Research Article

Wissem Jedidi*, Kholoud Basalim, and Safa Bridaa

Three classes of decomposable distributions

https://doi.org/10.1515/math-2020-0124
received October 20, 2020; accepted December 7, 2020

Abstract: In this work, we refine the results of Sendov and Shan [*New representation theorems for completely monotone and Bernstein functions with convexity properties on their measures*, J. Theor. Probab. **28** (2015), 1689–1725] on subordinators obtained by the class of Bernstein functions stable by the Mellin-Euler differential operator $I - x\frac{\mathrm{d}}{\mathrm{d}x}$ by giving a stochastic interpretation, proving monotonicity properties of the related distributions and providing additional extensions.

Keywords: Bernstein functions, complete monotonicity, concavity, convexity, infinite divisibility, Laplace transform, Mellin-Euler differential operator, stable distribution, subordinators

MSC 2020: 60E07, 60E15

1 Introduction

Recall that the class $\mathcal{CM}$ of *completely monotone functions* corresponds to those infinitely differentiable functions $f : (0, \infty) \to (0, \infty)$ s.t. $(-1)^n f^{(n)} \geq 0$. By Bernstein characterisation $f \in \mathcal{CM}$, if, and only if, f is the Laplace transform of some measure:

$$f(\lambda) = \int\limits_{[0,\infty)} e^{-\lambda x}\nu(\mathrm{d}x), \quad \lambda > 0.$$

The class $\mathcal{BF}$ of *Bernstein functions* corresponds to non-negative antiderivatives of completely monotone function, i.e. to those functions ϕ represented by:

$$\phi(\lambda) \coloneqq q + \mathrm{d}\lambda + \int\limits_{(0,\infty)} (1 - e^{-\lambda x})\mu(\mathrm{d}x), \quad \lambda \geq 0, \tag{1}$$

where $q \geq 0$ is called the *killing rate*, $d \geq 0$ is called the *drift term* and μ is a positive measure on $(0, \infty)$ which integrates $x \wedge 1$, called the *Lévy measure*. By integration by parts, ϕ is equivalently represented by

$$\phi(\lambda) = q + \mathrm{d}\lambda + \lambda \int\limits_0^\infty e^{-x\lambda} l(x)\,\mathrm{d}x = q + \mathrm{d}\lambda + \lambda^2 \int\limits_0^\infty e^{-u\lambda} k(u)\,\mathrm{d}u, \tag{2}$$

where we have used the notation l for the nonincreasing function and k for the concave function given by

$$l(x) \coloneqq \mu(x, \infty), \quad \text{we sometimes also use the notation } \bar{\mu}(x) \coloneqq \mu(x, \infty) \tag{3}$$

*** Corresponding author: Wissem Jedidi,** King Saud University, Department of Statistics & OR, College of Sciences, Riyadh 11451, Kingdom of Saudi Arabia, e-mail: wjedidi@ksu.edu.sa
Kholoud Basalim: King Saud University, Department of Statistics & OR, College of Sciences, Riyadh 11451, Kingdom of Saudi Arabia, e-mail: kbasalim@ksu.edu.sa
Safa Bridaa: Université de Tunis El Manar, Faculté des Sciences de Tunis, Laboratoire d'Analyse Mathématiques et Applications LR11ES11, 2092, Tunis, Tunisia, e-mail: bridaasafa@gmail.com

$$k(u) := \int_0^u l(x)\,\mathrm{d}x. \tag{4}$$

In this work, we refine and extend the results obtained by Sendov and Shan [1] on a class of Bernstein functions that we denote here by $\mathcal{BF}_\Theta$, and which is defined by

$$\mathcal{BF}_\Theta = \{\varphi \in \mathcal{BF}, \ \text{ s.t. } \ \phi(\lambda) - \lambda\phi'(\lambda) \in \mathcal{BF}, \ \text{ for every } \ c \in (0,1)\}.$$

Our contribution in this work consists in the following:

(a) The class $\mathcal{BF}_\Theta$ is demystified by proving the existence of densities enjoying a monotony property for the Lévy measures and for the transition of the associated subordinators, if we restrict ourselves in the subclass of the stabilisators of $\mathcal{BF}_\Theta$. We provide a full characterisation by the stochastic point of view. More precisely, a positive and infinitely divisible random variable X has its Bernstein function $\phi \in \mathcal{BF}_\Theta$, if, and only if, X is embedded into a subordinator $(X_t)_{t\geq 0}$, i.e. $X = X_1$, and for all $c \in (0,1)$, there exists a random variable Z_c such that we have the identity in law

$$cX \overset{d}{=} X_c + Z_c \quad (X_c \text{ and } Z_c \text{ independent}). \tag{5}$$

This decomposability property is linked to the operators θ_c defined by

$$\theta_c\varphi(\lambda) = \varphi(c\lambda) - c\varphi(\lambda), \quad c \in (0,1),$$

Operating on the class $\mathcal{BF}$ and is new in the literature. We denote by **RD** the class of distributions satisfying (5) and call it the class *reverse self-decomposable* distributions. Observe that this class is very close (in its formalism) to the famous class **SD** of *self-decomposable* ones, formed by those distributions associated with random variables X = satisfying the following: for every $c \in (0,1)$, there exists a r.v. Y_c such that we have the identity in law

$$X \overset{d}{=} cX + Y_c \quad (X \text{ and } Y_c \text{ independent}).$$

(b) Observe that rare are the cases in which we are able to deduce the shape of the distribution $\mathbb{P}(X_t \in \mathrm{d}x)$ for arbitrary subordinators $(X_t)_{t\geq 0}$. The class $\mathcal{BF}^1_\Theta$, introduced by Sendov and Shan [1], consists of those of Bernstein functions ϕ such that $1 - e^{-t\phi} \in \mathcal{BF}_\Theta$ for all $t > 0$. In this paper, we show that $\mathcal{BF}^1_\Theta$ is in bijection with the class of subordinators $(X_t)_{t\geq 0}$, such that for every $t > 0$, the distribution of X_t is of the form

$$\mathbb{P}(X_t \in \mathrm{d}x) = \frac{p_t(x)}{x^2}, \quad x > 0, \quad \text{where } \ p_t \ \text{ is nondecreasing.}$$

As a consequence, we provide a satisfying answer to the open problem raised by Sendov and Shan [1, Open Problem 4.1] in case where $(X_t)_{t\geq 0}$ is an α-stable subordinator: what is the critical value $\alpha_0 \in (2/3,1)$ such that $\lambda \mapsto \lambda^\alpha \in \mathcal{BF}^1_\Theta$ for all $\alpha \in (0,\alpha_0]$?

(c) We also introduce the class $\mathcal{BF}^2_\Theta$ of Bernstein functions ϕ such that $e^{-t\phi} - 1 + t\phi \in \mathcal{BF}_\Theta$ for all $t > 0$, and provide monotonicity properties for the finite-dimensional distributions of the associated subordinators. Other results of stochastic nature are also deduced.

(d) Finally, we generalise the class of reverse self-decomposable distributions via the operator υ_c defined by

$$\upsilon_c\varphi(\lambda) = \varphi(c\lambda) + \varphi((1-c)\lambda) - \varphi(\lambda), \quad c \in (0,1).$$

2 Some operators of interest

We need to introduce some formalism in order to clarify the characterisation of three classes of distributions that will be studied later on.

Definition 2.1. For differentiable functions $\psi : (0, \infty) \to \mathbb{R}$, the Mellin (or Euler) differential operator Ξ and its companion Θ are defined by

$$\Xi\psi(\lambda) = \lambda\psi'(\lambda) \text{ and } \Theta = I - \Xi \ (I = \text{Identity}).$$

For $c \in (0, 1)$, the difference operators ξ_c, θ_c, υ_c are defined by

$$\begin{aligned} \xi_c\psi(\lambda) &= \psi(\lambda) - \psi(c\lambda), \\ \theta_c\psi(\lambda) &= \psi(c\lambda) - c\psi(\lambda), \\ \upsilon_c\psi(\lambda) &= \psi(c\lambda) + \psi((1-c)\lambda) - \psi(\lambda). \end{aligned}$$

For a set $\mathcal{A}$ of functions $\phi : (0, \infty) \to \mathbb{R}$, we denote by $\mathcal{A}_\Delta$ and $\mathcal{A}_\delta$ the subsets

$$\mathcal{A}_\Delta = \{\phi \in \mathcal{A} \text{ such that } \Delta\phi \in \mathcal{A}\}, \quad \text{if } \Delta = \Xi, \Theta, \tag{6}$$

$$\mathcal{A}_\delta = \bigcap_{c\in(0,1)} \{\phi \in \mathcal{A} \text{ such that } \delta_c\phi \in \mathcal{A}\}, \quad \text{if } \delta_c = \xi_c, \theta_c, \upsilon_c. \tag{7}$$

Remark 2.2. We have the remarkable facts:

(i) The difference operators are obtained from integrals of the differential operators:

$$\xi_c\psi(\lambda) = \int_c^1 \frac{\Xi\psi(\lambda x)}{x^2}\,\mathrm{d}x \quad \text{and} \quad \theta_c\psi(\lambda) = \int_c^1 \frac{\Theta\psi(\lambda x)}{x^2}\,\mathrm{d}x. \tag{8}$$

(ii) The differential operators are obtained from limits of the difference operators:

$$\Xi\psi(\lambda) = \lim_{c\to 1-} \frac{1 - e^{-\xi_c\psi(\lambda)}}{1-c} \quad \text{and} \quad \Theta\psi(\lambda) = \lim_{c\to 1-} \frac{1 - e^{-\theta_c\psi(\lambda)}}{1-c}. \tag{9}$$

(iii) Note that

$$\psi(0+) := \lim_{u\to 0+} \psi(u) < \infty \iff \frac{\Xi\psi(x)}{x} \text{ is integrable at } 0$$

and

$$\lim_{u\to\infty} \frac{\psi(u)}{u} < \infty \iff \frac{\Theta\psi(x)}{x^2} \text{ is integrable at } \infty.$$

Then, under the last integrability conditions, we have the inversion formulae:

$$\psi(\lambda) = \psi(0+) + \int_0^\lambda \frac{\Xi\psi(x)}{x}\,\mathrm{d}x, \tag{10}$$

$$\psi(\lambda) = \lambda\left(\lim_{u\to\infty} \frac{\psi(u)}{u} + \int_\lambda^\infty \frac{\Theta\psi(x)}{x^2}\,\mathrm{d}x\right). \tag{11}$$

3 Some tools on Bernstein function and infinite divisibility

Next result on the stability and closure properties will be used several times in the sequel:

Proposition 3.1. [2, Corollaries 1.6 and 3.9]

1. ***Stability properties***: *If $f \in \mathcal{CM}$ and $\phi, \varphi \in \mathcal{BF}$, then $f \circ \phi \in \mathcal{CM}$ and $\phi \circ \varphi \in \mathcal{BF}$.*
2. ***Closure by pointwise limits***: *If $(f_n)_n \subset \mathcal{CM}$ (resp. $(\phi_n)_n \subset \mathcal{BF}$) and if $\lim_{n\to\infty} f_n(\lambda) = f(\lambda)$ (resp. $\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda)$) exists for every $\lambda > 0$, then $f \in \mathcal{CM}$ (resp. $\phi \in \mathcal{BF}$).*

See [2] for more details on completely monotone and Bernstein functions. The following two propositions are easy to obtain but useful, and they will be used several times in the sequel:

Proposition 3.2. *Let* $f : (0,\infty) \to (0,\infty)$. *Then*
1. *If* f *is concave, then it is nondecreasing.*
2. f *is concave if, and only if,* $x \mapsto xf(1/x)$ *is concave.*

The second statement is Lemma 2.2 in [3]. We were not able to find a reference for the first statement, which is probably known in the literature, and we propose this simple proof.

