r[AA^-BB^-, BB^-AA^-] &= 2\min\{r(A), r(B)\} + r[A, B] - r(A) - r(B) \\ &= r[A, B] - |r(A) - r(B)|, \end{aligned} \tag{4.141}$$

$$\min_{A^-,B^-} r\,[AA^-BB^-, BB^-AA^-] = 2r(A) + 2r(B) - 2r\,[A, B] + r\,[A, B] - r(A) - r(B) \\ = r(A) + r(B) - r\,[A, B]. \tag{4.142}$$

Combining (4.136) with (4.141) and (4.142) leads to (4.137) and (4.138), respectively. Setting the both sides of (4.137) and (4.138) equal to zero leads to Results (b) and (c). □

5 Bounds of ranks of some matrix pencils composed by two idempotent matrices

Since the rank of a matrix expression is a function of the variable entries in the expression, which may vary with respect to the different choices of the entries, people wish to know the exact upper and lower bounds of the ranks (maximum and minimum ranks) of the matrix expression under various assumptions. By the definitions of generalized inverses, both AA^- and A^-A are always idempotent for any A^- of a singular matrix A. In this case, people are interested in the performance of idempotent matrices associated with the generalized inverses and their operations [7,11,12,21,29]. It has been realized since the seminal work in [19] that matrix rank formulas are the powerful tool to characterize matrix equalities that involve generalized inverses. In this section, the author approaches the ranks of the following four characteristic matrices:

$$\lambda I_m + AA^- \pm BB^-, \quad \lambda I_m + AA^- \pm C^-C \tag{5.1}$$

associated with the idempotent matrices A^-, B^-, and C^-, where λ is a scalar. It is obvious that the ranks of the four matrix expressions in (5.1) are all functions of λ, A^-, B^-, and C^-. Thus, we are interested in the maximum and minimum ranks as well as rank distributions of the four matrix expressions. To determine the two ranks, the following results will be used.

Lemma 5.1. [30] *The maximum and minimum ranks of the linear matrix-valued function $A - B_1X_1C_1 - B_2X_2C_2$ with respect to the two variable matrices X_1 and X_2 are given by*

$$\max_{X_1,X_2} r(A - B_1X_1C_1 - B_2X_2C_2) = \min\left\{ r\,[A, B_1, B_2],\ r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix},\ r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix},\ r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} \right\}, \tag{5.2}$$

$$\min_{X_1,X_2} r(A - B_1X_1C_1 - B_2X_2C_2) = r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r\,[A, B_1, B_2] + \max\{s_1, s_2\}, \tag{5.3}$$

where

$$s_1 = r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}.$$

Lemma 5.2. [6,10,25] *Let $A \in \mathbb{C}^{m\times n}$. Then, the general expressions of A^-, AA^-, and A^-A can be written as*

$$A^- = A^\dagger + F_AU + VE_A, \quad AA^- = AA^\dagger + AVE_A, \quad A^-A = A^\dagger A + F_AUA, \tag{5.4}$$

where $U, V \in \mathbb{C}^{n\times m}$ are arbitrary.

Theorem 5.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times p}$ be given.*

(a) *If $\lambda \neq 0, -1, -2$, then the following two formulas hold*

$$\max_{A^-,B^-} r(\lambda I_m + AA^- + BB^-) = m, \tag{5.5}$$

$$\min_{A^-,B^-} r(\lambda I_m + AA^- + BB^-) = \max\{m + r(A) - r[A, B], m + r(B) - r[A, B]\}. \tag{5.6}$$

In particular, the following results hold.

(i) *There always exist A^- and B^-, such that $\lambda I_m + AA^- + BB^-$ is nonsingular.*

(ii) *$\lambda I_m + AA^- + BB^-$ is nonsingular for all A^- and B^- $\Leftrightarrow$. The rank of $\lambda I_m + AA^- + BB^-$ is invariant for all A^- and B^- $\Leftrightarrow r[A, B] = r(A) = r(B) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B)$.*

(iii) *There do not exist A^- and B^-, such that $\lambda I_m + AA^- + BB^- = 0$.*

(b) *The following two formulas hold*

$$\max_{A^-,B^-} r(AA^- + BB^-) = r[A, B], \tag{5.7}$$

$$\min_{A^-,B^-} r(AA^- + BB^-) = \max\{r(A), r(B)\}. \tag{5.8}$$

In particular, the following results hold.

(i) *There exist A^- and B^-, such that $AA^- + BB^-$ is nonsingular $\Leftrightarrow r[A, B] = m$.*

(ii) *$AA^- + BB^-$ is nonsingular for all A^-* and *B^- $\Leftrightarrow r(A) = m$ or $r(B) = m$.*

(iii) *There exist A^- and B^-, such that $AA^- + BB^- = 0 \Leftrightarrow AA^- + BB^- = 0$ for all A^- and $B^- \Leftrightarrow [A, B] = 0$.*

(iv) *The rank of $AA^- + BB^-$ is invariant for all A^- and B^- $\Leftrightarrow \mathscr{R}(A) \supseteq \mathscr{R}(B)$ or $\mathscr{R}(A) \subseteq \mathscr{R}(B)$.*

(c) *The following two formulas hold*

$$\max_{A^-,B^-} r(-I_m + AA^- + BB^-) = m - |r(A) - r(B)|, \tag{5.9}$$

$$\min_{A^-,B^-} r(-I_m + AA^- + BB^-) = m + r(A) + r(B) - 2r[A, B]. \tag{5.10}$$

In particular, the following results hold.

(i) *There exist A^- and B^-, such that $-I_m + AA^- + BB^-$ is nonsingular $\Leftrightarrow r(A) = r(B)$.*

(ii) *$-I_m + AA^- + BB^-$ is nonsingular for all A^- and B^- $\Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B)$.*

(iii) *There exist A^- and B^-, such that $AA^- + BB^- = I_m \Leftrightarrow r[A, B] = r(A) + r(B) = m$.*

(iv) *The rank of $-I_m + AA^- + BB^-$ is invariant for all A^- and B^- $\Leftrightarrow \mathscr{R}(A) \supseteq \mathscr{R}(B)$ or $\mathscr{R}(A) \subseteq \mathscr{R}(B)$.*

(d) *The following two formulas hold*

$$\max_{A^-,B^-} r(-2I_m + AA^- + BB^-) = m + r[A, B] - r(A) - r(B), \tag{5.11}$$

$$\min_{A^-,B^-} r(-2I_m + AA^- + BB^-) = \max\{m - r(A), m - r(B)\}. \tag{5.12}$$

In particular, the following results hold.

