Research Article

Chao Ma, Hao Liang, Qimiao Xie*, and Pengcheng Wang

Some inequalities on the spectral radius of nonnegative tensors

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Abstract: The eigenvalues and the spectral radius of nonnegative tensors have been extensively studied in recent years. In this paper, we investigate the analytic properties of nonnegative tensors and give some inequalities on the spectral radius.

Keywords: nonnegative tensor, spectral radius, continuity, Hadamard product

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1 Introduction

In recent years, problems related to tensors have drawn much people's attention. As a generalization of matrix theory, fruitful research achievements have been made in topics such as tensor decomposition, tensor eigenvalues and structured tensors [1–3]. Tensors also have wide applications in quantum entanglement, higher order Markov chains, magnetic resonance imaging, machine learning, data analysis, polynomial optimization, nonlinear optimization, hypergraph partitioning, etc. [4–19].

In 2005, Qi [20] and Lim [21] independently defined the concept of eigenvalues of tensors. In 2008, Chang et al. established the Perron–Frobenius theorem for nonnegative tensors [22]. In 2010, Yang and Yang introduced the definition of spectral radius of tensors [23]. Some bounds on the spectral radius of nonnegative tensors are given in [24–28].

In the proof of Theorem 2.3 in [23], the authors considered the sequence of nonnegative tensors and gave the limit formula regarding the spectral radius. Note that the result holds when the sequence is monotonic. We want to know whether the result still holds when the sequence is not monotonic and try to investigate the continuity of the spectral radius of nonnegative tensors by some inequalities. Recently, Sun et al. generalized some inequalities on the spectral radius of the Hadamard product of nonnegative matrices to nonnegative tensors [29]. Their beautiful results make us interested in the further study of the Hadamard product of tensors.

In this paper, we mainly investigate the analytic properties of the spectral radius of nonnegative tensors. We discuss the continuity of the spectral radius by means of limit formulas as well as tensor inequalities involving norms. We also give some inequalities on the spectral radius of the Hadamard product of nonnegative tensors. These results can be seen as a generalization of the existing inequalities on the spectral radius of nonnegative matrices.

*** Corresponding author: Qimiao Xie,** College of Ocean Science and Engineering, Shanghai Maritime University, Shanghai 201306, China, e-mail: qmxie@shmtu.edu.cn
Chao Ma: School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China; Department of Mathematics, Shanghai Maritime University, Shanghai 201306, China
Hao Liang: Department of Mathematics, East China Normal University, Shanghai 200241, China
Pengcheng Wang: College of Ocean Science and Engineering, Shanghai Maritime University, Shanghai 201306, China

The paper is organized as follows. In Section 2, we collect some definitions, notations and helpful lemmas. In Section 3, we discuss the continuity of the spectral radius. In Section 4, we give some inequalities on the spectral radius involving the Hadamard product.

2 Preliminaries

A real mth order n-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a multiarray of real entries $a_{i_1 i_2 \cdots i_m}$, where $i_j \in \{1, 2,...,n\}$ for $j \in \{1, 2,...,m\}$. When $m = 2$, $\mathcal{A}$ is a matrix of order n. The set of all mth order n-dimensional real tensors is denoted as $T_{m,n}$. Throughout this paper, we assume that $m, n \geq 2$.

A tensor is said to be nonnegative (positive) if each of its entry is nonnegative (positive). Denote by O the zero tensor, and by $\mathcal{J}$ the tensor with each entry equal to 1. For a tensor $\mathcal{A}$, $\mathcal{A} \geq (>) O$ implies that $\mathcal{A}$ is nonnegative (positive). For two tensors $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \geq \mathcal{B}$ or $\mathcal{B} \leq \mathcal{A}$ implies that $\mathcal{A} - \mathcal{B}$ is nonnegative. Let $|\mathcal{A}|$ be the tensor obtained from $\mathcal{A}$ by taking the absolute values of the entries. Then, $|\mathcal{A}|$ is nonnegative and $\mathcal{A} \leq |\mathcal{A}|$. $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in T_{m,n}$ is said to be reducible, if there is a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{i_1 i_2 \cdots i_m} = 0, \quad \forall \ i_1 \in I, \quad \forall \ i_2, i_3, \ldots, i_m \notin I.$$

A tensor is said to be irreducible if it is not reducible.

