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Research Article

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A study on strongly convex hyper S-subposets in hyper S-posets

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Abstract: In this paper, we study various strongly convex hyper S-subposets of hyper S-posets in detail. To begin with, we consider the decomposition of hyper S-posets. A unique decomposition theorem for hyper S-posets is given based on strongly convex indecomposable hyper S-subposets. Furthermore, we discuss the properties of minimal and maximal strongly convex hyper S-subposets of hyper S-posets. In the sequel, the concept of hyper C-subposets of a hyper S-poset is introduced, and several related properties are investigated. In particular, we discuss the relationship between greatest strongly convex hyper S-subposets and hyper C-subposets of hyper S-posets. Moreover, we introduce the concept of bases of a hyper S-poset and give out the sufficient and necessary conditions of the existence of the greatest hyper C-subposets of a hyper S-poset by the properties of bases.

Keywords: hyper S-poset, strongly convex hyper S-subposet, hyper C-subposet, greatest hyper C-subposet, base

MSC 2020: 20N20, 06F05, 20M30

1 Introduction

For a semigroup S, a *left S-act* is a nonempty set A equipped with a mapping $S \times A \to A$, $(s, a) \rightsquigarrow sa$, such that $(st)a = s(ta)$ (and $1 \cdot a = a$ if S is a monoid with identity 1) for all $a \in A$ and $s, t \in S$. Right S-acts are defined analogously. As we have seen in [1,2], S-acts (also called S-systems) play an important role not only in studying properties of semigroups or monoids but also in other mathematical areas, such as graph theory and algebraic automata theory. Partially ordered acts over a partially ordered semigroup S, or S-posets, appear naturally in the study of mappings between posets (see [3]). Recall that if $(S, \cdot, \leq)$ is an ordered semigroup, then a *left S-poset A* is a left S-act A equipped with a partial order $\leq_A$ and, in addition, for all $s, t \in S$ and $a, b \in A$, $a \leq_A b$ and $s \leq t$ imply $sa \leq_A tb$. Right S-posets can be defined analogously. Preliminary work on S-posets was carried out by Fakhruddin in the 1980s (see [4,5]) and continued in recent papers, for example, see [6–14]. Also see [15] for an overview.

Algebraic hyperstructures (or hypersystems), particularly hypergroups, were originally proposed in 1934 by a French mathematician Marty [16], at the 8th Congress of Scandinavian Mathematicians. As we know, the composition of two elements in a group is an element, whereas the composition of two elements in a hypergroup is a nonempty set. The law characterizing such a structure is called the multi-valued operation, or hyperoperation and the theory of the algebraic structures endowed with at least one

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multi-valued operation is known as the Hyperstructure Theory. So far, hyperstructures have been widely investigated from the theoretical point of view and for their applications to many branches of pure and applied mathematics, for example, see [17–19]. In particular, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays, many authors have studied different aspects of semihypergroups, for instance, Anvariyeh et al. [20], Davvaz [21], Davvaz and Leoreanu [22], Hila et al. [23] and Leoreanu [24], also see [25,26]. It is worth pointing out that Heidari and Davvaz [27] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. The work on ordered semihypergroup theory can be found in [28–35]. In particular, Tang et al. [32] defined and studied the hyper S-posets over an ordered semihypergroup, and extended some results on S-posets to hyper S-posets. As a further study of hyper S-poset theory, we attempt in the present paper to study the strongly convex hyper S-subposets of hyper S-posets in detail.

The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic notions and results from the hyperstructure theory which will be used throughout this paper. In Section 3, the decomposition of hyper S-posets is discussed. Particularly, we prove that every hyper S-poset can be uniquely decomposable into a disjoint union of strongly convex indecomposable hyper S-subposets. In Section 4, the properties of minimal and maximal strongly convex hyper S-subposets of hyper S-posets are discussed. Furthermore, we define and investigate the a-maximal strongly convex hyper S-subposets of a hyper S-poset. In Section 5, the concept of hyper C-subposets of hyper S-posets is introduced, and several related properties are investigated. In particular, we discuss the relationship between greatest strongly convex hyper S-subposets and hyper C-subposets of hyper S-posets. In the sequel, we introduce the concept of bases of a hyper S-poset and give out the sufficient and necessary conditions of the existence of the greatest hyper C-subposets of a hyper S-poset in terms of bases. Some conclusions are given in Section 6.

2 Preliminaries and some notations

In this section, we present some definitions and results which will be used throughout this paper.

Let S be a nonempty set. A mapping $\circ : S \times S \to P^*(S)$, where $P^*(S)$ denotes the family of all nonempty subsets of S, is called a *hyperoperation* or *hypercomposition* on S. The image of the pair (x, y) is denoted by $x \circ y$. The couple $(S, \circ)$ is called a *hyperstructure*. In the above definition, if $x \in S$ and A, B are nonempty subsets of S, then $A \circ B$ is defined by

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

Also $A \circ x$ is used for $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$. A hyperstructure $(S, \circ)$ is a *semihypergroup* [17] if for all $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

An *ordered semihypergroup* (also called *po-semihypergroup* in [27]) $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ with an order relation $\leq$ which is compatible with the hyperoperation $\circ$, meaning that for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \leq a \circ y$ and $x \circ a \leq y \circ a$. Here, if $A, B \in P^*(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \leq B$ instead of $\{a\} \leq B$. Clearly, every ordered semigroup can be regarded as an ordered semihypergroup. Also see [31]. Throughout this paper, unless otherwise mentioned, S denotes an ordered semihypergroup.

A nonempty subset A of an ordered semihypergroup S is called a *left* (resp. *right*) *hyperideal* of S if (1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$), and (2) if $a \in A$, $b \leq a$ with $b \in S$, then $b \in A$. If A is both a left and a right hyperideal of S, then it is called a *hyperideal* of S (see [27]). An element a of S is called a *zero element* of S if $x \circ a = a \circ x = \{a\}$, $a \leq x$ for all $x \in S$ and denote it by 0.

We now recall the notion of hyper S-acts over semihypergroups from [36].

Let $(S, \circ)$ be a semihypergroup and A a nonempty set. If we have a mapping $\mu : S \times A \to P^*(A)|(s, a) \mapsto \mu(s, a) := s * a \in P^*(A)$, called the *hyper action of* S (or the *S-hyperaction*) on A, such that $(s \circ t) * a = s * (t * a)$, for all $a \in A$, $s, t \in S$, where

$$(\forall T \subseteq S)\ T * a = \bigcup_{t \in T} t * a; \quad (\forall B \subseteq A)\ s * B = \bigcup_{b \in B} s * b,$$

then we call A a *left hyper S-act* (also called *left S-hypersystem* in [36]). Right hyper S-acts can be defined analogously, and in this paper we will often use the term hyper S-act to mean left hyper S-act.

Furthermore, Tang et al. introduced the concept of hyper S-posets over an ordered semihypergroup.

Definition 2.1. [32] Let $(S, \circ, \leq)$ be an ordered semihypergroup. A *left hyper S-poset* $(A, \leq_A)$, often denoted ${}_S A$ (or briefly A), is a left hyper S-act A equipped with a partial order $\leq_A$ and, in addition, for all $s, t \in S$ and $a, b \in A$, if $s \leq t$ then $s * a \leq_A t * a$, and if $a \leq_A b$ then $s * a \leq_A s * b$. Here, $s * a$ stands for the result of the hyper action of s on a, and if $A_1, A_2 \in P^*(A)$, then we say that $A_1 \leq_A A_2$ if for every $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $a_1 \leq_A a_2$.

Analogously, we can define a right hyper S-poset A_S. Throughout this paper, we use the term hyper S-poset to mean left hyper S-poset. It is easily seen that every S-poset over an ordered semigroup can be regarded as a hyper S-poset over an ordered semihypergroup, and an ordered semihypergroup S is a hyper S-poset with respect to the hyperoperation of S, denoted by ${}_S S$. Also see [32].

Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. For $\varnothing \neq H \subseteq A$, we define

$$(H] := \{t \in A \mid t \leq_A h \text{ for some } h \in H\},$$
$$[H) := \{t \in A \mid h \leq_A t \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$, $[a)$ instead of $(\{a\}]$, $[\{a\})$, respectively. Clearly, $B \subseteq (B]$, $((B]] = (B]$ for any nonempty subset B of A, and if $B, C \subseteq A$, $B \subseteq C$, then $(B] \subseteq (C]$. A hyper S-poset A is called *unitary* if $A = (S * A]$, where $S * A = \bigcup_{s \in S, a \in A} s * a$.