(i) *There exist A^- and B^-, such that $-2I_m + AA^- + BB^-$ is nonsingular $\Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$.*

(ii) *$-2I_m + AA^- + BB^-$ is nonsingular for all A^- and B^- $\Leftrightarrow A = 0$ or $B = 0$.*

(iii) *There exist A^- and B^-, such that $AA^- + BB^- = 2I_m \Leftrightarrow AA^- + BB^- = 2I_m$ for all A^- and B^- $\Leftrightarrow r(A) = r(B) = m$.*

(iv) *The rank of $-2I_m + AA^- + BB^-$ is invariant for all A^- and B^- $\Leftrightarrow \mathscr{R}(A) \supseteq \mathscr{R}(B)$ or $\mathscr{R}(A) \subseteq \mathscr{R}(B)$.*

Proof. By (5.4),

$$\lambda I_m + AA^- + BB^- = \lambda I_m + AA^\dagger + BB^\dagger + AV_1E_A + BV_2E_B, \tag{5.13}$$

where $V_1 \in \mathbb{C}^{n\times m}$ and $V_2 \in \mathbb{C}^{p\times m}$ are arbitrary. Applying (5.2) and (5.3) to (5.13) and simplifying gives

$$\begin{aligned}
&\max_{V_1,V_2} r(\lambda I_m + AA^{\dagger} + BB^{\dagger} + AV_1E_A + BV_2E_B)\\
&= \min\left\{ r[\lambda I_m + AA^{\dagger} + BB^{\dagger}, A, B], \quad r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} \\ E_A \\ E_B \end{bmatrix}, \right.\\
&\quad \left. r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} & A \\ E_B & 0 \end{bmatrix}, \quad r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} & B \\ E_A & 0 \end{bmatrix} \right\}\\
&= \min\left\{ r[\lambda I_m, A, B], \quad r\begin{bmatrix} (\lambda+2)I_m \\ E_A \\ E_B \end{bmatrix}, \quad r\begin{bmatrix} (\lambda+1)I_m & A \\ E_B & 0 \end{bmatrix}, \quad r\begin{bmatrix} (\lambda+1)I_m & B \\ E_A & 0 \end{bmatrix} \right\},
\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}
&\min_{V_1,V_2} r(\lambda I_m + AA^{\dagger} + BB^{\dagger} + AV_1E_A + BV_2E_B)\\
&= r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} \\ E_A \\ E_B \end{bmatrix} + r[\lambda I_m + AA^{\dagger} + BB^{\dagger}, A, B]\\
&\quad + \max\left\{ r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} & A \\ E_B & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} & A & B \\ E_B & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} & A \\ E_A & 0 \\ E_B & 0 \end{bmatrix}, \right.\\
&\quad \left. r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} & B \\ E_A & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} & A & B \\ E_A & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^{\dagger} + BB^{\dagger} & B \\ E_A & 0 \\ E_B & 0 \end{bmatrix} \right\}\\
&= r\begin{bmatrix} (\lambda+2)I_m \\ E_A \\ E_B \end{bmatrix} + r[\lambda I_m, A, B] + \max\left\{ r\begin{bmatrix} (\lambda+1)I_m & A \\ E_B & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A & B \\ E_B & 0 & 0 \end{bmatrix} - r\begin{bmatrix} (\lambda+1)I_m & A \\ E_A & 0 \\ E_B & 0 \end{bmatrix}, \right.\\
&\quad \left. r\begin{bmatrix} (\lambda+1)I_m & B \\ E_A & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A & B \\ E_A & 0 & 0 \end{bmatrix} - r\begin{bmatrix} (\lambda+1)I_m & B \\ E_A & 0 \\ E_B & 0 \end{bmatrix} \right\}.
\end{aligned} \tag{5.15}$$

Substituting different values of λ into the formulas in (5.14) and (5.15) and simplifying yield the rank formulas in (5.5)–(5.12), respectively. The facts in (a)–(d) follow from setting the rank formulas in (5.5)–(5.12) equal to m and 0 and applying Lemma 2.1, respectively. □

Theorem 5.4. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times p}$ be given.*

(a) *If $\lambda \neq 1, 0, -1$, then the following two formulas hold*

$$\max_{A^-,B^-} r(\lambda I_m + AA^- - BB^-) = m, \tag{5.16}$$

$$\min_{A^-,B^-} r(\lambda I_m + AA^- - BB^-) = \max\{m + r(A) - r[A, B], m + r(B) - r[A, B]\}. \tag{5.17}$$

In particular, the following results hold.

(i) *There always exist A^- and B^-, such that $\lambda I_m + AA^- - BB^-$ is nonsingular.*

(ii) *$\lambda I_m + AA^- - BB^-$ is nonsingular for all A^- and B^- ⇔ the rank of $\lambda I_m + AA^- - BB^-$ is invariant for all A^- and B^- ⇔ $r[A, B] = r(A) = r(B)$ ⇔ $\mathscr{R}(A) = \mathscr{R}(B)$.*

(iii) *There do not exist A^- and B^-, such that $\lambda I_m + AA^- - BB^- = 0$.*

(b) *The following two formulas hold*

$$\max_{A^-,B^-} r(I_m + AA^- - BB^-) = \min\{m, m + r(A) - r(B)\}, \tag{5.18}$$

$$\min_{A^-,B^-} r(I_m + AA^- - BB^-) = m + r(A) - r[A, B]. \tag{5.19}$$

In particular, the following results hold.

(i) *There exist* A^- *and* B^-, *such that* $I_m + AA^- - BB^-$ *is nonsingular* $\Leftrightarrow r(A) = r(B)$.

(ii) $I_m + AA^- - BB^-$ *is nonsingular for all* A^- *and* $B^- \Leftrightarrow \quad \Leftrightarrow \mathscr{R}(A) \supseteq \mathscr{R}(B)$.

(iii) *There exist* A^- *and* B^-, *such that* $BB^- - AA^- = I_m \Leftrightarrow BB^- - AA^- = I_m$ *holds for all* A^- *and* $B^- \Leftrightarrow A = 0$ *and* $r(B) = m$.

(iv) *The rank of* $I_m + AA^- - BB^-$ *is invariant for all* A^- *and* $B^- \Leftrightarrow \mathscr{R}(A) \supseteq \mathscr{R}(B)$ *or* $\mathscr{R}(A) \subseteq \mathscr{R}(B)$.

(c) *The following two formulas hold*

$$\max_{A^-,B^-} r(AA^- - BB^-) = \min\{r[A, B], m + r[A, B] - r(A) - r(B)\}, \tag{5.20}$$

$$\min_{A^-,B^-} r(AA^- - BB^-) = \max\{r[A, B] - r(A), r[A, B] - r(B)\}. \tag{5.21}$$

In particular, the following results hold.

(i) *There exist* A^- *and* B^-, *such that* $AA^- - BB^-$ *is nonsingular* $\Leftrightarrow r[A, B] = r(A) + r(B) = m$.