The inner product of $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) \in T_{m,n}$, denoted by $\langle \mathcal{A}, \mathcal{B} \rangle$, is defined as follows:

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} b_{i_1 i_2 \cdots i_m}.$$

The Frobenius norm of $\mathcal{A}$ is defined and denoted as $||\mathcal{A}||_F = \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle}$. Denote by $\mathbb{R}^n$ the set of real vectors of dimension n. $\mathbb{R}^n_+(\mathbb{R}^n_{++})$ represents the cone $\{\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \mid x_i \geq (>) 0, \ i = 1, 2, \ldots, n\}$.

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in T_{m,n}$, and let $\mathbf{x} = (x_1, x_2,...,x_n)^T$ be a complex vector of dimension n. Then, $\mathcal{A}\mathbf{x}^{m-1}$ is a vector of dimension n with its ith component as

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

for $i = 1, 2,...,n$. Denote by $\mathbf{0}$ the zero vector. A complex number λ is called an eigenvalue of $\mathcal{A}$ if it together with $\mathbf{x} \neq \mathbf{0}$ forms a solution to the following system of homogeneous polynomial equations:

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathbf{x}^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T$. The nonzero vector $\mathbf{x}$ is called an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ. The spectral radius of $\mathcal{A}$ is defined and denoted as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

It is well known that the Perron–Frobenius theorem is a fundamental result for nonnegative matrices [30, p. 123]. Chang, Pearson and Zhang generalized this theorem to nonnegative tensors. Yang and Yang gave some further results on the Perron–Frobenius theorem for nonnegative tensors. We summarize some of their results as follows.

Lemma 2.1. [22] *Let $\mathcal{A} \in T_{m,n}$ be nonnegative. Then, there exists $\lambda_0 \geq 0$ and $\mathbf{x}_0 \in \mathbb{R}^n_+$ such that $\mathcal{A}\mathbf{x}_0^{m-1} = \lambda_0 \mathbf{x}_0^{[m-1]}$.*

Lemma 2.2. [22] *Let $\mathcal{A} \in T_{m,n}$ be irreducible nonnegative. Then, there exists $\lambda_0 > 0$ and $\mathbf{x}_0 \in \mathbb{R}^n_{++}$ such that $\mathcal{A}\mathbf{x}_0^{m-1} = \lambda_0 \mathbf{x}_0^{[m-1]}$. Moreover, if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \lambda_0$. If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \le \lambda_0$.*

Lemma 2.3. [23] *Let $\mathcal{A}, \mathcal{B} \in T_{m,n}$ be nonnegative. Then,*
(i) $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector corresponding to it;
(ii) If λ is an eigenvalue of $\mathcal{A}$ with a positive eigenvector, then $\lambda = \rho(\mathcal{A})$;
(iii) If $\mathcal{A} \le \mathcal{B}$, then $\rho(\mathcal{A}) \le \rho(\mathcal{B})$;
(iv) $\rho(\mathcal{A}) = \max_{x \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}} \min_{x_i \neq 0} \frac{(\mathcal{A}\mathbf{x}^{m-1})_i}{x_i^{m-1}}$;
(v) If $\mathbf{x} \in \mathbb{R}^n_{++}$, then $\rho(\mathcal{A}) \le \max_{1 \le i \le n} \frac{(\mathcal{A}\mathbf{x}^{m-1})_i}{x_i^{m-1}}$;
(vi) Let $\mathcal{A}_k = \mathcal{A} + \frac{1}{k}\mathcal{J}$, $k = 1, 2, \ldots$ Then, $\lim_{k\to\infty} \rho(\mathcal{A}_k) = \rho(\mathcal{A})$.

3 Continuity of the spectral radius of nonnegative tensors

Suppose $\mathcal{A} \in T_{m,n}$, and $\{\mathcal{A}_k\}_{k=1}^{\infty}$ is a set of mth order n-dimensional tensors. $\lim_{k\to\infty} \mathcal{A}_k = \mathcal{A}$ means $\lim_{k\to\infty} (\mathcal{A}_k)_{i_1 i_2 \cdots i_m} = (\mathcal{A})_{i_1 i_2 \cdots i_m}$ for any $i_1, i_2, \ldots, i_m$.