Let $(A, \leq_A)$ be a hyper S-poset and B a nonempty subset of A. B is called a *hyper S-subposet* of A if for any $b \in B$, $s \in S$, $s * b \subseteq B$, denoted by $B \leq_A A$. Furthermore, B is called *strongly convex* [32] if $a \in A$, $b \in B$ and $a \leq_A b$ imply $a \in B$, equivalently, $B = (B]$. It is easily seen that every left hyperideal of an ordered semihypergroup S is a strongly convex hyper S-subposet of hyper S-poset ${}_S S$. In fact, a nonempty subset I of an ordered semihypergroup S is a strongly convex hyper S-subposet of hyper S-poset ${}_S S$ if and only if I is a left hyperideal of S. A strongly convex hyper S-subposet B of a hyper S-poset A is called *proper* if $B \neq A$.

Let $(S, \circ, \leq)$ be an ordered semihypergroup with zero 0 and $(A, \leq_A)$ a hyper S-poset. A is called *central hyper S-poset* if there exists unique element $\theta \in A$ such that $0 * a = \{\theta\} = s * \theta$ and $\theta \leq_A a$ for all $a \in A$, $s \in S$. In a central hyper S-poset A, θ is called *zero element* of A. Moreover, it is easy to check that $\{\theta\}$ is a strongly convex hyper S-subposet of A. A hyper S-poset A without zero is called *S-simple* if it does not contain proper strongly convex hyper S-subposets.

Lemma 2.2. *Let A be a hyper S-poset and $\{A_i \mid i \in I\}$ a family of strongly convex hyper S-subposets of A. Then $\bigcup_{i \in I} A_i$ is a strongly convex hyper S-subposet of A and $\bigcap_{i \in I} A_i$ is also a strongly convex hyper S-subposet of A if $\bigcap_{i \in I} A_i \neq \varnothing$.*

Proof. The proof is straightforward verification, and hence we omit the details. □

Let now A be a hyper S-poset and B a nonempty subset of A. We denote

$$\Omega = \{C \mid C \text{ is a strongly convex hyper } S\text{-subposet of } A \text{ containing } B\}.$$

Clearly, Ω is not empty since $A \in \Omega$. Let $L(B) = \bigcap_{C \in \Omega} C$. It is clear that $L(B) \neq \varnothing$ because $B \subseteq L(B)$. By Lemma 2.2, $L(B)$ is a strongly convex hyper S-subposet of A. Moreover, $L(B)$ is the smallest strongly convex hyper S-subposet of A containing B. $L(B)$ is called the *strongly convex hyper S-subposet of A generated by B*. For $B = \{a\}$, let $L(a)$ denote the strongly convex hyper S-subposet of A generated by $\{a\}$.

A strongly convex hyper S-subposet C of a hyper S-poset A is called *cyclic* if $C = L(a)$ for some $a \in A$.

Lemma 2.3. *Let A be a hyper S-poset and B a nonempty subset of A. Then $L(B) = (B \cup S * B] = (B] \cup (S * B]$. In particular, for any $a \in A$, $L(a) = (a \cup S * a] = (a] \cup (S * a]$.*

Proof. Let $D = (B \cup S * B]$. It is easy to see that D is a strongly convex hyper S-subposet of A and $B \subseteq D$. Furthermore, if C is a strongly convex hyper S-subposet of A and $B \subseteq C$, then $s * a \in C$ for any $a \in B$, $s \in S$, and we have $B \cup S * B \subseteq C$ and consequently $(B \cup S * B] \subseteq (C]$. Also, since C is strongly convex, we have $D = (B \cup S * B] \subseteq (C] = C$, that is, $D \subseteq C$. Therefore, $L(B) = (B \cup S * B] = (B] \cup (S * B]$. □

For strongly convex hyper S-subposets, we also have the following interesting result.

Proposition 2.4. *Let $(A, \leq_A)$ be a hyper S-poset. Every hyper S-subposet of A is coincided with its strongly convex hyper S-subposets if and only if for $a \leq_A b$ in A implies $a = b$ or $a \in S * b$.*

Proof. $\Rightarrow$. If $a \leq_A b$ in A, then $a \in (b \cup S * b]$. Let $B = \{b\} \cup S * b$. Then, clearly, B is a hyper S-subposet of A. By hypothesis, $(B] = B$. Therefore, $a \in B = \{b\} \cup S * b$ and consequently $a = b$ or $a \in S * b$.

$\Leftarrow$. Let B be a hyper S-subposet of A. If $a \leq_A b \in B$, then $a = b$ or $a \in S * b$. Since B is a hyper S-subposet of A, we have $a = b$ or $a \in S * B \subseteq B$. Consequently, $(B] = B$. Hence, B is indeed a strongly convex hyper S-subposet of A. □

The following lemma is obvious.

Lemma 2.5. *Let A be a hyper S-poset and B a strongly convex hyper S-subposet of A. Then $(S * a]$ is a strongly convex hyper S-subposet of A for $a \in B$ and $(S * a] \subseteq B$.*

The reader is referred to [2,18,37] for notation and terminology not defined in this paper.

3 Decomposition of hyper *S*-posets

In this section, we consider the decomposition of hyper S-posets. In particular, a unique decomposition theorem for hyper S-posets is given based on strongly convex indecomposable hyper S-subposets.

Definition 3.1. Let A be a hyper S-poset. A is called *decomposable* if there exist nonempty strongly convex hyper S-subposets A_1, A_2 of A such that $A = A_1 \tilde{\cup} A_2$, where $A = A_1 \tilde{\cup} A_2$ means that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \varnothing$. Otherwise, A is called *indecomposable*.

Lemma 3.2. *Let A be a hyper S-poset and $a \in A$. Then $L(a)$ is a strongly convex indecomposable hyper S-subposet of A.*

Proof. Assume that $L(a)$ is a strongly convex decomposable hyper S-subposet of A. Then there exist nonempty strongly convex hyper S-subposets A_1, A_2 of A such that $L(a) = A_1 \tilde{\cup} A_2$, and we have $a \in A_1$ or $a \in A_2$. If $a \in A_1$, then, since A_1 is strongly convex and $A_1 \cap A_2 = \varnothing$, we have $A_1 = L(a)$, $A_2 = \varnothing$. This is a contradiction. Similarly, if $a \in A_2$, then we can also obtain a contradiction. □

Lemma 3.3. *Let A be a hyper S-poset and $\{A_i \mid i \in I\}$ a family of strongly convex indecomposable hyper S-subposets of A. If $\bigcap_{i \in I} A_i \neq \varnothing$, then $\bigcup_{i \in I} A_i$ is also a strongly convex indecomposable hyper S-subposet of A.*

Proof. By Lemma 2.2, $\bigcup_{i \in I} A_i$ is a strongly convex hyper S-subposet of A. Assume that $\bigcup_{i \in I} A_i = M \tilde{\cup} N$, where M and N are strongly convex hyper S-subposets of $\bigcup_{i \in I} A_i$. Let $a \in \bigcap_{i \in I} A_i$ and assume that $a \in M$. Then $a \in M \cap A_i$ for all $i \in I$. Since A_i is indecomposable and $A_i = (M \cap A_i) \cup (N \cap A_i)$, and, by Lemma 2.2,

$M \cap A_i$ and $N \cap A_i$ are both strongly convex hyper S-subposets of hyper S-poset A_i, it follows that $N \cap A_i = \varnothing$ for all $i \in I$. Consequently, $N = \varnothing$. Hence, $\bigcup_{i \in I} A_i$ is indeed a strongly convex indecomposable hyper S-subposet of A. □

In the following, we give a unique decomposition theorem for hyper S-posets based on strongly convex indecomposable hyper S-subposets.