(ii) $AA^- - BB^-$ *is nonsingular for all* A^- *and* $B^- \Leftrightarrow r(A) = m$ *and* $B = 0$, *or* $A = 0$ *and* $r(B) = m$.

(iii) *There exist* A^- *and* B^-, *such that* $AA^- = BB^- \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B)$.

(iv) $AA^- = BB^-$ *holds for all* A^- *and* $B^- \Leftrightarrow [A, B]{=}0$ *or* $r[A, B] = r(A) + r(B) - m$.

(v) *The rank of* $AA^- - BB^-$ *is invariant for all* A^- *and* $B^- \Leftrightarrow A = 0$ *or* $B = 0$ $r(A) = m$ *or* $r(B) = m$.

Proof. By (5.4),

$$\lambda I_m + AA^- - BB^- = \lambda I_m + AA^\dagger - BB^\dagger + AV_1E_A - BV_2E_B, \tag{5.22}$$

where $V_1 \in \mathbb{C}^{n\times m}$ and $V_2 \in \mathbb{C}^{p\times m}$ are arbitrary. Applying (5.2) and (5.3) to (5.22) and simplifying give

$$\begin{aligned}
\max_{V_1,V_2} r(\lambda I_m + AA^\dagger - BB^\dagger + AV_1E_A - BV_2E_B) &= \min\left\{ r[\lambda I_m + AA^\dagger - BB^\dagger, A, B],\ r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger \\ E_A \\ E_B \end{bmatrix},\right.\\
&\qquad \left. r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger & A \\ E_B & 0 \end{bmatrix},\ r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger & B \\ E_A & 0 \end{bmatrix} \right\}\\
&= \min\left\{ r[\lambda I_m, A, B],\ r\begin{bmatrix} \lambda I_m \\ E_A \\ E_B \end{bmatrix},\ r\begin{bmatrix} (\lambda-1)I_m & A \\ E_B & 0 \end{bmatrix},\ r\begin{bmatrix} (\lambda+1)I_m & B \\ E_A & 0 \end{bmatrix} \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\min_{V_1,V_2} r(\lambda I_m + AA^\dagger - BB^\dagger + AV_1E_A - BV_2E_B)\\
&= r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger \\ E_A \\ E_B \end{bmatrix} + r[\lambda I_m + AA^\dagger - BB^\dagger, A, B]\\
&\quad + \max\left\{ r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger & A \\ E_B & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger & A & B \\ E_B & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger & A \\ E_A & 0 \\ E_B & 0 \end{bmatrix},\right.\\
&\quad \left. r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger & B \\ E_A & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger & A & B \\ E_A & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger - BB^\dagger & B \\ E_A & 0 \\ E_B & 0 \end{bmatrix} \right\}\\
&= r\begin{bmatrix} \lambda I_m \\ E_A \\ E_B \end{bmatrix} + r[\lambda I_m, A, B] + \max\left\{ r\begin{bmatrix} (\lambda-1)I_m & A \\ E_B & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A & B \\ E_B & 0 & 0 \end{bmatrix} - r\begin{bmatrix} (\lambda-1)I_m & A \\ E_A & 0 \\ E_B & 0 \end{bmatrix},\right.\\
&\quad \left. r\begin{bmatrix} (\lambda+1)I_m & B \\ E_A & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A & B \\ E_A & 0 & 0 \end{bmatrix} - r\begin{bmatrix} (\lambda+1)I_m & B \\ E_A & 0 \\ E_B & 0 \end{bmatrix} \right\}.
\end{aligned}$$

Substituting different values of λ into the above formulas and simplifying yield the rank formulas required. □

Theorem 5.5. *Let $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{p\times m}$ be given.*

(a) *If $\lambda \neq 0, -1, -2$, then the following two formulas hold*

$$\max_{A^-,C^-} r(\lambda I_m + AA^- + C^-C) = m,$$
$$\min_{A^-,C^-} r(\lambda I_m + AA^- + C^-C) = \max\{m - r(CA), r(A) + r(C) - r(CA)\}.$$

In particular, the following results hold.

(i) *There always exist A^- and C^-, such that $\lambda I_m + AA^- + C^-C$ is nonsingular.*
(ii) *$\lambda I_m + AA^- + C^-C$ is nonsingular for all A^- and C^- ⇔ the rank of $\lambda I_m + AA^- + C^-C$ is invariant for all A^- and C^- ⇔ $r[A, C] = r(A) = r(C)$ ⇔ $CA = 0$ and $r(A) + r(C) = m$.*
(iii) *There do not exist A^- and C^-, such that $\lambda I_m + AA^- + C^-C = 0$.*

(b) *The following two formulas hold*

$$\max_{A^-,C^-} r(AA^- + C^-C) = \min\{m, r(A) + r(C)\},$$
$$\min_{A^-,C^-} r(AA^- + C^-C) = r(A) + r(C) - r(CA).$$

In particular, the following results hold.

(i) *There exist A^- and C^-, such that $AA^- + C^-C$ is nonsingular ⇔ $r(A) + r(C) \geq m$.*
(ii) *$AA^- + C^-C$ is nonsingular for all A^- and C^- ⇔ $r(CA) = r(A) + r(C) - m$.*
(iii) *There exist A^- and C^-, such that $AA^- + C^-C = 0$ ⇔ $AA^- + C^-C = 0$ for all A^- and C^- ⇔ $A = 0$ and $C = 0$.*
(iv) *The rank of $AA^- + C^-C$ is invariant for all A^- and C^- ⇔ $CA = 0$ or $r(CA) = r(A) + r(C) - m$.*

(c) *The following two formulas hold*

$$\max_{A^-,C^-} r(-I_m + AA^- + C^-C) = \min\{m + r(CA) - r(A), m + r(CA) - r(C)\},$$
$$\min_{A^-,C^-} r(-I_m + AA^- + C^-C) = \max\{m + r(CA) - r(A) - r(C), r(CA)\}.$$

In particular, the following results hold.

(i) *There exist A^- and C^-, such that $-I_m + AA^- + C^-C$ is nonsingular ⇔ $r(CA) = r(A) = r(C)$.*
(ii) *$-I_m + AA^- + C^-C$ is nonsingular for all A^- and C^- ⇔ $A = 0$ and $C = 0$, or $r(A) = r(C) = m$.*
(iii) *There exist A^- and C^-, such that $AA^- + C^-C = I_m$ ⇔ $CA = 0$ and $r(A) + r(C) = m$.*
(iv) *$AA^- + C^-C = I_m$ cannot hold for all A^- and C^-.*
(v) *The rank of $-I_m + AA^- + C^-C$ is invariant for all A^- and C^- ⇔ $A = 0$ or $C = 0$ or $r(A) = m$ or $r(C) = m$.*

(d) *The following two formulas hold*

$$\max_{A^-,C^-} r(-2I_m + AA^- + C^-C) = \min\{m, 2m - r(A) - r(C)\},$$
$$\min_{A^-,C^-} r(-2I_m + AA^- + C^-C) = m - r(CA).$$

In particular, the following results hold.

