Research Article

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Skew-symmetric and essentially unitary operators via Berezin symbols

https://doi.org/10.1515/math-2020-0117 received February 3, 2020; accepted November 12, 2020

Abstract: We characterize skew-symmetric operators on a reproducing kernel Hilbert space in terms of their Berezin symbols. The solution of some operator equations with skew-symmetric operators is studied in terms of Berezin symbols. We also studied essentially unitary operators via Berezin symbols.

Keywords: skew-symmetric operator, composition operator, reproducing kernel Hilbert space, Berezin symbol, essentially unitary operator

MSC 2020: 47A58, 47B32

1 Introduction

We denote by \mathcal{H} a complex separable infinite dimensional Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . Sometimes the letter \mathcal{H} will denote the so-called reproducing kernel Hilbert space (RKHS) over some set Ω .

Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be skew-symmetric, if $T = -T^*$.

It is easy to check, for example, that the Volterra integral operator $(V_0f)(x) = \int_{-x}^x f(t) dt$ is a skew-symmetric operator on the space $L^2(-1, 1)$. Also, for any self-adjoint operator T on \mathcal{H} the operator iT is skew-symmetric.

As it is known, many classical results in the matrix theory deal with complex symmetric matrices (i.e., $T = T^t$) and skew-symmetric matrices (i.e., $T = -T^t$). These concepts appear naturally in a variety of applications such as complex analysis, functional analysis (including operator theory), and even quantum mechanics.

In [1], Zagorodnyuk studied the polar decomposition of skew-symmetric operators and obtained some basic properties of skew-symmetric operators. Later, Li and Zhu [2] also noted that an important way to investigate the structure of skew-symmetric operators is to characterize the skew-symmetry of concrete class of operators. For example, Zagorodnyuk [3] studied the skew-symmetry of cyclic operators. Li and Zhu [4] studied the skew-symmetry of normal operators and gave two structure theorems of skew-symmetric normal operators. For more information about skew-symmetric operators, see, for instance, Zhu [5], Li and Zhu [6], and references therein.

In the present article, we characterize skew-symmetric operators on an RKHS in terms of their Berezin symbols. We also study in terms of the Berezin symbols the solvability of some operator equations with skew-symmetric operators. Note that such an approach, apparently, was initiated by the second author

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in [7]. We also give in terms of the Berezin number a sufficient condition providing essential unitarity of essentially invertible operators.

2 Reproducing kernel Hilbert space and some properties

A reproducing kernel Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that evaluation $f \to f(\lambda)$ at any point of Ω is a continuous functional on \mathcal{H} . The Riesz representation theorem ensures that an RKHS \mathcal{H} has a reproducing kernel, that is, for every $\lambda \in \Omega$ there is a unique element $k_{\mathcal{H},\lambda} \in \mathcal{H}$ for which $f(\lambda) = \langle f, k_{\mathcal{H},\lambda} \rangle$ for all $f \in \mathcal{H}$. We call the function $k_{\mathcal{H},\lambda}$ the reproducing kernel at λ . The following proposition gives a way to compute the reproducing kernels (see, for instance, Aronzajn [8], Halmos [9], and Stroethoff [10]).

Proposition 1. If $\{e_j\}_{j\geq 1}$ is an orthonormal basis for the RKHS $\mathcal{H}=\mathcal{H}(\Omega)$, then

$$k_{\mathcal{H},\lambda} = \sum_{j=1}^{\infty} \overline{e_j(\lambda)} e_j,$$

where the convergence is in H. In particular,

$$k_{\mathcal{H},\lambda}(z) = \sum_{j=1}^{\infty} \overline{e_j(\lambda)} e_j(z), \quad z \in \Omega.$$

It follows from the aforementioned proposition that $k_{\mathcal{H},z}(\lambda) = \overline{k_{\mathcal{H},\lambda}(z)}$. Writing $k_{\mathcal{H}}(z,\lambda) = k_{\mathcal{H},\lambda}(z)$, we have $k_{\mathcal{H}}(\lambda, z) = \overline{k_{\mathcal{H}}(z, \lambda)}$ for all $z, \lambda \in \Omega$. The norm of $k_{\mathcal{H}, \lambda}$ is easily determined: $\|k_{\mathcal{H}, \lambda}\|^2 = \langle k_{\mathcal{H}, \lambda}, k_{\mathcal{H}, \lambda} \rangle = k_{\mathcal{H}, \lambda}(\lambda)$. The function

$$\widehat{k}_{\mathcal{H},\lambda} = \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|} = \frac{k_{\mathcal{H},\lambda}}{(k_{\mathcal{H},\lambda}(\lambda))^{1/2}}$$

is called the normalized reproducing kernel at λ .

Definition 2.1. Let T be a bounded linear operator on \mathcal{H} , the Berezin symbol of T is defined by $\widetilde{T}(\lambda) = 0$ $\langle T\hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda} \rangle$ for $\lambda \in \Omega$.

Note that the Berezin symbol \widetilde{T} is a complex-valued bounded function, because $|\widetilde{T}(\lambda)| \leq ||T||$ for all $\lambda \in \Omega$. Let ber(*T*) denote the Berezin number of *T* defined by

$$\operatorname{ber}(T) \coloneqq \sup_{\lambda \in \Omega} |\widetilde{T}(\lambda)| = \|\widetilde{T}\|_{L^{\infty}(\Omega)}.$$

For any $T \in \mathcal{B}(\mathcal{H})$, we can write

$$(Tf)(z) = K\langle Tf, k_{\mathcal{H},z} \rangle = \langle f, T^*k_{\mathcal{H},z} \rangle.$$

Thus, *T* is uniquely determined by the function $T^*k_{\mathcal{H},z}(\lambda)$ on $\Omega \times \Omega$, also by the function

$$T(z,\lambda) \coloneqq \frac{\overline{T^*k_{\mathcal{H},z}(\lambda)}}{k_{\mathcal{H},z}(\lambda)} = \frac{\langle Tk_{\mathcal{H},\lambda}, k_z \rangle}{\langle k_{\mathcal{H},\lambda}, k_{\mathcal{H},z} \rangle} = \frac{Tk_{\mathcal{H},\lambda}(z)}{k(z,\lambda)}$$

defined at all points, where $k(z, \lambda) \neq 0$. It follows from the definition that the mapping $T \to T(z, \lambda)$ is linear, $T^*(z,\lambda) = \overline{T(\lambda,z)}$, and $I(z,\lambda) = 1$, where *I* is the identity operator and 1 the constant function one. Also, if the elements of \mathcal{H} are continuous (smooth, holomorphic), then so is $T(z, \overline{\lambda})$ in each variable, at all points where $k_{\mathcal{H}}(z, \overline{\lambda}) \neq 0.$

Assume now that:

*the functions $T(z, \lambda)$ are uniquely determined by their restrictions T(z, z) to the diagonal.