Theorem 3.4. *Every hyper S-poset A can be uniquely decomposable into a disjoint union of strongly convex indecomposable hyper S-subposets, that is,* $A = \dot{\bigcup}_{i \in I} A_i$, *where each* A_i, $i \in I$, *is a strongly convex indecomposable hyper S-subposet of A.*

Proof. By Lemma 3.2, for any $a \in A$, $L(a)$ is a strongly convex indecomposable hyper S-subposet of A. For $x \in A$, let $D_x = \{B \mid B$ be a strongly convex indecomposable hyper S-subposet of A, $x \in B\}$. Then $D_x \neq \varnothing$ since $L(x) \in D_x$ and $\bigcap_{B \in D_x} B \neq \varnothing$. By Lemma 3.3, $A_x = \bigcup_{B \in D_x} B$ is a strongly convex indecomposable hyper S-subposet of A. Obviously, for any $x, y \in A$, either $A_x = A_y$ or $A_x \cap A_y = \varnothing$. Hence, we can define a relation ρ on A by

$$x\rho y \Leftrightarrow A_x = A_y.$$

Then ρ is an equivalence relation. Taking a representative subset C of equivalence classes of ρ, then $A = \dot{\bigcup}_{x \in C} A_x$ is the decomposition of hyper S-poset A in which A_x is a strongly convex indecomposable hyper S-subposet of A for every $x \in C$. Furthermore, suppose that $A = \dot{\bigcup}_{i \in I} B_i$ and $A = \dot{\bigcup}_{j \in J} C_j$, where each B_i and C_j is a strongly convex indecomposable hyper S-subposet of A. For any $i \in I$, taking $b_0 \in B_i$, then there exists $j \in J$ such that $b_0 \in C_j$. We consider two subsets of $B_i : B_i' = B_i \cap C_j$ and $B_i'' = B_i \cap (A \backslash C_j)$. Clearly, $B_i = B_i' \dot{\cup} B_i''$, and if B_i' and B_i'' are not empty, then they are strongly convex hyper S-subposets of B_i. But B_i is indecomposable, and thus $B_i'' = \varnothing$, which implies that $B_i \subseteq C_j$. Similarly, for j there exists $i' \in I$ such that $C_j \subseteq B_{i'}$. Consequently, $B_i \subseteq Cj \subseteq B_{i'}$. Thus, $B_i = C_j$. In a similar way, for any $j \in J$, there exists $i \in I$ such that $C_j = B_i$. Thus, the decomposition is unique. □

Remark 3.5. For the above decomposition: $A = \dot{\bigcup}_{i \in I} A_i$, if $a, b \in A$, and $a \leq_A b \in A_i$ for some $i \in I$, then, since A_i is a strongly convex indecomposable hyper S-subposet of A, $a \in A_i$. Thus, the order relation on $A = \dot{\bigcup}_{i \in I} A_i$ is as follows: $a \leq_A b$ in $A = \dot{\bigcup}_{i \in I} A_i$ if and only if $a \leq_A b$ in A_i for some $i \in I$. In this particular decomposition of A we call A_i a *strongly convex indecomposable component* of A for all $i \in I$.

By Theorem 3.4, we immediately obtain the following corollary:

Corollary 3.6. *Every ordered semihypergroup can be uniquely decomposable into a disjoint union of left hyperideals.*

4 Minimal and maximal strongly convex hyper S-subposets of hyper S-posets

In this section, we discuss the properties of minimal and maximal strongly convex hyper S-subposets of hyper S-posets. In particular, we give some characterizations of minimal and maximal strongly convex hyper S-subposets.

Definition 4.1. Let S be an ordered semihypergroup without zero and A a hyper S-poset over S. A strongly convex hyper S-subposet L of A is called *minimal* if there does not exist strongly convex hyper S-subposet B of A such that $B \subset L$. Equivalently, for any strongly convex hyper S-subposet B of A, if $B \subseteq L$, then $B = L$.

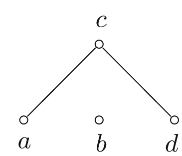
Example 4.2. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":

$\circ$	a	b	c	d
a	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$
d	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$

$$\leq :=\{(a, a), (a, c), (b, b), (c, c), (d, c), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, c), (d, c)\}.$$

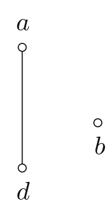


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{a, b, d\}$ defined by the following order:

$$\leq_A := \{(a, a), (b, b), (d, d), (d, a)\}.$$

We give the covering relation "$\prec_A$" and the figure of A:

$$\prec_A = \{(d, a)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as the aforementioned hyperoperation table. Let $L = \{a, d\}$. We can easily verify that L is a minimal strongly convex hyper S-subposet of A.

Theorem 4.3. *Let S be an ordered semihypergroup without zero and let L be a strongly convex hyper S-subposet of a hyper S-poset A. Then the following statements are equivalent:*
(1) *L is minimal.*
(2) *$(S * a] = L$ for all $a \in L$.*
(3) *$L(a) = L$ for all $a \in L$.*
(4) *L is S-simple.*

Proof. (1) $\Rightarrow$ (2). Let L be a minimal strongly convex hyper S-subposet of A. For every $a \in L$, let $B = (S * a]$. Then, by Lemma 2.5, B is a strongly convex hyper S-subposet of A and $B \subseteq L$. Since L is minimal, we have $L = B = (S * a]$.

(2) $\Rightarrow$ (3). Suppose that $(S * a] = L$ for all $a \in L$. Then we have

$$L(a) = (a \cup S * a] = (a] \cup (S * a] = (a] \cup L = L.$$

(3) $\Rightarrow$ (4). Assume that $L(a) = L$ for all $a \in L$. Let B be a strongly convex hyper S-subposet of L. Then for $b \in B$, we have $L = L(b) \subseteq B \subseteq L$. Therefore, $B = L$, that is, L does not contain proper strongly convex hyper S-subposets. Thus, L is S-simple.

(4) ⇒ (1). Suppose that L is S-simple. Let B be a strongly convex hyper S-subposet of A such that $B \subseteq L$. Then B is a strongly convex S-subposet of L. Since L is S-simple, we have $B = L$. Hence, L is minimal. □

Theorem 4.4. *Let S be an ordered semihypergroup without zero and A a hyper S-poset which has proper strongly convex hyper S-subposets. Then every proper strongly convex hyper S-subposet of A is minimal if and only if A contains exactly one proper strongly convex hyper S-subposet or A contains exactly two proper strongly convex hyper S-subposets B_1, B_2 such that $A = B_1 \tilde{\cup} B_2$.*

Proof. (Necessity) Let J be a proper strongly convex hyper S-subposet of A. Then, by hypothesis, J is a minimal strongly convex hyper S-subposet of A. Then we have the following two cases:

Case 1. Let $A = L(a)$ for all $a \in A\backslash J$, where $A\backslash J$ is the complement of J in A. Assume that K is also a proper strongly convex hyper S-subposet of A and $K \neq J$. Then, since J is minimal, we have $K\backslash J \neq \varnothing$, and there exists $a \in K\backslash J \subseteq A\backslash J$. Thus, $A = L(a) \subseteq K \subseteq A$, and so $A = K$, which is impossible. Hence, $K = J$. Therefore, in this case, J is the unique proper strongly convex hyper S-subposet of A.

Case 2. Let $A \neq L(a)$ for some $a \in A\backslash J$. Then $L(a) \neq J$ and $L(a)$ is a minimal strongly convex hyper S-subposet of A. By Lemma 2.2, $L(a) \cup J$ is a strongly convex hyper S-subposet of A. Since every proper strongly convex hyper S-subposet of A is minimal and $J \subset L(a) \cup J$, we have $A = L(a) \cup J$. Also, since $L(a) \cap J \subset L(a)$ and $L(a)$ is a minimal strongly convex hyper S-subposet of A, by Lemma 2.2 we get $L(a) \cap J = \varnothing$. Furthermore, let K be an arbitrary proper strongly convex hyper S-subposet of A. Then, by hypothesis, K is a minimal strongly convex hyper S-subposet of A. We observe that $K = K \cap A = (K \cap L(a)) \cup (K \cap J)$. If $K \cap J \neq \varnothing$, then, since K and J are also minimal strongly convex hyper S-subposets of A, we have $K = J$. If $K \cap L(a) \neq \varnothing$, then $K = L(a)$. Hence, in this case, A contains exactly two proper strongly convex hyper S-subposets $L(a)$ and J such that $A = L(a) \cup J$ and $L(a) \cap J = \varnothing$.

(Sufficiency) Let A contain exactly one proper strongly convex hyper S-subposet L. Then it is not difficult to see that L is minimal. Now, suppose that A contains exactly two proper strongly convex hyper S-subposets B_1, B_2 such that $A = B_1 \cup B_2$ and $B_1 \cap B_2 = \varnothing$. Let B be a strongly convex hyper S-subposet of A such that $B \subseteq B_1$. Then $B \subseteq B_1 \subset A$, and so B is a proper strongly convex hyper S-subposet of A. Since $B \subseteq B_1$ and $B_1 \cap B_2 = \varnothing$, we have $B \neq B_2$. By hypothesis, we have $B = B_1$. Hence, B_1 is minimal. In the same way, we can show that B_2 is also minimal. □

Now, we discuss maximal strongly convex hyper S-subposets of hyper S-posets.