Proof. Since f has necessarily nonincreasing slopes:

$$\frac{f(y)-f(x)}{y-x} \geq \frac{f(z)-f(x)}{z-x}, \quad 0 \leq x < y < z,$$

and since $f(z) > 0$, we obtain

$$f(y) \geq f(x) + \frac{y-x}{z-x}(f(z)-f(x)) > \frac{z-y}{z-x} f(x),$$

for fixed x, y and arbitrary big values of z. It is suffice to let $z \to \infty$. □

Proposition 3.3. *Let* $f : (0,\infty) \to (0,\infty)$ *and* $\phi : [0,\infty) \to (0,\infty)$. *Then the following holds true.*
(1) $f \in \mathcal{CM} \Leftrightarrow f(c) - f(.+c) \in \mathcal{BF}$, *for every* $c > 0$;
(2) $\Theta f \in \mathcal{CM} \Leftrightarrow \Theta(f(c) - f(.+c)) \in \mathcal{BF}$, *for every* $c > 0$;
(3) $\Theta f \in \mathcal{CM} \Rightarrow f \in \mathcal{CM}$ and $\lambda \mapsto f(\lambda) = \int_1^\infty \Theta f(\lambda x)\frac{dx}{x^2}$;
(4) $\phi \in \mathcal{BF} \Rightarrow \lim_{\lambda\to 0+} \Xi\phi(\lambda) = 0$;
(5) $\Xi\phi \in \mathcal{BF} \Rightarrow \phi \in \mathcal{BF}$.

Proof.
(1) The equivalence is obtained by differentiation and by the closure property in Proposition 3.1.
(2) Elementary computation gives that

$$\Theta(x \mapsto (f(c) - f(x+c)))(\lambda) = (\Theta f)(c) - (\Theta f)(\lambda + c), \quad \lambda \geq 0.$$

Then, using (1), we have the equivalences:

$$\begin{aligned}(\Theta f)(c) - (\Theta f)(.+c) \in \mathcal{BF}, \quad \forall c > 0 &\Leftrightarrow (\Theta f)(.+c) \in \mathcal{CM}, \quad \forall c > 0 \\ &\Leftrightarrow \Theta f \in \mathcal{CM}.\end{aligned}$$

(3) The implication is justified by the facts that $\mathcal{CM}$ is a convex cone, and essentially by (11): since Θf is a nonincreasing function, then $\Theta f(x)/x^2$ is integrable at infinity and necessarily $l = \lim_{+\infty} f(x)/x$ exists. To check that $l = 0$, observe that f is necessarily often differentiable (like Θf) and $-xf'' = (\Theta f)'$. Assuming $\lim_{+\infty} f(x)/x = l \in (0,\infty)$, we would have $\lim_{+\infty} f'(x) = \lim_{+\infty} f(x)/x - \lim_{+\infty}\Theta f(x)/x = l$ (since $\Theta f \in \mathcal{CM}$, then $\lim_{+\infty}\Theta f(x)/x = 0$). Then, we would also have $\lim_{+\infty} xf''(x) = -\lim_{+\infty}(\Theta f)'(x) = 0$. The latter gives a contradiction between the behaviour of f' and f''.
(4) Assume ϕ is represented by (1). For every $x > 0$, $\lambda \in (0,1)$, we have $\lambda x e^{-\lambda x} \leq 1 \wedge (\lambda x) \leq 1 \wedge x$, and the dominated convergence theorem ensures that

$$\lim_{\lambda\to 0+} \lambda\phi'(\lambda) = \lim_{\lambda\to 0+}\left[\mathrm{d}\lambda + \int_{(0,\infty)} \lambda x e^{-\lambda x}\,\mu(\mathrm{d}x)\right] = 0.$$

(5) Certainly, $\lambda \mapsto \phi'(\lambda) = \Xi\phi(\lambda)/\lambda \in \mathcal{CM}$. Since $\phi \geq 0$, then $\phi \in \mathcal{BF}$. □

The injectivity of the Laplace transform ensures that the distribution of a positive random variable X is entirely characterised by its Laplace transform $\mathbb{E}[e^{-\lambda X}] = \int_{[0,\infty)} e^{-\lambda x}\,\mathbb{P}(X \in \mathrm{d}x)$, $\lambda \geq 0$. The class of cumulant functions is denoted by $\mathcal{CF}$ and defined by

$$\mathcal{CF} := \{\phi_X(\lambda) = -\log \mathbb{E}[e^{-\lambda X}], X \geq 0\}.$$

Remark 3.4. By point (1) of Proposition 3.3, we have that

$$\phi \in \mathcal{CF} \Leftrightarrow \phi(0) = 0 \quad \text{and} \quad 1 - e^{-\phi} \in \mathcal{BF}.$$

Using Proposition 3.1, it is then evident that
(i) $\phi \in \mathcal{BF} \Leftrightarrow 1 - e^{-t\phi} \in \mathcal{BF}, \quad \forall t > 0$;
(ii) $t\phi \in \mathcal{CF}, \forall t > 0 \Leftrightarrow \phi(0) = 0$ and $1 - e^{-t\phi} \in \mathcal{BF}, \quad \forall t > 0$.

From now on, we adopt the notation

$$X \sim \mathbf{C}, \quad \text{if the distribution of } X \text{ belongs to the class of distributions } \mathbf{C}.$$

Recall that a random variable X has an infinitely divisible distribution ($X \sim \mathbf{ID}$) if for every $n \in \mathbb{N}^*$, there exists sequence of independent and identically distributed random variables $X_{n,1}, \ldots, X_{n,n}$ (called the nth folds of X), such that

$$X \overset{d}{=} X_{n,1} + \cdots + X_{n,n},$$

c.f. [4]. Actually, the set infinitely divisible distributions on $[0,\infty)$ are in bijection with the class of Bernstein functions null at 0, namely, infinite divisibility of a non-negative r.v. X is entirely characterised by the fact that its cumulant ϕ is a Bernstein function and every non-negative infinitely divisible r.v. X is embedded into a subordinator $(X_t)_{t\geq 0}$ (i.e. an increasing Lévy process, see [5]), this means that

$$X \overset{d}{=} X_1 \quad \text{and} \quad \mathbb{E}[e^{-\lambda X_t}] = e^{-t\phi(\lambda)}, \quad \lambda \geq 0.$$

4 Revisiting the class of self-decomposable distributions

A proper subclass of **ID** is the well-known class **SD** of self-decomposable distributions, also known as the Lévy class, and introduced by Lévy in 1937. In the specialised literature, the notation $\mathcal{L}$ is frequent for the class **SD** (see [6] for instance). This class forms a natural extension of the class of stable laws and its importance stems from the fact that it arises in limit theorems for sums of independent variables. Without using the fact that **SD** $\subset$ **ID** (frequently required in the literature), self-decomposability could be defined as follows:

Definition 4.1. (Sato [7, Definition 15.1]) We say that X is self-decomposable, and we denote $X \sim$ **SD**, if for each $c \in (0, 1)$, there exists a non-negative random variable Y_c, independent of X, such that we have the identity in distribution

$$X \overset{d}{=} cX + Y_c. \tag{12}$$

A reformulation of (12) property in terms of cumulant functions is as follows:

$$X \sim \mathbf{SD} \Leftrightarrow \lambda \mapsto \frac{\mathbb{E}[e^{-\lambda X}]}{\mathbb{E}[e^{-\lambda cX}]} = e^{-\xi_c \phi_X(\lambda)} \in \mathcal{CM}, \quad \forall c \in (0, 1) \Leftrightarrow \phi_X \in \mathcal{CF}_\xi, \tag{13}$$

where we used the notations (6) and (7). For more account on self-decomposability, we suggest [4] and [8].

For the sake of clarity, we provide a full characterisation of the class **SD**. With the notations (6) and (7), we have:

Theorem 4.2. *Let X be a non-negative r.v. with cumulant function ϕ. We have the equivalences*
(1) $X \in$ **SD**;
(2) $\xi_{c_n}\phi \in \mathcal{CF}$, *for some sequence* $c_n \in (0,1)$ *such that* $c_n \to 1$;
(3) $\Xi\phi \in \mathcal{BF}$;
(4) $\xi_c\phi \in \mathcal{BF}$, *i.e.* $Y_c \sim$ **ID** *in* (12), *for every* $c \in (0,1)$;
(5) $\phi \in \mathcal{BF}$ *and its Lévy measure is of the form* $x^{-1}l(x)\,\mathrm{d}x$, *where l is nondecreasing.*

As an immediate consequence we get the following.

Corollary 4.3. *We have*

$$\mathcal{CF}_\xi = \{\phi \in \mathcal{BF}_\xi,\ \ s.t.\ \phi(0) = 0\} \ \text{and}\ \mathcal{BF}_\xi = \mathcal{BF}_\Xi \subset \mathcal{BF}.$$

Proof of Theorem 4.2.
(1) $\Rightarrow$ (2): Just apply (13).
(2) $\Rightarrow$ (3): Since $e^{-\xi_{c_n}\phi} \in \mathcal{CM}$, then, by Proposition 3.3(1), obtain $(1 - e^{-\xi_{c_n}\phi})/(1 - c_n) \in \mathcal{BF}$. Then, use Proposition 3.1 together with (9) and obtain that

$$\Xi\phi = \lim_{n\to\infty} \frac{1 - e^{-\xi_{c_n}\phi}}{1 - c_n} \in \mathcal{BF}.$$

(3) $\Leftrightarrow$ (4): Use (8) and the fact that $\mathcal{BF}$ is a convex cone. For the converse, use (9).
(3) $\Rightarrow$ (5): Observe that there is no drift in $\Xi\phi$, then use representation (2) for $\Xi\phi$ and (1) for ϕ and obtain that ϕ' is represented by both these expressions

$$\phi'(\lambda) = \frac{\Xi\phi(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda x} l(x)\,\mathrm{d}x \quad \text{and} \quad \phi'(\lambda) = \int_{(0,\infty)} e^{-\lambda x} x\mu(\mathrm{d}x), \quad \lambda > 0,$$

for some decreasing function l. Then conclude by Laplace inversion.
(5) $\Rightarrow$ (1): Use twice (2), make a change of variable and get the following representation that meets (1): for every $c \in (0,1)$,

$$\xi_c\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) \frac{l(x) - l(x/c)}{x}\,\mathrm{d}x \in \mathcal{BF} \subset \mathcal{CF},$$

and the latter is (13). □

5 The class of reverse-decomposable distributions

We propose in this section a new class of infinitely divisible distributions denoted by **RD**, which is the dual in some sense of the class **SD**. We shall provide the counterpart of Theorem 4.2 for this class.

Definition 5.1. (Reverse decomposable distributions)
(i) Recall

$$\mathcal{CF}_\theta := \bigcap_{c\in(0,1)} \mathcal{CF}_{\theta_c} = \{\phi \in \mathcal{CF},\ \ \text{s.t.}\ \ \theta_c\phi \in \mathcal{CF},\ \ \text{for every } c \in (0,1)\}.$$

A non-negative **ID** random variable X, with cumulant function ϕ_X, is said "reverse-decomposable," and we denote $X \sim$ **RD** if $\phi_X \in \mathcal{CF}_\theta$, or equivalently if

$$\lambda \mapsto \frac{\mathbb{E}[e^{-\lambda cX}]}{\mathbb{E}[e^{-\lambda X}]^c} = e^{\theta_c(\phi_X)(\lambda)} \text{ is completely monotone for all } c \in (0,1).$$

(ii) We denote by

$$\mathcal{BF}_\theta := \bigcap_{c\in(0,1)} \mathcal{BF}_{\theta_c} = \{\phi \in \mathcal{BF}, \text{ s.t. } \theta_c\phi \in \mathcal{BF}, \text{ for every } c \in (0,1)\}$$

and by

$$\mathcal{BF}_\Theta := \{\phi \in \mathcal{BF}, \text{ s.t. } \Theta\phi \in \mathcal{BF}\}.$$

In order to clarify the structure of **RD**, we need several analytic results.

5.1 Analytic properties of the classes $\mathcal{CF}_\theta$ and $\mathcal{BF}_\Theta$

We start with the following two propositions.