Lemma 3.1. *Let $\mathcal{A} \in T_{m,n}$ be nonnegative, and let $\{\mathcal{E}_k\}_{k=1}^{\infty}$ be a set of mth order n-dimensional nonnegative tensors with $\lim_{k\to\infty} \mathcal{E}_k = \mathcal{O}$. If $\mathcal{A}$ has an eigenvalue with a positive eigenvector corresponding to it, then $\lim_{k\to\infty} \rho(\mathcal{A} + \mathcal{E}_k) = \rho(\mathcal{A})$.*

Proof. By Lemma 2.3 (ii), there exists $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n_{++}$ such that $\mathcal{A}\mathbf{x}^{m-1} = \rho(\mathcal{A})\mathbf{x}^{[m-1]}$. Then, by Lemma 2.3 (iii) and (v),

$$\rho(\mathcal{A}) \le \rho(\mathcal{A} + \mathcal{E}_k) \le \max_{1 \le i \le n} \frac{((\mathcal{A} + \mathcal{E}_k)\mathbf{x}^{m-1})_i}{x_i^{m-1}} = \rho(\mathcal{A}) + \max_{1 \le i \le n} \frac{(\mathcal{E}_k \mathbf{x}^{m-1})_i}{x_i^{m-1}}.$$

Since $\lim_{k\to\infty} \mathcal{E}_k = \mathcal{O}$, $\lim_{k\to\infty} \max_{1 \le i \le n} \frac{(\mathcal{E}_k x^{m-1})_i}{x_i^{m-1}} = 0$. Thus, $\lim_{k\to\infty} \rho(\mathcal{A} + \mathcal{E}_k) = \rho(\mathcal{A})$. □

Lemma 3.2. *Let $\mathcal{A} \in T_{m,n}$ be nonnegative, and let $\{\mathcal{E}_k\}_{k=1}^{\infty}$ be a set of mth order n-dimensional nonnegative tensors such that $\mathcal{E}_k \le \mathcal{A}$ for $k = 1, 2, \ldots$ with $\lim_{k\to\infty} \mathcal{E}_k = \mathcal{O}$. Then, $\lim_{k\to\infty} \rho(\mathcal{A} - \mathcal{E}_k) = \rho(\mathcal{A})$.*

Proof. By Lemma 2.3 (i), there exists nonzero $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n_{+}$ such that $\mathcal{A}\mathbf{y}^{m-1} = \rho(\mathcal{A})\mathbf{y}^{[m-1]}$. Then, by Lemma 2.3 (iii) and (iv),

$$\rho(\mathcal{A}) \ge \rho(\mathcal{A} - \mathcal{E}_k) \ge \min_{y_i \neq 0} \frac{((\mathcal{A} - \mathcal{E}_k)\mathbf{y}^{m-1})_i}{y_i^{m-1}} = \rho(\mathcal{A}) - \max_{y_i \neq 0} \frac{(\mathcal{E}_k \mathbf{y}^{m-1})_i}{y_i^{m-1}}.$$

Since $\lim_{k\to\infty} \mathcal{E}_k = \mathcal{O}$, $\lim_{k\to\infty} \max_{y_i \neq 0} \frac{(\mathcal{E}_k \mathbf{y}^{m-1})_i}{y_i^{m-1}} = 0$. Thus, $\lim_{k\to\infty} \rho(\mathcal{A} - \mathcal{E}_k) = \rho(\mathcal{A})$. □

Theorem 3.3. *Let $\mathcal{A} \in T_{m,n}$ be nonnegative, and suppose $\mathcal{A}$ has an eigenvalue with a positive eigenvector corresponding to it. Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a set of mth order n-dimensional nonnegative tensors with $\lim_{k\to\infty} \mathcal{A}_k = \mathcal{A}$. Then, $\lim_{k\to\infty} \rho(\mathcal{A}_k) = \rho(\mathcal{A})$.*

Proof. Let $\mathcal{A} - \mathcal{A}_k = \mathcal{E}_k^+ - \mathcal{E}_k^-$, $k = 1, 2, \ldots$, where

