Research Article

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The adaptive finite element method for the Steklov eigenvalue problem in inverse scattering

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Abstract: In this study, for the first time, we discuss the posteriori error estimates and adaptive algorithm for the non-self-adjoint Steklov eigenvalue problem in inverse scattering. The differential operator corresponding to this problem is non-self-adjoint and the associated weak formulation is not H^1-elliptic. Based on the study of Armentano et al. [Appl. Numer. Math. **58** (2008), 593–601], we first introduce error indicators for primal eigenfunction, dual eigenfunction, and eigenvalue. Second, we use Gårding's inequality and duality technique to give the upper and lower bounds for energy norm of error of finite element eigenfunction, which shows that our indicators are reliable and efficient. Finally, we present numerical results to validate our theoretical analysis.

Keywords: Steklov eigenvalues in inverse scattering, a posteriori error estimates, reliability and efficiency, adaptive algorithm

MSC 2010: 65N25, 65N30

1 Introduction

Steklov eigenvalue problems are extensively applied to surface waves (see, e.g., [1,2]), mechanical oscillators immersed in a viscous fluid (see [3,4]), the vibration modes of a structure in contact with an incompressible fluid (see [5,6]), the inverse scattering theory to reconstruct the index of refraction of inhomogeneous media (see [7]), etc. Many studies have been carried out on numerical methods for the Steklov eigenvalue problems (see, e.g., [6,8–19]). In recent years, researchers have paid attention to the Steklov eigenvalue problem in inverse scattering (see, e.g., [7,20–22]).

Adaptive finite element methods are favored in the current scientific and engineering computations. For a given tolerance, adaptive finite element methods require less degrees of freedom. So far, many excellent works on the a posteriori error estimates and adaptive algorithm have been summarized in previous studies [23–25]. There are some studies on the a posteriori error estimates and adaptive finite element methods for self-adjoint Steklov eigenvalue problems (see, e.g., [11–13,17,26–28]). As far as we know, up to now, there does not exist any study on the a posteriori error estimates and adaptive finite methods for the non-self-adjoint Steklov eigenvalue problem considered in this study. Therefore, deriving the a posteriori error estimates for the problem is still a new topic.

In this study, we first propose error indicators for primal eigenfunction, dual eigenfunction, and eigenvalue of the non-self-adjoint Steklov eigenvalue problem in inverse scattering. The theoretical

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analysis in this study primarily refers to [11]. But different from what Armentano and Patra discussed in [11], the differential operator corresponding to this problem is non-self-adjoint and the associated weak formulation is not H^1-elliptic, which leads to the difficulty of theoretical analysis. We use Gårding's inequality and duality technique to solve this difficulty. Second, we give upper and lower bounds for the energy norm of error of finite element solution, which shows that our indicators are reliable and efficient. Finally, based on the error indicators, we design an adaptive algorithm and present numerical examples to show the efficiency of our algorithm. Numerical results coincide with our theoretical analysis.

For the basic theory of finite element, we refer to [24,29–33]. In this study, without special explanation, C denotes the constant independent of mesh size, which may stand for different values at its different occurrences.

2 Preliminary

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with Lipschitz boundary $\Gamma = \partial\Omega$. We consider the following Steklov eigenvalue problem:

$$\Delta u + k^2 n(x) u = 0, \quad \text{in } \Omega, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} + \lambda u = 0, \quad \text{on } \Gamma, \tag{2.2}$$

where ν denotes the unit outward normal to Γ, k is the wavenumber, and $n(x)$ is the index of refraction. Assume that $n(x)$ is a bounded complex value function given by

$$n(x) = n_1(x) + i\frac{n_2(x)}{k},$$

where $i = \sqrt{-1}$, $n_1(x) \geq \delta > 0$, and $n_2(x) \geq 0$ are bounded and properly smooth.

Here, we use the standard notation for Sobolev spaces $H^s(\Omega)$ with norm $\|\cdot\|_{H^s(\Omega)}$ and $H_0^1(\Omega) = \{v \in H^1(\Omega): v|_\Gamma = 0\}$. Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be defined as follows:

$$a(u, v) = \int_\Omega (\nabla u \cdot \nabla \bar{v} - k^2 n(x) u \bar{v}),$$

$$b(u, v) = \int_\Gamma u \bar{v}.$$

The variational formulation associated with (2.1)–(2.2) is given by: find $(\lambda, u) \in \mathbb{C} \times H^1(\Omega)$, $u \neq 0$, such that

$$a(u, v) = -\lambda b(u, v), \quad \forall\, v \in H^1(\Omega), \tag{2.3}$$

$$\|u\|_{L^2(\Gamma)} = 1. \tag{2.4}$$

Let π_h be a family of triangulations of Ω such that any two triangles in π_h share at most a vertex or an edge. h_E denotes the diameter of element E. Let $h = \max_{E \in \pi_h}\{h_E\}$ be the diameter of π_h. We assume that the family of triangulations π_h satisfies a minimal angle condition, i.e., there exists a constant $\sigma > 0$ such that $\frac{h_E}{\rho_E} \leq \sigma$, where ρ_E is the diameter of the largest circle contained in E.

$V_h = \{v \in H^1(\Omega) : v|_E \in P_1, \forall\, E \in \pi_h\}$, where P_1 denotes the space of linear polynomials. Then, the discrete variational formulation associated with (2.3)–(2.4) is given by: find $(\lambda_h, u_h) \in \mathbb{C} \times V_h$, $u_h \neq 0$, such that

$$a(u_h, v) = -\lambda_h b(u_h, v), \quad \forall\, v \in V_h, \tag{2.5}$$

$$\|u_h\|_{L^2(\Gamma)} = 1. \tag{2.6}$$

Consider the dual problem (2.3)–(2.4): find $(\lambda^*, u^*) \in \mathbb{C} \times H^1(\Omega)$, $u^* \neq 0$, such that

$$a(v, u^*) = -\overline{\lambda^*} b(v, u^*), \quad \forall\, v \in H^1(\Omega), \tag{2.7}$$

$$\|u^*\|_{L^2(\Gamma)} = 1. \tag{2.8}$$

Note that the primal and dual eigenvalues are connected via $\lambda = \overline{\lambda^*}$.

The discrete variational formulation associated with (2.7)–(2.8) is given by: find $(\lambda_h^*, u_h^*) \in \mathbb{C} \times V_h$, $u_h^* \neq 0$, such that

$$a(v, u_h^*) = -\overline{\lambda_h^*} b(v, u_h^*), \quad \forall\, v \in V_h, \tag{2.9}$$

$$\|u_h^*\|_{L^2(\Gamma)} = 1. \tag{2.10}$$

Note that the primal and dual eigenvalues are connected via $\lambda_h = \overline{\lambda_h^*}$.

For convenience of reading, we introduce the following notation. Let λ be the jth eigenvalue of (2.3)–(2.4) with the ascent $\alpha = 1$ and the algebraic multiplicity q, i.e., $\lambda = \lambda_i = \cdots = \lambda_{i+q-1}$. Then, there are q eigenvalues $\lambda_{j,h}(j = i, i+1, \ldots, i+q-1)$ of (2.5)–(2.6) converging to λ. Let $M(\lambda)$ be the space spanned by all eigenfunctions corresponding to the eigenvalue λ. Let $M_h(\lambda)$ be the space spanned by all eigenfunctions of (2.5)–(2.6) corresponding to the eigenvalues $\lambda_{j,h}(j = i, i+1, \ldots, i+q-1)$. For the dual problems (2.7)–(2.8) and (2.9)–(2.10), the definitions of $M^*(\lambda^*)$ and $M_h^*(\lambda^*)$ are made similar to $M(\lambda)$ and $M_h(\lambda)$, respectively.

Thanks to Theorem 3.2 in [20] and Theorem 2.1 in [21], the following lemmas are valid.

Lemma 2.1. *Let $M(\lambda) \subset H^{1+r}(\Omega)$. Assume that λ and λ_h are the jth eigenvalue of* (2.3)–(2.4) *and* (2.5)–(2.6), *respectively. There exists $h_0 \in (0,1)$ such that, if $h \leq h_0$, then*

$$|\lambda - \lambda_h| \leq Ch^{2r}. \tag{2.11}$$

Let $u_h \in M_h(\lambda)$, $\|u_h\|_{L^2(\Gamma)} = 1$, then there exists $u \in M(\lambda)$ such that

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^r, \tag{2.12}$$

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{3r/2}, \tag{2.13}$$

$$\|u - u_h\|_{L^2(\Gamma)} \leq Ch^{3r/2}, \tag{2.14}$$

where $r < \pi/\omega$ (with ω being the largest inner angle of Ω) and $r = 1$ if Ω is convex.

Similar conclusions hold for the finite element approximation of dual problem (2.7)–(2.8).