Definition 4.5. Let A be a hyper S-poset. A proper strongly convex hyper S-subposet L of A is said to be *maximal* if there does not exist proper strongly convex hyper S-subposet B of A such that $L \subset B$. Equivalently, if for any strongly convex hyper S-subposet B of A such that $L \subseteq B$, then $B = A$.

Lemma 4.6. *Let A be a hyper S-poset and L a proper strongly convex hyper S-subposet of A. Then the following statements are equivalent:*
(1) *L is maximal.*
(2) *$L(a) \cup L = A$ for all $a \in A\backslash L$.*

Theorem 4.7. *Let A be a hyper S-poset and L a proper strongly convex hyper S-subposet of A. Then L is maximal if and only if one and only one of the following two conditions is satisfied:*
(1) *$A\backslash L = \{a\}$ for some $a \in A$.*
(2) *$A\backslash L \subseteq (S * a]$ for all $a \in A\backslash L$.*

Proof. (Necessity) Assume that L is a maximal strongly convex hyper S-subposet of A. Then we consider the following two cases:

Case 1. Let $(S * a] \subseteq L$ for some $a \in A\backslash L$. Then we have

$$L \cup (a] = (L \cup (S * a]) \cup (a] = L \cup ((S * a] \cup (a]) = L \cup L(a).$$

Then, by Lemma 2.2, $L \cup (a]$ is a strongly convex hyper S-subposet of A. On the other hand, since $a \in A\backslash L$, we have $L \subset L \cup (a]$. Also, since L is maximal, we have $L \cup (a] = A$. Thus, $A\backslash L \subseteq (a]$. To show that $A\backslash L = \{a\}$, let $x \in A\backslash L$. Then $x \leq_A a$ and so $(S * x] \subseteq (S * a] \subseteq L$. From $(S * x] \subseteq L$ and $x \in A\backslash L$, a similar argument shows that $A\backslash L \subseteq (x]$. Thus, we have $a \leq_A x$, and so $x = a$. Hence, we have shown that $A\backslash L = \{a\}$. In this case, the property (1) holds.

Case 2. Let $(S * a] \nsubseteq L$ for all $a \in A\backslash L$. In this case, we show that the property (2) holds. In fact, let $a \in A\backslash L$. Since $(S * a]$ is a strongly convex hyper S-subposet of A, by Lemma 2.2, we obtain that $L \cup (S * a]$ is also a strongly convex hyper S-subposet of A. On the other hand, since $(S * a] \in L$, we have $L \subset L \cup (S * a]$. Thus, since L is maximal, $L \cup (S * a] = A$. Hence, $A\backslash L \subseteq (S * a]$ for all $a \notin A\backslash L$.

(Sufficiency) Let T be a strongly convex hyper S-subposet of A such that $L \subset T$. Then $T\backslash L \neq \varnothing$. If $A\backslash L = \{a\}$ for some $a \in A$, then $T\backslash L \subseteq A\backslash L = \{a\}$. Thus, we have $T\backslash L = \{a\}$, and so $T = L \cup \{a\} = A$. Hence, L is a maximal strongly convex hyper S-subposet of A. If $A\backslash L \subseteq (S * a]$ for all $a \in A\backslash L$, then for any $x \in T\backslash L$, $A\backslash L \subseteq (S * x] \subseteq (S * T] \subseteq (T] = T$. Thus, $A = (A\backslash L) \cup L \subseteq T \cup T = T$. Therefore, L is a maximal strongly convex hyper S-subposet of A. □

Now we give a classification of hyper S-poset by maximal strongly convex hyper S-subposets. We denote the union of all proper strongly convex hyper S-subposets of a hyper S-poset A by U. Furthermore, we have the following lemma:

Lemma 4.8. *Let A be a hyper S-poset. Then $A = U$ if and only if A is not a cyclic hyper S-poset, that is, $A \neq L(a)$ for all $a \in A$.*

Proof. (Necessity) Suppose that there exists $a \in A$ such that $A = L(a)$. Then, since $a \in A = U$, there exists some proper strongly convex hyper S-subposet B of A such that $a \in B$. It thus follows that $A = L(a) \subseteq B$, which is a contradiction.

(Sufficiency) Clearly, $U \subseteq A$. On the other hand, by hypothesis, $L(a) \neq A$ for any $a \in A$, thus $L(a)$ is a proper strongly convex hyper S-subposet of A. Therefore, $a \in L(a) \subseteq U$, which implies that $A \subseteq U$. □

Theorem 4.9. *Let A be a hyper S-poset over an ordered semihypergroup S. Then one and only one of the following four conditions is satisfied:*
(1) *A is S-simple.*
(2) *$A \neq L(a)$ for all $a \in A$.*
(3) *There exists $a \in A$ such that $A = L(a)$, $U = A\backslash\{a\}$, and U is the unique maximal strongly convex hyper S-subposet of A.*
(4) *$A\backslash U = \{x \in A \mid (S * x] = A\}$ and U is the unique maximal strongly convex hyper S-subposet of A.*

Proof. Assume that S is not S-simple. Then there exists a proper strongly convex hyper S-subposet L of S, and so $U \neq \varnothing$. By Lemma 2.2, U is a strongly convex hyper S-subposet of A. We consider the following two cases:

Case 1. Let $U = A$. By Lemma 4.8, we have $L(a) \neq A$ for all $a \in A$. In this case, the condition (2) is satisfied.

Case 2. Let $U \neq A$. Then U is a maximal strongly convex hyper S-subposet of A. Moreover, since U is the union of all proper strongly convex hyper S-subposets of A, clearly U is the unique maximal strongly convex hyper S-subposet of A. By Theorem 4.7, one and only one of the following two conditions is satisfied:
(i) $A\backslash U = \{a\}$ for some $a \in A$.
(ii) $A\backslash U \subseteq (S * a]$ for all $a \in A\backslash U$.

Assume $A\backslash U = \{a\}$ for some $a \in A$. Then, in this case, the condition (3) is satisfied. In fact, we have:
(1) $L(a) = A$. Indeed, let $L(a) \neq A$. Then $L(a)$ is a proper strongly convex hyper S-subposet of A, and we have $a \in L(a) \subseteq U$, which is a contradiction. Thus, $L(a) = A$.
(2) $U = A\backslash\{a\}$. Indeed, since $A\backslash U = \{a\}$, we have $U = A\backslash\{a\}$.

Now, let $A\backslash U \subseteq (S * a]$ for all $a \in A\backslash U$. Then, in this case, the condition (4) is satisfied. To show that $A\backslash U = \{x \in A \mid (S * x] = A\}$, let $x \in A\backslash U$. Then, by hypothesis, $x \in A\backslash U \subseteq (S * x]$, and we have $(x] \subseteq (S * x]$. Thus, $L(x) = (x] \cup (S * x] = (S * x]$. Also, since $x \notin U$, we have $L(x) = A$. Hence, $A = L(x) = (S * x]$. Conversely, let $x \in A$ be such that $(S * x] = A$. If $x \in U$, then, since $U \neq A$, $L(x) \subseteq U \subset A$. It is impossible, since $L(x) = (x] \cup (S * x] = (x] \cup A = A$. Hence, $x \in A\backslash U$. Thus, $A\backslash U = \{x \in A \mid (S * x] = A\}$. □

Let S be an ordered semihypergroup and A a hyper S-poset. Then we define an equivalence relation "$\mathcal{L}$" on S as follows:

$$(a, b) \in \mathcal{L} \text{ if and only if } L(a) = L(b).$$

We denote the $\mathcal{L}$-class containing a by L^a and assign a partial order relation "$\preccurlyeq$" on the $\mathcal{L}$-classes as follows:

$$L^a \preccurlyeq L^b \text{ if and only if } L(a) \subseteq L(b).$$

Theorem 4.10. *Let A be a hyper S-poset and $\varnothing \neq L \subseteq A$. Then the following statements are equivalent:*
(1) *L is a maximal strongly convex hyper S-subposet of A.*
(2) *$A\backslash L$ is an $\mathcal{L}$-class of A.*
(3) *$A\backslash L$ is a maximal $\mathcal{L}$-class of A.*

Proof. (1) ⇒ (2). Assume that L is a maximal strongly convex hyper S-subposet of A. We claim that $A\backslash L$ is an $\mathcal{L}$-class. To prove our claim, let a, b be arbitrary elements of $A\backslash L$. If $a \notin L(b)$, then, clearly, $a \notin L(b) \cup L$. By Lemma 2.2, $L(b) \cup L$ is a strongly convex hyper S-subposet of A. By hypothesis, $L(b) \cup L = L$. It thus implies that $b \in L$, which is a contradiction. Hence, $a \in L(b)$. Similarly, it can be shown that $b \in L(a)$. Thus, $L(a) = L(b)$. On the other hand, let $a \in A\backslash L$. It is not difficult to show that $L(a) \neq L(x)$ for any $x \in L$. Therefore, $S\backslash L$ is an $\mathcal{L}$-class.