(i) *There exist A^- and C^-, such that $-2I_m + AA^- + C^-C$ is nonsingular ⇔ $r(A) + r(C) \leq m$.*
(ii) *$-2I_m + AA^- + C^-C$ is nonsingular for all A^- and C^- ⇔ $CA = 0$.*
(iii) *There exist A^- and C^-, such that $AA^- + C^-C = 2I_m$ ⇔ the rank of $-I_m + AA^- + C^-C$ is invariant for all A^- and C^- ⇔ $r(A) = r(C) = m$.*
(iv) *The rank of $-2I_m + AA^- + C^-C$ is invariant for all A^- and C^- ⇔ $CA = 0$ or $r(CA) = r(A) + r(C) - m$.*

Proof. By (5.4),

$$\lambda I_m + AA^- + C^-C = \lambda I_m + AA^\dagger + C^\dagger C + AV_1E_A + F_CV_2C, \tag{5.23}$$

where $V_1 \in \mathbb{C}^{n\times m}$ and $V_2 \in \mathbb{C}^{m\times p}$ are arbitrary. Applying (5.2) and (5.3) to (5.23) and simplifying give

$$\begin{aligned}
&\max_{V_1,V_2} r(\lambda I_m + AA^\dagger + C^\dagger C + AV_1E_A + F_CV_2C)\\
&= \min\left\{ r[\lambda I_m + AA^\dagger + C^\dagger C, A, F_C],\ r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C \\ E_A \\ C \end{bmatrix},\right.\\
&\left. r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C & A \\ C & 0 \end{bmatrix},\ r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C & F_C \\ E_A & 0 \end{bmatrix} \right\}\\
&= \min\left\{ r[(\lambda+1)I_m, A, F_C],\ r\begin{bmatrix} (\lambda+1)I_m \\ E_A \\ C \end{bmatrix},\ r\begin{bmatrix} \lambda I_m & A \\ C & 0 \end{bmatrix},\ r\begin{bmatrix} (\lambda+2)I_m & F_C \\ E_A & 0 \end{bmatrix} \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\min_{V_1,V_2} r(\lambda I_m + AA^\dagger + C^\dagger C + AV_1E_A + F_CV_2C)\\
&= r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C \\ E_A \\ C \end{bmatrix} + r[\lambda I_m + AA^\dagger + C^\dagger C, A, F_C]\\
&+ \max\left\{ r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C & A \\ C & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C & A & F_C \\ C & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C & A \\ E_A & 0 \\ C & 0 \end{bmatrix},\right.\\
&\left. r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C & F_C \\ E_A & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C & A & F_C \\ E_A & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger + C^\dagger C & F_C \\ E_A & 0 \\ C & 0 \end{bmatrix} \right\}\\
&= r\begin{bmatrix} (\lambda+1)I_m \\ E_A \\ C \end{bmatrix} + r[(\lambda+1)I_m, A, F_C] + \max\left\{ r\begin{bmatrix} \lambda I_m & A \\ C & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A & F_C \\ C & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A \\ E_A & 0 \\ C & 0 \end{bmatrix},\right.\\
&\left. r\begin{bmatrix} (\lambda+2)I_m & F_C \\ E_A & 0 \end{bmatrix} - r\begin{bmatrix} (\lambda+1)I_m & A & F_C \\ E_A & 0 & 0 \end{bmatrix} - r\begin{bmatrix} (\lambda+1)I_m & F_C \\ E_A & 0 \\ C & 0 \end{bmatrix} \right\}.
\end{aligned}$$

Substituting different values of λ into the above formulas and simplifying yield the rank formulas required. □

Theorem 5.6. *Let $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{p\times m}$ be given.*

(a) *If $\lambda \neq 1, 0, -1$, then the following two formulas hold*

$$\max_{A^-,C^-} r(\lambda I_m + AA^- - C^-C) = m,$$
$$\min_{A^-,C^-} r(\lambda I_m + AA^- - C^-C) = \max\{m - r(CA), r(A) + r(C) - r(CA)\}.$$

In particular, the following results hold.

(i) *There always exist A^- and C^-, such that $\lambda I_m + AA^- - C^-C$ is nonsingular.*

(ii) *$\lambda I_m + AA^- - C^-C$ is nonsingular for all A^- and C^- ⇔ the rank of $\lambda I_m + AA^- - C^-C$ is invariant for all A^- and C^- ⇔ $CA = 0$ and $r(A) + r(C) = m$.*

(iii) *There do not exist A^- and C^-, such that $\lambda I_m + AA^- - C^-C = 0$.*

(b) *The following two formulas hold*

$$\max_{A^-,C^-} r(AA^- - C^-C) = m - |r(A) + r(C) - m|,$$
$$\min_{A^-,C^-} r(AA^- - C^-C) = r(A) + r(C) - 2r(CA).$$

In particular, the following results hold.

(i) *There exist* A^- *and* C^-, *such that* $AA^- - C^-C$ *is nonsingular* $\Leftrightarrow r(A) = r(C) = m$.

(ii) $AA^- - C^-C$ *is nonsingular for all* A^- *and* $C^- \Leftrightarrow CA = 0$ *and* $r(A) + r(C) = m$.

(iii) *There exist* A^- *and* C^-, *such that* $AA^- = C^-C \Leftrightarrow r(CA) = r(A) = r(C)$.

(iv) $AA^- = C^-C$ *holds for all* A^- *and* $C^- \Leftrightarrow r(A) = r(C) = m$.

(v) *The rank of* $AA^- - C^-C$ *is invariant for all* A^- *and* $C^- \Leftrightarrow CA = 0$ *or* $r(A) + r(C) = m$.

(c) *The following two formulas hold*

$$\max_{A^-, C^-} r(-I_m + AA^- - C^-C) = m - r(A) + r(CA),$$
$$\min_{A^-, C^-} r(-I_m + AA^- - C^-C) = \max\{m - r(A), r(C)\}.$$

In particular, the following results hold.

(i) *There exist* A^- *and* C^-, *such that* $-I_m + AA^- - C^-C$ *is nonsingular* $\Leftrightarrow r(A) = r(CA)$.

(ii) $-I_m + AA^- - C^-C$ *is nonsingular for all* A^- *and* $C^- \Leftrightarrow A = 0$ *or* $r(C) = m$.

(iii) *There exist* A^- *and* C^-, *such that* $AA^- - C^-C = I_m \Leftrightarrow AA^- - C^-C = I_m$ *holds for all* A^- *and* $C^- \Leftrightarrow r(A) = m$ *and* $C = 0$.