Lemma 2.2. *Let $M^*(\lambda^*) \subset H^{1+r}(\Omega)$. Assume that λ^* and λ_h^* are the jth eigenvalue of* (2.7)–(2.8) *and* (2.9)–(2.10), *respectively. There exists $h_0 \in (0,1)$ such that, if $h \leq h_0$, then*

$$|\lambda^* - \lambda_h^*| \leq Ch^{2r}. \tag{2.15}$$

Let $u_h^* \in M_h^*(\lambda^*)$, $\|u_h^*\|_{L^2(\Gamma)} = 1$, then there exists $u^* \in M^*(\lambda^*)$ such that

$$\|u^* - u_h^*\|_{H^1(\Omega)} \leq Ch^r, \tag{2.16}$$

$$\|u^* - u_h^*\|_{L^2(\Omega)} \leq Ch^{3r/2}, \tag{2.17}$$

$$\|u^* - u_h^*\|_{L^2(\Gamma)} \leq Ch^{3r/2}. \tag{2.18}$$

3 *A posteriori* error estimates

In this section, we introduce error indicator for the non-self-adjoint Steklov eigenvalue problem in inverse scattering and prove that, up to higher order terms, it is equivalent to the energy norm of the error, i.e., it provides upper and lower bounds for the error of finite element solution. First of all, the local *a posteriori* error indicator η_E for u_h will be given. We use the symbol ε_E to represent the set of edges of E, $E \in \pi_h$. Define $\varepsilon = \cup_{E\in\pi_h}\varepsilon_E$ and $\varepsilon_\Gamma = \{l \in \varepsilon; l \subset \Gamma\}$. We decompose $\varepsilon = \varepsilon_\Omega \cup \varepsilon_\Gamma$, where $\varepsilon_\Omega = \varepsilon \backslash \varepsilon_\Gamma$. The following value of $\left[\!\left[\frac{\partial v_h}{\partial \nu_l}\right]\!\right]_l$ is independent of the choice of ν_l:

$$\left[\!\left[\frac{\partial v_h}{\partial \nu_l}\right]\!\right]_l = \left(\nabla\left(v_h|_{E_1}\right) - \nabla\left(v_h|_{E_2}\right)\right)\cdot\nu_l, \quad l \in \varepsilon_\Omega,$$

where E_1 and E_2 are the elements sharing edge l, the unit normal ν_l points outwards E_2. Define

$$J_l(\lambda_h, u_h) = \begin{cases} \dfrac{1}{2}\left[\!\left[\dfrac{\partial u_h}{\partial \nu_l}\right]\!\right]_l, & l \in \varepsilon_\Omega, \\ -\lambda_h u_h - \dfrac{\partial u_h}{\partial \nu_l}, & l \in \varepsilon_\Gamma, \end{cases}$$

$$J_l^*(\lambda_h^*, u_h^*) = \begin{cases} \dfrac{1}{2}\left[\!\left[\dfrac{\partial u_h^*}{\partial \nu_l}\right]\!\right]_l, & l \in \varepsilon_\Omega, \\ -\lambda_h^* u_h^* - \dfrac{\partial u_h^*}{\partial \nu_l}, & l \in \varepsilon_\Gamma. \end{cases}$$

The local error indicator is given by

$$\eta_E(\lambda_h, u_h, n(x), k) = \left\{k^4 h_E^2 \|n(x)u_h\|_{L^2(E)}^2 + \sum_{l\in\varepsilon_E} |l| \|J_l\|_{L^2(l)}^2\right\}^{\frac{1}{2}}, \tag{3.1}$$

and the global error indicator is given by

$$\eta_{\Omega}(\lambda_h, u_h, n(x), k) = \left(\sum_{E \in \pi_h} \eta_E^2(\lambda_h, u_h, n(x), k) \right)^{\frac{1}{2}}. \tag{3.2}$$

Similarly, we give the local error indicator and the global error indicator of the dual eigenfunction as follows:

$$\eta_E^*(\lambda_h^*, u_h^*, n(x), k) = \left\{ k^4 h_E^2 \|n(x) u_h^*\|_{L^2(E)}^2 + \sum_{l \in \varepsilon_E} |l| \|J_l^*(\lambda_h^* u_h^*)\|_{L^2(l)}^2 \right\}^{\frac{1}{2}}, \tag{3.3}$$

$$\eta_{\Omega}^*(\lambda_h^*, u_h^*, n(x), k) = \left(\sum_{E \in \pi_h} (\eta_E^*(\lambda_h^*, u_h^*, \overline{n(x)}, k))^2 \right)^{\frac{1}{2}}. \tag{3.4}$$

For convenience of reading, in the next discussion, we simplify $J_l(\lambda_h,u_h)$, $\eta_E(\lambda_h,u_h,n(x),k)$, and $\eta_\Omega(\lambda_h,u_h,n(x),k)$ as J_l, η_E, and η_Ω if without specification, respectively. Similarly, we simplify $J_l^*(\lambda_h^*, u_h^*)$, $\eta_E^*(\lambda_h^*, u_h^*, n(x), k)$, and $\eta_\Omega^*(\lambda_h^*, u_h^*, n(x), k)$ as J_l^*, η_E^*, and η_Ω^*, respectively.

Second, for analyzing the reliability and efficiency of the error indicators, we make some preparations. Let (λ,u) and (λ_h,u_h) be the eigenpair of (2.3)–(2.4) and (2.5)–(2.6), respectively. On one hand, it is easy to know that $e = u-u_h$ satisfies

$$\int_{\Omega} (\nabla e \cdot \nabla \bar{v} - k^2 n(x) e \bar{v}) = -\int_{\Gamma} (\lambda u \bar{v} - \lambda_h u_h \bar{v}), \quad \forall\ v \in V_h. \tag{3.5}$$

On the other hand, for any $v \in V_h$, it follows from Green formula and $\Delta u_h = 0$ on E that $\int_E \nabla u_h \cdot \nabla \bar{v} = \int_{\partial E} \frac{\partial u_h}{\partial \nu} \cdot \bar{v}$. Thus,

$$a(u_h, v) = \sum_E \left\{ \int_E \nabla u_h \cdot \nabla \bar{v} - k^2 \int_E n(x) u_h \bar{v} \right\} = \sum_E \left\{ \int_{\partial E} \frac{\partial u_h}{\partial \nu} \cdot \bar{v} - k^2 \int_E n(x) u_h \bar{v} \right\}.$$

Combining the aforementioned equality with (2.3), we conclude

$$\begin{aligned} \int_{\Omega} (\nabla e \cdot \nabla \bar{v} - k^2 n(x) e \bar{v}) &= a(u, v) - a(u_h, v) \\ &= -\int_{\Gamma} \lambda u \bar{v} - \sum_E \left(\int_{\partial E} \frac{\partial u_h}{\partial \nu} \cdot \bar{v} - k^2 \int_E n(x) u_h \bar{v} \right) \\ &= \sum_E \left\{ k^2 \int_E n(x) u_h \bar{v} + \sum_{l \in \varepsilon_E \cap \varepsilon_\Gamma} \int_l \left(-\lambda_h u_h - \frac{\partial u_h}{\partial \nu_l} \right) \cdot \bar{v} \right. \\ &\quad \left. + \frac{1}{2} \sum_{l \in \varepsilon_E \cap \varepsilon_\Omega} \int_l \left[\!\left[\frac{\partial u_h}{\partial \nu_l} \right]\!\right]_l \bar{v} \right\} - \int_{\Gamma} (\lambda u - \lambda_h u_h) \bar{v}. \end{aligned} \tag{3.6}$$

In addition, we will use the following estimates for the interpolation operator of Clement $I_h : H^1(\Omega) \to V_h$:

$$\|v - I_h v\|_{L^2(E)} \le C h_E \|v\|_{H^1(\hat{E})}, \tag{3.7}$$

$$\|v - I_h v\|_{L^2(l)} \le C|l|^{\frac{1}{2}} \|v\|_{H^1(\hat{l})}, \tag{3.8}$$

where $\hat{E}$ and $\hat{l}$ is the union of all elements sharing at least a vertex with E and l, respectively.

It is easy to verify that $a(\cdot,\cdot)$ satisfies Gårding's inequality [31], i.e., there exist constants $0 < M < \infty$ (M is large enough) and $\alpha_0 > 0$ such that

$$\mathrm{Re}\{a(v,v)\} + M\|v\|^2_{L^2(\Omega)} \ge \alpha_0 \|v\|^2_{H^1(\Omega)}, \quad \forall\, v \in H^1(\Omega), \tag{3.9}$$

where both M and α_0 are dependent on k and $\|n(x)\|_{L^\infty(\Omega)}$.

Third, we will use inequality (3.9) and the arguments in [11] to prove the following theorem. The theorem shows that the global error indicator provides upper bound of the error in the energy norm up to two higher order terms, which guarantees the reliability of indicator.