(2) ⇒ (3). Let $A\backslash L$ be an $\mathcal{L}$-class, denoted by L^a. Then $A\backslash L$ is a maximal $\mathcal{L}$-class of A. In fact, if there exists an $\mathcal{L}$-class L^c such that $L^a \prec L^c$, then $L(a) \subset L(c)$, which implies that $c \notin A\backslash L$, i.e., $c \in L$. Thus, $a \in L(a) \subset L(c) \subseteq L$, which is a contradiction. Hence, $A\backslash L$ is a maximal $\mathcal{L}$-class of A.

(3) ⇒ (1). Let $A\backslash L$ be a maximal $\mathcal{L}$-class of A. First we show that L is a strongly convex hyper S-subposet of A. Let $a \in L$ and $x \in S$. Then $x * a \subseteq L$. In fact, if $x * a \nsubseteq L$, then there exists $b \in x * a$ such that $b \notin L$, that is, $b \in A\backslash L$. By hypothesis, $L(a) \subseteq L(b)$. On the other hand, the reverse inclusion is obvious. Thus, $L(a) = L(b)$. This implies that $a \in A\backslash L$, which is a contradiction. Now suppose that $a \in L$, $b \in A$ such that $b \leq a$. Then $L(b) \subseteq L(a)$. We claim that $b \in L$. In fact, if $b \in A\backslash L$, then, by hypothesis, $L(a) \subseteq L(b)$. Hence, $L(a) = L(b)$. This shows that $a \in A\backslash L$, which is impossible. Therefore, L is a strongly convex hyper S-subposet of A. Furthermore, if there exists a proper strongly convex hyper S-subposet L_1 of A such that $L \subset L_1$, then we can pick $c \in L_1\backslash L$. Since $L(x) = L(c)$ for any $x \in A\backslash L$, we have $A\backslash L \subseteq L(c) \subseteq L_1$. Thus, $A = (A\backslash L) \cup L \subseteq L_1$, which is a contradiction. It thus follows that L is a maximal strongly convex hyper S-subposet of A. □

The remainder of this section is to discuss the a-maximal strongly convex hyper S-subposet of a hyper S-poset by terms of the equivalence relation $\mathcal{L}$.

Definition 4.11. Let A be a hyper S-poset over an ordered semihypergroup S and $a \in A$. A strongly convex hyper S-subposet L is called a-maximal if L is a maximal hyper S-subposet of A with respect to not containing the element a.

It is not difficult to see that if there exists an a-maximal strongly convex hyper S-subposet of A, then it is unique.

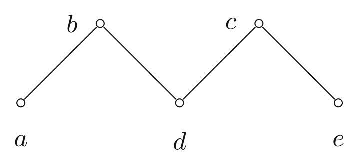
Example 4.12. We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation "$\circ$" and the order "$\leq$".

∘	a	b	c	d	e
a	$\{b,d\}$	$\{b,d\}$	$\{d\}$	$\{d\}$	$\{d\}$
b	$\{b,d\}$	$\{b,d\}$	$\{d\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c\}$

$$\leq\, : \,= \{(a,a),(a,b),(b,b),(c,c),(d,b),(d,c),(d,d),(e,c),(e,e)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec\; = \{(a,b),(d,b),(d,c),(e,c)\}.$$

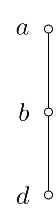


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{a, b, d\}$ defined by the following order:

$$\leq_A \,:= \{(a,a),(b,a),(b,b),(d,a),(d,b),(d,d)\}.$$

We give the covering relation "$\prec_A$" and the figure of A:

$$\prec_A = \{(b,a),(d,b)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as the aforementioned hyperoperation table. Let $L = \{b, d\}$. One can easily verify that L is a strongly convex hyper S-subposet of A, and it is a-maximal.

Let L, L_1 be strongly convex hyper S-subposets of a hyper S-poset A. L_1 is called a *cover* of L if $L \neq L_1$ and L_1 is the smallest strongly convex hyper S-subposet containing L. Let L be a strongly convex hyper S-subposet of a hyper S-poset A and $a \in A$, we denote the intersection of all strongly convex hyper S-subposets of A containing L and a by L^*. Then L^* is a strongly convex hyper S-subposet of A by Lemma 2.2, and we have:

Proposition 4.13. *Let A be a hyper S-poset over an ordered semihypergroup S. Then a strongly convex hyper S-subposet L of A is a-maximal if and only if L^* is a cover of L.*

Proof. $\Rightarrow$. If L is a-maximal and J is a strongly convex hyper S-subposet of A such that $L \subset J \subseteq L^*$, then $a \in J$, and we have $\{a\} \cup L \subseteq J$, which implies that $L^* \subseteq J$. Consequently, $J = L^*$.

$\Leftarrow$. Let L^* be a cover of L. If K is a strongly convex hyper S-subposet of A which does not contain a such that $L \subset K$, then $a \in L^* \subseteq K$. This is a contradiction. Hence, L is a-maximal. □

Lemma 4.14. *Let A be a hyper S-poset over an ordered semihypergroup S. If L is an a-maximal strongly convex hyper S-subposet of A, then the following statements hold:*
(1) $A\backslash L = \bigcup\{L^x \mid x \in A\backslash L\}$.
(2) *If $C = \{L^x \mid x \in A\backslash L\}$, then $L^*\backslash L$ is the least element of the set C with respect to the ordering $\preccurlyeq$ on $\mathcal{L}$-classes, and any L^x $(x \in L)$ is not greater than any element L^x in C.*

Proof.
(1) Obviously, $A\backslash L \subseteq \bigcup\{L^x \mid x \in A\backslash L\}$. To obtain the reverse inclusion, we have to show that $L^x \subseteq A\backslash L$ holds for all $x \in A\backslash L$. In fact, if $y \in L^x$, then $y \in A\backslash L$. Otherwise, if $y \notin A\backslash L$, then $y \in L$. Thus, we obtain that $x \in L(x) = L(y) \subseteq L$, which is a contradiction.
(2) We first prove that $L^*\backslash L$ is an $\mathcal{L}$-class. It is clear that $a \in L^*\backslash L$. Let $x \in L^*\backslash L$. Then L is also x-maximal. If $x \notin L(a)$, then $x \notin L(a) \cup L \neq L$, which contradicts the fact that L is a x-maximal strongly convex hyper S-subposet of A. Thus, $L(x) \subseteq L(a)$. Similarly, we can show that $L(a) \subseteq L(x)$. Therefore, $L(a) = L(x)$. Moreover, if $y \in A\backslash L^*$, then $L(y) \neq L(a)$. In fact, if $y \in L^x$, then $y \in L(y) = L(x) \subseteq L^*$, which is a contradiction. Consequently, $L^*\backslash L$ is an $\mathcal{L}$-class.

Now, consider $y \in A\backslash L^*$. Clearly, $L(a) \subseteq L(y)$, otherwise, it can be obtained that $a \notin L(y)$. This leads to $y \in L(y) \subseteq L$ since L is an a-maximal strongly convex hyper S-subposet of A. This is also a contradiction. Thus, $L^*\backslash L$ is the least element of C.

Furthermore, we claim that any L^x $(x \in L)$ is not greater than any element L^x in C. To prove our claim, it is enough to prove that $L^x \preccurlyeq L^*\backslash L$ for any $x \in L$. Assume that $L^*\backslash L \preccurlyeq L^x$ for some $x \in L$. Then, we have

$$a \in L(a) \subseteq L(x) \subseteq L.$$

But this is clearly impossible. The lemma is proved. □

Theorem 4.15. *Let A be a hyper S-poset and L a strongly convex hyper S-subposet of A. Then there exists $a \in A$ such that L is a-maximal strongly convex hyper S-subposet of A if and only if $A\backslash L$ contains the least $\mathcal{L}$-class among all the $\mathcal{L}$-classes contained in $A\backslash L$.*

Proof. $\Rightarrow$. This part is clear by Lemma 4.14.