This is the case, for example, whenever the functions $T(z, \lambda)$ *are* holomorphic in z and $\overline{\lambda}$, by a well-known classical theorem in complex analysis (see, for example, Folland [11] and Stroethoff [12, Lemma 2.3]). The following uniqueness lemma is due to the second author (see [13, Lemma 2]).

Lemma 2. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be an RKHS whose elements are functions on some set Ω . If \mathcal{H} satisfies condition (*), then the correspondence $T \leftrightarrow \widetilde{T}$ is one-to-one, i.e., T = 0 if and only if $\widetilde{T}(\overline{\lambda}) = 0$ for all $\lambda \in \Omega$.

Definition 2.2. The RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ is said to be standard (see Nordgren and Rosenthal [14]) if the underlying set Ω is a subset of a topological space and the boundary $\partial\Omega$ is nonempty and has the property that $\{\hat{k}_{\mathcal{H},\lambda_n}\}$ converges weakly to 0 whenever $\{\lambda_n\}$ is a sequence in Ω that converges to a point in $\partial\Omega$.

The common Hardy, Bergman, and Fock Hilbert spaces are standard in this sense (see Stroethoff [10]).

For a compact operator K on the standard RKHS \mathcal{H} , it is clear that $\lim_{n\to\infty} \widetilde{K}(\lambda_n) = 0$ whenever $\{\lambda_n\}$ converges to a point in $\partial\Omega$, since compact operators send weakly convergent sequences into strongly convergent ones. In this sense, the Berezin symbol of a compact operator on a standard RKHS vanishes on the boundary.

3 Operator equations on C*-algebras

The equations AX = C and XB = D for operators, including square and rectangular matrices, have a long history. In particular, the first equation has applications in the control theory. For more information and application about these equations, see, for example, [15] by Dajic and Koliha.

In the present section, we study the solution of the operator equations TX = K + Y and $XT = L + \mathbb{Z}$ (where K, L are compact and Y, \mathbb{Z} are skew-symmetric operators) in some Engliš C^* -operator algebras of operators on the reproducing kernel Hilbert spaces, including the Hardy space $H^2 = H^2(\mathbb{D})$ over the unit disc \mathbb{D} of the complex plane \mathbb{C} . First, we need some notations and preliminaries.

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$||f||_2^2 := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < +\infty.$$

Since $\{z^n\}_{n\geq 0}$ is an orthonormal basis in H^2 , it is easy to see from Proposition 1 that $k_{H^2,\lambda}(z)=\frac{1}{1-\bar{\lambda}z}$, and hence $\hat{k}_{H^2,\lambda}=\frac{k_{H^2,\lambda}}{\|k_{H^2,\lambda}\|}=\frac{(1-\|\lambda\|^2)^{1/2}}{1-\bar{\lambda}z}$. The symbol $H^\infty=H^\infty(\mathbb{D})$ denotes the Banach algebra of bounded and analytic functions on the unit disc \mathbb{D} equipped with the norm $\|f\|_\infty=\sup\{|f(z)|:z\in\mathbb{D}\}$. It is convenient to establish a natural embedding of the space H^2 in the space $L^2=L^2(\mathbb{T})$ by associating to each function $f\in H^2$ its radial boundary values $(bf)(\zeta):=\lim_{r\to \mathbb{T}}f(r\zeta)$, which (by the Fatou Theorem [16]) exist for almost all $\zeta\in\mathbb{T}=\partial\mathbb{D}$, where m is the normalized Lebesgue measure on \mathbb{T} . Then we have

$$H^2 = \{ f \in L^2 : \hat{f}(n) = 0, n < 0 \},$$

where $\hat{f}(n) \coloneqq \int_{\mathbb{T}} \overline{\zeta}^n f(\zeta) \mathrm{d}m(\zeta)$ is the Fourier coefficient of the function f.

If $\varphi \in L^{\infty} = L^{\infty}(\mathbb{T})$, then the Toeplitz operator T_{φ} on H^2 is defined by $T_{\varphi}f = P_{+}(\varphi f)$, where $P_{+}: L^2(\mathbb{T}) \to H^2$ is the Riesz projection (orthogonal projection). The harmonic extension of function $\varphi \in L^{\infty}$ is defined by $\widetilde{\varphi}$:

$$\widetilde{\varphi}(re^{i\theta}) = \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(e^{it}) \frac{1 - r^2}{1 + r^2 - 2 \cos(\theta - t)} dt, \quad re^{i\theta} \in \mathbb{D}.$$

The following lemma is well known (see, for instance, Engliš [17] and Zhu [19]).

Lemma 3. If T_{φ} is a Toeplitz operator on H^2 , then $\widetilde{T}_{\varphi}(\lambda) = \widetilde{\varphi}(\lambda)$ for all $\lambda \in \mathbb{D}$, i.e., $\widetilde{T}_{\varphi} = \widetilde{\varphi}$.

The main lemma for our further discussions is the following lemma due to Englis [17, Theorem 6] and Karaev [18, Lemma 1.1], which shows that the normalized reproducing kernels $\hat{k}_{H^2,\lambda}$ of H^2 are, loosely speaking, asymptotic eigenfunctions for any Toeplitz operator T_{φ} , $\varphi \in L^{\infty}(\mathbb{T})$.