Theorem 3.1. *The following conclusions are valid:*
(a) *there exists a constant C such that*

$$\|u - u_h\|_{H^1(\Omega)} \le C\left\{\eta_\Omega + (M + |\lambda|)\|u - u_h\|_{L^2(\Omega)} + |\lambda - \lambda_h|\right\}, \tag{3.10}$$

(b) *there exists a constant C such that*

$$\|u^\star - u_h^\star\|_{H^1(\Omega)} \le C\left\{\eta_\Omega^\star + (M + |\lambda^\star|)\|u^\star - u_h^\star\|_{L^2(\Omega)} + |\lambda^\star - \lambda_h^\star|\right\}. \tag{3.11}$$

Proof. Take $v = e^I$ in (3.5), where e^I denotes the Clement interpolation of e, then

$$\int_\Omega (\nabla e \cdot \nabla \overline{e^I} - k^2 n(x) e \overline{e^I}) = -\int_\Gamma (\lambda u - \lambda_h u_h)\overline{e^I},$$

which together with (3.6) yields

$$\begin{aligned}\int_\Omega (\nabla e \cdot \nabla \bar{e} - k^2 n(x) e\bar{e}) &= \int_\Omega (\nabla e \cdot \nabla(\bar{e} - \overline{e^I}) - k^2 n(x) e(\bar{e} - \overline{e^I})) + \int_\Gamma (\nabla e \cdot \nabla \overline{e^I} - k^2 n(x) e \overline{e^I}) \\ &= \sum_E \left\{k^2 \int_E n(x) u_h (\bar{e} - \overline{e^I}) + \sum_{l \in \varepsilon_E} \int_l J_l(\bar{e} - \overline{e^I})\right\} - \int_\Gamma (\lambda u - \lambda_h u_h)\bar{e}.\end{aligned} \tag{3.12}$$

Noting that $e = u - u_h$, and using the triangle inequality and Schwarz's inequality, we have

$$\left|\int_\Gamma (\lambda u - \lambda_h u_h)\bar{e}\right| = \left|\int_\Gamma \lambda e\bar{e} + \int_\Gamma (\lambda - \lambda_h) u_h \bar{e}\right| \le |\lambda| \|e\|^2_{L^2(\Gamma)} + |\lambda - \lambda_h| \|e\|_{L^2(\Gamma)}. \tag{3.13}$$

From (3.9), (3.12), and (3.13), we derive that

$$\begin{aligned}\alpha_0 \|e\|^2_{H^1(\Omega)} \le |a(e,e)| + M\|e\|^2_{L^2(\Omega)} &\le \left|\sum_E \left\{k^2 \int_E n(x) u_h (\bar{e} - \overline{e^I}) + \sum_{l \in \varepsilon_E} \int_l J_l(\bar{e} - \overline{e^I}\right\}\right| \\ &+ |\lambda| \|e\|^2_{L^2(\Gamma)} + |\lambda - \lambda_h| \|e\|_{L^2(\Gamma)} + M\|e\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.14}$$

Using Schwarz's inequality, (3.7) and (3.8), and that the triangulation satisfies the minimum angle condition we obtain

$$\begin{aligned}
&\left|\sum_E \left\{k^2 \int_E n(x)u_h(\bar{e} - \overline{e^I}) + \sum_{l\in\varepsilon_E} \int_l J_l(\bar{e} - \overline{e^I})\right\}\right| \\
&\le \sum_E \left\{k^2\|n(x)u_h\|_{L^2(E)}\|e - e^I\|_{L^2(E)} + \sum_{l\in\varepsilon_E} \|J_l\|_{L^2(l)}\|e - e^I\|_{L^2(l)}\right\} \\
&\le C\sum_E \left\{k^2 h_E\|e\|_{H^1(\hat{E})}\|n(x)u_h\|_{L^2(E)} + \sum_{l\in\varepsilon_E} |l|^{\frac{1}{2}}\|e\|_{H^1(\hat{l})}\|J_l\|_{L^2(l)}\right\} \\
&\le C\left\{\sum_E \left(k^4 h_E^2\|n(x)u_h\|^2_{L^2(E)} + \sum_{l\in\varepsilon_E} |l|\|J_l\|^2_{L^2(l)}\right)\right\}^{\frac{1}{2}} \|e\|_{H^1(\Omega)}.
\end{aligned} \tag{3.15}$$

Using the trace theorems, we get

$$\|e\|^2_{L^2(\Gamma)} \le C\|e\|_{L^2(\Omega)}\|e\|_{H^1(\Omega)}, \tag{3.16}$$

$$\|e\|_{L^2(\Gamma)} \le C\|e\|_{H^1(\Omega)}. \tag{3.17}$$

Substituting (3.15), (3.16), and (3.17) into (3.14) and noting $\|e\|_{L^2(\Omega)} \le C\|e\|_{H^1(\Omega)}$, we deduce (3.10).

Denoting $e^\star = u^\star - u_h^\star$ and using the similar arguments to (3.10), we can derive (3.11) easily. Hence proved. □

Finally, we use the arguments in [11] to prove the efficiency of our indicator for practical adaptive refinement. To achieve our goal, we must prove that the local indicator is bounded by the error in the energy norm and higher order terms.

Now, we will prove that the following Lemmas 3.1 and 3.2 are true, which will be used to estimate the first term of η_E.

Lemma 3.1. *There exists a constant C such that*

$$k^2 h_E\|n(x)u_h\|_{L^2(E)} \le C\left(\|\nabla e\|_{L^2(E)} + k^2 h_E\|n(x)e\|_{L^2(E)} + h_E^2\|n(x)\|_{W^{1,\infty}(E)}\|u_h\|_{L^2(E)}\right). \tag{3.18}$$

Proof. Let $\delta_{1,E}$, $\delta_{2,E}$, and $\delta_{3,E}$ denote the barycentric coordinates of $E \in \pi_h$. Define the cubic bubble b_E by

$$b_E = \begin{cases} \delta_{1,E}\delta_{2,E}\delta_{3,E} & in\ \ E, \\ 0 & in\ \ \Omega\backslash E. \end{cases}$$

Let $v_E \in P_4(E) \cap H_0^1(E)$. Then, there exists a unique v_E such that

$$\int_E \frac{\int_E \overline{n(x)}}{|E|} v_E \bar{w} = k^2 h_E^2 \int_E |n(x)|^2 u_h \bar{w}, \quad \forall\ w \in P_1(E). \tag{3.19}$$

We select $v_E = \sum_{i=1}^{3} \alpha_i \varphi_i$, where $\varphi_i = \delta_{i,E} b_E$ $(i = 1,2,3)$. According to $\int_E \delta_{1,E}^{k_1}\delta_{2,E}^{k_2}\delta_{3,E}^{k_3} = \frac{k_1!\,k_2!\,k_3!}{(k_1+k_2+k_3+2)!} 2!|E|$ and noting that

$$\int_E \delta_{1,E}^{k_1}\delta_{2,E}^{k_2}\delta_{3,E}^{k_3} = \begin{cases} \frac{12}{7!}|E|, & (k_1, k_2, k_3) \text{ is any possible permutation of } (3,1,1); \\ \frac{8}{7!}|E|, & (k_1, k_2, k_3) \text{ is any possible permutation of } (2,2,1), \end{cases}$$

then equality (3.19) can be written as:

$$\frac{\int_E \overline{n(x)}}{|E|} \cdot \frac{4}{7!}|E| \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = k^2 h_E^2 \begin{pmatrix} \int_E |n(x)|^2 u_h \delta_{1,E} \\ \int_E |n(x)|^2 u_h \delta_{2,E} \\ \int_E |n(x)|^2 u_h \delta_{3,E} \end{pmatrix}.$$

Obviously, the determinant of coefficient matrix is not equal to zero. Therefore, v_E exists uniquely. The solution of this linear system satisfies that

$$|\alpha_i| \le \frac{C}{|E|} \max_{1\le i\le 3} \left| k^2 h_E^2 \int_E |n(x)|^2 u_h \delta_{i,E} \right|,$$

then

$$\begin{aligned} \max_{1\le i\le 3} |\alpha_i| &\le \frac{C}{|E|} \max_{1\le i\le 3} \left| k^2 h_E^2 \int_E |n(x)|^2 u_h \delta_{i,E} \right| \\ &\le \frac{C}{|E|} \max_{1\le i\le 3} k^2 h_E^2 \|n(x) u_h\|_{L^2(E)} \left(\int_E \delta_{i,E}^2 \right)^{\frac{1}{2}} \\ &\le \frac{C}{|E|} |E|^{\frac{1}{2}} k^2 h_E^2 \|n(x) u_h\|_{L^2(E)}, \end{aligned} \tag{3.20}$$

where C is dependent on $\|n(x)\|_{L^\infty(\Omega)}$.

Since $v_E = \sum_{i=1}^{3} \alpha_i \varphi_i$, we get

$$\|v_E\|_{L^2(E)}^2 = \int_E \left(\sum_{i=1}^{3} \alpha_i \varphi_i \right) \left(\overline{\sum_{i=1}^{3} \alpha_i \varphi_i} \right) = \sum_{i=1}^{3} |\alpha_i|^2 \int_E \varphi_i^2 + \sum_{1\le i<j\le 3} 2\,\mathrm{Re}(\alpha_i \alpha_j) \int_E \varphi_i \varphi_j. \tag{3.21}$$

Combining (3.21), $\int_E \varphi_i \varphi_j \le C|E|\,(i, j = 1, 2, 3)$, and (3.20), we deduce

$$\|v_E\|_{L^2(E)} \le C|E|^{\frac{1}{2}} \max_{1\le i\le 3} |\alpha_i| \le Ck^2 h_E^2 \|n(x) u_h\|_{L^2(E)}. \tag{3.22}$$

Thus,

$$\|v_E\|_{L^2(E)} + h_E \|\nabla v_E\|_{L^2(E)} \le C\|v_E\|_{L^2(E)} \le Ck^2 h_E^2 \|n(x) u_h\|_{L^2(E)}. \tag{3.23}$$

Taking $w = u_h$ in (3.19), we deduce

$$k^2 h_E^2 \|n(x)u_h\|_{L^2(E)}^2 = \int_E \frac{\int_E \overline{n(x)}}{|E|} \overline{u_h} v_E = \int_E \frac{\int_E n(x)}{|E|} u_h \overline{v_E}.$$