$$(\mathcal{E}_k^+)_{i_1 i_2 \cdots i_m} = \begin{cases} (\mathcal{A} - \mathcal{A}_k)_{i_1 i_2 \cdots i_m}, & \text{if } (\mathcal{A} - \mathcal{A}_k)_{i_1 i_2 \cdots i_m} > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\mathcal{E}_k^+ \geq 0$, $\mathcal{E}_k^- \geq 0$, $\mathcal{A} - \mathcal{E}_k^+ \geq 0$, $\lim_{k\to\infty}\mathcal{E}_k^+ = 0$, $\lim_{k\to\infty}\mathcal{E}_k^- = 0$. Since $\mathcal{A}_k = \mathcal{A} + \mathcal{E}_k^- - \mathcal{E}_k^+ \leq \mathcal{A} + \mathcal{E}_k^-$, by Lemma 2.3 (iii), $\rho(\mathcal{A}_k) \leq \rho(\mathcal{A} + \mathcal{E}_k^-)$. Then, by Lemma 3.1, $\lim_{k\to\infty}\rho(\mathcal{A}_k) \leq \lim_{k\to\infty}\rho(\mathcal{A} + \mathcal{E}_k^-) = \rho(\mathcal{A})$. Since $\mathcal{A}_k \geq \mathcal{A}_k - \mathcal{E}_k^- = \mathcal{A} - \mathcal{E}_k^+$, by Lemma 2.3 (iii), $\rho(\mathcal{A}_k) \geq \rho(\mathcal{A} - \mathcal{E}_k^+)$. Then, by Lemma 3.2, $\lim_{k\to\infty}\rho(\mathcal{A}_k) \geq \lim_{k\to\infty}\rho(\mathcal{A} - \mathcal{E}_k^+) = \rho(\mathcal{A})$. This completes the proof. ☐

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in T_{m,n}$. If the entries $a_{i_1 i_2 \cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a symmetric tensor. The set of all mth order n-dimensional real symmetric tensors is denoted as $S_{m,n}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. Define

$$\mathcal{A}\mathbf{x}^m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

Lemma 3.4. *Let $\mathcal{A} \in T_{m,n}$ be positive, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_{++}^n$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}_+^n$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}_+^n$. Then,*

$$\mathcal{A}\mathbf{x}^m \prod_{i_1, i_2, \ldots, i_m = 1}^{n} (v_{i_1} u_{i_2} \cdots u_{i_m})^{\frac{a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}}{\mathcal{A}x^m}} \leq \mathbf{v}^T(\mathcal{A}\mathbf{u}^{m-1}) \prod_{i_1, i_2, \ldots, i_m = 1}^{n} (x_{i_1} x_{i_2} \cdots x_{i_m})^{\frac{a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}}{\mathcal{A}x^m}}.$$

Proof. Since the function $f(t) = \log t$ is concave on $(0, +\infty)$, we have

$$\log\left(\sum_{i_1, i_2, \ldots, i_m = 1}^{n} \frac{a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}}{\mathcal{A}\mathbf{x}^m} \cdot \frac{v_{i_1} u_{i_2} \cdots u_{i_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}}\right) \geq \sum_{i_1, i_2, \ldots, i_m = 1}^{n} \frac{a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}}{\mathcal{A}\mathbf{x}^m} \log \frac{v_{i_1} u_{i_2} \cdots u_{i_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}}.$$

Then,

$$\log \frac{\mathbf{v}^T(\mathcal{A}\mathbf{u}^{m-1})}{\mathcal{A}\mathbf{x}^m} \geq \log \prod_{i_1, i_2, \ldots, i_m = 1}^{n} \left(\frac{v_{i_1} u_{i_2} \cdots u_{i_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}}\right)^{\frac{a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}}{\mathcal{A}\mathbf{x}^m}},$$

which implies

$$\frac{\mathbf{v}^T(\mathcal{A}\mathbf{u}^{m-1})}{\mathcal{A}\mathbf{x}^m} \geq \prod_{i_1, i_2, \ldots, i_m = 1}^{n} \left(\frac{v_{i_1} u_{i_2} \cdots u_{i_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}}\right)^{\frac{a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}}{\mathcal{A}\mathbf{x}^m}}.$$

This completes the proof. ☐

Lemma 3.5. *Let $\mathcal{A} \in S_{m,n}$ be nonnegative, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$. Then, $\mathcal{A}\mathbf{x}^m \leq \rho(\mathcal{A}) \sum_{i=1}^n x_i^m$.*