Proposition 5.2. *If $c \in (0,1)$ and if $\phi \in \mathcal{BF}$, then the functions $\Theta\phi$ and $\theta_c\phi$ are non-negative, nondecreasing and both functions $x \mapsto \theta_c\phi(x)/x^2$, $\Theta\phi(x)/x^2$ are integrable at ∞.*

Proof. Note that the derivative of $(\Theta\phi)' = -\lambda\phi''$ is non-negative and, by Proposition 3.3(4), we have $\lim_{\lambda\to 0+}\Theta\phi(\lambda) = \phi(0) \geq 0$. We deduce that $\Theta\phi$ is non-negative and nondecreasing. Furthermore, using inversion formula (11), we get the integrability condition for $\Theta\phi$. The assertion on $\theta_c\phi$ is obtained by the representation (8). □

Proposition 5.3. *Let $\phi : [0,\infty) \to [0,\infty)$. The following holds:*
(1) If $\Theta\phi \in \mathcal{BF}$ and has no drift term, then $\phi \in \mathcal{BF}$;
(2) If $\theta_c\phi \in \mathcal{BF}$ for some $c \in (0,1)$, then $\phi \in \mathcal{BF}$.
(3) If ϕ is nondecreasing and differentiable function on $(0,\infty)$ and if $\theta_{c_n}\phi \in \mathcal{CF}$ for some sequence $c_n \in (0,1)$ such that $c_n \to 1$, then $\Theta\phi \in \mathcal{BF}$.

Proof.
(1) Since there is no drift term in $\Theta\phi$, then $x \mapsto \Theta\phi(x)/x^2$ is integrable at ∞, and representation (2) for $\Theta\phi$ gives

$$\frac{\Theta\phi(x)}{x^2} = \int_0^\infty e^{-xu}\left(\int_0^x l_0(t)\,dt\right)du, \quad x > 0,$$

with some nonincreasing function l_0. The inversion formula (11) gives a representation of type (2) for ϕ:

$$\phi(\lambda) = \lambda\left(\lim_{u\to\infty}\frac{\phi(u)}{u} + \int_0^\infty e^{-\lambda x} l(x)\right), \quad l(x) := \frac{1}{x}\int_0^x l_0(t)\,dt, \ \lambda \geq 0,$$

and note that $x \mapsto l(x) = \int_0^1 l_0(xt)\,dt$ is nonincreasing. Thus, ϕ meets the representation (2) of a Bernstein function.

(2) For every integer $n \geq 1$ and real number $\lambda \geq 0$, we have

$$\theta_{c^n}\phi(\lambda) = \theta_c\phi(c^{n-1}\lambda) + c\theta_{c^{n-1}}\phi(\lambda).$$

By induction, we see that for every $n \in \mathbb{N}$, the function $\theta_{c^n}\phi$ belongs to $\mathcal{BF}$, or equivalently

$$\lambda \mapsto \phi(\lambda) - c^n\phi\left(\frac{\lambda}{c^n}\right) \in \mathcal{BF}.$$

The next step is to prove that $\varphi(\lambda) = \lim_{n\to\infty} c^n \phi(\lambda/c^n)$ is of the form

$$\varphi(\lambda) = K\lambda, \quad \text{for every } \lambda \geq 0 \text{ and some } K \geq 0. \tag{14}$$

Using Proposition 3.1 and taking the limit as $n \to \infty$, the latter will give that $\lambda \mapsto \phi(\lambda) - K\lambda \in \mathcal{BF}$ and hence, $\phi \in \mathcal{BF}$. In order to prove the claim (14), note that for every fixed $\lambda \geq 0$, the increments of sequence $u_n = \phi(\lambda) - c^n\phi(\lambda/c^n)$ are of the form

$$u_{n+1} - u_n = c^n \theta_c \phi\left(\frac{\lambda}{c^{n+1}}\right) \geq 0.$$

The sequence u_n, being nondecreasing and bounded by $\phi(\lambda)$, is convergent. Then, the function φ is well defined on $[0, \infty)$ and satisfies

$$\varphi(0) = 0, \quad \varphi(c^m\lambda) = c^m\varphi(\lambda), \quad \text{for every } \lambda \geq 0, m \in \mathbb{N}. \tag{15}$$

Every $\lambda \in (0, 1]$ could be written in the form $\lambda = c^x$, $x > 0$, and plugging in (15) with $m = 1$, we see that the function $f(x) = \phi(c^x)$ satisfies the iterative functional equation

$$f(x+1) = cf(x), \quad x > 0.$$

By [9, Theorem 2.4.2, p. 72], the unique solution f is necessarily of the form Kc^x, with $K \geq 0$. At this stage, we have proved that

$$f(\log\lambda) = \varphi(\lambda) = K\lambda, \quad \text{for every } \lambda \in (0, 1] \text{ and some } K \geq 0. \tag{16}$$

If $\lambda > 1$, there exits $p_0 \in \mathbb{N}$ such that $c^{p_0}\lambda \leq 1$. By (16), and by (15) with $m = p_0$, we obtain

$$Kc^{p_0}\lambda = \varphi(c^{p_0}\lambda) = c^{p_0}\varphi(\lambda).$$

Simplifying the last identity, we get (14).

(3) As in the proof of Theorem 4.2, use (9) to get that $(1 - e^{-\theta_{c_n}\phi})/(1 - c_n)$ is a sequence of Bernstein functions, then apply Proposition 3.1 and retrieve

$$\Theta\phi = \lim_{n\to\infty} \frac{1 - e^{-\theta_{c_n}\phi}}{1 - c_n} \in \mathcal{BF}. \qquad \square$$

Remark 5.4. Recall the classes $\mathcal{CF}_\theta$ and $\mathcal{BF}_\Theta$ of Definition 7.2. Proposition 5.3 shows that actually

$$\mathcal{CF}_\theta = \bigcap_{c\in(0,1)} \{\phi \in \mathcal{BF}, \ \text{s.t. } \phi(0) = 0 \text{ and } \theta_c\phi \in \mathcal{CF}\} \text{ and } \mathcal{CF}_\theta \subset \mathcal{BF}_\Theta.$$

5.2 Improving the results of Sendov and Shan [1], full characterisation of $\mathcal{CF}_\theta$ and $\mathcal{BF}_\Theta$

The right tail function $\bar{\mu}$ of Lévy measure μ and its reflected function $\overleftarrow{\mu}$ are denoted by

$$\bar{\mu} := \mu(x, \infty), \quad \overleftarrow{\mu}(x) := \mu\left(\frac{1}{x}, \infty\right), \quad x > 0. \tag{17}$$

Sendov and Shan [1] showed that

$$\Theta\phi \in \mathcal{BF} \text{ if, and only if, } x \mapsto x\bar{\mu}(x) \text{ is a concave function,}$$

and called such measures μ *measures with harmonically concave tail*. We complete their work and provide the complete characterisation of $\mathcal{BF}_\Theta$ as follows:

Theorem 5.5. *Let ϕ be a Bernstein function represented by* (1), *associated with the Lévy measure μ, and the functions $\overline{\mu}$, $\overleftarrow{\mu}$ be given by* (17). *Then, we have the equivalence between the following assertions.*

(1) *$x\overline{\mu}(x)$ is concave;*
(2) *$\overleftarrow{\mu}(x)$ is concave;*
(3) *μ has a density function in the form*

$$\mu(\mathrm{d}x) = \frac{p(x)}{x^2}\,\mathrm{d}x, \quad \textit{where } p \textit{ is nondecreasing};$$

(4) *$\theta_c\overleftarrow{\mu}$ is positive and nondecreasing for every $c \in (0,1)$;*
(5) *$\theta_c\phi \in \mathcal{BF}$, for every $c \in (0,1)$;*
(6) *$\theta_{c_n}\phi \in \mathcal{CF}$, for some sequence $c_n \in (0,1)$ such that $c_n \to 1$ as $n \to \infty$;*
(7) *$\phi \in \mathcal{BF}_\Theta$;*
(8) *there exists $\varphi \in \mathcal{BF}$ (with drift term equal to 0), s.t. $\phi(\lambda) = \lambda\int_\lambda^\infty \frac{\varphi(x)}{x^2}\,\mathrm{d}x$.*

Under any of the latter, we have the representation

$$\Theta\phi(\lambda) = \phi(0) + \int\limits_{(0,\infty)} (1 - e^{-\lambda x})\frac{\mathrm{d}p(x)}{x}. \tag{18}$$

Note that if $\phi \in \mathcal{BF}_\Theta$, then certainly $\phi'(0) = +\infty$. Indeed, by point (3) of Theorem 5.5, and because p is nondecreasing, one has

$$\phi'(0) \geq \int\limits_{(0,\infty)} x\mu(\mathrm{d}x) \geq \int\limits_0^\infty \frac{p(x)}{x}\,\mathrm{d}x \geq \int\limits_1^\infty \frac{p(1)}{x}\,\mathrm{d}x = +\infty.$$

With the same argument as for Corollary 4.3, also note that Remark 5.4 and Theorem 5.5 yield:

Corollary 5.6. *$\mathcal{CF}_\theta = \mathcal{BF}_\theta \cap \{\phi,\ s.t.\ \phi(0) = 0\}$ and $\mathcal{BF}_\theta = \mathcal{BF}_\Theta \subset \mathcal{BF}$.*

Example 5.7. We propose two examples:

(i) If $\phi \in \mathcal{BF}_\Theta$, then $\lambda \mapsto \phi(\lambda^\alpha) \in \mathcal{BF}_\Theta$, for all $0 < \alpha \leq 1$. Indeed,

$$\Theta(\lambda \mapsto \phi(\lambda^\alpha)) = (1-\alpha)\phi(\lambda^\alpha) + (\Theta\phi)(\lambda^\alpha),$$

and the two functions in r.h.s are in $\mathcal{BF}$ since they are obtained by the composition with the stable Bernstein function $\lambda \mapsto \lambda^\alpha$.

(ii) Next example is less trivial: the function

$$\phi_0(\lambda) = e^{-\sqrt{\lambda}} - 1 + \sqrt{\lambda} \quad \text{belongs to } \mathcal{BF}_\Theta.$$

To see that, note that for every $\lambda \geq 0$

$$\sqrt{\lambda} = \int\limits_0^\infty (1 - e^{-\lambda x})\frac{1}{2\sqrt{\pi}x^{3/2}}\,\mathrm{d}x \quad \text{and} \quad 1 - e^{-\sqrt{\lambda}} = \int\limits_0^\infty (1 - e^{-\lambda x})\frac{e^{-1/4x}}{2\sqrt{2\pi}x^{3/2}}\,\mathrm{d}x$$

so that ϕ_0 is represented by

$$\phi_0(\lambda) = \int\limits_0^\infty (1 - e^{-\lambda x})\frac{p(x)}{x^2}\,\mathrm{d}s, \quad x \mapsto p(x) = \sqrt{x}\left(1 - \frac{e^{-1/4x}}{2}\right) \text{ is nondecreasing.}$$

Proof of Theorem 5.5. By Proposition 3.2, observe that $x \mapsto x\overline{\mu}(x)$ is concave and non-negative, hence is a nondecreasing function. Thus, the function $x \mapsto \overleftarrow{\mu}(x)/x := \overline{\mu}(1/x)/x$ is nonincreasing.

(1) $\Leftrightarrow$ (2): The equivalence is due to [3, Lemma 2.2].