Taking $v = v_E$ in (3.6) and combining the aforementioned equality, we conclude

$$\begin{aligned} k^4 h_E^2 \|n(x)u_h\|_{L^2(E)}^2 &= \int_E (\nabla e \cdot \nabla \overline{v_E} - k^2 n(x) e \overline{v_E}) - k^2 \left(\int_E n(x) u_h \overline{v_E} - \int_E \frac{\int_E n(x)}{|E|} u_h \overline{v_E} \right) \\ &\le \left| \int_E (\nabla e \cdot \nabla \overline{v_E} - k^2 n(x) e \overline{v_E}) \right| + k^2 \left| \int_E \left(n(x) - \frac{\int_E n(x)}{|E|} \right) u_h \overline{v_E} \right| . \end{aligned} \tag{3.24}$$

We estimate the first term on the right-hand side of (3.24) by using Schwarz's inequality and (3.23).

$$\begin{aligned} \left| \int_E (\nabla e \cdot \overline{\nabla v_E} - k^2 n(x) e \overline{v_E}) \right| &\le C \left(\|\nabla e\|_{L^2(E)} \|\nabla v_E\|_{L^2(E)} + k^2 h_E \|n(x)e\|_{L^2(E)} \frac{1}{h_E} \|v_E\|_{L^2(E)} \right) \\ &\le C \left(\|\nabla e\|_{L^2(E)} + k^2 h_E \|n(x)e\|_{L^2(E)} \right) \left(\|\nabla v_E\|_{L^2(E)} + \frac{1}{h_E} \|v_E\|_{L^2(E)} \right) \\ &\le C k^2 h_E \left(\|\nabla e\|_{L^2(E)} + k^2 h_E \|n(x)e\|_{L^2(E)} \right) \|n(x)u_h\|_{L^2(E)}. \end{aligned} \tag{3.25}$$

We estimate the second term on the right-hand side of (3.24) by using Schwarz's inequality and (3.22).

$$\begin{aligned} \left| \int_E \left(n(x) - \frac{\int_E n(x)}{|E|} \right) u_h \overline{v_E} \right| &\le \left\| \left(n(x) - \frac{\int_E n(x)}{|E|} \right) u_h \right\|_{L^2(E)} \|v_E\|_{L^2(E)} \\ &\le \left\| n(x) - \frac{\int_E n(x)}{|E|} \right\|_{L^\infty(E)} \|u_h\|_{L^2(E)} \|v_E\|_{L^2(E)} \\ &\le C k^2 h_E^3 \|n(x)\|_{W^{1,\infty}(E)} \|u_h\|_{L^2(E)} \|n(x)u_h\|_{L^2(E)}. \end{aligned} \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.24), we derive (3.18). □

The following Lemma 3.2 will be used to estimate the second term of η_E.

Lemma 3.2. *The following conclusions are valid:*
(*a*) *for $l \in \varepsilon_E \cap \varepsilon_\Gamma$, there exists a constant C such that*

$$|l|^{\frac{1}{2}} \|J_l\|_{L^2(l)} \le C \left(\|\nabla e\|_{L^2(E)} + k^2 h_E \|n(x)e\|_{L^2(E)} + h_E^2 \|n(x)\|_{W^{1,\infty}(E)} \|u_h\|_{L^2(E)} + |l|^{\frac{1}{2}} \|\lambda u - \lambda_h u_h\|_{L^2(l)} \right). \tag{3.27}$$

(*b*) *for $l \in \varepsilon_E \cap \varepsilon_\Omega$, let $E_1, E_2 \in \pi_h$ be the two triangles sharing l. Then, there exists a constant C such that*

$$|l|^{\frac{1}{2}} \|J_l\|_{L^2(l)} \le C \left(\|\nabla e\|_{L^2(E_1 \cup E_2)} + k^2 h_E \|n(x)e\|_{L^2(E_1 \cup E_2)} + h_E^2 \|n(x)\|_{W^{1,\infty}(E_1 \cup E_2)} \|u_h\|_{L^2(E_1 \cup E_2)} \right). \tag{3.28}$$

Proof. Using similar arguments to Lemma 3.2 in [11], we complete the proof. For $l \in \varepsilon_E \cap \varepsilon_\Omega$, we denote by E_1 and E_2 the two triangles sharing l and we enumerate the vertices of E_1 and E_2 such that the vertices of l are numbered first. Then, we introduce the edge-bubble function b_l

$$b_l = \begin{cases} \delta_{1,E_i}\delta_{2,E_i} & \text{in } E_i, \\ 0 & \text{in } \Omega \backslash E_1 \cup E_2. \end{cases}$$

(a) For $l \in \varepsilon_E \cap \varepsilon_\Gamma$, let $v_l \in P_3(E)$ be the function such that $v_{l|l'} = 0$ for $l' \in \varepsilon_E$, $l \neq l'$ and

$$\begin{cases} \int_l v_l \bar{w} = |l| \int_l J_l \bar{w}, & \forall\, w \in P_1(E), \\ \|v_l\|_{L^2(l)} \leq C|l| \|J_l\|_{L^2(l)}. \end{cases}$$

Then, such v_l exists and is unique. Actually, let $v_l = \sum_{i=1}^{2} \beta_i \psi_i$ where $\psi_i = \delta_{i,E} b_l$ and $\delta_{1,E}$, $\delta_{2,E}$ are the barycentric coordinates associated with the vertices of l. Then, using that $\int_l \delta_{1,E}^{k_1} \delta_{2,E}^{k_2} = \frac{k_1!\,k_2!}{(k_1+k_2+1)!}|l|$ to solve β_1, β_2, we know

$$|\beta_i| \leq \frac{C}{|l|} \max \left| \int_l |l| J_l \delta_{i,E} \right|,$$

and

$$\max_{1 \leq i \leq 2} |\beta_i| \leq C \|J_l\|_{L^2(l)} \max_{1 \leq i \leq 2} \|\delta_{i,E}\|_{L^2(l)} \leq C|l|^{\frac{1}{2}} \|J_l\|_{L^2(l)}. \tag{3.29}$$

Using the similar arguments to (3.22), we get

$$\|v_l\|_{L^2(l)} \leq C|l|^{\frac{1}{2}} \max_{1 \leq i \leq 2} |\beta_i| \leq C|l| \|J_l\|_{L^2(l)}. \tag{3.30}$$

Analogously, combining (3.29) we obtain

$$\|v_l\|_{L^2(E)} \leq C h_E \max_{1 \leq i \leq 2} |\beta_i| \leq C|l|^{\frac{3}{2}} \|J_l\|_{L^2(l)}, \tag{3.31}$$

furthermore,

$$\|v_l\|_{L^2(E)} + h_E \|\nabla v_l\|_{L^2(E)} \leq C|l|^{\frac{3}{2}} \|J_l\|_{L^2(l)}. \tag{3.32}$$

Since $|l| \|J_l\|_{L^2(l)}^2 = \int_l v_l \overline{J_l} = \int_l J_l \overline{v_l}$, it follows from (3.6) that

$$|l| \|J_l\|_{L^2(l)}^2 = \int_E (\nabla e \cdot \nabla \overline{v_l} - k^2 n(x) e \overline{v_l}) + \int_l (\lambda u - \lambda_h u_h) \overline{v_l} - k^2 \int_E n(x) u_h \overline{v_l}.$$

Using Schwarz's inequality and combining (3.30) and (3.31) with (3.32), we conclude

$$|l| \|J_l\|_{L^2(l)}^2 \leq C \Big(\|\nabla e\|_{L^2(E)} + k^2 h_E \|n(x) e\|_{L^2(E)} + |l|^{\frac{1}{2}} \|\lambda u - \lambda_h u_h\|_{L^2(l)} + k^2 h_E \|n(x) u_h\|_{L^2(E)} \Big) |l|^{\frac{1}{2}} \|J_l\|_{L^2(l)}. \tag{3.33}$$

From (3.33) and Lemma 3.1 we conclude (3.27).

(b) For $l \in \varepsilon_E \cap \varepsilon_\Omega$, let $E_1, E_2 \in \pi_h$ be the two triangles sharing l. Let $v_l \in H_0^1(E_1 \cup E_2)$ be an edge-bubble function such that $v_l|_{E_i} \in P_2$, $i = 1,2$, and

$$\begin{cases} \int_l v_l \bar{w} = |l| \int_l J_l \bar{w}, & \forall\, w \in P_0(E); \\ \|v_l\|_{L^2(l)} \le C|l|\|J_l\|_{L^2(l)}. \end{cases}$$

Using similar arguments to (a), we deduce

$$\|v_l\|_{L^2(E)} + h_E\|\nabla v_l\|_{L^2(E)} \le C|l|^{\frac{3}{2}}\|J_l\|_{L^2(l)}.$$

$|l|\|J_l\|^2_{L^2(l)} = \int_l \overline{J_l} v_l = \int_l J_l \overline{v_l}$ and (3.6) imply that

$$|l|\|J_l\|^2_{L^2(l)} = \int_{E_1\cup E_2} (\nabla e\cdot\overline{\nabla v_l} - k^2 n(x) e\overline{v_l}) - k^2 \int_{E_1\cup E_2} n(x) u_h \overline{v_l},$$

thus,

$$|l|\|J_l\|^2_{L^2(l)} \le C(\|\nabla e\|_{L^2(E_1\cup E_2)} + k^2 h_E\|n(x)e\|_{L^2(E_1\cup E_2)} + h_E\|n(x)u_h\|_{L^2(E_1\cup E_2)})|l|^{\frac{1}{2}}\|J_l\|_{L^2(l)}. \tag{3.34}$$

From (3.34) and Lemma 3.1, we conclude (3.28). □

The following theorem provides the lower bound for the error and high-order terms.