Lemma 4. Let $\varphi \in L^{\infty}(\mathbb{T})$ and let $\widetilde{\varphi}$ be its harmonic extension into \mathbb{D} (which is the Berezin symbol of T_{φ}). Then $T_{\omega}\hat{k}_{H^2,\lambda} - \widetilde{T}_{\omega}(\lambda)\hat{k}_{\lambda} \to 0$ radially, i.e.,

$$\lim_{r \to 1^{-}} \| T_{\varphi} \hat{k}_{H^{2}, re^{it}} - \widetilde{\varphi}(re^{it}) \hat{k}_{H^{2}, re^{it}} \| = 0$$

for almost all $t \in (0, 2\pi)$.

The following set is defined by Engliš in [17, (6) in Section 3]

$$\varepsilon_{H^2} := \{ T \in \mathcal{B}(\mathcal{H}) : \|T\hat{k}_{H^2,\lambda}\|^2 - |\widetilde{T}(\lambda)|^2 \text{ and } \|T^*\hat{k}_{H^2,\lambda}\|^2 - |\widetilde{T}^*(\lambda)|^2 \to 0 \text{ radially} \}$$

(note that $|\widetilde{T}(\lambda)| = |\widetilde{T}^*(\lambda)|$ for all $\lambda \in \mathbb{D}$). Engliš proved (see [17, Section 3]) that ε_{H^2} is a C^* -algebra. We call ε_{H^2} the Engliš algebra of operators on the Hardy space H^2 . It follows from Lemma 4 that for any $\varphi \in L^{\infty}(\mathbb{T})$, $T_{\varphi} \in \mathcal{E}_{H^2}$. Here we study the solution of the operator equations TX = K + Y and XT = K + Y in the set

$$\mathcal{A}_{\mathcal{H}} := \{ A \in \mathcal{B}(\mathcal{H}) : \|A\hat{k}_{\mathcal{H},\lambda}\|^2 - |\widetilde{A}(\lambda)|^2 \to 0 \text{ as } \lambda \to \partial\Omega \}$$
 (1)

and in the Engliš algebra $\varepsilon_{\mathcal{H}}$, where *K* is compact and *Y* is a skew-symmetric operator, which is defined on an RKHS $\mathcal{H} = \mathcal{H}(\Omega)$, as follows:

$$\varepsilon_{\mathcal{H}} := \{ A \in \mathcal{B}(\mathcal{H}) : \|A\hat{k}_{\mathcal{H},\lambda}\|^2 - |\widetilde{A}(\lambda)|^2 \text{ and } \|A^*\hat{k}_{\mathcal{H},\lambda}\|^2 - |\widetilde{A}^*(\lambda)|^2 \to 0 \text{ as } \lambda \to \partial\Omega \}.$$

As in the case $\mathcal{H} = H^2$, it can be shown that actually the set $\varepsilon_{\mathcal{H}}$ is a C^* - algebra (see the proof of (A1) in Section 3 of Englis's paper [17]). We also characterize the skew-symmetric operators in terms of Berezin symbols.

Proposition 5. Let $\mathcal{H}(\Omega)$ be an RKHS such that the functions $T(z,\lambda)$ satisfy condition (*) whenever T is a bounded linear operator on $\mathcal{H} = \mathcal{H}(\Omega)$. Then the operator T is skew-symmetric if and only if Re $\widetilde{T} = 0$, i.e., if its Berezin symbol \tilde{T} is a purely imaginary complex-valued function.

Proof. According to Lemma 2, $T^* = -T$ if and only if $\widetilde{T}^*(\lambda) = -\widetilde{T}(\lambda)$ for all $\lambda \in \Omega$, or equivalently $\widetilde{T}^* + \widetilde{T}(\lambda) = 0$ for all $\lambda \in \Omega$. This means that

$$\overline{\langle T\hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda}\rangle} + \langle T\hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda}\rangle = 0$$

(because $\widetilde{T}^* = \overline{\widetilde{T}}$), and hence

$$2\operatorname{Re}\langle T\hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda}\rangle = 0$$

for all $\lambda \in \Omega$. Consequently, $T^* = -T$ if and only if $\operatorname{Re} \widetilde{T} = 0$ for all $\lambda \in \Omega$, as desired.

Theorem 6. Let $T, K, Y \in \mathcal{B}(\mathcal{H}(\Omega))$ be operators such that Y is skew-symmetric.

(i) If $\widetilde{K}(\lambda) \to 0$ as $\lambda \to \partial \Omega$ and $X \in A_{\mathcal{H}}$ is a solution of the equation

$$TX = K + Y, (2)$$

then

$$\lim_{\lambda\to\partial\Omega}\operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}(\lambda))=0.$$

(ii) If $T^* \in A_{\mathcal{H}}$ and $X \in B(\mathcal{H}(\Omega))$ is a solution of equation (2) with $\widetilde{K}(\lambda) \to 0$ as $\lambda \to \partial \Omega$, then

$$\lim_{\lambda \to \partial \Omega} \operatorname{Re}(\widetilde{X}(\lambda) \widetilde{T}(\lambda)) = 0.$$

(iii) Let $\mathcal{H}(\Omega)$ be an RKHS such that the functions $T(z,\lambda)$ satisfy condition (*) whenever T is a bounded linear operator on $\mathcal{H} = \mathcal{H}(\Omega)$. If $X \in B(\mathcal{H}(\Omega))$ is a solution of the equation TX = K + Y such that Range $(X) \perp Range(T^*)$, then K is skew-symmetric.

Proof.