(iv) *The rank of* $-I_m + AA^- - C^-C$ *is invariant for all* A^- *and* $C^- \Leftrightarrow CA = 0$ *or* $r(CA) = r(A) + r(C) - m$.

Proof. By (5.4),

$$\lambda I_m + AA^- - C^-C = \lambda I_m + AA^\dagger - C^\dagger C + AV_1E_A - F_CV_2C, \tag{5.24}$$

where $V_1 \in \mathbb{C}^{n\times m}$ and $V_2 \in \mathbb{C}^{m\times p}$ are arbitrary. Applying (5.2) and (5.3) to (5.24) and simplifying give

$$\max_{V_1,V_2} r(\lambda I_m + AA^\dagger - C^\dagger C + AV_1E_A - F_CV_2C) = \min\left\{ r[\lambda I_m + AA^\dagger - C^\dagger C, A, F_C],\ r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C \\ E_A \\ C \end{bmatrix},\right.$$
$$\left. r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C & A \\ C & 0 \end{bmatrix},\ r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C & F_C \\ E_A & 0 \end{bmatrix} \right\}$$
$$= \min\left\{ r[(\lambda-1)I_m, A, F_C],\ r\begin{bmatrix} (\lambda+1)I_m \\ E_A \\ C \end{bmatrix},\ r\begin{bmatrix} \lambda I_m & A \\ C & 0 \end{bmatrix},\ r\begin{bmatrix} \lambda I_m & F_C \\ E_A & 0 \end{bmatrix} \right\},$$

and

$$\min_{V_1,V_2} r(\lambda I_m + AA^\dagger - C^\dagger C + AV_1E_A - F_CV_2C)$$
$$= r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C \\ E_A \\ C \end{bmatrix} + r[\lambda I_m + AA^\dagger - C^\dagger C, A, F_C]$$
$$+ \max\left\{ r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C & A \\ C & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C & A & F_C \\ C & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C & A \\ E_A & 0 \\ C & 0 \end{bmatrix},\right.$$
$$\left. r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C & F_C \\ E_A & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C & A & F_C \\ E_A & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m + AA^\dagger - C^\dagger C & F_C \\ E_A & 0 \\ C & 0 \end{bmatrix} \right\}$$
$$= r\begin{bmatrix} (\lambda+1)I_m \\ E_A \\ C \end{bmatrix} + r[(\lambda-1)I_m, A, F_C] + \max\left\{ r\begin{bmatrix} \lambda I_m & A \\ C & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A & F_C \\ C & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A \\ E_A & 0 \\ C & 0 \end{bmatrix},\right.$$
$$\left. r\begin{bmatrix} \lambda I_m & F_C \\ E_A & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & A & F_C \\ E_A & 0 & 0 \end{bmatrix} - r\begin{bmatrix} \lambda I_m & F_C \\ E_A & 0 \\ C & 0 \end{bmatrix} \right\}.$$

Substituting different values of λ into the above formulas and simplifying yield the rank formulas required. □

Setting $C = A$ in Theorems 5.5 and 5.6 leads to the following corollaries.

Corollary 5.7. *Let $A \in \mathbb{C}^{m\times m}$ be given.*

(a) *If $\lambda \neq 0, -1, -2$, then the following two formulas hold*

$$\max_{A^-} r(\lambda I_m + AA^- + A^-A) = m,$$
$$\min_{A^-} r(\lambda I_m + AA^- + A^-A) = \max\{m - r(A^2), 2r(A) - r(A^2)\}.$$

In particular, the following results hold.

(i) *There always exists an A^-, such that $\lambda I_m + AA^- + A^-A$ is nonsingular.*

(ii) *$\lambda I_m + AA^- + A^-A$ is nonsingular for all A^- $\Leftrightarrow$ the rank of $\lambda I_m + AA^- + A^-A$ is invariant for all A^- $\Leftrightarrow$ $A^2 = 0$ and $2r(A) = m$.*

(iii) *There does not exist an A^-, such that $\lambda I_m + AA^- + A^-A = 0$.*

(b) *The following two formulas hold*

$$\max_{A^-} r(AA^- + A^-A) = \min\{m, 2r(A)\},$$
$$\min_{A^-} r(AA^- + A^-A) = 2r(A) - r(A^2).$$

In particular, the following results hold.

(i) *There exists an A^-, such that $AA^- + A^-A$ is nonsingular $\Leftrightarrow$ $2r(A) \geq m$.*

(ii) *$AA^- + A^-A$ is nonsingular for all A^- $\Leftrightarrow$ $r(A^2) = 2r(A) - m$.*

(iii) *There exists an A^-, such that $AA^- + A^-A = 0 \Leftrightarrow AA^- + A^-A = 0$ for all A^- $\Leftrightarrow A = 0$.*

(iv) *The rank of $AA^- + A^-A$ is invariant for all A^- $\Leftrightarrow$ $A^2 = 0$ or $r(A^2) = 2r(A) - m$.*

(c) *The following two formulas hold*

$$\max_{A^-} r(-I_m + AA^- + A^-A) = m + r(A^2) - r(A),$$
$$\min_{A^-} r(-I_m + AA^- + A^-A) = \max\{m + r(A^2) - 2r(A), r(A^2)\}.$$

In particular, the following results hold.

(i) *There exists an A^-, such that $-I_m + AA^- + A^-A$ is nonsingular $\Leftrightarrow$ $r(A^2) = r(A)$.*

(ii) *$-I_m + AA^- + A^-A$ is nonsingular for all A^- $\Leftrightarrow$ $A = 0$ or $r(A) = m$.*

(iii) *There exists an A^-, such that $AA^- + A^-A = I_m \Leftrightarrow A^2 = 0$ and $2r(A) = m$.*

(iv) *$AA^- + A^-A = I_m$ cannot hold for all A^-.*

(v) *The rank of $-I_m + AA^- + A^-A$ is invariant for all A^- $\Leftrightarrow$ $A = 0$ or $r(A) = m$.*

(d) *The following two formulas hold*

$$\max_{A^-} r(-2I_m + AA^- + A^-A) = \min\{m, 2m - 2r(A)\},$$
$$\min_{A^-} r(-2I_m + AA^- + A^-A) = m - r(A^2).$$

In particular, the following results hold.

(i) *There exists an A^-, such that $-2I_m + AA^- + A^-A$ is nonsingular $\Leftrightarrow$ $2r(A) \leq m$.*

(ii) *$-2I_m + AA^- + A^-A$ is nonsingular for all A^- $\Leftrightarrow$ $A^2 = 0$.*

(iii) *There exists an A^-, such that $AA^- + A^-A = 2I_m \Leftrightarrow$ the rank of $-I_m + AA^- + A^-A$ is invariant for all A^- $\Leftrightarrow$ $r(A) = m$.*

(iv) *The rank of $-2I_m + AA^- + A^-A$ is invariant for all A^- $\Leftrightarrow$ $A^2 = 0$ or $r(A^2) = 2r(A) - m$.*

Corollary 5.8. Let $A \in \mathbb{C}^{m\times m}$ be given.