(1) ⇒ (3): If $x \mapsto g(x) := x\overline{\mu}(x)$ is nondecreasing, then

$$\mathrm{d}(g(x)) = \overline{\mu}(x)\mathrm{d}x - x\mu(\mathrm{d}x) \tag{19}$$

is a positive measure. We deduce that μ admits a density function, which we write in the form $p(x)/x^2$, where p is a non-negative measurable function. Observing that g is concave, differentiable and is represented by

$$g(x) = x\int_x^\infty \frac{p(t)}{t^2}\,\mathrm{d}t,$$

obtain from

$$x \mapsto g'(x) = \int_x^\infty \frac{p(t)}{t^2}\,\mathrm{d}t - \frac{p(x)}{x} \quad \text{is nonincreasing,}$$

that

$$\mathrm{d}(-g'(x)) = \frac{\mathrm{d}p(x)}{x} - \frac{p(x)}{x^2}\,\mathrm{d}x + \frac{p(x)}{x^2}\,\mathrm{d}x = \frac{\mathrm{d}p(x)}{x}. \tag{20}$$

Finally, deduce that the measure dp is a positive, i.e. p is nondecreasing.
(3) ⇒ (4): For $x > 0$, define

$$\overline{\mu}_c(x) := \frac{1}{c}\theta_c(\overleftarrow{\overline{\mu}})\left(\frac{1}{x}\right) = \frac{\overline{\mu}(x/c)}{c} - \overline{\mu}(x)\left(\frac{1}{c}\int_{\frac{x}{c}}^\infty \frac{p(t)}{t^2}\,\mathrm{d}t - \int_x^\infty \frac{p(t)}{t^2}\,\mathrm{d}t\right) \tag{21}$$

and by the change variable $t = s/c$, retrieve the representation

$$\overline{\mu}_c(x) = \int_x^\infty \frac{p(s/c) - p(s)}{s^2}\,\mathrm{d}s.$$

Since p is nondecreasing, then $\overline{\mu}_c$ is non-negative and nonincreasing, which is equivalent to the claim on $\theta_c(\overleftarrow{\overline{\mu}})$.
(4) ⇒ (5): Recall the form (2) of ϕ and that $\overline{\mu}_c$ is given in (21). By the change of variable $x = y/c$, write

$$\theta_c\phi(\lambda) = \phi(c\lambda) - c\phi(\lambda) = q(1-c) + c\lambda\int_0^\infty e^{-c\lambda x}\overline{\mu}(x)\mathrm{d}x - c\lambda\int_0^\infty e^{-\lambda x}\overline{\mu}(x)\mathrm{d}x = q(1-c) + c\lambda\int_0^\infty e^{-y\lambda}\overline{\mu}_c(y)\mathrm{d}y.$$

Since $\overline{\mu}_c$, given by (21), is nonincreasing, then $\theta_c\phi$ has also the representation (2) of a Bernstein function.
(5) ⇒ (6): Just recall that $\mathcal{BF} \subset \mathcal{CF}$ and use the definition of the class $\mathcal{BF}_\Theta$ in 5.1.
(6) ⇒ (7): This is point (3) of Proposition 5.3.
(7) ⇔ (8): Take $d = \lim_{+\infty}\phi(u)/u$, $\varphi = \Theta\phi$ and use inversion formula (11).
(8) ⇒ (1): Using representation (2) for $\Theta\phi$, write

$$\phi'(\lambda) = d + \int_0^\infty e^{-x\lambda}\overline{\mu}(x)\mathrm{d}x - \lambda\int_0^\infty e^{-\lambda x}x\overline{\mu}(x)\mathrm{d}x$$

so that

$$\Theta\phi(\lambda) = \phi(\lambda) - \lambda\phi'(\lambda) = q + \lambda^2\int_0^\infty e^{-x\lambda}x\overline{\mu}(x)\mathrm{d}x. \tag{22}$$

By representation (2), conclude that $x \mapsto x\overline{\mu}(x)$ is concave.

For the last assertion, use representations (1) and (2) of ϕ and write

$$\Theta\phi(\lambda) = q + \lambda \int_{(0,\infty)} e^{-\lambda x}(\bar{\mu}(x)\mathrm{d}x - x\mu(\mathrm{d}x)),$$

then conclude with (19), (20) and representation (2) for $\Theta\phi$. □

Remark 5.8. Due to the linearity of the operator Θ and due to Proposition 3.1, we retrieve the following properties for the class $\mathcal{BF}_\Theta$:

(i) The class $\mathcal{BF}_\Theta$ is a convex cone, i.e. if $(\phi_u)_{u\in U}$ is a family of $\mathcal{BF}_\Theta$ and ν is a measure on the indexation set U, then, modulo the existence of the integral, we have

$$\lambda \mapsto \int_U \phi_u(\lambda)\nu(\mathrm{d}u) \in \mathcal{BF}_\Theta.$$

(ii) The class $\mathcal{BF}_\Theta$, like $\mathcal{BF}$, is closed under pointwise limits.

Point (3) in the next theorem shows a nice fact:

Proposition 5.9. *We have the equivalence between the following assertions.*

(1) $\phi \in \mathcal{BF}_\Theta$;

(2) $\lambda \mapsto \phi_\alpha(\lambda) := \phi(\lambda^\alpha) \in \mathcal{BF}_\Theta$ *for every* $\alpha \in (0,1)$;

(3) ϕ *is differentiable and for every* $a > 0$,

$$\lambda \mapsto \psi_a(\lambda) := a\phi(\lambda) - \lambda(\phi(\lambda + a) - \phi(\lambda)) \in \mathcal{BF}.$$

Proof. (1) ⇔ (2): It is suffice to write

$$\Theta\phi_\alpha(\lambda) = \phi(\lambda^\alpha) - \alpha\lambda^\alpha\phi'(\lambda^\alpha) = (1-\alpha)\phi(\lambda^\alpha) + \alpha(\Theta\phi)(\lambda^\alpha).$$

For the converse, conclude by letting $\alpha \to 1-$ and by the closure property of the class $\mathcal{BF}_\Theta$ in Remark 5.8.
(1) ⇒ (3) By Theorem 5.5, there is no loss of generality to assume that $\phi \in \mathcal{BF}_\Theta$ has the characteristics $\left(0, 0, \frac{p(x)}{x^2}\mathrm{d}x\right)$, with p nondecreasing. Using (2), write

$$\begin{aligned}\psi_a(\lambda) &= a\lambda \int_0^\infty e^{-\lambda x} l_a(x)\mathrm{d}x \\ l_a(x) &:= \int_x^\infty \frac{p(t)}{t^2}\mathrm{d}t - \varepsilon(ax)\frac{p(x)}{x}, \quad \varepsilon(x) := \frac{1-e^{-x}}{x} \in (0,1), \quad x > 0,\end{aligned}$$

and by (20), it is clear that $l_a(x) \geq \int_x^\infty \frac{p(t)}{t^2} - \frac{p(x)}{x} \geq 0$. It remains to show that l_a is nonincreasing, but this is easily seen by the fact that

$$\kappa(x) = 1 - \varepsilon(x) + x\varepsilon'(x) \geq 0, \quad \forall x > 0$$

and by the differentiation of $-l_a$, which gives the positive measure:

$$-\mathrm{d}(l_a)(x) = \kappa(ax)\frac{p(x)}{x^2}\mathrm{d}x + \varepsilon(ax)\frac{\mathrm{d}p(x)}{x}.$$

(3) ⇒ (1): For the sufficient part, use the closure property of the class $\mathcal{BF}$ in Proposition 3.1, and note that $\Theta\phi = \lim_{a\to 0+}\phi_a/a$. □

The class $\mathcal{CBF}$ of complete Bernstein functions (resp. $\mathcal{TBF}$ of Thorin-Bernstein function) consists in those Bernstein function ϕ s.t. the associated Lévy measure μ has a density function $m \in \mathcal{CM}$ (resp. $xm(x) \in \mathcal{CM}$). Let Δ be the operator defined on $\mathcal{BF}$ by:

$$\Delta\phi(\lambda) = \lambda^2 \int\limits_{(0,\infty)} e^{-\lambda x}\phi(x)\mathrm{d}x, \quad \phi \in \mathcal{BF}. \tag{23}$$

The following nice fact gives an additional interest to the class $\mathcal{BF}_\Theta$:

Proposition 5.10. *The operator Δ is a bijection from $\mathcal{BF}$ (resp. $\mathcal{BF}_\Theta$) to $\mathcal{CBF}$ (resp. $\mathcal{TBF}$).*

Proof. In [2, Theorem 6.2], it is stated that the operator Δ is a bijection from $\mathcal{BF}$ to $\mathcal{CBF}$. Observe that if $\phi \in \mathcal{BF}_\Theta$, then, by Theorem 5.5, there exist $d, q \geq 0$ and a nondecreasing function p such that

$$\phi(x) = d + q\lambda + \int\limits_{(0,\infty)} (1 - e^{-xu})\frac{p(u)}{u^2}\,\mathrm{d}u, \quad x \geq 0$$

and elementary computations give that

$$\Delta\phi(\lambda) = q + \mathrm{d}\lambda + \int\limits_{(0,\infty)} \frac{\lambda}{\lambda + u}\frac{p(u)}{u}\,\mathrm{d}u, \quad \lambda \geq 0,$$

meets the representation of a $\mathcal{TBF}$-function as given in [2, Theorem 8.2]. The converse is obtained by reversing the calculus. □

5.3 Stochastic interpretation of the class $\mathcal{BF}_\Theta$

Recall that a non-negative r.v. $X \sim$ **ID** is embedded into a subordinator $(X_t)_{t\geq 0}$, i.e. $X = X_1$. From this observation we retrieve a simple characterisation of **RD**:

Theorem 5.11. *Let X be a non-negative random variable with cumulant function $\phi_X(\lambda) = -\log \mathbb{E}[e^{-\lambda X}]$, $\lambda \geq 0$. Then, the following assertions are equivalent.*

(1) $X \sim$ **RD**;

(2) $X \sim$ **ID** *and is embedded into a subordinator $(X_t)_{t\geq 0}$ such that we have the identities in distribution:*

$$cX_t \stackrel{d}{=} X_{ct} + Z_{c,t}, \quad \textit{for every } c \in (0,1) \textit{ and } t > 0, \tag{24}$$

where $Z_{c,t}$ is a non-negative infinitely divisible random variable independent of X_c;

(3) $\phi_X \in \mathcal{BF}_\Theta$.

Remark 5.12. In Remark 5.8, we have noted that $\mathcal{BF}_\Theta$ is a convex cone, i.e. it is stable by mixture of families of $\mathcal{BF}_\Theta$ by a measure μ. In case where μ is a discrete measure, say $\nu = \delta_1 + \delta_2 + \cdots + \delta_n$, the latter is stochastically interpreted as follows: If $X_1, X_2, \ldots, X_n \sim$ **ID** are independent and associated functions $\phi_1, \phi_2, \ldots, \phi_n \in \mathcal{BF}_\Theta$, i.e. there exists an independent family of r.v.'s $Z_{1,c}, Z_{2,c}, \ldots, Z_{n,c}$ associated with $X_1, X_2, \ldots, X_n$ via the identity (24), then $S_n := X_1 + X_2 + \cdots + X_n \in$ **RD** and the r.v.

$$Z_{1,c} + Z_{2,c} + \cdots + Z_{n,c} \text{ is associated with } S_n \text{ via (24).}$$

Another stochastic interpretation is as follows. By Theorem 5.5, we know that the Lévy measure of a function $\phi \in \mathcal{BF}_\Theta$ is represented by:

$$\frac{\mu(\mathrm{d}x)}{\mathrm{d}x} = \frac{p(x)}{x^2}\,\mathrm{d}x, \quad p \text{ nondecreasing.}$$

Recall that any Bernstein function ϕ is associated with a subordinator $(X_t)_t \geq 0$, possibly killed with a rate $q = \phi(0)$. The so-called harmonic and harmonic potential measures are defined by:

$$U(\mathrm{d}x) := \int\limits_{(0,\infty)} \mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t, \quad H(\mathrm{d}x) := \int\limits_{(0,\infty)} \mathbb{P}(X_t \in \mathrm{d}x)\frac{\mathrm{d}t}{t}, \tag{25}$$

and the exponential functional of the subordinator $(X_t)_{t\geq 0}$ is given by the stochastic integral $I = \int_0^\infty e^{-X_t}\mathrm{d}t$. The measure

$$W(\mathrm{d}x) = \mathrm{d}\delta_0(\mathrm{d}x) + (q + \bar{\mu}(x))\mathrm{d}x$$

is also of a particular interest. In [10], it is shown that the measure U is infinitely divisible (in the sense of the convolution), whereas W is not in general, but we have the following:

Proposition 5.13. *For any Bernstein function ϕ, the following assertions are equivalent:*
(1) $\lambda \mapsto \frac{1}{\lambda} - \frac{\phi'}{\phi}(\lambda)$ is completely monotone;
(2) $\lambda \mapsto (\phi(\lambda)/\lambda)^t$ is completely monotone for all $t \geq 0$;
(3) $W(\mathrm{d}x)$ is an infinitely divisible measure;
(4) For every $a > 0$, $a\phi(a)^{-1}e^{-ax}W(\mathrm{d}x)$ is an infinitely divisible distribution;
(5) The exponential functional I is such that $\log I$ is infinitely divisible;
(6) The harmonic potential measure H has a density function of the form $\frac{\rho(x)}{x}$, $x > 0$, with $\rho(x) \in [0, 1]$.

Corollary 5.14. *Any function $\phi \in \mathcal{BF}_\Theta$ satisfies the assertions of* Proposition 5.13.