(i) Let $X \in A_{\mathcal{H}}$ satisfy (2), that is, TX = K + Y. Then $X^*T^* = K^* + Y^* = K^* - Y$. Since $\widetilde{TX}(\lambda) = \widetilde{K}(\lambda) + \widetilde{Y}(\lambda)$ and $\widetilde{X^*T^*}(\lambda) = \widetilde{K}^*(\lambda) - \widetilde{Y}(\lambda)$, we get $\widetilde{TX}(\lambda) + \widetilde{X^*T^*}(\lambda) = \overline{K^* + K}(\lambda)$ for all $\lambda \in \Omega$. Since $\widetilde{TX}(\lambda) = \overline{TX}(\lambda)$, the latter means that

$$2 \operatorname{Re} (\widetilde{TX}(\lambda)) = \widetilde{K^* + K}(\lambda),$$

and hence

Re
$$\langle TX\hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda} \rangle = \frac{1}{2} \overline{(K^* + K)}(\lambda)$$
 (3)

for all $\lambda \in \Omega$. Then we have

$$\frac{1}{2}(\widetilde{K^* + K})(\lambda) = \operatorname{Re}\left[\langle X\hat{k}_{\mathcal{H},\lambda} - \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}, T^*\hat{k}_{\mathcal{H},\lambda}\rangle + \widetilde{X}(\lambda)\langle\hat{k}_{\mathcal{H},\lambda}, T^*\hat{k}_{\mathcal{H},\lambda}\rangle\right]$$

$$= \operatorname{Re}\langle X\hat{k}_{\mathcal{H},\lambda} - \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}, T^*\hat{k}_{\mathcal{H},\lambda}\rangle + \operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}(\lambda)),$$

from which we have that

$$|\operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}(\lambda))| \leq \frac{1}{2} |\widetilde{K^* + K}(\lambda)| + |\langle X\hat{k}_{\mathcal{H},\lambda} - \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}, T^*\hat{k}_{\mathcal{H},\lambda}\rangle|$$

$$\leq \frac{1}{2} |\widetilde{K^* + K}(\lambda)| + ||T|| ||X\hat{k}_{\mathcal{H},\lambda} - \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}||_{\mathcal{H}}$$

$$(4)$$

for all $\lambda \in \Omega$. On the other hand, by considering that $X\hat{k}_{\mathcal{H},\lambda} - \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda} \perp \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}$, we have

$$\|X\hat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}}^2 = \|X\hat{k}_{\mathcal{H},\lambda} - \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}}^2 + |\widetilde{X}(\lambda)|^2,$$

and hence

$$\|X\hat{k}_{\mathcal{H},\lambda}-\widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}}^2 = \|X\hat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}}^2 - |\widetilde{X}(\lambda)|^2$$

for all $\lambda \in \Omega$. This shows that

$$\lim_{\lambda \to \zeta \in \partial \Omega} (\|X \hat{k}_{\mathcal{H}, \lambda}\|_{\mathcal{H}}^2 - |\widetilde{X}(\lambda)|^2) = 0$$

if and only if

$$\lim_{\lambda \to \zeta \in \partial \Omega} (\|X\hat{k}_{\mathcal{H},\lambda} - \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}}) = 0.$$

So, since

$$\widetilde{K^* + K}(\lambda) \to 0$$
 as $\lambda \to \zeta \in \partial \Omega$,

we have from the inequality (4) that

$$\lim_{\lambda \to \zeta \in \partial\Omega} (\operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}(\lambda))) = 0,$$

which implies the desired result.

(ii) It follows from (3) that

$$\operatorname{Re}\left[\langle X\hat{k}_{\mathcal{H},\lambda}, T^*\hat{k}_{\mathcal{H},\lambda} - \widetilde{T}^*(\lambda)\hat{k}_{\mathcal{H},\lambda}\rangle\right] + \operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}(\lambda)) = \frac{1}{2}\widetilde{(K^* + K)}(\lambda),$$

and therefore,

$$|\operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}(\lambda))| \leq ||X|| \, ||T^*\hat{k}_{\mathcal{H},\lambda} - \widetilde{T}^*(\lambda)\hat{k}_{\mathcal{H},\lambda}|| + \frac{1}{2} ||\widetilde{(K^* + K)}(\lambda)||$$

for all $\lambda \in \Omega$. Now the result follows immediately from this inequality. This proves (ii).

(iii) It follows from (3) that

$$\operatorname{Re}\langle X\hat{k}_{\mathcal{H},\lambda}, T^*\hat{k}_{\mathcal{H},\lambda}\rangle = \frac{1}{2}\widetilde{(K^*+K)}(\lambda)$$

for all $\lambda \in \Omega$, and since $X\mathcal{H} \perp T^*\mathcal{H}$, we have

$$\widetilde{K^* + K}(\lambda) = 0$$

for all $\lambda \in \Omega$. This implies by Lemma 2 that

$$K^* = -K,$$

as desired. This proves the theorem.

Remark 1. Assertion (ii) in Theorem 3 shows that the necessary condition

$$\operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}(\lambda)) \to 0 \text{ as } \lambda \to \partial\Omega$$

in (i) is not in general a sufficient condition for a solution X of (2) being from the class $\mathcal{A}_{\mathcal{H}}$.

Corollary 7. Let $\mathcal{H}(\Omega)$ be an RKHS such that the functions $T(z,\lambda)$ satisfy condition (*) whenever T is a bounded linear operator on \mathcal{H} . Let $T, K, Y \in \mathcal{B}(\mathcal{H})$ be operators such that Y is skew-symmetric and K is not skew-symmetric. Then the operator equation TX = K + Y has no solution with the property that $Range(X) \perp Range(T^*)$.

Remark 2. It is easy to see from the proof of Theorem 6 that the same results can be obtained for the operator equation XT = K + Y; we omit them.

