Research Article

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Approximation properties of tensor norms and operator ideals for Banach spaces

https://doi.org/10.1515/math-2020-0116
received June 12, 2019; accepted November 17, 2020

Abstract: For a finitely generated tensor norm α, we investigate the α-approximation property (α-AP) and the bounded α-approximation property (bounded α-AP) in terms of some approximation properties of operator ideals. We prove that a Banach space X has the λ-bounded $\alpha_{p,q}$-AP ($1 \le p, q \le \infty$, $1/p + 1/q \ge 1$) if it has the λ-bounded g_p-AP. As a consequence, it follows that if a Banach space X has the λ-bounded g_p-AP, then X has the λ-bounded w_p-AP.

Keywords: approximation property, tensor norm, Banach operator ideal

MSC 2020: 46B28, 46B45, 47L20

1 Introduction

The main subjects of this paper originate from the classical approximation properties for Banach spaces, which was systematically investigated by Grothendieck [1]. A Banach space X is said to have the *approximation property* (AP) if

$$\mathrm{id}_X \in \overline{\mathcal{F}(X, X)}^{\tau_c},$$

where id_X is the identity map on X, $\mathcal{F}$ is the ideal of finite rank operators and τ_c is the topology of uniform convergence on compact sets.

Let X and Y be Banach spaces. We denote by $X \otimes Y$ the algebraic tensor product of X and Y. The normed space $X \otimes Y$ equipped with a norm α is denoted by $X \otimes_\alpha Y$ and its completion is denoted by $X \hat{\otimes}_\alpha Y$. The basic two norms on $X \otimes Y$ are the *injective norm* ε and the *projective norm* π which are defined as follows.

$$\varepsilon(u; X, Y) \coloneqq \sup\left\{\left|\sum_{j=1}^{n} x^*(x_j)y^*(y_j)\right| : x^* \in B_{X^*}, y^* \in B_{Y^*}\right\},$$

where $\sum_{j=1}^{n} x_j \otimes y_j$ is any representation of u and we denote by B_Z the closed unit ball of a normed space Z.

$$\pi(u; X, Y) \coloneqq \inf\left\{\sum_{j=1}^{n} \|x_j\|\|y_j\| : u = \sum_{j=1}^{n} x_j \otimes y_j, n \in \mathbb{N}\right\}.$$

It is well known that a Banach space X has the AP if and only if for every Banach space Y, the natural map

$$J_\pi : Y \hat{\otimes}_\pi X \to Y \hat{\otimes}_\varepsilon X$$

is injective (cf. [2, Theorem 5.6]). This equivalent statement can be naturally extended to tensor norms. For basic definitions and general background of the theory of tensor norms, we refer to [2,3]. For a finitely

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generated tensor norm α, a Banach space X is said to have the α-AP if for every Banach space Y, the natural map

$$J_\alpha : Y \hat{\otimes}_\alpha X \to Y \hat{\otimes}_\varepsilon X$$

is injective (cf. [2, Section 21.7]). It is well known that if a Banach space X has the AP, then it has the α-AP for every finitely generated tensor norm α (cf. [2, Proposition 21.7(1)]).

Some of the well-known tensor norms can be obtained from the tensor norm $\alpha_{p,q}$ ($1 \le p, q \le \infty$, $1/p + 1/q \ge 1$), which was introduced by Lapresté [4]. For $1 \le p < \infty$, $\ell_p^w(X)$ stands for the Banach space of all X-valued weakly p-summable sequences endowed with the norm $\|\cdot\|_p^w$. Let $1 \le r \le \infty$ with $1/r = 1/p + 1/q - 1$. For $u \in X \otimes Y$, let

$$\alpha_{p,q}(u) := \inf\left\{\|(\lambda_j)_{j=1}^n\|_r \|(x_j)_{j=1}^n\|_{q^*}^w \|(y_j)_{j=1}^n\|_{p^*}^w : u = \sum_{j=1}^n \lambda_j x_j \otimes y_j,\, n \in \mathbb{N}\right\},$$