Proof. Since $\Theta\phi \in \mathcal{BF}$, then

$$\lambda \mapsto \frac{\Theta\phi(\lambda)}{\lambda} = \frac{\phi(\lambda) - \lambda\phi'(\lambda)}{\lambda} \in \mathcal{CM}$$

and multiplying by the completely monotone function $1/\phi$, we get the result. □

Proof of Proposition 5.13.
(1) ⇒ (2): Writing

$$\left(\frac{\phi(\lambda)}{\lambda}\right)^t = e^{-t(\log\lambda - \log\phi)},$$

note that $\log\lambda - \log\phi$ is an extended Bernstein function, whose derivative is $\frac{1}{\lambda} - \frac{\phi}{\phi'}$. The assertion stems from [2, Theorem 5.11, p. 53].
(2) ⇒ (3): By representation (1), the Laplace transform of the measure $W(\mathrm{d}x)$ equals

$$\mathcal{L}_W(\lambda) = \frac{q}{\lambda} + d + \int\limits_0^\infty e^{-\lambda x}\bar{\mu}(x)\mathrm{d}x = \frac{\phi(\lambda)}{\lambda}, \quad \lambda > 0.$$

Since for every positive integer n, $(\mathcal{L}_W)^{\frac{1}{n}}$ is the Laplace transform of some positive measure W_n, i.e. the following convolution equation holds:

$$W = W_n * \cdots * W_n, \quad n \text{ times}.$$

(3) ⇒ (4): is justified by the fact that the measure $a\phi(a)^{-1}e^{-ax}W(\mathrm{d}x)$ is a probability measure whose Laplace transform is

$$\frac{a}{\phi(a)}\frac{\phi(\lambda + a)}{\lambda + a}.$$

(4) ⇔ (5) ⇔ (6) ⇔ (1): These assertions are stated in [10, Theorem 1.5, p. 715]. □

5.4 What about $\mathcal{BF}_{\Xi\circ\Theta}$, $\mathcal{BF}_{\Theta\circ\Xi}$ and $\mathcal{BF}_{\Theta}\cap\mathcal{BF}_{\Xi}$?

The differential operators Ξ and its companion Θ were defined in (2.1)

$$\Xi\phi(\lambda) = \lambda\phi'(\lambda) \quad \text{and} \quad \Theta = I - \Xi \quad (I = \text{Identity}).$$

Observe that the operators Θ and Ξ commute:

$$(\Theta\circ\Xi)(\phi)(\lambda) = (\Xi\circ\Theta)(\phi)(\lambda) = -\lambda^2\phi'',$$

and then, $\mathcal{BF}_{\Xi\circ\Theta} = \mathcal{BF}_{\Theta\circ\Xi}$. The last class is not void since it contains all stable Bernstein functions $\lambda\mapsto\lambda^\alpha$, $0<\alpha<1$. By representation (2),

$$(\Xi\circ\Theta)(\phi)\in\mathcal{BF} \Leftrightarrow \lambda\mapsto\lambda^2\phi''(\lambda) = \lambda^2\int_0^\infty e^{-\lambda x}x^2\mu(\mathrm{d}x) = \lambda^2\int_0^\infty e^{-\lambda x}k(x)\mathrm{d}x,$$

for some concave function k and it is clear that

$$\mathcal{BF}_{\Xi\circ\Theta} = \mathcal{BF}_{\Theta\circ\Xi} = \left\{\phi\in\mathcal{BF},\ \text{ s.t. }\ \mu(\mathrm{d}x) = \frac{k(x)}{x^2}\mathrm{d}x,\ \ k\ \text{ concave}\right\}.$$

On the other hand,

$$\mathcal{BF}_{\Xi}\cap\mathcal{BF}_{\Theta} = \left\{\phi\in\mathcal{BF},\ \text{ s.t. }\ \mu(\mathrm{d}x) = \frac{k(x)}{x^2}\mathrm{d}x,\quad x\mapsto\frac{k(x)}{x}\nearrow\ \text{and}\ k\searrow\right\}.$$

By Proposition 3.2, if a non-negative function on $(0,\infty)$, $k(x) = xl(x)$, is concave, then it is necessarily nonincreasing and $xk(1/x) = l(1/x)$ is also concave, and then nondecreasing. We deduce that

$$\mathcal{BF}_{\Xi\circ\Theta}\subset\mathcal{BF}_{\Xi}\cap\mathcal{BF}_{\Theta}.$$

It is then natural to ask the question: whether it is true that $\mathcal{BF}_{\Xi\circ\Theta} = \mathcal{BF}_{\Xi}\cap\mathcal{BF}_{\Theta}$? The assertion fails, and a counterexample is given by the function $k(x) = x^{3/2}(1-e^{-x})$, which is not concave but satisfies the requirements for $\mathcal{BF}_{\Xi}\cap\mathcal{BF}_{\Theta}$.

6 The class $\mathcal{BF}^1_\Theta$

We now introduce a more refined class than $\mathcal{BF}_\Theta$, the class $\mathcal{BF}^1_\Theta$, that will provide the behaviour of the transitions of the subordinator behind the involved Bernstein functions.

6.1 Definition of $\mathcal{BF}^1_\Theta$ and analytic results

Definition 6.1. [1] We denote by $\mathcal{BF}^1_\Theta$ the class of functions $\phi:[0,\infty)\to[0,\infty)$, differentiable on $(0,\infty)$, such that

$$\Theta(e^{-t\phi}) := e^{-t\phi}(1+t\Xi\phi)\in\mathcal{CM},\quad \text{for all }\ t>0. \tag{26}$$

The class $\mathcal{BF}^1_\Theta$ has also been considered by Sendov and Shan [1] and denoted there by $\mathcal{H}_{\mathcal{BF}}$. We start by some enlightenment on the structure of $\mathcal{BF}^1_\Theta$. The linearity of Θ leads to the following observation that will be used several times.

$$\Theta\phi = \lim_{t\to0+}\frac{\Theta(1-e^{-t\phi})}{t}. \tag{27}$$

Proposition 6.2. *Let $\phi : [0, \infty) \to [0, \infty)$ be differentiable on $(0, \infty)$ such that $\phi(\lambda) > 0$ if $\lambda > 0$. Then*
(1) $\Xi\phi \geq 0$ *and* $\Theta\phi \geq 0$ *if, and only if,* $0 \leq \Theta(e^{-t\phi}) \leq 1$, *for all* $t > 0$;
(2) *If* $\phi \in \mathcal{BF}$, *then* $0 \leq \Theta(1 - e^{-t\phi}) \leq 1$, *for all* $t > 0$.

Proof.
(1) For the sufficiency part, fix $\lambda > 0$ and denote $\rho_\lambda(t) = \Theta(e^{-t\phi})(\lambda)$. Observe that ρ_λ is nonincreasing because its derivative is

$$\rho'_\lambda(t) = -[\Theta\phi(\lambda) + t\phi(\lambda)\Xi\phi(\lambda)]\, e^{-t\phi(\lambda)} \leq 0$$

and conclude with the facts that $\rho_\lambda(0) = 1$ and $\rho_\lambda(+\infty) = 0$. For the necessity part, use representation (26) to deduce that $\Xi\phi = \lim_{t\to\infty} e^{-t\phi}\Theta(e^{-t\phi})/t \geq 0$ and use (27) for $\Theta\phi$.
(2) Certainly $\Xi\phi \geq 0$ and Proposition 5.2 ensure that $\Theta\phi \geq 0$, so point (1) applies. □

Proposition 6.3. *We have the (strict) inclusion* $\mathcal{BF}^1_\Theta \subset \mathcal{BF}_\Theta$, *and the equivalence between the following assertions:*
(1) $\phi \in \mathcal{BF}^1_\Theta$;
(2) $\Theta(f \circ \phi) \in \mathcal{CM}$, *for all* $f \in \mathcal{CM}$;
(3) $\Theta(1 - e^{-t\phi}) \in \mathcal{BF}$, *for all* $t > 0$;
(4) $\Theta(\varphi \circ \phi) \in \mathcal{BF}$, *for all* $\varphi \in \mathcal{BF}$.

Proof. The inclusion is obtained by (27) and by Proposition 3.1. The inclusion is strict because the identity function belongs to $\mathcal{BF}_\Theta$ but not to $\mathcal{BF}^1_\Theta$. For the equivalences, use Proposition 3.3:

$$\phi \in \mathcal{BF}^1_\Theta \Leftrightarrow \Theta(e^{-t\phi}) \in \mathcal{CM}, \forall t > 0 \Leftrightarrow \Theta(1 - e^{-t\phi}) = 1 - \Theta(e^{-t\phi}) \in \mathcal{BF}, \forall t > 0,$$

then use the fact that kernels e^{-tx} and $1 - e^{-tx}$, $t, x \geq 0$, respectively, generate the convex cones $\mathcal{CM}$ and $\mathcal{BF}$ in order to retrieve the stability property of $\mathcal{BF}^1_\Theta$ by composition on the left. □

Remark 6.4.
(i) Due to the linearity of the operator Θ, it is not difficult to see that the class $\mathcal{BF}^1_\Theta$ inherits from $\mathcal{BF}$ its closure property by pointwise limits given by Proposition 3.1.
(ii) Assume $\phi^{1/\alpha} \in (\mathcal{BF}^1_\Theta)$ for $\alpha \in (0, 1)$. Proposition 6.3, applied with the composition with the Bernstein function $\lambda \mapsto \lambda^\alpha$, gives that necessarily $\phi \in \mathcal{BF}^1_\Theta$.

Next result gives a simple sufficient condition for a function to belong to the class $\mathcal{BF}^1_\Theta$:

Proposition 6.5. *If* $\phi \in \mathcal{BF}_\Theta$ *and* $(\Xi\phi)^2 \in \mathcal{BF}$, *then* $\phi \in \mathcal{BF}^1_\Theta$.

Proof. First note that $\lambda \mapsto \varphi_1(\lambda) := \sqrt{\lambda} - \log(1 + \sqrt{\lambda}) \in \mathcal{BF}$ because its derivative $\varphi_1'(\lambda) = (2(1 + \sqrt{\lambda}))^{-1}$ is completely monotone. Also,

$$\Theta(e^{-t\phi}) = e^{-t\phi}(1 + t\Xi\phi) = e^{-\Phi_t}, \quad \Phi_t = t\phi - \log(1 + t\Xi\phi)$$

and by Proposition 3.1 on the compositions, obtain

$$\Phi_t = t\Theta\phi + \varphi_1((t\Xi\phi)^2) \in \mathcal{BF}, \forall t > 0 \Rightarrow e^{-\Phi_t} \in \mathcal{CM}, \forall t > 0 \Leftrightarrow \phi \in \mathcal{BF}^1_\Theta.$$ □

Example 6.6. For every $\alpha \in \left(0, \frac{1}{2}\right]$, the stable Bernstein function $\lambda \mapsto \lambda^\alpha$ satisfies Proposition 5.5, and hence belongs to $\mathcal{BF}^1_\Theta$. Certainly, this not true for $\alpha = 1$, and this motivates the investigation made in Section 6.4 on the determination of the maximal value $\alpha_0 \in [1/2, 1)$ for which $\lambda \mapsto \lambda^\alpha \in \mathcal{BF}^1_\Theta$.

6.2 A stochastic result closely related to the class $\mathcal{BF}^1_\Theta$

The following result is very close to the requirements on functions in $\mathcal{BF}^1_\Theta$:

Proposition 6.7. *Let X be a positive random variable with cumulant function $\phi(\lambda) = -\log \mathbb{E}[e^{-\lambda X}]$, $\lambda \geq 0$, and define for every $c > 0$, the functions*

$$f_c(x) = ce^{\phi(c)}e^{-cx}\mathbb{P}(X \leq x) \quad \textit{and} \quad g_c(x) = c\frac{e^{\phi(c)}}{e^{\phi(c)}-1}e^{-cx}\mathbb{P}(X > x), \quad x > 0.$$

(1) *The functions f_c and g_c are probability density functions associated with two independent random variables, say $X_{(c)}$ and $Y_{(c)}$, satisfying the identity in distribution:*

$$\frac{\mathbf{E}}{c} \stackrel{d}{=} \mathbf{B}_c X_{(c)} + (1-\mathbf{B}_c)Y_{(c)},$$

where $\mathbf{E}$ has the standard exponential distribution and $\mathbf{B}_c$ is Bernoulli distributed, with parameter $e^{-\phi(c)}$ and is independent of $X_{(c)}$ and $Y_{(c)}$.