Proposition 8. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be an RKHS, and let T_1 , T_2 , K_1 , K_2 , Y_1 , $Y_2 \in \mathcal{B}(\mathcal{H})$ be operators such that $\widetilde{K}_1(\lambda) \to 0$, $\widetilde{K}_2(\lambda) \to 0$ as $\lambda \to \partial \Omega$, Y_1 and Y_2 are the skew-symmetric operators. If $X \in \varepsilon_{\mathcal{H}}$ satisfies the equations

$$T_1 X = K_1 + Y_1 \tag{5}$$

and

$$XT_2 = K_2 + Y_2,$$
 (6)

then

$$\lim_{\lambda \to \partial \Omega} |\operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}_{1}(\lambda))| = 0$$

and

$$\lim_{\lambda \to \partial \Omega} |\operatorname{Re}(\widetilde{X}(\lambda) \, \widetilde{T}_2(\lambda))| = 0.$$

Proof. The proof is similar to the proof of (i) in Theorem 6. Indeed, if $X \in \mathcal{E}_{\mathcal{H}}$ is a common solution of equations (5) and (6), then as in the proof of (i) in Theorem 6, we obtain that

$$|\operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}_{1}(\lambda))| \leq \left| \frac{\widetilde{K_{1}^{*}+K_{1}}(\lambda)}{2} \right| + \|T_{1}\| \|X\hat{k}_{\mathcal{H},\lambda} - \widetilde{X}(\lambda)\hat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}}$$

and

$$|\operatorname{Re}(\widetilde{X}(\lambda)\widetilde{T}_{2}(\lambda))| \leq \left| \begin{array}{c} \widetilde{K_{2}^{*} + K_{2}}(\lambda) \\ 2 \end{array} \right| + \|T_{2}\| \|X^{*}\widehat{k}_{\mathcal{H},\lambda} - \widetilde{X}^{*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}}.$$

Since $X \in \mathcal{E}_{\mathcal{H}}$ and $\widetilde{K}_i(\lambda)$, $\widetilde{K}_i^*(\lambda) \to 0$ (i = 1, 2) as $\lambda \to \partial \Omega$, the desired results are immediate from these inequalities. This proves the proposition.

Corollary 9. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a standard RKHS, and let T_1 , T_2 , K_1 , K_2 , Y_1 , $Y_2 \in \mathcal{B}(\mathcal{H})$ be operators such that K_1 , K_2 are compact and Y_1 , Y_2 are skew-symmetric. If $X \in \mathcal{E}_{\mathcal{H}}$ satisfies equations (5) and (6), then

$$\max \left\{ \lim_{\lambda \to \partial \Omega} |\operatorname{Re} \left(\widetilde{X}(\lambda) \, \widetilde{T}_1(\lambda) \right)|, \, \lim_{\lambda \to \partial \Omega} |\operatorname{Re} \left(\widetilde{X}(\lambda) \, \widetilde{T}_2(\lambda) \right)| \right\} = 0.$$

The proof of the following corollary is immediate from Lemmas 3 and 4 and Proposition 8.

Corollary 10. Let $\mathcal{H} = H^2$ in Proposition 8. If the Toeplitz operator T_{φ} with the symbol $\varphi \in L^{\infty}(\mathbb{T})$ is a common solution of (5) and (6), then

$$\max \left\{ \lim_{\lambda \to \partial \Omega} | \operatorname{Re} \left(\widetilde{\varphi}(\lambda) \widetilde{T}_1(\lambda) \right) |, \lim_{\lambda \to \partial \Omega} | \operatorname{Re} \left(\widetilde{\varphi}(\lambda) \widetilde{T}_2(\lambda) \right) | \right\} = 0.$$

Remark 3. Note that the Berezin symbol of $\widetilde{\varphi}$ of a function $\varphi \in L^{\infty}(\mathbb{D}, dA)$ is defined to be the Berezin symbol of the Toeplitz operator T_{φ} on the Bergman space $L_a^2 = L_a^2(\mathbb{D})$ with the normalized reproducing kernel $\hat{k}_{a,\lambda}(z) := \frac{1-|\lambda|^2}{(1-\overline{\lambda}z)^2}$. In other words, $\widetilde{\varphi} = \widetilde{T}_{\varphi}$. Because $\langle T_{\varphi}\hat{k}_{a,\lambda}, \hat{k}_{a,\lambda} \rangle = \langle P(\varphi\hat{k}_{a,\lambda}), \hat{k}_{a,\lambda} \rangle = \langle \varphi\hat{k}_{a,\lambda}, \hat{k}_{a,\lambda} \rangle$, we obtain the formula

$$\widetilde{\varphi}(\lambda) = \int\limits_{\mathbb{D}} \varphi(z) |\hat{k}_{a,\lambda}(z)|^2 \mathrm{d}A(z),$$

where $dA(z) = \frac{dxdy}{\pi}$ is the normalized Lebesgue area measure and P is the Bergman projection.

The Berezin symbol of a function in $L^{\infty}(\mathbb{D}, dA)$ often plays the same important role in the theory of Bergman spaces as the harmonic extension of a function in $L^{\infty}(\partial \mathbb{D})$ plays in the theory of Hardy spaces.