Theorem 3.2. *There exists a constant C such that*
(*a*) *If $\partial E \cap \Gamma = \varnothing$ for $E \in \pi_h$, then*

$$\eta_E \le C\left\{\|e\|_{H^1(E^u)} + k^2 h_E\|n(x)e\|_{L^2(E^u)} + h_E^2\|n(x)\|_{W^{1,\infty}(E^u)}\|u_h\|_{L^2(E^u)}\right\}, \tag{3.35}$$

where E^u denotes the union of E and the triangles sharing an edge with E.

(*b*) *If $\partial E \cap \Gamma \ne \varnothing$ for $E \in \pi_h$, then*

$$\eta_E \le C\left\{\|e\|_{H^1(E)} + k^2 h_E\|n(x)e\|_{L^2(E)} + h_E^2\|n(x)\|_{W^{1,\infty}(E)}\|u_h\|_{L^2(E)} + \sum_{l\in\varepsilon_E\cap\varepsilon_\Gamma} |l|^{\frac{1}{2}}\|\lambda u - \lambda_h u_h\|_{L^2(l)}\right\}. \tag{3.36}$$

Proof. It follows from Lemmas 3.1 and 3.2. □

There are similar conclusions for the dual problem.

Lemma 3.3. *There exists a constant C such that*

$$k^2 h_E\|n(x)u_h^\star\|_{L^2(E)} \le C\left(\|\nabla e^\star\|_{L^2(E)} + k^2 h_E\|n(x)e^\star\|_{L^2(E)} + h_E^2\|n(x)\|_{W^{1,\infty}(E)}\|u_h^\star\|_{L^2(E)}\right). \tag{3.37}$$

Lemma 3.4.
(*a*) *For $l \in \varepsilon_E \cap \varepsilon_\Gamma$, there exists a constant C such that*

$$|l|^{\frac{1}{2}}\|J_l^\star\|_{L^2(l)} \le C\left(\|\nabla e^\star\|_{L^2(E)} + k^2 h_E\|n(x)e^\star\|_{L^2(E)} + h_E^2\|n(x)\|_{W^{1,\infty}(E)}\|u_h^\star\|_{L^2(E)} + |l|^{\frac{1}{2}}\|\lambda^\star u^\star - \lambda_h^\star u_h^\star\|_{L^2(l)}\right). \tag{3.38}$$

(*b*) *For* $l \in \varepsilon_E \cap \varepsilon_\Omega$, *let* $E_1, E_2 \in \pi_h$ *be the two triangles sharing* l. *Then, there exists a constant* C *such that*

$$|l|^{\frac{1}{2}}\|J_l^\star\|_{L^2(l)} \le C\left(\|\nabla e^\star\|_{L^2(E_1\cup E_2)} + k^2 h_E\|n(x)e^\star\|_{L^2(E_1\cup E_2)} + h_E^2\|n(x)\|_{W^{1,\infty}(E_1\cup E_2)}\|u_h^\star\|_{L^2(E_1\cup E_2)}\right). \tag{3.39}$$

Theorem 3.3. *There exists a constant* C *such that*
(*a*) *If* $\partial E \cap \Gamma = \emptyset$ *for* $E \in \pi_h$, *then*

$$\eta_E^\star \le C\left\{\|e^\star\|_{H^1(E^u)} + k^2 h_E\|n(x)e^\star\|_{L^2(E^u)} + h_E^2\|n(x)\|_{W^{1,\infty}(E^u)}\|u_h^\star\|_{L^2(E^u)}\right\}. \tag{3.40}$$

(*b*) *If* $\partial E \cap \Gamma \ne \emptyset$ *for* $E \in \pi_h$, *then*

$$\eta_E^\star \le C\left\{\|e^\star\|_{H^1(E)} + k^2 h_E\|n(x)e^\star\|_{L^2(E)} + h_E^2\|n(x)\|_{W^{1,\infty}(E)}\|u_h^\star\|_{L^2(E)} + \sum_{l\in\varepsilon_E\cap\varepsilon_\Gamma}|l|^{\frac{1}{2}}\|\lambda^\star u^\star - \lambda_h^\star u_h^\star\|_{L^2(l)}\right\}. \tag{3.41}$$

Referring to Lemma 9.1 in [29], *we deduce the following lemma.*

Lemma 3.5. *Let* (λ,u) *and* $(\lambda^\star,u^\star)$ *be the eigenpair of* (2.3)–(2.4) *and* (2.5)–(2.6), *respectively. Then, for any* v, $v^\star \in H^1(\Omega)$, $b(v,v^\star) \ne 0$, *the generalized Rayleigh quotient satisfies*

$$\frac{a(v,v^\star)}{-b(v,v^\star)} - \lambda = \frac{a(v-u,v^\star-u^\star)}{-b(v,v^\star)} + \lambda\frac{b(v-u,v^\star-u^\star)}{-b(v,v^\star)}. \tag{3.42}$$

Proof. From (2.3) and (2.7), we get

$$\begin{aligned} &a(v-u,v^\star-u^\star) + \lambda b(v-u,v^\star-u^\star)\\ &\quad= a(v,v^\star) - a(v,u^\star) - a(u,v^\star) + a(u,u^\star) + \lambda(b(v,v^\star) - b(v,u^\star) - b(u,v^\star) + b(u,u^\star))\\ &\quad= a(v,v^\star) + \lambda b(v,v^\star), \end{aligned}$$

dividing both sides by $-b(v,v^\star)$, we obtain the desired conclusion. □

From Lemma 4.1 in [34] and Lemma 4.2 in [35], we have the following hypothesis. Hypothesis *H*0 $b(u_h,u_h^\star)$ has a positive lower bound uniformly with respect to h.

Theorem 3.4. *Let* (λ_h,u_h) *be an approximate eigenpair of* (2.5)–(2.6), *which converges to the eigenpair* (λ,u). *Under hypothesis* H0, *we have*

$$|\lambda_h - \lambda| \le C(\eta_\Omega^2 + (\eta_\Omega^\star)^2). \tag{3.43}$$

Proof. From Lemma 3.5, we get

$$\lambda_h - \lambda = \frac{a(u_h-u,u_h^\star-u^\star)}{-b(u_h,u_h^\star)} + \lambda\frac{b(u_h-u,u_h^\star-u^\star)}{-b(u_h,u_h^\star)}.$$

According to hypothesis *H*0 and combining Lemmas 2.1 and 2.2, we derive

$$
\begin{aligned}
|\lambda_h - \lambda| &\le C|a(u_h - u, u_h^\star - u^\star) + b(u_h - u, u_h^\star - u^\star)| \\
&\le C\left\{|u_h - u|_{H^1(\Omega)}\|u_h^\star - u^\star\|_{H^1(\Omega)} + \|u_h - u\|_{L^2(\Gamma)}\|u_h^\star - u^\star\|_{L^2(\Gamma)}\right\} \\
&\le C\|u_h - u\|_{H^1(\Omega)}\|u_h^\star - u^\star\|_{H^1(\Omega)} \\
&\le C\left\{\|u_h - u\|^2_{H^1(\Omega)} + \|u_h^\star - u^\star\|^2_{H^1(\Omega)}\right\}.
\end{aligned}
$$

Substituting (3.10) and (3.11) into the aforementioned inequality and ignoring higher order terms, we conclude (3.43). □

4 Edge residual error estimator

In this section, for primal eigenfunction and dual eigenfunction, we propose simpler global error indicators which are equivalent to the errors up to higher order terms, respectively.

$$
\hat{\eta}_\Omega = \left\{\sum_E \hat{\eta}_E^2\right\}^{1/2}, \quad \hat{\eta}_\Omega^\star = \left\{\sum_E (\hat{\eta}_E^\star)^2\right\}^{1/2},
$$

where $\hat{\eta}_E = \left(\sum_{l\in\varepsilon_E} |l|\|J_l\|^2_{L^2(l)}\right)^{1/2}$ and $\hat{\eta}_E^\star = \left(\sum_{l\in\varepsilon_E} |l|\|J_l^\star\|^2_{L^2(l)}\right)^{1/2}$. Actually, they are obtained by omitting the first term in the residual error indicator given in (3.1) and (3.3), respectively. Next, we prove that they are reliable and efficient up to higher order terms. For any $P \in N_\Omega$, we define $\Omega_P = \cup\{E \in \pi_h : P \in E\}$.