(a) *If $\lambda \neq 1, 0, -1$, then the following two formulas hold*

$$\max_{A^-} r(\lambda I_m + AA^- - A^-A) = m,$$
$$\min_{A^-} r(\lambda I_m + AA^- - A^-A) = \max\{m - r(A^2), 2r(A) - r(A^2)\}.$$

In particular, the following results hold.

(i) *There always exists an* A^-, *such that* $\lambda I_m + AA^- - A^-A$ *is nonsingular.*

(ii) $\lambda I_m + AA^- - A^-A$ *is nonsingular for all* $A^- \Leftrightarrow$ *the rank of* $\lambda I_m + AA^- - A^-A$ *is invariant for all* $A^- \Leftrightarrow A^2 = 0$ *and* $2r(A) = m$.

(iii) *There does not exist an* A^-, *such that* $\lambda I_m + AA^- - A^-A = 0$.

(b) *The following two formulas hold*

$$\max_{A^-} r(AA^- - A^-A) = m - |2r(A) - m|,$$
$$\min_{A^-} r(AA^- - A^-A) = 2r(A) - 2r(A^2).$$

In particular, the following results hold.

(i) *There exists an* A^-, *such that* $AA^- - A^-A$ *is nonsingular* $\Leftrightarrow 2r(A) = m$.

(ii) $AA^- - A^-A$ *is nonsingular for all* $A^- \Leftrightarrow A^2 = 0$ *and* $2r(A) = m$.

(iii) *There exists an* A^-, *such that* $AA^- = A^-A \Leftrightarrow r(A^2) = r(A)$.

(iv) $AA^- = A^-A$ *holds for all* $A^- \Leftrightarrow r(A) = m$.

(v) *The rank of* $AA^- - A^-A$ *is invariant for all* $A^- \Leftrightarrow A = 0$ *or* $2r(A) = m$.

(c) *The following two formulas hold*

$$\max_{A^-} r(-I_m + AA^- - A^-A) = m - r(A) + r(A^2),$$
$$\min_{A^-} r(-I_m + AA^- - A^-A) = \max\{m - r(A), r(A)\}.$$

In particular, the following results hold.

(i) *There exists an* A^-, *such that* $-I_m + AA^- - A^-A$ *is nonsingular* $\Leftrightarrow r(A^2) = r(A)$.

(ii) $-I_m + AA^- - A^-A$ *is nonsingular for all* $A^- \Leftrightarrow A = 0$ *or* $r(A) = m$.

(iii) *There exists an* A^-, *such that* $AA^- - A^-A = I_m \Leftrightarrow AA^- - A^-A = I_m$ *holds for all* $A^- \Leftrightarrow A = 0$ *and* $r(A) = m$.

(iv) *The rank of* $-I_m + AA^- - A^-A$ *is invariant for all* $A^- \Leftrightarrow A^2 = 0$ *or* $r(A^2) = 2r(A) - m$.

Finally, the author presents a group of rank formulas associated with the products of two block matrices and their generalized inverses.

Corollary 5.9. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, *and* $D \in \mathbb{C}^{l\times n}$ *be given, and denote* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *and* $N = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. *If* $\lambda \neq 1, 0, -1$, *then the following two formulas hold*

$$\max_{M^-, N^-} r(\lambda I_{m+l} + MM^- - NN^-) = m + l, \tag{5.25}$$

$$\min_{M^-, N^-} r(\lambda I_{m+l} + MM^- - NN^-) = m + l - r[A, B] - r[C, D] + \max\{r(M), r(A) + r(D)\}. \tag{5.26}$$

In particular, the following formulas hold

$$\max_{M^-, N^-} r(I_{m+l} + MM^- - NN^-) = \min\{m + l, m + l + r(M) - r(A) - r(D)\}, \tag{5.27}$$

$$\min_{M^-, N^-} r(I_{m+l} + MM^- - NN^-) = m + l + r(M) - r[A, B] - r[C, D], \tag{5.28}$$

$$\max_{M^-, N^-} r(MM^- - NN^-) = r[A, B] + r[C, D] + \min\{0, m + l - r(M) - r(A) - r(D)\}, \tag{5.29}$$

$$\min_{M^-, N^-} r(MM^- - NN^-) = r[A, B] + r[C, D] - \min\{r(M), r(A) + r(D)\}. \tag{5.30}$$

Setting the both sides of (5.26)–(5.30) equal to $m + l$ or equal to zero will lead to a group of interesting facts on the singularity, the non-singularity, the rank invariance, and the equalities associated with the matrix expressions, which I leave as exercises for the reader.

6 Concluding remarks

The author has presented a large number of known and novel algebraic identities that are composed by idempotent matrices and established many explicit formulas for calculating the ranks and ranges of idempotent matrices by means of the BMM and EBMOs. The author has also presented a variety of applications of these identities and formulas in dealing with a variety of issues associated with idempotent matrices under various assumptions. Note that all the preceding results and facts are presented in elementary and analytical forms that fall in the scope of elementary linear algebra and matrix theory, so that the work is accepted easily for all readers with a background of college mathematics, and thus, the author hopes that they can be used in the teaching and research of idempotent matrix problems and also hopes that part of the contents in this survey paper can be chosen as constructive materials in the compilation of new textbooks and handbooks on linear algebra and matrix mathematics. Furthermore, this work shows that the equalities and inequalities of matrices have been one of the most attractive and fruitful research topics in linear algebra and also demonstrates that there still exist many simple and interesting problems on fundamental objects in linear algebra for which we can make deeper exploration and find out various novel and intrinsic conclusions.

It is expected that more algebraic identities composed by idempotent matrices and more analytical formulas for calculating the ranks of matrix expressions can be established by using various tricky block matrix constructions, which we believe will bring a great increase of classic knowledge in linear algebra and will also provide useful tools to deal with various challenging problems in matrix analysis and applications. As a continuation of this research, the author proposes some problems for consideration:

(I) Establish analytical formulas for calculating the ranks of the following general matrix expressions:

$$\begin{aligned}
&A_1A_1^- \pm A_2A_2^- \pm \cdots \pm A_kA_k^-,\\
&[A_1A_1^-[A_2, \ldots, A_k], \ldots, A_kA_k^-[A_1, \ldots, A_{k-1}]],\\
&[A_1, A_2, \ldots, A_k][A_1, A_2, \ldots, A_k]^- - A_1A_1^- - A_2A_2^- - \cdots - A_kA_k^-,\\
&[A_1, A_2, \ldots, A_k] - [A_1, A_2, \ldots, A_k]\begin{bmatrix} A_1^- \\ A_2^- \\ \vdots \\ A_k^- \end{bmatrix}[A_1, A_2, \ldots, A_k]\\
&\quad = [(A_2A_2^- + \cdots + A_kA_k^-)A_1, \ldots, (A_1A_1^- + \cdots + A_{k-1}A_{k-1}^-)A_k]
\end{aligned}$$

with respect to the multiple generalized inverses in them.