$\Leftarrow$. Let L^a be the least $\mathcal{L}$-class in $C = \{L^x \mid x \in A\backslash L\}$. Then $a \notin L$. If L is not a-maximal strongly convex hyper S-subposet of A, then, by Zorn's lemma, there exists an a-maximal strongly convex hyper S-subposet K such that $L \subset K$. Let $b \in K\backslash L$. By hypothesis, $L(a) \subset L(b)$. Thus, $a \in L(a) \subseteq L(b) \subseteq K$ which contradicts to K is a-maximal. Therefore, L is an a-maximal strongly convex hyper S-subposet of A. □

Theorem 4.16. *Let A be a hyper S-poset over an ordered semihypergroup S. If $A \neq (S * A]$, then the a-maximal strongly convex hyper S-subposet of A is of the form $A\backslash[a)$ for all $a \in A\backslash(S * A]$.*

Proof. For $a \in A\backslash(S * A]$, we first show that $A\backslash[a)$ is a strongly convex hyper S-subposet of A. If $b \in A\backslash[a)$ and $s \in S$, then $s * b \subseteq A\backslash[a)$. In fact, if there exists $x \in s * b$ such that $x \notin A\backslash[a)$, then $a \leq_A x \in s * b \subseteq S * A$, that is, $a \in (S * A]$. This is impossible. Furthermore, if $x \in A\backslash[a)$ and $y \in A$ such that $y \leq_A x$, then $y \in A\backslash[a)$. Indeed, if $y \in [a)$, then $a \leq_A y \leq_A x$, and thus $x \in [a)$, which is a contradiction. Therefore, $A\backslash[a)$ is a strongly convex hyper S-subposet of A with $a \notin A\backslash[a)$. Moreover, if there exists a strongly convex hyper S-subposet K with $a \notin K$ such that $A\backslash[a) \subset K$, then there exists $b \in K$, $b \notin A\backslash[a)$. This implies that $a \leq_A b$ and so $a \in K$. Impossible. Consequently, $A\backslash[a)$ is a-maximal strongly convex hyper S-subposet of A. Also, a-maximal strongly convex hyper S-subposet of A is unique. Thus, the a-maximal strongly convex hyper S-subposet of A is of the form $A\backslash[a)$. □

By Theorem 4.16, we immediately obtain the following corollary:

Corollary 4.17. *Let A be a hyper S-poset over an ordered semihypergroup S. If A contains no maximal strongly convex hyper S-subposets, then A is a unitary hyper S-poset, that is, $A = (S * A]$.*

5 Hyper C-subposets of hyper S-posets

In this section, we define and study the hyper C-subposets of hyper S-posets. In particular, we discuss the relationship between greatest strongly convex hyper S-subposet and hyper C-subposets of hyper S-posets.

Definition 5.1. Let A be a hyper S-poset over an ordered semihypergroup S. A proper strongly convex hyper S-subposet L of A is called a *hyper C-subposets* of A if $L \subseteq (S * (A\backslash L)]$.

Example 5.2. Consider the hyper S-poset A given in Example 4.12. Let $L = \{b, d\}$. Then $(S * (A\backslash L)] = (S * a] = (\{b, d\}] = (L] = L$. It thus implies that L is a hyper C-subposet of A.

Theorem 5.3. *Let A be a hyper S-poset. If A contains two proper strongly convex hyper S-subposets L_1 and L_2 such that $A = L_1 \cup L_2$. Then L_1 and L_2 are not hyper C-subposets of A.*

Proof. Let $L_1 \cup L_2 = A$. Then $A\backslash L_2 \subseteq L_1$ and $A\backslash L_1 \subseteq L_2$. Hence, L_1, L_2 are not hyper C-subposets of A. Indeed, if L_1 is a hyper C-subposet of A, then

$$L_1 \subseteq (S * (A\backslash L_1)] \subseteq (S * L_2] \subseteq (L_2] = L_2.$$

Since $L_1 \cup L_2 = A$, we have $L_2 = A$, which is impossible. Thus, L_1 is not a hyper C-subposet of A. In a similar way, we can show that L_2 is also not a hyper C-subposet of A. □

Corollary 5.4. *Let A be a hyper S-poset. If A contains more than one maximal strongly convex hyper S-subposets of A, then all maximal strongly convex hyper S-subposets of A are not hyper C-subposet of A.*

Proof. Let M_1 and M_2 be two different maximal strongly convex hyper S-subposets of A. Then $M_1 \cup M_2 = A$. By Theorem 5.3, M_1, M_2 are not hyper C-subposets of A. □

Corollary 5.5. *Let A be a hyper S-poset. If A contains maximal strongly convex hyper S-subposet L and L is a hyper C-subposet of A, then L is the greatest proper strongly convex hyper S-subposet of A.*

Proof. Suppose that L is maximal, and let K be a strongly convex hyper S-subposet of A such that $K \nsubseteq L$. Then $L \subset L \cup K$, thus $L \cup K = A$, by Theorem 5.3, L is not a hyper C-subposet of A. □

Corollary 5.6. *Let A be a hyper S-poset. If A contains maximal strongly convex hyper S-subposet L and L is a hyper C-subposet of A, then $A = L(a)$ for all $a \in A\backslash L$.*

Proof. Assume that L is a maximal strongly convex hyper S-subposet of A. Then, by Theorem 4.6, $A = L \cup L(a)$ for all $a \in A\backslash L$. If $L(a) \neq A$, then L is not C-subposet of A by Theorem 5.3. □

Proposition 5.7. *Let A be a hyper S-poset and L a maximal strongly convex hyper S-subposet of A. If K is a hyper C-subposet of A, then $K \subseteq L$.*

Proof. By Lemma 2.2, $K \cup L$ is a strongly convex hyper S-subposet of A. Clearly, $L \subseteq K \cup L$. If $L = K \cup L$, then $K \subseteq L$. If $L \subset K \cup L$, since L is maximal, then $K \cup L = A$, and thus $A\backslash K \subseteq L$. Therefore, $K \subseteq (S * (A\backslash K)] \subseteq (S * L] \subseteq L$. □

Proposition 5.8. *Let A be a hyper S-poset. If L_1, L_2 are two hyper C-subposets of A, then $L_1 \cap L_2$ is a hyper C-subposet of A.*

Proof. From the relation $L_1 \subseteq (S * (A\backslash L_1)]$, we have

$$L_1 \cap L_2 \subseteq L_1 \subseteq (S * (A\backslash L_1)] \subseteq (S * (A\backslash (L_1 \cap L_2))].$$

Therefore, $L_1 \cap L_2$ is a hyper C-subposet of A. □

Proposition 5.9. *Let A be a hyper S-poset. If L_1, L_2 are two hyper C-subposets of A, then $L_1 \cup L_2$ is a hyper C-subposet of A.*

Proof. By hypothesis, we have

$$L_1 \subseteq (S * (A \backslash L_1)], L_2 \subseteq (S * (A \backslash L_2)].$$

Then $L_1 \cup L_2 \subseteq (S * (A\backslash(L_1 \cup L_2))]$. In fact, let $x \in L_1$. Then $L_1 \subseteq (S * (A\backslash L_1)]$ implies that there exists $a \in A\backslash L_1$ such that $x \in (S * a]$. There are two cases to be considered:
Case 1. If $a \in A\backslash(L_1 \cup L_2)$, then $x \in (S * (A\backslash(L_1 \cup L_2))]$.
Case 2. Let $a \in (A\backslash L_1) \cap L_2$. Then $a \in L_2 \subseteq (S * (A\backslash L_2)]$. Thus, there exists $b \in A\backslash L_2$ such that $a \in (S * b]$. We claim that b does not belong to L_1. Otherwise, we would have $a \in (S * b] \subseteq (S * L_1] \subseteq L_1$, which is a contradiction. Therefore, $b \in A\backslash L_1$ and $b \in A\backslash L_2$, which imply that

$$b \in (A\backslash L_1) \cap (A\backslash L_2) = A\backslash(L_1 \cup L_2).$$

Thus, we have

$$x \in (S * a] \subseteq (S * (S * b]] \subseteq (S * b] \subseteq (S * (A\backslash(L_1 \cup L_2))].$$

Hence, $L_1 \subseteq (S * (A\backslash(L_1 \cup L_2))]$. Similarly, it can be shown that $L_2 \subseteq (S * (A\backslash(L_1 \cup L_2))]$. Thus, we have $L_1 \cup L_2 \subseteq (S * (A\backslash(L_1 \cup L_2))]$. □

If we consider the empty set $\varnothing$ as a hyper C-subposet, then, by Propositions 5.8 and 5.9, we immediately obtain the following corollary:

Corollary 5.10. *Let A be a hyper S-poset. Then the set of all hyper C-subposets of A is a sublattice of the lattice of all strongly convex hyper S-subposets of A.*

In the following, we discuss the relationship between greatest strongly convex hyper S-subposet and hyper C-subposets of hyper S-posets.