Lemma 4.1. *For any $P \in N_\Omega$, we have that*

$$
\sum_{E\subset\Omega_P} k^4 h_E^2 \|n(x)u_h\|^2_{L^2(E)} \le C\left(\sum_{l\in\varepsilon_{\Omega_P}} |l|\|J_l\|^2_{L^2(l)} + |\Omega_P|^2\|\nabla u_h\|^2_{L^2(\Omega_P)}\right), \tag{4.1}
$$

$$
\sum_{E\subset\Omega_P} k^4 h_E^2 \|n(x)u_h^\star\|^2_{L^2(E)} \le C\left(\sum_{l\in\varepsilon_{\Omega_P}} |l|\|J_l^\star\|^2_{L^2(l)} + |\Omega_P|^2\|\nabla u_h^\star\|^2_{L^2(\Omega_P)}\right). \tag{4.2}
$$

Proof. Using similar arguments to Lemma 4.1 in [11], we complete the proof. Let $I_0(u_h)$ be the $L^2(\Omega_P)$ projection of u_h onto the constants, i.e.,

$$
\int_{\Omega_P} u_h \bar{w} = \int_{\Omega_P} I_0(u_h)\bar{w}, \quad \forall \ w \text{ is constant } in \ \Omega_P.
$$

Obviously,

$$
\begin{aligned}
\sum_{E\subset\Omega_P} k^4 h_E^2 \|n(x)u_h\|^2_{L^2(E)} &\le C|\Omega_P|\|k^2 n(x)u_h\|^2_{L^2(\Omega_P)} \\
&\le C|\Omega_P|\left\{\|I_0(k^2 n(x)u_h)\|^2_{L^2(\Omega_P)} + \|k^2 n(x)u_h - I_0(k^2 n(x)u_h)\|^2_{L^2(\Omega_P)}\right\}.
\end{aligned} \tag{4.3}
$$

Next, we estimate the first term on the right-hand side of (4.3). Let Φ_P be the corresponding Lagrange basis function with supp $\Phi_P = \Omega_P$, then

$$\begin{aligned} \|I_0(k^2n(x)u_h)\|^2_{L^2(\Omega_P)} &= |I_0(k^2n(x)u_h)|^2|\Omega_P| = \frac{9}{|\Omega_P|}\left(\int_{\Omega_P} I_0(k^2n(x)u_h)\overline{\Phi_P}\right)^2 \\ &\le \frac{18}{|\Omega_P|}\left\{\left(\int_{\Omega_P}(I_0(k^2n(x)u_h) - k^2n(x)u_h)\overline{\Phi_P}\right)^2 + \left(\int_{\Omega_P} k^2n(x)u_h\overline{\Phi_P}\right)^2\right\}. \end{aligned} \tag{4.4}$$

From (2.5)–(2.6), we have

$$\int_{\Omega_P} k^2n(x)u_h\overline{\Phi_P} = \int_{\Omega_P}\nabla u_h\cdot\nabla\overline{\Phi_P} = \frac{1}{2}\sum_{l\in\varepsilon_\Omega\cap\varepsilon_{\Omega_P}}|l|\left[\!\left[\frac{\partial u_h}{\partial \nu_l}\right]\!\right]_l = \frac{1}{2}\sum_{l\in\varepsilon_\Omega\cap\varepsilon_{\Omega_P}}\int_l\left[\!\left[\frac{\partial u_h}{\partial \nu_l}\right]\!\right]_l. \tag{4.5}$$

The relationships (4.4) and (4.5) imply that

$$|\Omega_P|\|I_0(k^2n(x)u_h)\|^2_{L^2(\Omega_P)} \le C\left\{|\Omega_P|\|I_0(k^2n(x)u_h) - k^2n(x)u_h\|^2_{L^2(\Omega_P)} + \frac{1}{2}\sum_{l\in\varepsilon_\Omega\cap\varepsilon_{\Omega_P}}|l|\|J_l\|^2_{L^2(\Omega_P)}\right\}. \tag{4.6}$$

Substituting (4.6) into (4.5), we derive

$$\sum_{E\subset\Omega_P} k^4h_E^2\|n(x)u_h\|^2_{L^2(E)} \le C\left\{|\Omega_P|\|I_0(k^2n(x)u_h) - k^2n(x)u_h\|^2_{L^2(\Omega_P)} + \frac{1}{2}\sum_{l\in\varepsilon_\Omega\cap\varepsilon_{\Omega_P}}|l|\|J_l\|^2_{L^2(\Omega_P)}\right\}.$$

Inequality (4.1) can be concluded by using the standard estimate for the $L^2(\Omega_P)$ projection. □

The following theorem indicates that the indicator is globally reliable and locally efficient up to higher order terms.

Theorem 4.1. *There exists a constant C such that*

$$\|u - u_h\|_{H^1(\Omega)} \le C\left\{\eta_\Omega + (M + |\lambda|)\|u - u_h\|_{L^2(\Omega)} + |\lambda - \lambda_h| + h^2\|\nabla u_h\|_{L^2(\Omega)}\right\}, \tag{4.7}$$

and if $\partial E \cap \Gamma = \varnothing$ for $E \in \pi_h$, *then*

$$\hat{\eta}_E \le C\left\{\|e\|_{H^1(E^u)} + k^2h_E\|n(x)e\|_{L^2(E^u)} + h_E^2\|n(x)\|_{W^{1,\infty}(Eu)}\|u_h\|_{L^2(E^u)}\right\}, \tag{4.8}$$

if $\partial E \cap \Gamma \neq \varnothing$ for $E \in \pi_h$, *then*

$$\hat{\eta}_E \le C\left\{\|e\|_{H^1(E)} + k^2h_E\|n(x)e\|_{L^2(E)} + h_E^2\|n(x)\|_{W^{1,\infty}(E)}\|u_h\|_{L^2(E)} + \sum_{l\in\varepsilon_E\cap\varepsilon_\Gamma}\|\lambda u - \lambda_h u_h\|_{L^2(l)}\right\}. \tag{4.9}$$

Proof. It follows immediately from Theorem 3.1, Theorem 3.2, and Lemma 4.1. □

Theorem 4.2. *There exists a constant C such that*

$$\|u^\star - u_h^\star\|_{H^1(\Omega)} \le C\left\{\hat{\eta}_\Omega^\star + (M + |\lambda^\star|)\|u^\star - u_h^\star\|_{L^2(\Omega)} + |\lambda^\star - \lambda_h^\star| + h^2\|\nabla u_h^\star\|_{L^2(\Omega)}\right\}, \tag{4.10}$$

and if $\partial E \cap \Gamma = \varnothing$ *for* $E \in \pi_h$, *then*

$$\hat{\eta}_E^\star \le C\left\{\|e^\star\|_{H^1(E^u)} + k^2 h_E \|n(x)e^\star\|_{L^2(E^u)} + h_E^2 \|n(x)\|_{W^{1,\infty}(E^u)} \|u_h^\star\|_{L^2(E^u)}\right\}, \tag{4.11}$$

if $\partial E \cap \Gamma \neq \varnothing$ *for* $E \in \pi_h$, *then*

$$\hat{\eta}_E^\star \le C\left\{\|e^\star\|_{H^1(E)} + k^2 h_E \|n(x)e^\star\|_{L^2(E)} + h_E^2 \|n(x)\|_{W^{1,\infty}(E)} \|u_h\|_{L^2(E)} + \sum_{l \in \varepsilon_E \cap \varepsilon_\Gamma} \|\lambda^\star u^\star - \lambda_h^\star u_h^\star\|_{L^2(l)}\right\}. \tag{4.12}$$

Proof. It follows immediately from Theorem 3.1, Theorem 3.3, and Lemma 4.1. □

On adaptive mesh with local refinement, the convergence rate of eigenvalue is measured by the total number of degrees of freedom. There are some literature studies that have discussed the relationship between the mesh diameter and the degrees of freedom (e.g., see Remark 2.1 in [35]).

5 Adaptive algorithm and numerical experiments

First of all, we give the following Algorithm 5.1 which is fundamental and important and can be found in [36].

Algorithm 5.1.
Choose parameter $0 < \theta < 1$.
Step 1. Pick any initial mesh π_{h_0} with mesh size h_0.
Step 2. Solve (2.5)–(2.6) and (2.9)–(2.10) on π_{h_0} for discrete solutions (λ_{h_0}, u_{h_0}) and $(\lambda_{h_0}^\star, u_{h_0}^\star)$, respectively.
Step 3. Let $i = 0$.
Step 4. Compute the local indicator $\eta_E^2 + (\eta_E^\star)^2$ corresponding to λ_{h_i}.
Step 5. Construct $\hat{\pi}_{h_i} \subset \pi_{h_i}$ by Marking Strategy M and parameter θ.
Step 6. Refine π_{h_i} to get a new mesh $\pi_{h_{i+1}}$ by Procedure REFINE.
Step 7. Solve (2.5)–(2.6) and (2.9)–(2.10) on $\pi_{h_{i+1}}$ for discrete solution $(\lambda_{h_{i+1}}, u_{h_{i+1}})$ and $(\lambda_{h_{i+1}}^\star, u_{h_{i+1}}^\star)$, respectively.
Step 8. Let $i = i + 1$ and go to Step 4.

Marking Strategy M
Given parameter $0 < \theta < 1$.
Step 1. Construct a minimal subset $\hat{\pi}_{h_i}$ of π_{h_i} by selecting some elements in π_{h_i} such that

$$\sum_{E \in \hat{\pi}_{h_i}} (\eta_E^2 + (\eta_E^\star)^2) \ge \theta(\eta_\Omega^2 + (\eta_\Omega^\star)^2).$$

Step 2. Mark all the elements in $\hat{\pi}_{h_i}$.

Marking strategy M was first proposed by [37].

The procedure REFINE is some iterative or recursive bisection (see, e.g., [38,39]) of elements with the minimal refinement condition that marked elements are bisected at least once.