(II) Establish necessary and sufficient conditions for the following reverse order laws to hold for generalized inverses of matrix products associated with (3.1), (3.3)–(3.6), (3.11), and (3.87)–(3.91):

$$\begin{aligned}
(\alpha A + \beta B)^{(i,\,\ldots,\,j)} &= \lambda^{-1}(I_m + \mu_1 A)^{-1}(A + B)^{(s,\,\ldots,\,t)}(I_m + \mu_2 B)^{-1},\\
(\alpha A + \beta B)^{(i,\,\ldots,\,j)} &= \lambda^{-1}(I_m + \mu_2 B)^{-1}(A + B)^{(s,\,\ldots,\,t)}(I_m + \mu_1 A)^{-1},\\
(\alpha AB + \beta BA)^{(i,\,\ldots,\,j)} &= (A + B - I_m)^{(s_1,\,\ldots,\,t_1)}(\alpha A + \beta B)^{(s_2,\,\ldots,\,t_2)},\\
(\alpha AB + \beta BA)^{(i,\,\ldots,\,j)} &= (\beta A + \alpha B)^{(s_3,\,\ldots,\,t_3)}(A + B - I_m)^{(s_4,\,\ldots,\,t_4)},\\
(\alpha ABA + \beta BAB)^{(i,\,\ldots,\,j)} &= [(A + B - I_m)^2]^{(s_1,\,\ldots,\,t_1)}(\alpha A + \beta B)^{(s_2,\,\ldots,\,t_2)},\\
(\alpha ABA + \beta BAB)^{(i,\,\ldots,\,j)} &= (A + B - I_m)^{(s_3,\,\ldots,\,t_3)}(\beta A + \alpha B)^{(s_4,\,\ldots,\,t_4)}(A + B - I_m)^{(s_5,\,\ldots,\,t_5)},\\
(\alpha ABA + \beta BAB)^{(i,\,\ldots,\,j)} &= (\alpha A + \beta B)^{(s_6,\,\ldots,\,t_6)}[(A + B - I_m)^2]^{(s_7,\,\ldots,\,t_7)},\\
[\alpha(AB)^k + \beta(BA)^k]^{(i,\,\ldots,\,j)} &= [(A + B - I_m)^{2k-1}]^{(s_1,\,\ldots,\,t_1)}(\alpha A + \beta B)^{(s_2,\,\ldots,\,t_2)},\\
[\alpha(AB)^k + \beta(BA)^k]^{(i,\,\ldots,\,j)} &= (\beta A + \alpha B)^{(s_3,\,\ldots,\,t_3)}[(A + B - I_m)^{2k-1}]^{(s_4,\,\ldots,\,t_4)},\\
[\alpha(ABA)^k + \beta(BAB)^k]^{(i,\,\ldots,\,j)} &= [(A + B - I_m)^{2k}]^{(s_1,\,\ldots,\,t_1)}(\alpha A + \beta B)^{(s_2,\,\ldots,\,t_2)},\\
[\alpha(ABA)^k + \beta(BAB)^k]^{(i,\,\ldots,\,j)} &= (\alpha A + \beta B)^{(s_3,\,\ldots,\,t_3)}[(A + B - I_m)^{2k}]^{(s_4,\,\ldots,\,t_4)},\\
(I_m - \lambda AB)^{(i,\,\ldots,\,j)} &= (I_m + \beta B)(I_m + \alpha A + \beta B)^{(s,\,\ldots,\,t)}(I_m + \alpha A),\\
(I_m - \lambda BA)^{(i,\,\ldots,\,j)} &= (I_m + \alpha A)(I_m + \alpha A + \beta B)^{(s,\,\ldots,\,t)}(I_m + \beta B),
\end{aligned}$$

and

$$[\alpha(AB + AC) + \beta(BA + BC) + \gamma(CA + CB)]^{(i, \dots, j)} = (S - I_m)^{(s_1, \dots, t_1)}(\alpha A + \beta B + \gamma C)^{(s_2, \dots, t_2)},$$
$$[\alpha(BA + CA) + \beta(AB + CB) + \gamma(AC + BC)]^{(i, \dots, j)} = (\alpha A + \beta B + \gamma C)^{(s_1, \dots, t_1)}(S - I_m)^{(s_2, \dots, t_2)},$$
$$[\alpha(B + C)A(B + C) + \beta(A + C)B(A + C) + \gamma(A + B)C(A + B)]^{(i, \dots, j)}$$
$$= (S - I_m)^{(s_1, \dots, t_1)}(\alpha A + \beta B + \gamma C)^{(s_2, \dots, t_2)}(S - I_m)^{(s_3, \dots, t_3)},$$

where $A, B, C \in \mathbb{C}^{m\times m}$ are three idempotent matrices, and $S = A + B + C$.

On the other hand, it is well known that idempotents and generalized inverses of elements can symbolically be defined in rings and operator algebras, which have been recognized as important tools in the investigation of issues in these disciplines. Much to his regret, the author has nothing to say in this paper about idempotents and generalized inverses of elements in general algebraic structures. Nevertheless, it would be very attractive to be able to generalize by algebraic and deductive calculations of the preceding formulas, results, and facts to the problems that are somehow close to the complex matrix case, as they also encode more interesting issues on idempotents in general algebraic structures.

Acknowledgments: The author is grateful to an anonymous referee for his/her helpful comments and suggestions on this paper.