The Toeplitz algebra \mathcal{T} is the C^* -subalgebra of $\mathcal{B}(L_a^2)$ generated by $\{T_g:g\in H^\infty\}$. Let \mathcal{U} denote the C^* -subalgebra of $L^\infty(\mathbb{D},\,\mathrm{d}A)$ generated by H^∞ . As it is well known (see [20, Proposition 4.5]), \mathcal{U} equals the closed subalgebra of $L^\infty(\mathbb{D},\,\mathrm{d}A)$ generated by the set of bounded harmonic functions on \mathbb{D} . Although the map $u\mapsto T_u$ is not multiplicative on $L^\infty(\mathbb{D},\,\mathrm{d}A)$, the identities $T_u^*=T_{\overline{u}},\,T_uT_v=T_{uv}$, and $T_{\overline{v}}T_u=T_{\overline{v}u}$ hold for all $u\in L^\infty(\mathbb{D},\,\mathrm{d}A)$, and $v\in H^\infty$. This implies that \mathcal{T} equals the closed subalgebra of $\mathcal{B}(L_a^2)$ generated by the Toeplitz operators with bounded harmonic symbol, and that \mathcal{T} also equals the closed subalgebra of $\mathcal{B}(L_a^2)$ generated by $\{T_u:u\in\mathcal{U}\}$. The main goal of the paper [20] is to study the boundary behavior of the Berezin symbols of the operators in \mathcal{T} and of the functions in \mathcal{U} . Namely, the author's study shows (see [20, Theorem 2.11]) that if $S\in\mathcal{T}$, then $\widetilde{S}\in\mathcal{U}$. Also, they prove (see [20, Corollary 3.4]) that if $u\in\mathcal{U}$, then $\widetilde{u}=u$ has nontangential limit 0 at almost every point of $\partial\mathbb{D}$. Using similar techniques, they prove (see [20, Corollary 3.7]) that if $u\in\mathcal{U}$, then the function $\lambda\mapsto \|T_{u-\widetilde{u}(\lambda)}\hat{k}_{a,\lambda}\|_2$ has nontangential limit 0 at almost every point of $\partial\mathbb{D}$. Theorem 3.10 in [20] describes the functions $u\in\mathcal{U}$ such that $\widetilde{u}(\lambda)=u(\lambda)\to 0$ as $\lambda\mapsto\partial\mathbb{D}$. The aforementioned assertions show that the function $\lambda\mapsto \|T_{u-\widetilde{u}(\lambda)}\hat{k}_{a,\lambda}\|_2$ has nontangential limit 0 at almost every point of $\partial\mathbb{D}$. Since this property is mainly used in the proof of Corollary 10, the same results also can be proved for the Bergman space Toeplitz operators T_u with $u\in\mathcal{U}$, which we omit.

Remark 4. As an application of Proposition 5 and also for its usefulness note the following: if $\varphi \in H^{\infty}$ is a non-constant function and θ is a nontrivial analytic map of $\mathbb D$ onto itself (i.e., $\theta(\mathbb D) \subset \mathbb D$ and $\theta \neq \text{constant}$), then we can construct a weighted composition operator with symbols φ , θ ,

$$W_{\omega,\theta}f := T_{\omega}C_{\theta}f = \varphi f \circ \theta = \varphi f(\theta(z))$$

for f in the Hardy space H^2 , where T_{ω} is an analytic Toeplitz operator and C_{θ} is a composition operator on H^2 .

Note that $T_{\varphi}^* = T_{\overline{\varphi}}$, co-analytic Toeplitz operator. However, in the theory of composition operator determination of the adjoint is a problem of some interest. For example, this question is not trivial even for the composition operator C_{z^2} ; for more discussion about the adjoint of composition operators, see [21–26] and references therein. Thus, in particular, the investigation of skew-symmetric weighted composition operators and the operators of the form $C^*_{\theta}T_{\varphi}$ with $\varphi\in L^{\infty}$ is not in general a trivial question, while for the symbol θ with $\theta(0) = 0$, C_{θ} is obviously non skew-symmetric. In fact, since the set $\{k_{\lambda} : \lambda \in \mathbb{D}\}$ spans H^2 , it is easy to see that $C_{\theta}^* = -C_{\theta}$ if and only if $C_{\theta}^* k_{\lambda} = -C_{\theta} k_{\lambda}$ for all $\lambda \in \mathbb{D}$. Equivalently, $C_{\theta}^* = -C_{\theta}$ if and only if (see (7) below)

$$\frac{1}{1-\overline{\theta(\lambda)}z}=-\frac{1}{1-\overline{\lambda}\theta(z)}\ (\forall \lambda\in\mathbb{D}).$$

But, since $\theta(0) = 0$, for $\lambda = 0$ this equality does not hold, and hence $C_{\theta}^* \neq -C_{\theta}$, that is, C_{θ} is not skew-symmetric.

More generally, every composition operator C_{θ} on H^2 (or on the Bergman space $L_a^2(\mathbb{D})$) is not skewsymmetric. In fact, for $\lambda = 0$ we have that

$$Re((\widetilde{C}_{\theta})(0)) = Re\langle C_{\theta} \mathbf{1}, \mathbf{1} \rangle = \langle \mathbf{1}, \mathbf{1} \rangle = 1,$$

hence, by Proposition 5, C_{θ} cannot be skew-symmetric.

There is also a direct proof without using Proposition 5. In fact, as before, $C_{\theta}^* = -C_{\theta}$ if and only if $C_{\theta}^*k_{H^2,\lambda} = -C_{\theta}k_{H^2,\lambda}$ (or $C_{\theta}^*k_{a,\lambda} = -C_{\theta}k_{a,\lambda}$) for all $\lambda \in \mathbb{D}$. Equivalently, $C_{\theta}^* = -C_{\theta}$ if and only if $\frac{1}{1-\overline{\theta(\lambda)}z} = -\frac{1}{1-\overline{\lambda}\theta(z)}$ $(\text{or } \frac{1}{(1-\overline{\theta(\lambda)}z)^2}=-\frac{1}{(1-\overline{\lambda}\theta(z))^2}) \text{ for all } \lambda \in \mathbb{D} \text{ and } z \in \mathbb{D}. \text{ In particular, for } \lambda=0 \text{ and } z=\theta(0), \text{ we have } \frac{1}{1-|\theta(0)|^2}=-1$ (or $\frac{1}{(1-|\theta(0)|^2)^2}=-1$), which is impossible. This shows that every composition operator C_θ on the Hardy and Bergman spaces is not skew-symmetric.

However, in the following examples we demonstrate usefulness and application of Proposition 5, since $C_{\theta}^* k_{\lambda}$ has an explicit expression:

$$C_{\theta}^* k_{\lambda} = \frac{1}{1 - \overline{\theta(\lambda)}z} \ (\lambda \in \mathbb{D}). \tag{7}$$

Indeed, for any $u \in \mathbb{D}$, we have:

$$(C_{\theta}^*k_{\lambda})(\mu) = \langle C_{\theta}^*k_{\lambda}, k_{\mu} \rangle = \langle k_{\lambda}, C_{\theta}k_{\mu} \rangle = \left\langle k_{\lambda}, \frac{1}{1 - \overline{\mu}\theta(z)} \right\rangle = \overline{\left\langle \frac{1}{1 - \overline{\mu}\theta(z)}, k_{\lambda} \right\rangle} = \overline{\frac{1}{1 - \overline{\mu}\theta(\lambda)}} = \overline{\frac{1}{1 - \overline{\mu}\theta(\lambda)}} = \overline{\frac{1}{1 - \overline{\mu}\theta(\lambda)}},$$

which proves (7).