Proof. By Lemma 2.3 (vi), it suffices to consider the case when $\mathcal{A}$ is positive. First suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_{++}^n$. Note that Lemma 2.2 implies that $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a positive eigenvector corresponding to it. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}_{++}^n$ be the eigenvector corresponding to $\rho(\mathcal{A})$, and let $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}_{++}^n$ with $v_i = \frac{x_i^m}{u_i^{m-1}}$, $i = 1, 2, \ldots, n$. Since $\mathcal{A}$ is symmetric,

$$\prod_{i_1, i_2, \ldots, i_m = 1}^{n} (v_{i_1} u_{i_2} \cdots u_{i_m})^{\frac{a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}}{\mathcal{A}\mathbf{x}^m}} = \prod_{i=1}^{n} (v_i u_i^{m-1})^{\frac{x_i (\mathcal{A}\mathbf{x}^{m-1})_i}{\mathcal{A}\mathbf{x}^m}},$$

$$\prod_{i_1,i_2,\ldots,i_m=1}^{n} (x_{i_1}x_{i_2}\cdots x_{i_m})^{\frac{a_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}}{\mathcal{A}\mathbf{x}^m}} = \prod_{i=1}^{n} (x_i^m)^{\frac{x_i(\mathcal{A}\mathbf{x}^{m-1})_i}{\mathcal{A}\mathbf{x}^m}}.$$

Thus,

$$\prod_{i_1,i_2,\ldots,i_m=1}^{n} (v_{i_1}u_{i_2}\cdots u_{i_m})^{\frac{a_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}}{\mathcal{A}\mathbf{x}^m}} = \prod_{i_1,i_2,\ldots,i_m=1}^{n} (x_{i_1}x_{i_2}\cdots x_{i_m})^{\frac{a_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}}{\mathcal{A}\mathbf{x}^m}}.$$

By Lemma 3.4, $\mathcal{A}\mathbf{x}^m \le \mathbf{v}^T(\mathcal{A}\mathbf{u}^{m-1}) = \mathbf{v}^T\rho(\mathcal{A})\mathbf{u}^{[m-1]} = \rho(\mathcal{A})\sum_{i=1}^n v_i u_i^{m-1} = \rho(\mathcal{A})\sum_{i=1}^n x_i^m$.

Next suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$. For $k = 1, 2, \ldots$, let $\mathbf{x}^{(k)} = \mathbf{x} + \frac{1}{k}\mathbf{e}$, where $\mathbf{e}$ is the vector with each component equal to 1. Then, $\mathbf{x}^{(k)} \in \mathbb{R}_{++}^n$ and $\lim_{k\to\infty}\mathbf{x}^{(k)} = \mathbf{x}$. By what we have proved above, $\mathcal{A}(\mathbf{x}^{(k)})^m \le \rho(\mathcal{A})\sum_{i=1}^n \left(x_i + \frac{1}{k}\right)^m$ for $k = 1, 2, \ldots$. The conclusion holds when $k \to \infty$. □

Theorem 3.6. *Let $\mathcal{A}, \mathcal{B} \in S_{m,n}$ be nonnegative. Then, $\rho(\mathcal{A} + \mathcal{B}) \le \rho(\mathcal{A}) + \rho(\mathcal{B})$.*

Proof. By Lemma 2.3 (i), let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ be the eigenvector of $\mathcal{A} + \mathcal{B}$ corresponding to $\rho(\mathcal{A} + \mathcal{B})$ with $\sum_{i=1}^n x_i^m = 1$. Then, by Lemma 3.5, $\rho(\mathcal{A} + \mathcal{B}) = \rho(\mathcal{A} + \mathcal{B})\sum_{i=1}^n x_i^m = \sum_{i=1}^n x_i((\mathcal{A} + \mathcal{B})x^{m-1})_i = (\mathcal{A} + \mathcal{B})\mathbf{x}^m = \mathcal{A}\mathbf{x}^m + \mathcal{B}\mathbf{x}^m \le \rho(\mathcal{A})\sum_{i=1}^n x_i^m + \rho(\mathcal{B})\sum_{i=1}^n x_i^m = \rho(\mathcal{A}) + \rho(\mathcal{B})$. □

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. Denote by $\mathbf{x}^{\otimes m}$ the tensor in $T_{m,n}$ with its $(i_1, i_2, \ldots, i_m)$ entry as $x_{i_1}x_{i_2}\cdots x_{i_m}$.