Corollary 5.11. *Let A be a hyper S-poset containing only one maximal strongly convex hyper S-subposet L. If L is a hyper C-subposet of A, then L is the greatest strongly convex hyper S-subposet of A.*

Theorem 5.12. *Let A be a hyper S-poset over an ordered semihypergroup S. If A contains the greatest strongly convex hyper S-subposet L^*, then there exists $a \in A\backslash(S * A]$ such that $L^* \supseteq A\backslash[a)$ or L^* is a hyper C-subposet of A.*

Proof. Let L^* be the greatest strongly convex hyper S-subposet of A. Then $(S * (A\backslash L^*)] \subseteq L^*$ or $(S * (A\backslash L^*)] = A$. If L^* is not a hyper C-subposet of A, then $(S * (A\backslash L^*)] \subseteq L^*$, and we have

$$(S * A] = (S * (A\backslash L^*) \cup S * L^*] \subseteq ((S * (A\backslash L^*)] \cup (S * L^*]] \subseteq (L^* \cup L^*] = L^*.$$

Thus, $(S * A] \neq A$. Let $a \in A\backslash(S * A]$. By the proof of Theorem 4.16, $A\backslash[a)$ is a strongly convex hyper S-subposet of A. Since L^* is the greatest strongly convex hyper S-subposet of A, we have $L^* \supseteq A\backslash[a)$. □

In particular, we have the following corollary.

Corollary 5.13. *Let A be a hyper S-act over a semihypergroup S and L a maximal hyper S-subact of A. Then L is a hyper C-subact of A if and only if L is the greatest hyper S-subact of A.*

Theorem 5.14. *Let A be a hyper S-poset which contains proper hyper C-subposets. Then every proper strongly convex hyper S-subposet of A is a hyper C-subposet if and only if A satisfies just one of the following conditions:*

(1) *A contains the greatest strongly convex hyper S-subposet L^* and L^* is a hyper C-subposet of A.*
(2) *$A = (S * A]$, and for every proper strongly convex hyper S-subposet L, $a \in L$, there exists $b \in A \backslash L$ such that $L(a) \subset L(b)$.*

Proof. $\Rightarrow$. By hypothesis and Corollary 5.4, A contains at most one maximal strongly convex hyper S-subposet. Consequently, A contains at most one maximal $\mathcal{L}$-class by Theorem 4.10. The following two cases are considered:

Case 1. If A contains maximal $\mathcal{L}$-class L^a, then, by hypothesis, we have $A \backslash L^a \neq \varnothing$. By Theorem 4.10, $A \backslash L^a$ is a maximal strongly convex hyper S-subposet of A. By Corollary 5.5, it is a unique strongly convex hyper S-subposet of A, and thus $A \backslash L^a$ is the greatest strongly convex hyper S-subposet of A.

Case 2. Let A contain no maximal $\mathcal{L}$-class. Then, by Theorem 4.16, $A = (S * A]$. Now let L be a proper strongly convex hyper S-subposet of A, for any $a \in L$, we have $a \in L(a) \subseteq L \subseteq (S * (A \backslash L)]$. Thus, there exists $b \in A \backslash L$ such that $a \in (S * b] \subseteq L(b)$, and so $L(a) \subseteq L(b)$. If $L(a) = L(b)$, then $b \in L(a) \subseteq L$. Hence, $L(a) \subset L(b)$.

$\Leftarrow$. If A contains the greatest proper strongly convex hyper S-subposet L^* and L^* is a hyper C-subposet of A. Let L be a proper strongly convex hyper S-subposet. Then $L \subseteq L^* \subseteq (S * (A \backslash L^*)] \subseteq (S * (A \backslash L)]$, and thus L is a hyper C-subposet of A.

Let the condition (2) hold. If L is a proper strongly convex hyper S-subposet, $a \in L$, then there exists $b \in A \backslash L$ such that $L(a) \subset L(b)$. By $A = (S * A]$, there exists $d \in A$ such that $b \in (S * d]$, thus $L(a) \subset L(b) \subseteq (S * d]$ and $d \notin L$. In fact, if $d \in L$, then $b \in L(b) \subseteq (S * d] \subseteq L$. Impossible. Hence, $a \in L(a) \subseteq (S * d] \subseteq (S * (A \backslash L)]$, which implies that $L \subseteq (S * (A \backslash L)]$, that is, L is a hyper C-subposet of A. ☐

Definition 5.15. Let A be a hyper S-poset over an ordered semihypergroup S. A is called *C-simple* if it has no hyper C-subposets.

Theorem 5.16. *Let A be a hyper S-poset over an ordered semihypergroup S. Then A is C-simple if and only if A is the disjoint union of its minimal strongly convex hyper S-subposets.*

Proof. Let A contain no hyper C-subposets. Then for any $a \in A$, $L(a) = (S * a]$. In fact, if $a \notin (S * a]$, then $a \in A \backslash (S * a]$, and so $(S * a] \subseteq (S * (A \backslash (S * a])]$, therefore $(S * a]$ is a hyper C-subposet of A. Impossible. At the same time, $(S * a]$ is a minimal strongly convex hyper S-subposet of A. Otherwise, if there exists a strongly convex hyper S-subposet L of A such that $L \subset (S * a]$, then $a \notin L$, and so $L \subset (S * a] \subseteq (S * (A \backslash L)]$; therefore, L is a hyper C-subposet of A. Impossible. On the other hand, $A = \bigcup_{a \in A} L(a) = \bigcup_{a \in A} (S * a]$. Therefore, A is the union of its minimal strongly convex hyper S-subposets. Since the intersection of two strongly convex hyper S-subposets of A is also a strongly convex hyper S-subposet of A if the intersection is not empty, we have the union is disjoint.

Conversely, suppose that $A = \bigcup_{i \in I} L_i$, where L_i $(i \in I)$ are minimal strongly convex hyper S-subposets of A. Let $x, y \in L_i$. Then $L(x) \subseteq L_i$, $L(y) \subseteq L_i$. Since L_i is minimal, we have $L(x) = L_i = L(y)$. On the other hand, if $z \in A$ such that $L(z) = L(x)$, then $z \in L_i$. Hence, every L_i is just an $\mathcal{L}$-class. If $|I| = 1$, then A is simple; therefore, A is C-simple. Let $|I| > 1$ and L be a proper strongly convex hyper S-subposet of A. Then $L = \bigcup_{i \in J} L_i$ for some $J \subset I$. Hence, $A = L \cup \left(\bigcup_{i \in I \backslash J} L_i\right)$. By Theorem 5.3, L is not a hyper C-subposet of A. In other words, A is C-simple. ☐

In order to do further research on hyper C-subposets of hyper S-posets, we introduce the concept of bases of a hyper S-poset and give out the sufficient and necessary conditions of the existence of the greatest hyper C-subposets of a hyper S-poset in terms of bases.

Definition 5.17. Let A be a hyper S-poset over an ordered semihypergroup S and $\varnothing \neq B \subseteq A$. B is called a *base* of A if the following conditions are satisfied:
(1) $A = L(B)$.
(2) For $C \subseteq B$, if $A = L(C)$, then $C = B$.

In the following, we give an equivalent characterization of bases of hyper S-posets.

Theorem 5.18. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\varnothing \neq B \subseteq A$. Then B is a base of A if and only if the following conditions are satisfied:*
(1) *For any $a \in A$, there exists $b \in B$ such that $L(a) \subseteq L(b)$.*
(2) *For $b_1, b_2 \in B$, if $L^{b_1} \preccurlyeq L^{b_2}$, then $b_1 = b_2$.*

Proof. $\Rightarrow$ (1) Suppose that B is a base of A. Then $A = L(B) = (B \cup S * B]$, and thus for any $a \in A$, there exist $b_1, b_2 \in B$, $s \in S$ such that $a \leq_A b_1$ or $a \leq_A s * b_2$. If $a \leq_A b_1$, then, clearly, $L(a) \subseteq L(b_1)$. Let $a \leq_A s * b_2$. We claim that $L(a) \subseteq L(b_2)$. To prove our claim, let $x \in L(a) = (a \cup S * a]$. Then $x \leq_A a$ or $x \leq_A y * a$ for some $y \in S$. Thus, $x \leq_A a \leq_A s * b_2 \subseteq S * b_2$ or $x \leq_A y * a \leq_A y * (s * b_2) = (y \circ s) * b_2 \subseteq S * b_2$. It implies that $x \in (S * b_2] \subseteq L(b_2)$. Hence, $L(a) \subseteq L(b_2)$.