Next, we will report some numerical examples for Algorithm 5.1 to validate our theoretical results. For comparison, using linear finite element, we also solve the problem on uniform meshes. The discrete eigenvalue problems are solved in MATLAB 2016b on an Lenovo IdeaPad 500-14ACZ PC with 1.8 GHZ CPU and 8 GB RAM. Our program is compiled under the package of iFEM [40] and we take $\theta = 0.25$ in our program. In step 2 and step 7, we use the sparse solver "eigs" in MATLAB 2016b to solve the discrete eigenvalue problems. Numerical experiments are considered on the L-shaped domain $(-1,1)^2\backslash([0,1) \times (-1,0])$ and the square with a slit $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^2\Big\backslash\left\{0 \le x \le \frac{\sqrt{2}}{2}, y = 0\right\}$. The initial mesh is made up of congruent triangles with the mesh size $h_0 = \frac{\sqrt{2}}{32}$. We select the index of refraction $n(x) = 4$ or $n(x) = 4 + 4i$ when $k = 1$ (refer to [7]). When $n(x)$ is the real number, the first 30 eigenvalues are sorted in the descending order. When $n(x)$ is the complex number, the first 30 eigenvalues are sorted in the descending order of the absolute value of imaginary part. We select $n(x) = 4 + 2i, 4 + i$ when $k = 2$ and $k = 4$, respectively. For convenience and simplicity, the following notations are introduced in tables and figures.

- i: the ith iteration of Algorithm 5.1.
- λ^A_{j,h_i}: the jth eigenvalue from the ith iteration of the Adaptive Algorithm 5.1.
- $\lambda_{j,h}$: the jth eigenvalue in uniform meshes.
- dof: number of degrees of freedom.
- η^2: the a posteriori error estimator of approximate eigenvalue obtained by Algorithm 5.1.

In the case of $k = 1$, when $n(x) = 4$, we use the approximations $\lambda_1 \approx 2.53321363$, $\lambda_2 \approx 0.85778759$, and $\lambda_3 \approx 0.12452443$ as the reference values for the L-shaped domain. For the square with a slit, the first three reference eigenvalues are 1.48471191, 0.46173362, and −0.18417592, respectively. When $n(x) = 4 + 4i$, for the L-shaped domain, we take reference values as $\lambda_1 \approx 0.514287041 + 2.88232331i$, $\lambda_2 \approx 0.39703537 + 1.45898539i$, and $\lambda_3 \approx -0.07717876 + 1.04267799i$. For the square with a slit, the reference values are $\lambda_1 \approx 0.91930585 + 1.77078802i$, $\lambda_2 \approx 0.29263004 + 0.99987320i$, and $\lambda_3 \approx -0.26261456 + 0.75745031i$. In the case of $k \neq 1$, for the L-shaped domain, we take $\lambda_9 \approx -1.40118553 + 1.54057954i$ and $\lambda_{19} \approx -4.33350590 + 1.54440353i$ as the reference values when $k = 2$, $n(x) = 4 + 2i$ and $k = 4$, $n(x) = 4 + i$, respectively. For the square with a slit, we select $\lambda_2 \approx 2.19728089 + 5.85519972i$ and $\lambda_2 \approx 6.67501844 + 9.59727038i$ as the reference values, respectively. These reference values are computed by extrapolation.

For the L-shaped domain, using the linear finite element, we solve the first three eigenvalues on uniform meshes, the error curves are depicted in Figure 1. The error curves of eigenvalues obtained by Algorithm 5.1 are depicted in Figures 2 and 3. The numerical results are listed in Tables 1 and 2.

We know that the optimal convergence order of numerical eigenvalues on adaptive meshes can achieve $O(\text{dof}^{-1})$, and error curve of numerical eigenvalue should be parallel to the line with slope −1. In the case of $k = 1$, in Figure 1, we see that the error curves of the second eigenvalues are not parallel to the line with slope −1, which implies that the eigenfunction corresponding to the eigenvalue is of singularity and the convergence order of numerical eigenvalue computed on uniform meshes cannot achieve the optimal order. In Figure 2, the error curve of eigenvalue computed on adaptive meshes is parallel to the line with slope −1. Therefore, the numerical eigenvalue obtained by Algorithm 5.1 can achieve the optimal order. Using adaptive meshes is more effective than using uniform meshes when the eigenfunction corresponding to eigenvalue is of singularity. In addition, we know that the a posteriori error indicator of numerical eigenvalue is reliable and efficient as the error curve of numerical eigenvalue is basically parallel to the curve of error indicator. In the case of $k \neq 1$, the eigenfunctions corresponding to these eigenvalues are of singularity. In Figure 3, we also see that the numerical eigenvalues computed in adaptive meshes can achieve the optimal convergence order $O(\text{dof}^{-1})$.

For the square with a slit, in the case of $k = 1$, the error curves of the first three eigenvalues are depicted in Figure 4. The error curves of eigenvalues obtained by Algorithm 5.1 are depicted in Figures 5 and 6. The numerical results are listed in Tables 3 and 4. From the figures and tables, we obtain similar conclusion to the L-shaped domain, which coincides in theory results. We also see that the numerical eigenvalues computed in adaptive meshes can achieve the optimal convergence order $O(\text{dof}^{-1})$.

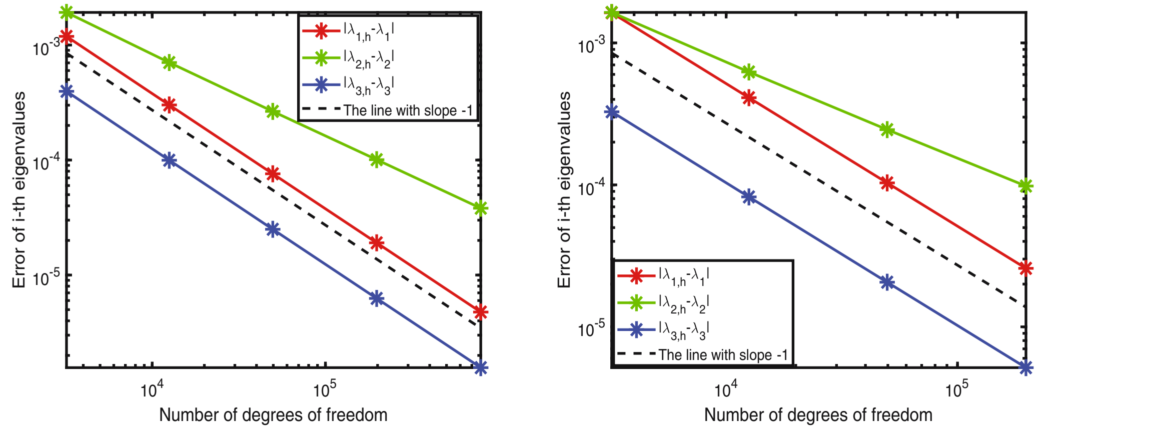

Figure 1: Error curves of the first three eigenvalues on uniform meshes for (2.1)–(2.2) on the L-shaped domain (the left: $k = 1$, $n(x) = 4$, the right: $k = 1$, $n(x) = 4 + 4i$).

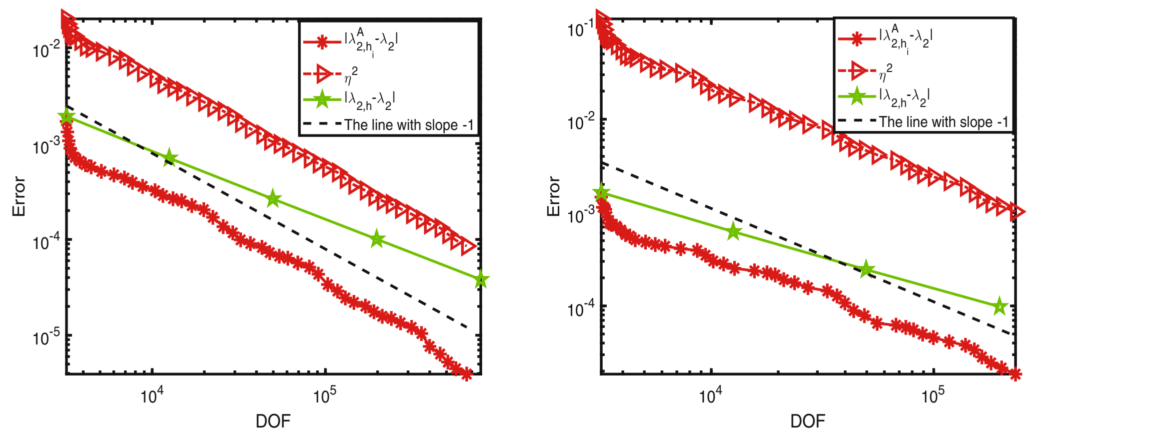

Figure 2: Error comparison of the second eigenvalue for (2.1)–(2.2) on the L-shaped domain (the left: $k = 1$, $n(x) = 4$, the right: $k = 1$, $n(x) = 4 + 4i$).