Lemma 3.7. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$ with $\sum_{i=1}^n x_i^m = 1$. Then, $\|\mathbf{x}^{\otimes m}\|_F \le n^{\frac{m}{2}-1}$.*

Proof. Since $\sum_{i=1}^n x_i^m = 1$ with $m \ge 2$ by the power mean inequality [31, p. 203], we have $\left(\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}\right)^{\frac{1}{2}} \le \left(\frac{x_1^m + x_2^m + \cdots + x_n^m}{n}\right)^{\frac{1}{m}} = \left(\frac{1}{n}\right)^{\frac{1}{m}}$. Then, $\|\mathbf{x}^{\otimes m}\|_F^2 = \sum_{i_1,i_2,\ldots,i_m=1}^n x_{i_1}^2 x_{i_2}^2\cdots x_{i_m}^2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^m \le n^{m-2}$. This completes the proof. □

Lemma 3.8. *Let $\mathcal{A} \in T_{m,n}$ be nonnegative. Then, $\rho(\mathcal{A}) \le n^{\frac{m}{2}-1}\|\mathcal{A}\|_F$.*

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$ be the eigenvector of $\mathcal{A}$ corresponding to $\rho(\mathcal{A})$ with $\sum_{i=1}^n x_i^m = 1$. By Lemma 3.7 and the Cauchy–Schwarz inequality, $\rho(\mathcal{A}) = \rho(\mathcal{A})\sum_{i=1}^n x_i^m = \sum_{i=1}^n x_i(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i,i_2,\ldots,i_m=1}^n a_{ii_2\cdots i_m}x_i x_{i_2}\cdots x_{i_m} = \langle \mathcal{A}, \mathbf{x}^{\otimes m}\rangle \le \|\mathcal{A}\|_F\|\mathbf{x}^{\otimes m}\|_F \le n^{\frac{m}{2}-1}\|\mathcal{A}\|_F$. □

Theorem 3.9. *Let $\mathcal{A}, \mathcal{B} \in S_{m,n}$ be nonnegative. Then, $|\rho(\mathcal{A}) - \rho(\mathcal{B})| \le n^{\frac{m}{2}-1}\|\mathcal{A} - \mathcal{B}\|_F$.*

Proof. Since $\mathcal{A} - \mathcal{B} \le |\mathcal{A} - \mathcal{B}|$, $\mathcal{A} \le |\mathcal{A} - \mathcal{B}| + \mathcal{B}$. By Theorem 3.6, $\rho(\mathcal{A}) \le \rho(|\mathcal{A} - \mathcal{B}| + \mathcal{B}) \le \rho(|\mathcal{A} - \mathcal{B}|) + \rho(\mathcal{B})$. Thus, $\rho(\mathcal{A}) - \rho(\mathcal{B}) \le \rho(|\mathcal{A} - \mathcal{B}|)$. Similarly, $\rho(\mathcal{B}) - \rho(\mathcal{A}) \le \rho(|\mathcal{A} - \mathcal{B}|)$. Then, by Lemma 3.8, $|\rho(\mathcal{A}) - \rho(\mathcal{B})| \le \rho(|\mathcal{A} - \mathcal{B}|) \le n^{\frac{m}{2}-1}\|\mathcal{A} - \mathcal{B}\|_F$. □

4 Some inequalities involving the Hadamard product

Let $\mathcal{A} = (a_{i_1i_2\cdots i_m})$, $\mathcal{B} = (b_{i_1i_2\cdots i_m}) \in T_{m,n}$. The Hadamard product of $\mathcal{A}$ and $\mathcal{B}$ is defined and denoted as $\mathcal{A}\circ\mathcal{B} = (a_{i_1i_2\cdots i_m}b_{i_1i_2\cdots i_m}) \in T_{m,n}$. Let $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ be nonnegative, and let α be a positive real number.

The αth Hadamard power of $\mathcal{A}$ is the tensor $\mathcal{A}^{[\alpha]} = (a^{\alpha}_{i_1 i_2 \cdots i_m})$. Similarly, for two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$, let $\mathbf{x} \circ \mathbf{y} = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)^T$; for $\mathbf{x} \in \mathbb{R}^n_+$ and $\alpha > 0$ let $\mathbf{x}^{[\alpha]} = (x_1^{\alpha}, x_2^{\alpha}, \ldots, x_n^{\alpha})$.