Example 1. Let $\varphi \in H^{\infty}$ be as in Remark 4. Then $W_{\varphi,\theta}f$ is skew-symmetric if and only if

$$\operatorname{Re}(\varphi(\lambda)(1-\lambda\overline{\theta(\lambda)}))=0 \ (\forall \lambda \in \mathbb{D}).$$

Proof. Let $\lambda \in \mathbb{D}$ be arbitrary. Then by using formula (7), we have:

$$\begin{split} \widetilde{W_{\varphi,\theta}}(\lambda) &= \langle T_{\varphi} C_{\theta} \hat{k}_{\lambda}, \, \hat{k}_{\lambda} \rangle = \langle C_{\theta} \hat{k}_{\lambda}, \, T_{\varphi}^{*} \hat{k}_{\lambda} \rangle = \langle C_{\theta} \hat{k}_{\lambda}, \, \overline{\varphi(\lambda)} \hat{k}_{\lambda} \rangle = (1 - |\lambda|^{2}) \varphi(\lambda) \langle k_{\lambda}, \, C_{\theta}^{*} k_{\lambda} \rangle \\ &= (1 - |\lambda|^{2}) \varphi(\lambda) \left\langle k_{\lambda}, \, \frac{1}{1 - \overline{\theta(\lambda)} z} \right\rangle = \frac{(1 - |\lambda|^{2}) \varphi(\lambda)}{1 - \overline{\lambda} \theta(\lambda)} = \frac{(1 - |\lambda|^{2})}{|1 - \overline{\lambda} \theta(\lambda)|^{2}} (\varphi(\lambda) (1 - \lambda \overline{\theta(\lambda)})), \end{split}$$

hence, $\operatorname{Re}(\widetilde{W_{\varphi,\theta}}(\lambda)) = 0$ if and only if $\operatorname{Re}(\varphi(\lambda)(1 - \lambda \overline{\theta(\lambda)})) = 0$. Since $\lambda \in \mathbb{D}$ is arbitrary, Proposition 5 works.

Example 2. Let $\varphi \in H^{\infty}$ be a non-constant function and θ be a non-trivial analytic self-map of the unit disc \mathbb{D} . Then $C^*_{\theta}T_{\varphi}$ is a skew-symmetric operator on H^2 if and only if $\operatorname{Re}(\widetilde{\Psi_{\lambda}}(\lambda)) = 0$ for all $\lambda \in \mathbb{D}$, where Ψ_{λ} denotes the function $\varphi(\zeta) \frac{1-\lambda \overline{\zeta}}{1-\lambda \overline{\theta(\zeta)}}$ on \mathbb{T} .

Proof. Indeed, we have:

$$\begin{split} \widehat{C_{\theta}^{*}T_{\varphi}}(\lambda) &= \left\langle C_{\theta}^{*}T_{\varphi}\widehat{k}_{\lambda},\, \widehat{k}_{\lambda}\right\rangle = \left\langle T_{\varphi}\widehat{k}_{\lambda},\, C_{\theta}\widehat{k}_{\lambda}\right\rangle = (1-|\lambda|^{2})\left\langle P_{+}\varphi k_{\lambda},\, \frac{1}{1-\overline{\lambda}\overline{\theta}(z)}\right\rangle \\ &= (1-|\lambda|^{2})\left\langle \varphi k_{\lambda},\, \frac{1}{1-\overline{\lambda}\overline{\theta}(z)}\right\rangle \;\; \text{(because }\; (1-\overline{\lambda}\theta(z))^{-1} \in H^{\infty}) \\ &= (1-|\lambda|^{2})\,\int_{\mathbb{T}} \frac{\varphi(\zeta)}{1-\lambda\overline{\zeta}}\, \frac{1}{1-\lambda\overline{\theta}(\zeta)} \;\mathrm{d}\zeta = \int_{\mathbb{T}} \varphi(\zeta)\, \frac{1-\lambda\overline{\zeta}}{1-\lambda\overline{\theta}(\zeta)}\, \frac{1-|\lambda|^{2}}{|1-\overline{\lambda}\zeta|^{2}} \;\mathrm{d}\zeta \\ &= \int_{\mathbb{T}} \varphi(\zeta)\, \frac{1-\lambda\overline{\zeta}}{1-\lambda\overline{\theta}(\zeta)}\, \frac{1-|\lambda|^{2}}{|\zeta-\lambda|^{2}} \;\mathrm{d}\zeta = \int_{\mathbb{T}} \Psi_{\lambda}(\zeta)\, \frac{1-|\lambda|^{2}}{|\zeta-\lambda|^{2}} \;\mathrm{d}\zeta. \end{split}$$

It is easy to see that $\Psi_{\lambda} \in L^{\infty}$ since $|\lambda \overline{\theta(\zeta)}| < 1$ for all $\lambda \in \mathbb{D}$. Hence, the last formula means that $\widehat{C_{\theta}^*T_{\varphi}}(\lambda) = \widehat{\Psi_{\lambda}}(\lambda)$, where $\widehat{\Psi_{\lambda}}$ denotes the harmonic extension of Ψ_{λ} into \mathbb{D} . Then, by Proposition 5, we deduce that $C_{\theta}^*T_{\varphi}$ is a skew-symmetric if and only if $\text{Re}(\widehat{\Psi_{\lambda}}(\lambda)) = 0$ for all $\lambda \in \mathbb{D}$, as required.