(2) *Assume X is infinitely divisible with Bernstein function ϕ, and let $(X_t)_{t\geq 0}$ be its embedding subordinator, i.e. $X \stackrel{d}{=} X_1$. Then, the r.v. $X_{(c)}$ is always infinitely divisible for every $c > 0$ and we have the equivalence between the following assertions:*
   - (i) *the r.v. $Y_{(c)}$ is infinitely divisible for every $c > 0$;*
   - (ii) *the function $\lambda \mapsto \frac{1}{\lambda}\frac{\Theta(1-e^{-\phi})(\lambda)}{1-e^{-\phi(\lambda)}} = \frac{1}{\lambda} - \frac{\phi'(\lambda)}{e^{\phi(\lambda)}-1}$ is completely monotone;*
   - (iii) *the harmonic renewal measure ν of the random walk $(X_k)_{k\geq 1}$ has a density κ on $(0,\infty)$ with respect to the measure $\mathrm{d}x/x$:*

$$\nu(\mathrm{d}x) := \sum_{k=1}^{\infty}\frac{1}{k}\mathbb{P}(X_k \in \mathrm{d}x) = \frac{\kappa(x)}{x}\,\mathrm{d}x \quad \textit{and} \quad 0 \leq \kappa(x) \leq 1.$$

Proof.

(1) It is not difficult to check that f_c and g_c are probability density functions. The main assertion is based on the following remark: Let $\mathbf{B}_p$ be a Bernoulli distributed random variable with parameter $p \in (0,1)$ independent of two independent real random variables X and Y, then the probability measure $p\mathbb{P}(X \in \mathrm{d}x) + (1-p)\mathbb{P}(Y \in \mathrm{d}x)$ is the distribution of the random variable

$$\mathbf{B}_p X + (1-\mathbf{B}_p)Y.$$

It is suffice to take $p = e^{-\phi(c)}$ and to note that

$$ce^{-cx} = pf_c(x) + (1-p)g_c(x), \quad x > 0.$$

(2) $X_{(c)} \sim \mathbf{ID}$ is due to the form the Laplace transform of $X_{(c)}$ is given by

$$\mathbb{E}[e^{-\lambda X_{(c)}}] = \frac{c}{\lambda + c}\frac{\mathbb{E}[e^{-(\lambda+c)X}]}{\mathbb{E}[e^{-cX}]} = e^{-\phi_c(\lambda)} \tag{28}$$

and $\lambda \mapsto \phi_c(\lambda) := \phi(\lambda + c) - \phi(c) + \log\left(1 + \frac{\lambda}{c}\right) \in \mathcal{BF}$. The statements on $Y_{(c)}$ are justified as follows: let φ be defined by

$$e^{-\varphi(\lambda)} := \int_0^\infty e^{-\lambda x}\mathbb{P}(X > x)\,\mathrm{d}x = \frac{1-e^{-\phi(\lambda)}}{\lambda} = \exp - [\log\lambda - \log(1-e^{-\phi(\lambda)})], \quad \lambda > 0.$$

Then

$$\varphi'(\lambda) = \frac{1}{\lambda} - \frac{\phi'(\lambda)e^{-\phi(\lambda)}}{1-e^{-\phi(\lambda)}} = \frac{\Theta(1-e^{\phi})(\lambda)}{\lambda(1-e^{\phi(\lambda)})}.$$

Observe that $\Theta(1-e^{\phi}) \in \mathcal{BF}$ implies $1-e^{-\phi} \in \mathcal{BF}$, $\lambda \mapsto \Theta(1-e^{\phi})(\lambda)/\lambda \in \mathcal{CM}$ and $(1-e^{-\phi})^{-1} \in \mathcal{CM}$. Then φ', as the product of the two last functions, is also completely monotone. The argument for (i) $\Leftrightarrow$ (ii) is finished because

$$\mathbb{E}[e^{-\lambda Y_{(c)}}] = \frac{c}{1-e^{-\phi(c)}} \frac{1-\mathbb{E}[e^{-(\lambda+c)X}]}{\lambda+c} = e^{-(\varphi(\lambda+c)-\varphi(c))}$$

and $\phi' \in \mathcal{CM} \Leftrightarrow \lambda \mapsto \varphi(\lambda+c)-\varphi(c) \in \mathcal{BF}$ for every $c>0$. For the equivalence (ii) $\Leftrightarrow$ (iii), just write

$$\frac{1}{\lambda} - \frac{\phi'(\lambda)}{e^{\phi(\lambda)}-1} = \frac{1}{\lambda} - \sum_{k=1}^{\infty} \phi'(\lambda)e^{-k\phi(\lambda)} = \frac{1}{\lambda} - \sum_{k=1}^{\infty} \frac{1}{k}\mathbb{E}[X_k e^{-\lambda X_k}] = \int_{(0,\infty)} e^{-\lambda x}(\mathrm{d}x - x\nu(\mathrm{d}x)). \qquad \square$$

Remark 6.8. The Esscher transform $X_{[c]}$ of X, $c>0$, whose distribution is given by

$$\mathbb{P}(X_{[c]} \in \mathrm{d}x) = e^{-cx}\mathbb{P}(X \in \mathrm{d}x)/\mathbb{E}[e^{-cX}],$$

is also infinitely divisible because

$$\mathbb{E}[e^{-\lambda X_{[c]}}] = e^{-(\phi(\lambda+c)-\phi(c))}$$

and $\phi(.+c)-\phi(c) \in \mathcal{BF}$. Take an r.v. $\mathbf{E}$ independent of $X_{[c]}$ and exponentially distributed. By (28), we have the identity in law

$$X_{(c)} \overset{d}{=} \frac{\mathbf{E}}{c} + X_{[c]}.$$

Corollary 6.9. *Assume $\phi \in \mathcal{BF}$, $\phi(0)=0$, be associated with the subordinator $(X_t)_{t\geq 0}$ and adopt the notations of* Proposition 6.7 *on the random variables X_t. Then, $(X_t)_{(c)}$ is always infinitely divisible for all $t, c>0$ and we have* (4) $\Rightarrow$ (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1):

(*1*) *$t\phi$ satisfies the conditions of* Proposition 5.13 *for every $t>0$;*

(*2*) *$(Y_t)_{(c)}$ is infinitely divisible for every $t, c>0$;*

(*3*) $\lambda \mapsto \frac{1}{\lambda}\frac{\Theta(1-e^{-t\phi})(\lambda)}{1-e^{-t\phi(\lambda)}} = \frac{1}{\lambda} - \frac{t\phi'(\lambda)}{e^{t\phi(\lambda)}-1} \in \mathcal{CM}$ *every $t>0$;*

(*4*) $\phi \in \mathcal{BF}^1_\Theta$.

Proof.
(4) $\Rightarrow$ (3): Observe that (*ii*) in Proposition 6.7 is implied by $\Theta(e^{-t\phi}) \in \mathcal{CM}$ or, equivalently, by $\Theta(1-e^{-t\phi}) \in \mathcal{BF}$.
(3) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (1): immediately follow by Proposition 6.7. $\square$

Conjecture. Let $(X_t)_{t\geq 0}$ be the gamma subordinator, i.e. X_t has the density and Bernstein functions f_t and ϕ_t:

$$f_t(x) = \frac{x^{t-1}}{\Gamma(t)}e^{-x}, \quad x>0 \quad \text{and} \quad \phi_t(\lambda) = t\log(1+\lambda), \quad \lambda \geq 0.$$

Observe that Corollary 6.9(iv) does not apply on $\phi_t = t\phi_1$ because ϕ_t is not even in $\mathcal{BF}_\Theta$:

$$\lambda \mapsto \Theta(\phi_t)(\lambda) = t\left[\log(1+\lambda) - \frac{\lambda}{1+\lambda}\right] \notin \mathcal{BF}.$$

Elementary calculus gives that the associated κ-function in Proposition 6.7 is given by

$$\kappa_t(x) := x\sum_{k\geq 1}\frac{f_{tk}(x)}{k} = te^{-x}(E_t(x^t)-1), \quad x>0,$$

where $E_t(u)$ stands for the Mittag-Leffler function. Also observe that $\kappa_1(x) = 1-e^{-x} \in [0,1]$ and then Theorem 6.7 applies. Due to the asymptotic [11, (3.5.7) p. 32], we surmise that

$$\kappa_t(x) \leq 1, \forall x \geq 0 \Leftrightarrow t \leq 1.$$

6.3 Stochastic interpretation of the class $\mathcal{BF}^1_\Theta$

The class $\mathcal{BF}^1_\Theta$ is related to harmonic concavity property of the finite-dimensional distributions of subordinators by the following corollary, which makes [1, Lemma 7.4] a straightforward consequence:

Corollary 6.10. *Let $\phi \in \mathcal{BF}$, $\phi(0) = 0$, associated with the subordinator $(X_t)_{t\geq 0}$. Then the following assertions are equivalent.*

(1) $\phi \in \mathcal{BF}^1_\Theta$;

(2) *There exists a family $(p_t)_{t\geq 0}$ of nondecreasing functions on $(0,\infty)$ s.t. such that, the distribution of X_t, $t > 0$, admits a density function represented by*

$$\frac{\mathbb{P}(X_t \in \mathrm{d}x)}{dx} = \frac{p_t(x)}{x^2}, \quad x > 0. \tag{29}$$

Proof. Using the representation

$$1 - e^{-t\phi(\lambda)} = 1 - \mathbb{E}[e^{-\lambda X_t}] = \int_{(0,\infty)} (1 - e^{-\lambda x})\mathbb{P}(X_t \in \mathrm{d}x),$$

and consider the probability measure $\mathbb{P}(X_t \in \mathrm{d}x)$ as the Lévy measure associated with the Bernstein function $1 - e^{-t\phi}$. Theorem 5.5 and Proposition 6.3 entail

$$\phi \in \mathcal{BF}^1_\Theta \Leftrightarrow \Theta(1 - e^{-t\phi}) \in \mathcal{BF},\ \forall t > 0 \Leftrightarrow \text{(29) is true for every } t > 0. \qquad \square$$

Corollary 6.11. *Let $\phi \in \mathcal{BF}^1_\Theta$, associated with the subordinator $(X_t)_{t\geq 0}$ and with the potential measure and harmonic potential measure U, H given by (25). Then,*

(1) *The measures U and H are absolutely continuous such that*

$$U(\mathrm{d}x) = \frac{u(x)}{x^2}\mathrm{d}x, \quad H(\mathrm{d}x) = \frac{h(x)}{x^2}\mathrm{d}x, \quad \textit{where } u, h \textit{ are nonincreasing}.$$

In particular, the ρ-function given by Proposition 5.13 *is s.t. $x \mapsto x\rho(x)$ is nondecreasing.*

(2) *Both functions $\Theta(1/\phi)$ and $\Theta(\phi'/\phi)$ are completely monotone.*

Proof.

(1) By Proposition 6.10, we have the representation

$$U(\mathrm{d}x) = \int_{(0,\infty)} \mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t = \int_{(0,\infty)} \frac{p_t(x)}{x^2}\mathrm{d}t = \frac{u(x)}{x^2}\mathrm{d}x,$$

$$H(\mathrm{d}x) = \int_{(0,\infty)} \mathbb{P}(X_t \in \mathrm{d}x)\frac{\mathrm{d}t}{t} := \int_{(0,\infty)} \frac{p_t(x)}{x^2}\frac{\mathrm{d}t}{t} := \frac{h(x)}{x^2}\mathrm{d}x,$$

where for every $t > 0$, p_t is nondecreasing. It is then obvious $x \mapsto u(x), h(x)$ are also nondecreasing.