Theorem 4.1. *Let $\mathcal{A} \in T_{m,n}$ be nonnegative. Then, $(\rho(\mathcal{A}^{[r]}))^{\frac{1}{r}} \le (\rho(\mathcal{A}^{[s]}))^{\frac{1}{s}}$, with $r \ge s > 0$.*

Proof. By Lemma 2.3 (vi), we may suppose $\mathcal{A}$ is positive. Let $\mathcal{A}^{[s]} = \mathcal{B} = (b_{i_1 i_2 \cdots i_m}) > O$, and let $t = \frac{r}{s} \ge 1$. It suffices to prove that $\rho(\mathcal{B}^{[t]}) \le (\rho(\mathcal{B}))^t$. By Lemma 2.2, there exists $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n_{++}$ such that $\mathcal{B}\mathbf{x}^{m-1} = \rho(\mathcal{B})\mathbf{x}^{[m-1]}$. Then, for $i = 1, 2, \ldots, n$,

$$\begin{aligned}(\mathcal{B}^{[t]}(\mathbf{x}^{[t]})^{m-1})_i &= \sum_{i_2,\ldots,i_m=1}^{n} b^t_{i,i_2\cdots i_m} x^t_{i_2} \cdots x^t_{i_m} \le \left(\sum_{i_2,\ldots,i_m=1}^{n} b_{i,i_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)^t \\ &= ((\mathcal{B}\mathbf{x}^{m-1})_i)^t = ((\rho(\mathcal{B})\mathbf{x}^{[m-1]})_i)^t = (\rho(\mathcal{B}))^t (x_i^t)^{m-1}.\end{aligned}$$

By Lemma 2.3 (v),

$$\rho(\mathcal{B}^{[t]}) \le \max_{1 \le i \le n} \frac{(\mathcal{B}^{[t]}(\mathbf{x}^{[t]})^{m-1})_i}{(x_i^t)^{m-1}} \le (\rho(\mathcal{B}))^t.$$

This completes the proof. □

Theorem 4.2. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k \in T_{m,n}$ be nonnegative. Then, $\rho(\mathcal{A}_1^{[\alpha_1]} \circ \mathcal{A}_2^{[\alpha_2]} \circ \cdots \circ \mathcal{A}_k^{[\alpha_k]}) \le (\rho(\mathcal{A}_1))^{\alpha_1} (\rho(\mathcal{A}_2))^{\alpha_2} \cdots (\rho(\mathcal{A}_k))^{\alpha_k}$ with $\alpha_i > 0$ and $\sum_{i=1}^k \alpha_i \ge 1$.*

Proof. By continuity of the spectral radius, we may suppose $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are all positive. Let $\alpha = \sum_{i=1}^k \alpha_i \ge 1$, and let $\mathcal{A}_i = (a^{(i)}_{i_1 i_2 \cdots i_m})$, $i = 1, 2, \ldots, k$. By Theorem 4.1,

$$\rho\left(\mathcal{A}_1^{[\alpha_1]} \circ \mathcal{A}_2^{[\alpha_2]} \circ \cdots \circ \mathcal{A}_k^{[\alpha_k]}\right) = \rho\left(\left(\mathcal{A}_1^{\left[\frac{\alpha_1}{\alpha}\right]} \circ \mathcal{A}_2^{\left[\frac{\alpha_2}{\alpha}\right]} \circ \cdots \circ \mathcal{A}_k^{\left[\frac{\alpha_k}{\alpha}\right]}\right)^{[\alpha]}\right) \le \left(\rho\left(\mathcal{A}_1^{\left[\frac{\alpha_1}{\alpha}\right]} \circ \mathcal{A}_2^{\left[\frac{\alpha_2}{\alpha}\right]} \circ \cdots \circ \mathcal{A}_k^{\left[\frac{\alpha_k}{\alpha}\right]}\right)\right)^{\alpha}. \tag{1}$$

By Lemma 2.2, for $i = 1, 2, \ldots, k$, there exist $\mathbf{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)})^T \in \mathbb{R}^n_{++}$ such that $\mathcal{A}_i(\mathbf{x}^{(i)})^{m-1} = \rho(\mathcal{A}_i)(\mathbf{x}^{(i)})^{[m-1]}$. Let $\mathcal{A} = \mathcal{A}_1^{\left[\frac{\alpha_1}{\alpha}\right]} \circ \mathcal{A}_2^{\left[\frac{\alpha_2}{\alpha}\right]} \circ \cdots \circ \mathcal{A}_k^{\left[\frac{\alpha_k}{\alpha}\right]}$, and let $\mathbf{x} = (\mathbf{x}^{(1)})^{\left[\frac{\alpha_1}{\alpha}\right]} \circ (\mathbf{x}^{(2)})^{\left[\frac{\alpha_2}{\alpha}\right]} \circ \cdots \circ (\mathbf{x}^{(k)})^{\left[\frac{\alpha_k}{\alpha}\right]} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n_{++}$. For $i = 1, 2, \ldots, n$,