4 On essential unitarity of essentially invertible operators

It is well known that unitary operators on a Hilbert space H can be characterized as invertible contractions with contractive inverses, i.e., as operators T with $||T|| \le 1$ and $||T^{-1}|| \le 1$. Recently, Sano and Uchiyama [27] proved that if T is an invertible operator on H such that $w(T) \le 1$ and $w(T^{-1}) \le 1$, then T is unitary (see also Stampfli [28, Corollary 1]); here w(T) denotes the numerical radius of T defined by

$$w(T) := \sup\{|\langle Tx, x \rangle| : x \in H \text{ and } ||x||_H = 1\}.$$

Ando and Li [29, Theorem 1.1] generalized the latter by using the so-called ρ -radius of operator $T \in \mathcal{B}(\mathcal{H})$ defined by

$$w_0(T) := \inf\{\mu > 0 : \mu^{-1}T \in C_0\},\$$

where C_{ρ} denotes the class of operators $T \in \mathcal{B}(\mathcal{H})$ which admits a unitary ρ -dilation, i.e., there is a unitary operator U on a subspace $\mathcal{K} \supset H$ such that $T^n = \rho P_H U^n | H$ for n = 1, 2, ..., where $P_H : \mathcal{K} \to H$ is the orthoprojection. When $\rho = 1$ and $\rho = 2$, this definition reduces to the operator norm and numerical radius, respectively. (For more details see also Garayev [7], and Sahoo et al. [30].)

We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is essentially invertible (or Fredholm) if there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that AT - I and TA - I are both compact operators, i.e.,

$$AT = I + K_1$$
 and $TA = I + K_2$

for some compact operators K_1 , K_2 on H. An essential inverse of T will be denoted as $T^{-1 \text{ ess}}$. Thus, the following problem naturally arises.

Problem 1

To find in terms of Berezin numbers of an essentially invertible operator T and its essential inverse T^{-1} ess the necessary and sufficient conditions under which T be an essentially unitary operator on RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ (i.e., $T^*T = I + K_1$ and $TT^* = I + K_2$ for some compact operators K_1 and K_2).

Note that the class of essentially unitary operators has not been extensively studied with respect to the class of unitary operators.

The present section, which is motivated by this question (see also [7]), gives in terms of the Berezin numbers of operators TT^* and $(TT^*)^{-1}$ ess a sufficient condition for essentially unitarity of the essentially invertible operator *T* on the RKHS. Our result improves the result of the paper [7, Theorem 1] where only the case $K_1 = K_2 = 0$ is considered.

Theorem 11. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a standard RKHS with the property (*) (Section 2) and $T \in \mathcal{B}(\mathcal{H})$ be an essentially invertible operator, associated with the compact operators K_1 and K_2 . If

$$||T^*\hat{k}_{\mathcal{H},\lambda}||^2 \le 1 + \operatorname{Re}\widetilde{K}_2(\lambda)$$

and

$$||T^{-1}ess\hat{k}_{\mathcal{H},\lambda}||^2 \le 1 + \operatorname{Re}\widetilde{K}_2(\lambda)$$

for all $\lambda \in \Omega$ which obviously imply that

$$\operatorname{ber}(TT^*) \leq \sup_{\lambda \in \Omega} (1 + \operatorname{Re} \widetilde{K}_2(\lambda))$$

and

$$\operatorname{ber}(T^{-1\operatorname{ess}} T^{-1\operatorname{ess}}) \leq \sup_{\lambda \in \Omega} (1 + \operatorname{Re} \widetilde{K}_2(\lambda)),$$

then T is an essentially unitary operator.

Proof. We have for all $\lambda \in \Omega$ that

$$\begin{split} \|(T^* - T^{-1 \, \text{ess}}) \hat{k}_{\mathcal{H},\lambda}\|^2 &= \langle (T^* - T^{-1 \, \text{ess}}) \hat{k}_{\mathcal{H},\lambda}, (T^* - T^{-1 \, \text{ess}}) \hat{k}_{\mathcal{H},\lambda} \rangle \\ &= \|T^* \hat{k}_{\mathcal{H},\lambda}\|^2 + \|T^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda}\|^2 - \langle T^* \hat{k}_{\mathcal{H},\lambda}, T^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda} \rangle - \langle T^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda}, T^* \hat{k}_{\mathcal{H},\lambda} \rangle \\ &= \|T^* \hat{k}_{\mathcal{H},\lambda}\|^2 + \|T^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda}\|^2 - \langle \hat{k}_{\mathcal{H},\lambda}, TT^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda} \rangle - \langle TT^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda} \rangle \\ &= \|T^* \hat{k}_{\mathcal{H},\lambda}\|^2 + \|T^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda}\|^2 - \langle \hat{k}_{\mathcal{H},\lambda}, (I + K_2) \hat{k}_{\mathcal{H},\lambda} \rangle - \langle (I + K_2) \hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda} \rangle \\ &= \|T^* \hat{k}_{\mathcal{H},\lambda}\|^2 + \|T^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda}\|^2 - 2 - \overline{\widetilde{K}_2(\lambda)} - \widetilde{K}_2(\lambda) \\ &= \|T^* \hat{k}_{\mathcal{H},\lambda}\|^2 + \|T^{-1 \, \text{ess}} \hat{k}_{\mathcal{H},\lambda}\|^2 - 2(1 + \operatorname{Re} \widetilde{K}_2(\lambda)). \end{split}$$

Hence, by considering conditions, we conclude from the latter that

$$||(T^* - T^{-1} ess) \hat{k}_{\mathcal{H},\lambda}||^2 \le 0,$$

and therefore $(T^* - T^{-1 \text{ ess}})k_{\mathcal{H},\lambda} = 0$ for all $\lambda \in \Omega$. Now by considering that $\{k_{\mathcal{H},\lambda} : \lambda \in \Omega\}$ is a total set, we deduce that $T^* = T^{-1}ess$, and hence $T^*T = I + K_1$ and $TT^* = I + K_2$, which proves the theorem.

Acknowledgement: This study was funded by Researchers Supporting Project number (RSP-2020/187), King Saud University, Riyadh, Saudi Arabia.

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