(2) The assertion stems from

$$\frac{1}{\phi(\lambda)} = \int_{[0,\infty)} e^{-\lambda x}U(\mathrm{d}x) \quad \text{and} \quad \frac{\phi'(\lambda)}{\phi(\lambda)} = \int_{(0,\infty)} e^{-\lambda x}xH(\mathrm{d}x), \quad \lambda > 0,$$

which entails

$$\Theta\left(\frac{1}{\phi}\right)(\lambda) = \int_{[0,\infty)} e^{-\lambda x}\frac{x+1}{x^2}u(x)\mathrm{d}x \quad \text{and} \quad \Theta\left(\frac{\phi'}{\phi}\right)(\lambda) = \int_{(0,\infty)} e^{-\lambda x}\frac{x+1}{x^2}h(x)\mathrm{d}x. \qquad \square$$

6.4 Answer to the problem of Sendov and Shan [1]: for which α does $\lambda^\alpha \in \mathcal{BF}^1_\Theta$?

In this section, we give a partial answer to an open problem raised by Sendov and Shan, [1, Open Problem 4.1], which is in strong relation with positive stable laws: what is the value $0 < \alpha_0 < 1$ for which it holds that

$$\lambda \mapsto \lambda^\alpha \in \mathcal{BF}^1_\Theta, \quad \text{for all } \alpha \leq \alpha_0? \tag{30}$$

The authors proved that (30) holds for $\alpha \leq 2/3$ and conjectured that

$$\alpha_0 = \frac{1}{\sqrt{2}} = 0.70710678118.$$

The importance of this problem is that the function $\phi(\lambda) = \lambda^\alpha$, $\lambda \geq 0$, is the Thorin-Bernstein function associated with the positive stable r.v. $\mathbf{S}_\alpha$ through the following representations:

$$\lambda^\alpha = \int_0^\infty (1 - e^{-\lambda x}) \frac{c_\alpha}{x^{\alpha+1}} \mathrm{d}x, \quad c_\alpha = \frac{\alpha}{\Gamma(1-\alpha)} = \Gamma(\alpha+1)\frac{\sin(\pi\alpha)}{\pi}, \qquad e^{-\lambda^\alpha} = \mathbb{E}[e^{-\lambda S_\alpha}] = \int_0^\infty e^{-\lambda s} f_\alpha(s)\,\mathrm{d}s, \tag{31}$$

where f_α is the probability density function of the r.v. $\mathbf{S}_\alpha$. Note that f_α is explicit only for the value $\alpha = 1/2$ and that

$$f_{1/2}(x) = \frac{e^{-1/4x}}{2\sqrt{2\pi}x^{3/2}}, \quad x > 0,$$

corresponds to the inverse-Gaussian distribution. In general, f_α is only evaluated by the series expansion given by [12, formula (2.4.8), p. 90]:

$$f_\alpha(x) = \frac{1}{\pi}\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \sin(\pi n\alpha) x^{-(n\alpha+1)}. \tag{32}$$

Note that (30) is equivalent to

$$\lambda \mapsto e^{-t\lambda^\alpha}(1 + \alpha t\lambda^\alpha) \in \mathcal{CM}, \quad \text{for all } \alpha \leq \alpha_0 \text{ and } t > 0$$

and also to

$$\lambda \mapsto e^{-\lambda^\alpha}(1 + \alpha\lambda^\alpha) \in \mathcal{CM}, \quad \text{for all } \alpha \leq \alpha_0.$$

Based on the result found in Simon [13], we assert that:

Theorem 6.12. *Let* $\alpha_1 = 0.688483504697$ *be the root of the function* $x \mapsto \sin^2(\pi x) - x$, $\frac{1}{2} \leq x \leq 1$. *If* $\alpha \leq \alpha_1$, *then* $\lambda \mapsto \lambda^\alpha \in \mathcal{BF}^1_\Theta$.

Proof. In [13, Lemma 2.3], it was shown that there exits an increasing function $R : [0, 1] \to [0, \infty]$ such that

$$\lambda \mapsto e^{-\lambda^\alpha}(s + \alpha\lambda^\alpha) \in \mathcal{CM} \Leftrightarrow \alpha \leq 1/2 \text{ or } s \geq R(\alpha),$$

where $R(\alpha) = \alpha$ if $\alpha \in [0, 1/2]$ and $1/4(1-\alpha) \leq R(\alpha) \leq \alpha/\sin^2(\pi\alpha)$ if $\alpha \in [1/2, 1]$. Taking $s = 1$, it is clear that (30) holds true if $\alpha \in [0, 1/2]$ or if $\alpha \in [1/2, 1]$ and $1 \geq \alpha/\sin^2(\pi\alpha)$. Thus, for $\alpha \in [1/2, \alpha_1]$, we always have $1 \geq \alpha/\sin^2(\pi\alpha)$. ☐

Recall that α_1 is given in Theorem 6.12 and that the density function f_α is given by (32). Theorem 6.12 immediately gives the following consequence.

Corollary 6.13. *If* $0 < \alpha \leq \alpha_1$, *then the following function is nondecreasing:*

$$x \mapsto x^2 f_\alpha(x) = \frac{1}{\pi}\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \sin(\pi n\alpha) x^{-(n\alpha-1)}, \quad x > 0.$$

7 The class $\mathcal{BF}_\Theta^2$

Definition 7.1. We denote by $\mathcal{BF}_\Theta^2$ the class of functions $\phi : [0, \infty) \to [0, \infty)$, such that

$$\lambda \mapsto e^{-t\phi} - 1 + t\phi \in \mathcal{BF}_\Theta \quad \text{for all } t > 0.$$

Proposition 7.2. *With the notation $\sqrt{B} = \{f, s.t. f^2 \in B\}$, we have the inclusions*

$$\sqrt{\mathcal{BF}_\Theta^1} \subset \mathcal{BF}_\Theta^2 \subset \sqrt{\mathcal{BF}_\Theta} \cap \mathcal{BF}_\Theta.$$

The first inclusion is strict because $\sqrt{\lambda} \in \mathcal{BF}_\Theta^2$ but $\lambda \notin \mathcal{BF}_\Theta^1$.

Proof.

(1) Example 5.7 ensures that $\lambda \mapsto \phi_0(\lambda) = e^{-\sqrt{\lambda}} - 1 + \sqrt{\lambda} \in \mathcal{BF}_\Theta$ and hence in $\mathcal{BF}$. Thus, if $\phi^2 \in \mathcal{BF}_\Theta^1$, then point (2) of Proposition 6.3 completes the first inclusion: $\phi_0(t^2\phi^2) = e^{-t\phi} - 1 + t\phi \in \mathcal{BF}_\Theta$. The inclusion is strict and a counter-example is given by $\phi(\lambda) = \sqrt{\lambda} \in \mathcal{BF}_\Theta^2$ because

$$\lambda \mapsto e^{-t\sqrt{\lambda}} - 1 + t\sqrt{\lambda} = \phi_0(t^2\lambda) \in \mathcal{BF}_\Theta, \quad \forall t > 0,$$

but $\phi^2(\lambda) = \lambda \notin \mathcal{BF}_\Theta^1$. The latter is true since the first derivative of $\Theta(1 - e^{-\lambda}) = 1 - e^{-\lambda}(1 + \lambda)$ equals to the noncompletely monotone $\lambda e^{-\lambda}$.

(2) Assume $\phi \in \mathcal{BF}_\Theta^2$, i.e. $\Theta(e^{-t\phi} - 1 + t\phi) \in \mathcal{BF}$, for all $t > 0$. From Proposition 3.1, and from next limits deduce the second inclusion:

$$\Theta(\phi) = \lim_{t\to\infty} \frac{1}{t}\Theta(e^{-t\phi} - 1 + t\phi) \in \mathcal{BF} \quad \text{and} \quad \Theta(\phi^2) = \lim_{t\to 0+} \frac{2}{t^2}\Theta(e^{-t\phi} - 1 + t\phi) \in \mathcal{BF}. \qquad \square$$

Next proposition gives a stochastic interpretation of the class $\mathcal{BF}_\Theta^2$.

Proposition 7.3. *Let $\phi \in \mathcal{BF}$ be associated with a subordinator $(X_t)_{t\geq 0}$. Then, the following assertions are equivalent.*

(1) $\phi \in \mathcal{BF}_\Theta^2$;

(2) the measure $\mu(\mathrm{d}x)$ and $\mathbb{P}(X_t \in \mathrm{d}x)$, $t > 0$, are absolutely continuous and the function

$$x \mapsto x^2\left(t\frac{\mu(\mathrm{d}x)}{\mathrm{d}x} - \frac{\mathbb{P}(X_t \in \mathrm{d}x)}{\mathrm{d}x}\right), \quad x > 0,$$

is non-negative and nondecreasing;

(3) the function $x \mapsto l_t(x) := t\overline{\mu}(x) - \mathbb{P}(X_t > x)$ is non-negative nonincreasing and $x \mapsto xl_t(x)$ is non-negative nondecreasing;

(4) the function $x \mapsto t\int_0^x u^2\mu(\mathrm{d}u) - \mathbb{E}[X_t^2 \imath_{(X_t\leq x)}]$ is non-negative and concave.

Proof. Without loss of generality, we may assume that ϕ has triplet of characteristics $(0, 0, \mu)$. By (2) and (22), we have the following representation valid for all $\lambda \geq 0$ and $t > 0$:

$$(e^{-t\phi} - 1 + t\phi)(\lambda) = \lambda \int_0^\infty e^{-\lambda x} l_t(x)\,\mathrm{d}x, \tag{33}$$

$$\Theta(e^{-t\phi} - 1 + t\phi)(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda x} x l_t(x)\,\mathrm{d}x, \tag{34}$$

where $x \mapsto l_t(x) := t\overline{\mu}(x) - \mathbb{P}(X_t > x)$.

(1) ⇒ (2): Theorem 5.5 and representation (34) entail that $\mathrm{d}(-l_t)(x) = t\mu(\mathrm{d}x) - \mathbb{P}(X_t \in \mathrm{d}x)$ is a positive measure which is absolutely continuous and

$$t\mu(\mathrm{d}x) - \mathbb{P}(X_t \in \mathrm{d}x) = \frac{q_t(x)}{x^2}\,\mathrm{d}x, \quad \text{with } q_t \text{ non-negative and nondecreasing}. \tag{35}$$

By Proposition 7.2, necessarily $\phi \in \mathcal{BF}_\Theta$ and $\mu(\mathrm{d}x)$ is absolutely continuous, and then, so is $\mathbb{P}(X_t \in \mathrm{d}x)$.

(2) ⇒ (3): Using (35), we have

$$x \mapsto l_t(x) = \int_x^\infty \frac{q_t(y)}{y^2}\,\mathrm{d}y \quad \text{is non-negative and nonincreasing}$$

and by the change of variable $y \mapsto xy$, we see that

$$x \mapsto xl_t(x) = \int_1^\infty \frac{q_t(xy)}{y^2}\,\mathrm{d}y \quad \text{is non-negative and nondecreasing.}$$

(3) ⇒ (4): By Theorem 5.5 and Corollary 6.10, we know that the measures μ and $\mathbb{P}(X_t \in \mathrm{d}x)$, $t > 0$, have density functions of the form

$$\mu(\mathrm{d}x) = \frac{p(x)}{x^2} \quad \text{and} \quad \mathbb{P}(X_t \in \mathrm{d}x) = \frac{p_t(x)}{x^2},$$

where both p and p_t are nondecreasing and, by representation (18), $\mathrm{d}p(x)/x$ and $\mathrm{d}p_t(x)/x$ are, respectively, the Lévy measures of $\Theta(\phi)$ and of $\Theta(1 - e^{-t\phi})$. Because $\phi \in \mathcal{BF}^2_\Theta$ and by representation (33), we obtain

$$\Theta(e^{-t\phi} - 1 + t\phi) = \int_{(0,\infty)} (1 - e^{-\lambda x})\frac{t\mathrm{d}p(x) - \mathrm{d}p_t(x)}{x} \in \mathcal{BF}, \quad \forall t > 0,$$

and the latter ensures that the measure $t\mathrm{d}p(x) - \mathrm{d}p_t(x)$ is positive, hence the function $x \mapsto tp(x) - p_t(x)$ is nondecreasing.

(2) ⇒ (1) is obtained by reading the last arguments from bottom upwards. The other equivalences are obtained by integration. □

Recall the constant c_α and the density function f_α are, respectively, given by (32) and (32). A straightforward consequence of Proposition 7.3 is the following result.

Corollary 7.4. *Let f_α, $\alpha \in (0, 1)$, be the density function of a positive stable distribution. Then, the following assertions are equivalent.*
(*1*) $0 < \alpha \le 1/2$;
(*2*) *The following function is non-negative and nondecreasing:*

$$x \mapsto x^2\left(\frac{c_\alpha}{x^{\alpha+1}} - f_\alpha(x)\right) = \frac{1}{\pi}\sum_{n=1}^{+\infty} (-1)^{n-1}\frac{\Gamma(n\alpha + 1)}{\Gamma(n + 1)}\,\sin(\pi n\alpha)x^{-(n\alpha-1)}, \quad x > 0.$$

8 Generalisation of reverse decomposability: convex decomposability

A natural extension of the class $\mathcal{BF}_\theta$ is given by the class $\mathcal{BF}_\upsilon$, which is described as follows.