$$\begin{aligned}(\mathcal{A}\mathbf{x}^{m-1})_i &= \sum_{i_2,\ldots,i_m=1}^{n} \left((a^{(1)}_{i,i_2\cdots i_m})^{\frac{\alpha_1}{\alpha}} \cdots (a^{(k)}_{i,i_2\cdots i_m})^{\frac{\alpha_k}{\alpha}} \right) \left((x^{(1)}_{i_2})^{\frac{\alpha_1}{\alpha}} \cdots (x^{(k)}_{i_2})^{\frac{\alpha_k}{\alpha}} \right) \cdots \left((x^{(1)}_{i_m})^{\frac{\alpha_1}{\alpha}} \cdots (x^{(k)}_{i_m})^{\frac{\alpha_k}{\alpha}} \right) \\ &= \sum_{i_2,\ldots,i_m=1}^{n} \left(a^{(1)}_{i,i_2\cdots i_m} x^{(1)}_{i_2} \cdots x^{(1)}_{i_m}\right)^{\frac{\alpha_1}{\alpha}} \cdots \left(a^{(k)}_{i,i_2\cdots i_m} x^{(k)}_{i_2} \cdots x^{(k)}_{i_m}\right)^{\frac{\alpha_k}{\alpha}} \\ &\le \left(\sum_{i_2,\ldots,i_m=1}^{n} a^{(1)}_{i,i_2\cdots i_m} x^{(1)}_{i_2} \cdots x^{(1)}_{i_m} \right)^{\frac{\alpha_1}{\alpha}} \cdots \left(\sum_{i_2,\ldots,i_m=1}^{n} a^{(k)}_{i,i_2\cdots i_m} x^{(k)}_{i_2} \cdots x^{(k)}_{i_m} \right)^{\frac{\alpha_k}{\alpha}} \\ &= ((\mathcal{A}_1(\mathbf{x}^{(1)})^{m-1})_i)^{\frac{\alpha_1}{\alpha}} \cdots ((\mathcal{A}_k(\mathbf{x}^{(k)})^{m-1})_i)^{\frac{\alpha_k}{\alpha}} = (\rho(\mathcal{A}_1)(x_i^{(1)})^{m-1})^{\frac{\alpha_1}{\alpha}} \cdots (\rho(\mathcal{A}_k)(x_i^{(k)})^{m-1})^{\frac{\alpha_k}{\alpha}} \\ &= (\rho(\mathcal{A}_1))^{\frac{\alpha_1}{\alpha}} \cdots (\rho(\mathcal{A}_k))^{\frac{\alpha_k}{\alpha}} \left((x_i^{(1)})^{\frac{\alpha_1}{\alpha}} \cdots (x_i^{(k)})^{\frac{\alpha_k}{\alpha}} \right)^{m-1} = (\rho(\mathcal{A}_1))^{\frac{\alpha_1}{\alpha}} \cdots (\rho(\mathcal{A}_k))^{\frac{\alpha_k}{\alpha}} x_i^{m-1}.\end{aligned}$$

Then, by Lemma 2.3 (v), $\rho(\mathcal{A}) \le \max_{1 \le i \le n} \frac{(\mathcal{A}\mathbf{x}^{m-1})_i}{x_i^{m-1}} \le (\rho(\mathcal{A}_1))^{\frac{\alpha_1}{\alpha}} \cdots (\rho(\mathcal{A}_k))^{\frac{\alpha_k}{\alpha}}$. Thus, by (1),

$$\rho\left(\mathcal{A}_1^{[\alpha_1]} \circ \cdots \circ \mathcal{A}_k^{[\alpha_k]}\right) \le \left(\rho\left(\mathcal{A}_1^{\left[\frac{\alpha_1}{\alpha}\right]} \circ \cdots \circ \mathcal{A}_k^{\left[\frac{\alpha_k}{\alpha}\right]}\right)\right)^{\alpha} = (\rho(\mathcal{A}))^{\alpha} \le (\rho(\mathcal{A}_1))^{\alpha_1} \cdots (\rho(\mathcal{A}_k))^{\alpha_k}.$$ □

5 Conclusion

In this paper, we focus on the analytic properties of the spectral radius of nonnegative tensors. First, we discuss the continuity of the spectral radius. Then, we give some inequalities on the spectral radius involving the Hadamard product. These results generalize some existing results on the spectral properties of nonnegative matrices to nonnegative tensors.

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