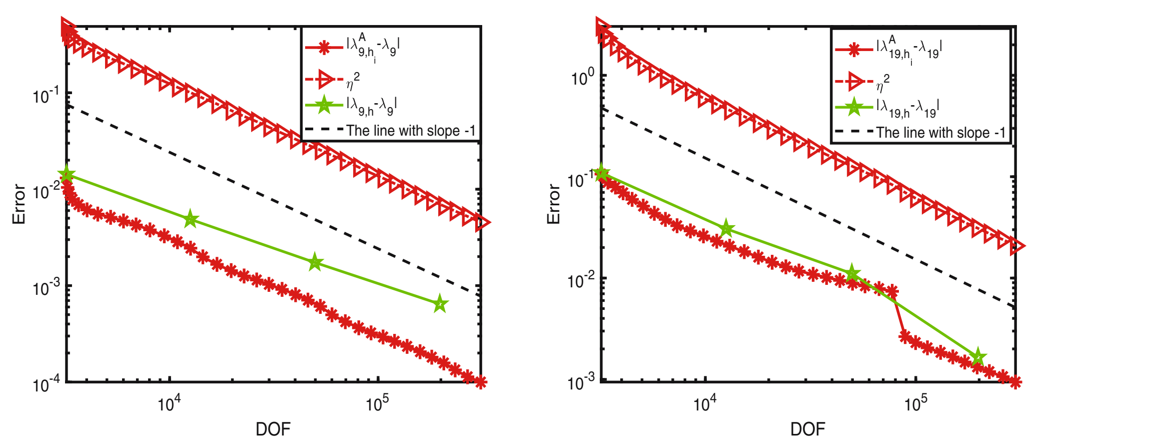

Figure 3: Error comparison of the eigenvalues for (2.1)–(2.2) on the L-shaped domain (the left: $k = 2$, $n(x) = 4 + 2i$, the right: $k = 4$, $n(x) = 4 + i$).

Table 1: The second eigenvalue for (2.1)–(2.2): L-shaped domain ($k = 1$, $n(x) = 4$)

i	j	dof	λ^A_{j,h_i}	dof	$\lambda_{j,h}$
16	2	11,536	0.857500	12,545	0.857083
27	2	47,688	0.857715	49,665	0.857523
38	2	1,90,616	0.857770	1,97,633	0.857687

Table 2: The second eigenvalue for (2.1)–(2.2): L-shaped domain ($k = 1$, $n(x) = 4 + 4i$)

i	j	dof	λ^A_{j,h_i}	dof	$\lambda_{j,h}$
20	2	11,365	0.3970403 + 1.4587041i	12,545	0.3966861 + 1.4584678i
32	2	48,797	0.3970372 + 1.4589067i	49,665	0.3968836 + 1.4587931i
43	2	1,81,128	0.3970354 + 1.4589609i	1,97,633	0.3969716 + 1.4589108i

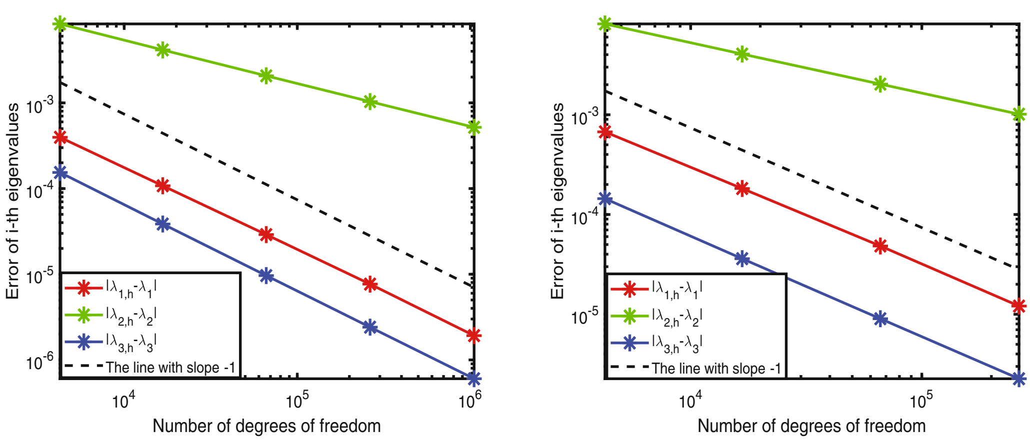

Figure 4: Error curves of the first three eigenvalues on uniform meshes for (2.1)–(2.2) on the square with a slit (the left: $k = 1$, $n(x) = 4$, the right: $k = 1$, $n(x) = 4 + 4i$).

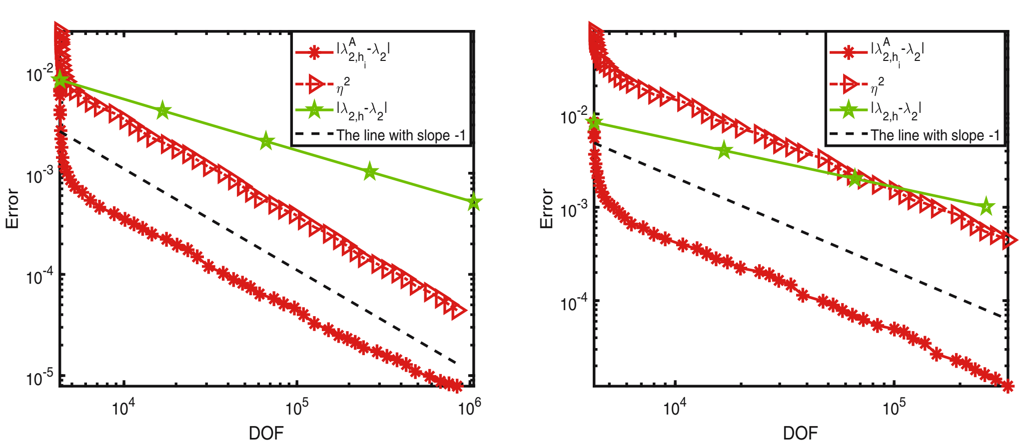

Figure 5: Error comparison of the second eigenvalue for (2.1)–(2.2) on the square with a slit (the left: $k = 1$, $n(x) = 4$, the right: $k = 1$, $n(x) = 4 + 4i$).

Table 3: The second eigenvalue for (2.1)–(2.2): the square with a slit ($k = 1$, $n(x) = 4$)

i	j	dof	λ^A_{j,h_i}	dof	$\lambda_{j,h}$
22	2	14,505	0.461483	16,705	0.457585
32	2	60,809	0.461670	66,177	0.459664
42	2	2,42,511	0.461715	2,63,425	0.460699

Table 4: The second eigenvalue for (2.1)–(2.2): the square with a slit ($k = 1$, $n(x) = 4 + 4i$)

i	j	dof	λ^A_{j,h_i}	dof	$\lambda_{j,h}$
22	2	15,370	0.292428 + 0.999679i	16,705	0.289067 + 0.997926i
33	2	64,492	0.292578 + 0.999827i	66,177	0.290846 + 0.998908i
44	2	2,59,623	0.292616 + 0.999865i	2,63,425	0.291737 + 0.999395i

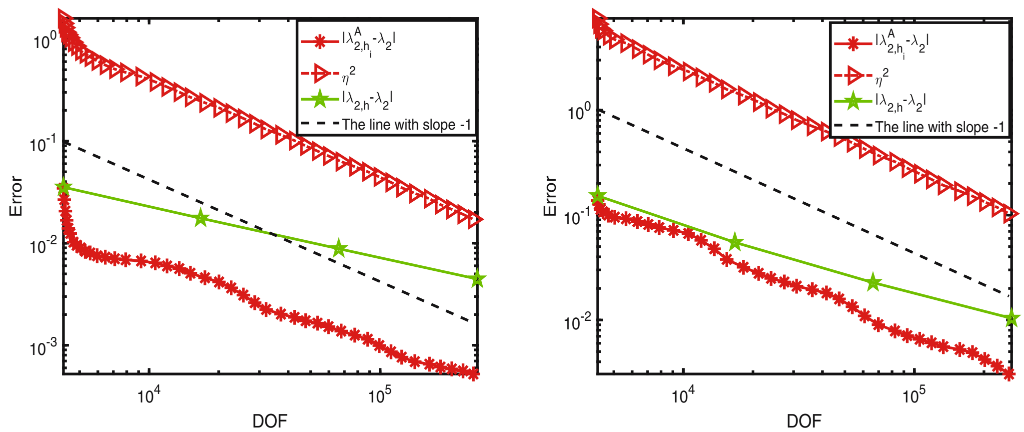

Figure 6: Error comparison of the eigenvalues for (2.1)–(2.2) on the square with a slit (the left: $k = 2$, $n(x) = 4 + 2i$, the right: $k = 4$, $n(x) = 4 + i$).

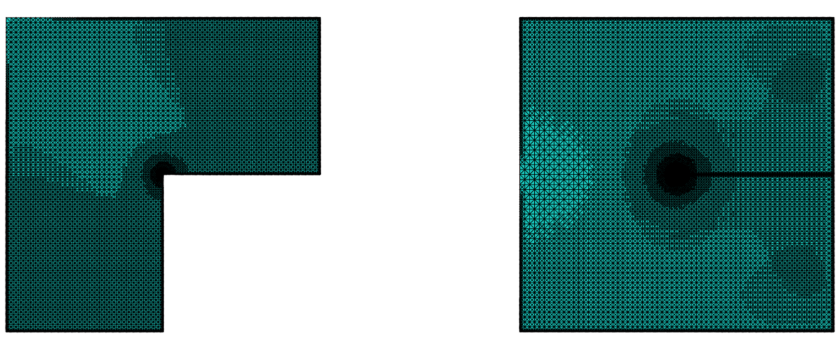

Figure 7: The adaptive meshes obtained by Algorithm 5.1 as $i = 25$, $k = 1$, and $n(x) = 4 + 4i$.

Finally, adaptively refined meshes obtained by Algorithm 5.1 are displayed in Figure 7 as $k = 1$ and $n(x) = 4 + 4i$. In theory, the errors of eigenfunctions on the elements around concave vertices should be greater than other elements. In Figure 7, we see that, on each domain, these elements around concave vertices are more refined than other elements, which coincides in theoretical results.

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