8.1 The class $\mathcal{BF}_\upsilon$ and its analytic properties

Definition 8.1. The class of convex decomposable cumulant functions, denoted by $\mathcal{BF}_\upsilon$, is defined by

$$\mathcal{BF}_\upsilon := \{\phi \in \mathcal{CF} \text{ s.t. } \upsilon_c\phi \in \mathcal{BF}, \quad \text{for every } c \in (0,\ 1)\},$$

where υ_c is the operator given in Definition 2.1.

Proposition 8.2. *If $\phi : [0,\ \infty) \to [0,\ \infty)$ is such that $\upsilon_c\phi \in \mathcal{BF}$ for some $c \in (0, 1)$, then $\phi \in \mathcal{BF}$. We actually have $\mathcal{BF}_\Theta \subset \mathcal{BF}_\upsilon \subset \mathcal{BF}$.*

Proof. Note that $\mathcal{BF}_\Theta \subset \mathcal{BF}_\upsilon$ because $\upsilon_c = \theta_c + \theta_{1-c}$. Since $\upsilon_c\phi = \theta_c\phi + \theta_{1-c}\phi \in \mathcal{BF}$ for any $c \in (0, 1)$, then

$$\upsilon_{\frac{1}{2}}\phi = 2\theta_{1/2}\phi \in \mathcal{BF}.$$

By Proposition 5.3 point (2), we deduce that $\phi \in \mathcal{BF}$. □

Unfortunately, we were not able to derive a differential operator associated with $\mathcal{BF}_\upsilon$, as we did for the identification $\mathcal{BF}_\theta = \mathcal{BF}_\Theta$ in Theorem 5.5, nevertheless we propose the following characterisation of $\mathcal{BF}_\upsilon$:

Theorem 8.3. *Let ϕ be a Bernstein function represented by* (2) *and associated with a Lévy measure μ. Recall $\overline{\mu}$ and $\overleftarrow{\overline{\mu}}$ are given by* (17)*. Then, we have equivalences:*

(1) $\upsilon_c\phi \in \mathcal{BF}_\upsilon$;

(2) *$\upsilon_c\overleftarrow{\overline{\mu}}$ is positive and nondecreasing for all $c \in (0, 1)$;*

(3) *The measure μ has the form*

$$\mu(\mathrm{d}x) = \frac{p(x)}{x^2}\,\mathrm{d}x \quad \textit{such that} \quad x \mapsto xp\left(\frac{1}{x}\right) \textit{ is subadditive};$$

(4) *$\phi(c_1\lambda) + \cdots + \phi(c_n\lambda) - \phi(\lambda) \in \mathcal{BF}_\upsilon$ for all $c_1, \ldots, c_n \ge 0$ s.t. $c_1 + c_2 + \cdots + c_n = 1$.*

Proof. (1) ⇒ (2): Use representation (2) of ϕ, make the appropriate changes of variable and use the operator υ_c in order to write

$$\upsilon_c\phi(\lambda) = \phi(c\lambda) + \phi(\bar{c}\lambda) - \phi(\lambda) = a + \lambda \int\limits_{(0,\infty)} e^{-x\lambda}\ (\upsilon_c\overleftarrow{\overline{\mu}})(1/x)\mathrm{d}x,$$

where

$$(\upsilon_c\overleftarrow{\overline{\mu}})(1/x) = \overline{\mu}(x/c) + \overline{\mu}(x/\bar{c}) - \overline{\mu}(x).$$

Since $\upsilon_c\phi \in \mathcal{BF}$, then using representation (1) for $\upsilon_c\phi(\lambda)$, deduce that $x \mapsto (\upsilon_c\overleftarrow{\overline{\mu}})(1/x)$ is right queue of a Lévy measure and hence $\upsilon_c\overleftarrow{\overline{\mu}}$ is positive and nonincreasing, so that $x \mapsto (\upsilon_c\overleftarrow{\overline{\mu}})(x)$ is positive and nondecreasing.

(2) ⇒ (3): It is clear that $\lim_{x\to 0}\overline{\mu}(1/x) = 0$ and by the assumption that $x \mapsto \overline{\mu}(1/x)$ subadditive nondecreasing. Deduce from [14, Theorem 16.2.1, p. 460] that $x \mapsto \overline{\mu}(1/x)$ is continuous on $(0, \infty)$, and then $\overline{\mu}$ is absolutely continuous. Thus, the measure μ can be represented in the form $\mu(\mathrm{d}x) = x^{-2}p(x)\mathrm{d}x$ with some measurable function p. Furthermore, since $(\upsilon_c\overleftarrow{\overline{\mu}})(1/x)$ is nonincreasing and since

$$\begin{aligned}(\upsilon_c\overleftarrow{\overline{\mu}})(1/x) &= \overline{\mu}(x/c) + \overline{\mu}(x/\bar{c}) - \overline{\mu}(x) = \int\limits_{\frac{x}{c}}^{\infty} \frac{p(t)}{t^2}\,\mathrm{d}t + \int\limits_{\frac{x}{\bar{c}}}^{\infty} \frac{p(t)}{t^2}\,\mathrm{d}t - \int\limits_{x}^{\infty} \frac{p(t)}{t^2}\,\mathrm{d}t \\ &= \int\limits_{x}^{\infty} \left[\frac{c}{s}p(s/c) + \frac{\bar{c}}{s}p(s/\bar{c}) - \frac{1}{s}p(s)\right]\frac{\mathrm{d}s}{s},\end{aligned} \tag{36}$$

deduce that the last integrand is positive for any s, and then $x \mapsto xp\left(\frac{1}{x}\right)$ is subadditive.

(3) ⇒ (4): As in the first implication of this proof, make the appropriate changes of variables in order to write

$$\phi(c_1\lambda) + \cdots + \phi(c_n\lambda) - \phi(\lambda) = q(n-1) + \lambda \int_0^\infty e^{-\lambda y}[\bar{\mu}(y/c_1) + \cdots + \bar{\mu}(y/c_n) - \bar{\mu}(y)]\,dy.$$

By the same argument as in (36), deduce that $\lambda \mapsto \phi(c_1\lambda) + \cdots + \phi(c_n\lambda) - \phi(\lambda) \in \mathcal{BF}_\upsilon$.

(4) ⇒ (1): Just take $n = 2$. □

Remark 8.4. The last results bring us to the following comments:

(i) If $\phi \in \mathcal{BF}_\upsilon$, then $\lim_{\lambda\to 0+} \phi'(\lambda) = +\infty$. To see this, do as in the proof of Proposition 5.3: if $\phi'(0) = \lim_{\lambda\to 0+} \phi'(\lambda)$ was finite, then Proposition 3.1 would give

$$\lambda \mapsto \lim_{c\to 1-} \frac{1 - e^{-\upsilon_c \phi(\lambda)}}{1-c} = \lambda(\phi'(0+) - \phi'(\lambda)) \in \mathcal{BF},$$

and then, $\phi'(0+) - \phi' \in \mathcal{CM}$, so that $\phi'' \in \mathcal{CM}$, which is impossible.

(ii) Assume p is nondecreasing. Then, [15, Remark 2.19] ensures that $x \mapsto xp(1/x)$ is subadditive and this comforts the inclusion $\mathcal{BF}_\upsilon \subset \mathcal{BF}_\Theta$ obtained in Proposition 8.2.

8.2 Stochastic interpretation of the class $\mathcal{BF}_\upsilon$

Let X be a non-negative random variable associated with a cumulant function $\phi \in \mathcal{BF}_\upsilon$. Proposition 8.2 shows that X is necessarily infinitely divisible, and Theorem 8.3 shows that $\phi \in \mathcal{BF}_\upsilon$ is equivalent to the identity in law:

$$cX + (1-c)X' \overset{d}{=} X + W_c, \quad \text{for every } c \in (0,1), \tag{37}$$

where, in the r.h.s, X and X' are i.i.d random variables and in the l.h.s, the random variable W_c, independent of X, is necessarily infinitely divisible because its cumulant function is the Bernstein function $\upsilon_c\phi$ given in Definition 2.1. It is then legitimate to introduce the subclass **CRD** of *convex reverse distributions* of random variable satisfying (37). A straightforward stochastic reformulation of Theorem 8.3 is as follows:

Proposition 8.5. *Assume $X \sim$ **CRD** and $X_1, \ldots, X_n$ are independent copies of X. Then, for all $\underline{c} = (c_1, \ldots, c_n) \in (0,1)^n$ such that $c_1 + \cdots + c_n = 1$, there exists a non-negative random variable in $W_{\underline{c}} \sim$ **ID**, independent from X such that*

$$c_1X_1 + \cdots + c_nX_n \overset{d}{=} X + W_{\underline{c}}, \quad \forall n \in \mathbb{N}.$$

9 Conclusion

We have revisited the class **SD** of self-decomposable distributions, introduced its dual class **RD** and then extended **RD** to the class **CRD**. We would like to stress that the idea of this work is inspired by the original idea of Sendov and Shan [1], who were the first to exhibit the analytic interest of the operator Θ in the context of completely monotone functions and Bernstein functions. Our work completes theirs with further analysis and essentially, it provides the stochastic interpretation and comparison between the Lévy measure of transitions of the related subordinator. We are confident that the classes **RD** and **CRD** will find their way, like **SD**, in the applications of the theory of Lévy processes and infinite divisible distributions.

Acknowledgment: The authors would like to thank the referees for their valuable comments and suggestions that improved the presentation of this paper. The work of the first author was supported by the "Research Supporting Project number (RSP-2020/162), King Saud University, Riyadh, Saudi Arabia."

References

[1] H. Sendov and S. Shan, *New representation theorems for completely monotone and Bernstein functions with convexity properties on their measures*, J. Theor. Probab. **28** (2015), 1689–1725, DOI: https://doi.org/10.1007/s10959-014-0557-9.

[2] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions. Theory and Applications*, 2 edn., De Gruyter Studies in Mathematics **37**, Walter de Gruyter & Co., Berlin, 2012.

[3] M. Merkle, *Reciprocally convex functions*, J. Math. Anal. Appl. **293** (2004), 210–218, DOI: https://doi.org/10.1016/j.jmaa.2003.12.021.

[4] L. Steutel and K. Van Harn, *Infinite Divisibility of Probability Distributions on the Real Line*, Marcel Dekker, New York, Basel, 2004.

[5] A. E. Kyprianou, *Fluctuations of Lévy Processes with Applications, Introductory Lectures*, 2 edn., Springer-Verlag, Berlin Heidelberg, 2006.

[6] Z. J. Jurek, *Limit distributions and one-parameter groups of linear operators on Banach spaces*, J. Multivariate Anal. **13** (1983), 578–604, DOI: https://doi.org/10.1016/0047-259X(83)90042-8.

[7] K. I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics **68**, Cambridge University Press, Cambridge, 1999.

[8] R. Aguech and W. Jedidi, *New characterizations of completely monotone functions and Bernstein functions, a converse to Hausdorff's moment characterization theorem*, Arab J. Math. Sci. **25** (2019), no. 1, 57–82, DOI: https://doi.org/10.1016/j.ajmsc.2018.03.001.

[9] M. Kuczma, B. Choczewski, and R. Ger, *Iterative Functional Equations*, Cambridge University Press, Cambridge, 1990.

[10] L. Alili, W. Jedidi, and V. Rivero, *On exponential functionals, harmonic potential measures and undershoots of subordinators*, ALEA Lat. Am. J. Probab. Math. Stat. **11** (2014), no. 2, 711–735.

[11] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer Monographs in Mathematics, Berlin, 2014.

[12] V. M. Zolotarev, *One Dimensional Stable Laws*, Translations of Mathematical Monographs, vol. 65, American Mathematical Society, Providence, 1986.

[13] T. Simon, *On the unimodality of power transformations of positive stable densities*, Math. Nachr. **285** (2012), no. 4, 497–506, DOI: https://doi.org/10.1002/mana.201000062.

[14] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, 2 edn., Birkhäuser, Basel, 2009.

[15] J. Matkowski, *Subadditive periodic functions*, Opusc. Math. **31** (2011), no. 1, 75–96, DOI: https://doi.org/10.7494/OpMath.2011.31.1.75.