(2) Let $b_1, b_2 \in B$ be such that $L^{b_1} \preccurlyeq L^{b_2}$. Then $L(b_1) \subseteq L(b_2)$, and we have $b_1 \in L(b_2)$. Assume that $a_1 \neq b_2$. Let $C = B \backslash \{b_1\}$. Since $b_1 \in L(b_2)$, it can be easily shown that $L(B) \subseteq L(C)$. Also, since B is a base of A, we have $A = L(B)$, and thus $L(C) = A$. By hypothesis, $C = B$, which is impossible.

$\Leftarrow$ Assume that the conditions (1) and (2) hold. By (1), we have

$$A = \bigcup_{a \in A} L(a) \subseteq \bigcup_{b \in B} L(b) = L(B) \subseteq A.$$

It implies that $A = L(B)$. Furthermore, suppose that there exists $C \subset B$ such that $A = L(C) = (C \cup S * C]$. Let $b \in B \backslash C$. Then there exist $c_1, c_2 \in C$, $s \in S$ such that $b \leq_A c_1$ or $b \leq_A s * c_2$, and thus $L(b) \subseteq L(c_1)$ or $L(b) \subseteq L(c_2)$, i.e., $L^b \preccurlyeq L^{c_1}$ or $L^b \preccurlyeq L^{c_2}$. By (2), we have $b = c_1$ or $b = c_2$. This is impossible. □

Remark 5.19. If B is a base of a hyper S-poset A, then for any $b \in B$, there exists a maximal $\mathcal{L}$-class L such that $b \in L$, and there exists the unique one element of B in every maximal $\mathcal{L}$-class.

Theorem 5.20. *Let A be a hyper S-poset and not C-simple. If there exists a base of A, then there exists the greatest hyper C-subposet L_c^* of A and in this case, $L_c^* = (S * A] \cap \hat{L}$, where $\hat{L}$ is the intersection of all maximal strongly convex hyper S-subposet of A.*

Proof. Let B be a base of A. By Remark 5.19 and Theorem 4.10, $\{A \backslash L^b \mid b \in B\}$ is the set of all maximal strongly convex hyper S-subposets of A. By Proposition 5.7, all hyper C-subposets of A are contained in every maximal strongly convex hyper S-subposet of A. Since A is not C-simple, we have $\hat{L} \neq \varnothing$. Hence, $\hat{L} = \bigcap_{b \in B}(A \backslash L^b) = A \backslash (\bigcup_{b \in B} L^b)$. Put $L_c^* = (S * A] \cap \hat{L}$. By Lemma 2.2, L_c^* is a strongly convex hyper S-subposet of A. For any $x \in L_c^*$, there exists $c \in A$ such that $x \in (S * c]$. Since $A = (B \cup S * B]$, we have $c \leq a$ for some $a \in B$ or $c \in (S * B]$, and thus

$$x \in (S * c] \subseteq (S * B] \subseteq (S * \left(\bigcup_{b \in B} L^b\right)] \subseteq (S * (A \backslash \hat{L})] \subseteq (S * (A \backslash L_c^*)],$$

which implies that L_c^* is a hyper C-subposet of A.

Furthermore, let K be a hyper C-subposet of A. Then K is contained in every maximal strongly convex hyper S-subposet of A by Proposition 5.7, and thus $K \subseteq \hat{L}$. Also, $K \subseteq (S * (A \backslash K)] \subseteq (S * A]$. Hence, $K \subseteq (S * A] \cap \hat{L} = L_c^*$. Therefore, L_c^* is the greatest hyper C-subposet of A. □

For the converse of the aforementioned theorem, we have the following theorem.

Theorem 5.21. *Let A be a hyper S-poset which contains the greatest hyper C-subposet L_c^*. If $A \neq (S * A]$ and any two elements in $A \backslash (S * A]$ are incomparable, then there exists a base of A.*

Proof. Note that any strongly convex hyper S-subposet of a hyper S-poset A is a union of L-classes. If $(S * A] \backslash L_c^* \neq \varnothing$, then $(S * A] \backslash L_c^*$ is a union of L-classes. Taking a representative subset C of $\mathcal{L}$-classes in $(S * A] \backslash L_c^*$. Put $B = C \cup (A \backslash (S * A])$. It is clear that $B \neq \varnothing$. We show that B is a base of A. In fact,

(1) For any $x \in A$, if $x \notin L_c^*$, then there exists $b \in B$ such that $L(b) = L(x)$. Let $x \in L_c^*$. Then

$$x \in L_c^* \subseteq (S * (A \backslash L_c^*)] = (S * (A \backslash (S * A])] \cup (S * ((S * A] \backslash L_c^*)].$$

If $x \in (S * (A \backslash (S * A])]$, then there exists $b \in A \backslash (S * A]$ such that $x \in (S * b]$, $b \in B$. Thus, $L(x) \subseteq (S * b] \subseteq L(b)$. If $x \in (S * ((S * A] \backslash L_c^*)]$, then there exists $c \in (S * A] \backslash L_c^*$ such that $x \in (S * c]$ and $L(c) = L(a)$ for some $a \in B$. Thus, $L(x) \subseteq (S * c] \subseteq L(c) = L(a)$.

(2) Let $b, e \in B$ such that $L^b \preccurlyeq L^e$. Then $b = e$. Indeed, since either $b \in (S * A] \backslash L_c^*$ or $b \in A \backslash (S * A]$, we need to consider the following two cases:

Case 1. If $b \in (S * A] \backslash L_c^*$, then there exists $a \in A$ such that $b \in (S * a]$. If $a \notin L(b)$, then $L(b) \subseteq (S * a] \subseteq (S * (A \backslash L(b)]$, thus $L(b)$ is a hyper C-subposet of A. Since L_c^* is the greatest hyper C-subposet, we have $L(b) \subseteq L_c^*$ and consequently $b \in L_c^*$. Impossible. Thus, $a \in L(b) = (b \cup S * b]$, which means that $a \leq_A b$ or $a \leq_A s * b$ for some $s \in S$. Hence, $L(b) \subseteq (S * a] \subseteq (S * (S * b]] \subseteq (S * b] \subseteq L(b)$, that is, $L(b) = (S * b]$.

Now, let $L^b \preccurlyeq L^e$ for some $e \in B$, then $L(b) = (S * b] \subseteq L(e) = (e \cup S * e]$ and consequently $L(b) = (S * b] \subseteq (S * e]$. If $e \notin L(b)$, $L(b) \subseteq (S * e] \subseteq (S * (A \backslash L(b))]$, thus $L(b)$ is a hyper C-subposet of A. Since L_c^* is the greatest hyper C-subposet of A, we have $b \in L(b) \subseteq L_c^*$. Impossible. Hence, $e \in L(b)$, and so $L^b = L^e$. On the other hand, $e \in L(b) = (S * b] \subseteq (S * A]$, thus $e \in (S * A] \backslash L_c^*$. Therefore, $b = e$.

Case 2. Let $b \in A \backslash (S * A]$. If there exists $e \in B$ such that $L^b \preccurlyeq L^e$, then $b \in L(b) \subseteq L(e) = (e \cup S * e]$. Since $b \notin (S * A]$, we have $b \leq_A e$. If $e \in (S * A]$, then $b \in L(b) \subseteq L(e) \subseteq (S * A]$, which contradicts to $b \in A \backslash (S * A]$. Hence, $e \in A \backslash (S * A]$, $b \in A \backslash (S * A]$ and $b \leq_A e$. Since any two elements in $A \backslash (S * A]$ are incomparable, it can be easily shown that $b = e$.

Therefore, by Theorem 5.18, B is a base of A. □

By Theorems 5.20 and 5.21, we immediately obtain the following corollary:

Corollary 5.22. *Let A be a hyper S-act over a semihypergroup S and not C-simple. Then A contains the greatest hyper C-subposet if and only if there exists a base of A.*

6 Conclusions

In this paper, a new link between algebraic hyperstructures and S-posets was initiated and as a result various strongly convex hyper S-subposets of hyper S-posets were defined and investigated in detail. In particular, the properties of maximal strongly convex hyper S-subposets and hyper C-subposets of hyper S-posets are discussed. We hope that the foundations that we made through this paper would offer foundation for further study of the hyper S-poset theory and can be used to get an insight into other types of hyperstructures.

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