References

[1] W. O. Amrein and K. B. Sinha, *On pairs of projections in a Hilbert space*, Linear Algebra Appl. **208/209** (1994), 425–435.
[2] T. W. Anderson and G. P. H. Styan, *Cochran's theorem, rank additivity and tripotent matrices*, in: G. Kallianpur, P. R. Krishnaiah, J. K. Ghosh (Eds.), Statistics and Probability: Essays in Honor of C.R. Rao, North-Holland, Amsterdam, 1982, pp. 1–23. [Reprinted in: G. P. H. Styan (Ed.), *The Collected Papers of T. W. Anderson: 1943–1985*, Wiley, New York, 1990, vol. 2, pp. 1307–1329.]
[3] J. Avron, R. Seiler, and B. Simon, *The index of a pair of projections*, J. Funct. Anal. **120** (1994), 220–237.
[4] C. S. Ballantine, *Products of idempotent matrices*, Linear Algebra Appl. **19** (1978), 81–86.
[5] M. Bataille, *Solution to a rank identity*, College Math. J. **35** (2004), 230.
[6] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.
[7] J. Benítez and V. Rakočević, *Applications of CS decomposition in linear combinations of two orthogonal projectors*, Appl. Math. Comput. **203** (2008), 761–769.
[8] S. Borac, *On the algebra generated by two projections*, J. Math. Phys. **36** (1995), 863–874.
[9] J. D. Botha, *Idempotent factorization of matrices*, Linear Multilinear Algebra. **40** (1996), 365–371.
[10] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*, SIAM, Philadephia, 2009.
[11] J. Cao and Y. Xue, *The characterizations and representations for the generalized inverses with prescribed idempotents in Banach algebras*, Filomat **27** (2013), 851–863.
[12] D. S. Djordjević and Y. Wei, *Outer generalized inverses in rings*, Commun. Algebra **33** (2005), 3051–3060.
[13] J. A. Erdos, *On products of idempotent matrices*, Glasgow Math. J. **8** (1967), 118–122.
[14] J. B. Hawkins and W. J. Kammerer, *A class of linear transformations which can be written as the product of projections*, Proc. Am. Math. Soc. **19** (1968), 739–745.
[15] R. E. Hartwig and M. S. Putcha, *When is a matrix a difference of two idempotents?* Linear Multilinear Algebra **26** (1990), 267–277.
[16] R. E. Hartwig and M. S. Putcha, *When is a matrix a sum of idempotents?* Linear Multilinear Algebra **26** (1990), 279–286.
[17] T. Kato, *Notes on Projections and Perturbation Theory*, Technical Report No. 9, Univ. Calif., Berkley, 1955.
[18] M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Prindle, Weber and Schmidt, Boston, 1969.
[19] G. Marsaglia and G. P. H. Styan, *Equalities and inequalities for ranks of matrices*, Linear Multilinear Algebra **2** (1974), 269–292.
[20] C. D. Meyer, *On the construction of solutions to the matrix equations $AX = A$ and $YA = A$*, SIAM Rev. **11** (1969), 612–615.
[21] B. Načevska and D. S. Djordjević, *Inner generalized inverses with prescribed idempotents*, Commun. Algebra **39** (2011), 634–646.
[22] K. Nishio, *The structure of a real linear combination of two projections*, Linear Algebra Appl. **66** (1985), 169–176.
[23] C. R. Rao, *Representations of best linear unbiased estimators in the Gauss-Markoff model with a singular dispersion matrix*, J. Multivariate Anal. **3** (1973), 276–292.
[24] C. R. Rao, *Linear Statistical Inference and Its Applications*, 2nd ed., Wiley, New York, 1973.
[25] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and Its Applications*, Wiley, New York, 1971.

[26] C. R. Rao, H. Toutenburg, Shalabh, and C. Heumann, *Linear Models and Generalizations: Least Squares and Alternatives*, 3rd ed., Springer, Berlin, 2008.
[27] B. Simon, *Unitaries permuting two orthogonal projections*, Linear Algebra Appl. **528** (2017), 436–441.
[28] G. P. H. Styan, *Schur complements and linear models*, in: T. Pukkila, S. Puntanen (Eds.), Proceedings of the First International Tampere Seminar on Linear Statistical Models and Their Applications, Department of Mathematical Science, University of Tampere, Finland, 1985, pp. 37–75.
[29] Y. Tian, *The maximal and minimal ranks of some expressions of generalized inverses of matrices*, Southeast Asian Bull. Math. **25** (2002), 745–755.
[30] Y. Tian, *Upper and lower bounds for ranks of matrix expressions using generalized inverses*, Linear Algebra Appl. **355** (2002), 187–214.
[31] Y. Tian, *Reverse order laws for the Drazin inverse of a triple matrix product*, Pub. Math. Debrecen **63** (2003), 261–277.
[32] Y. Tian, *A rank identity, Problem 753*, College Math. J. **34** (2003), 239. [Solution by M. Bataille & the proposer, College Math. J. **35** (2004), 230].
[33] Y. Tian, *Rank inequalities for idempotent matrices, Problem 772*, College Math. J. **35** (2004), 136. [Solution by L. Zhou & the proposer, College Math. J. **35** (2005), 162].
[34] Y. Tian, *Formulas for calculating the dimensions of the sums and the intersections of a family of linear subspaces with applications*, Beitr. Algebra Geom. **60** (2019), 471–485.
[35] Y. Tian, *On relationships between two linear subspaces and two orthogonal projectors*, Spec. Matrices **7** (2019), 142–212.
[36] Y. Tian and G. P. H. Styan, *Rank equalities for idempotent and involutory matrices*, Linear Algebra Appl. **335** (2001), 101–117.
[37] Y. Tian and G. P. H. Styan, *A new rank formula for idempotent matrices with applications*, Comment. Math. Univ. Carolin **43** (2002), 379–384.
[38] Y. Tian and G. P. H. Styan, *Rank equalities for idempotent matrices with applications*, J. Comput. Appl. Math. **191** (2006), 77–97.
[39] Y. Tian and G. P. H. Styan, *Cochran's statistical theorem revisited*, J. Statist. Plann. Inference **136** (2006), 2659–2667.
[40] Y. Tian and Y. Takane, *Some properties of projectors associated with the WLSE under a general linear model*, J. Multivariate Anal. **99** (2008), 1070–1082.
[41] Y. Tian and Y. Takane, *On V-orthogonal projectors associated with a semi-norm*, Ann. Inst. Stat. Math. **61** (2009), 517–530.
[42] G. Trenkler, *Characterizations of oblique and orthogonal projectors*, in: T. Calinski, R. Kala (Eds.), Proceedings of the International Conference on Linear Statistical Inference LINSTAT '93, Mathematics and Its Applications, vol. 306, Springer, Dordrecht, 1994, pp. 255–270.
[43] G. Trenkler, *On oblique and orthogonal projectors*, in: P. Brown, S. Liu, D. Sharma (Eds.), Contributions to Probability and Statistics: Applications and Challenges, Proceedings of the International Statistics Workshop, World Scientific Publishing Company, 2006, pp. 178–191.
[44] A. M. Vetoshkin, *Jordan form of the difference of projectors*, Comput. Math. Math. Phys. **54** (2014), 382–396.
[45] S. Zhao, H. Lü, Z. Lin, and Z. Yang, *Rank equalities for linear combinations of the power of product of idempotent matrices*, J. Northeast Norm. Univ. Nat. Sci. Edit. **44** (2012), 21–25.
[46] K. Zuo, X. Zhou, and S. Yu, *On some rank equalities of combinations of idempotent matrices with applications*, J. Huazhong Norm. Univ. Nat. Sci. **45** (2011), 366–370.