where p^* is the conjugate index of p. Then $\alpha_{p,q}$ is a finitely generated tensor norm and the *transposed tensor norm* $\alpha_{p,q}^t = \alpha_{q,p}$ (cf. [2, Proposition 12.5]). The special cases $g_p := \alpha_{p,1}$ and $d_p := \alpha_{1,p}$ are called the *Chevet-Saphar tensor norms* [5,6] and $\alpha_{1,1} = \pi$. The tensor norm $w_p := \alpha_{p,p^*}$ is also well known. Díaz et al. [7] studied the $\alpha_{p,q}$-AP in terms of certain approximation properties of operator ideals. As a consequence, it was shown that a Banach space X has the $\alpha_{p,q}$-AP if it has the $\alpha_{p,1}$-AP.

Let $\lambda \ge 1$. A Banach space X is said to have the λ-*bounded* AP if

$$\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X, X) : \|S\| \le \lambda\}}^{\tau_c}.$$

It is well known that a Banach space X has the λ-bounded AP if and only if for every Banach space Y, the natural map

$$I_\pi : Y \otimes_\pi X \to (Y^* \otimes_\varepsilon X^*)^*$$

satisfies $\pi(u; Y, X) \le \lambda\, \|I_\pi(u)\|_{(Y^* \otimes_\varepsilon X^*)^*}$ for every $u \in Y \otimes X$ (cf. [2, Corollary 16.3.2]). More generally, for a finitely generated tensor norm α, a Banach space X is said to have the λ-*bounded* α-AP if for every Banach space Y, the natural map

$$I_\alpha : Y \otimes_\alpha X \to (Y^* \otimes_{\alpha'} X^*)^*$$

satisfies $\alpha(u; Y, X) \le \lambda\, \|I_\alpha(u)\|_{(Y^* \otimes_{\alpha'} X^*)^*}$ for every $u \in Y \otimes X$ (cf. [2, Section 21.7]), where α' is the *dual tensor norm* (cf. [2]) of α. Note that $\pi' = \varepsilon$.

The main goal of this paper is to study the α-AP and the λ-bounded α-AP in terms of operator ideals. In Section 2, we extend the result of Díaz et al. [7], and in Section 3, we obtain some bounded versions of the results obtained in Section 2. As an application, it is shown that a Banach space X has the λ-bounded $\alpha_{p,q}$-AP if it has the λ-bounded $\alpha_{p,1}$-AP. Consequently, if X has the λ-bounded $\alpha_{p,1}$-AP, then X has the λ-bounded α_{p,p^*}-AP.

2 The α-approximation property

We denote by $[\mathcal{L}, \|\cdot\|]$ the ideal of all operators and refer to [2,8–10] for operator ideals and their some information. A tensor norm α is said to *be associated with* a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ if the canonical map $(\mathcal{A}(M, N), \|\cdot\|_{\mathcal{A}}) \to M^* \otimes_\alpha N$ is an isometry for all finite-dimensional normed spaces M and N. Let X and Y be Banach spaces. For $T \in \mathcal{L}(X, Y)$, let

$$\|T\|_{\mathcal{A}^{\max}} := \sup\{\|q_L^Y T I_M^X\| : \dim M, \dim Y/L < \infty\},$$

where $I_M^X : M \to X$ is the inclusion map and $q_L^Y : Y \to Y/L$ is the quotient map, and

$$\mathcal{A}^{\max}(X, Y) := \{T \in \mathcal{L}(X, Y) : \|T\|_{\mathcal{A}^{\max}} < \infty\}.$$

We call $[\mathcal{A}^{\max}, \|\cdot\|_{\mathcal{A}^{\max}}]$ the *maximal hull* of $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$. If $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}] = [\mathcal{A}^{\max}, \|\cdot\|_{\mathcal{A}^{\max}}]$, then $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is called *maximal*. If α is a finitely generated tensor norm, then its associated maximal Banach operator ideal is uniquely determined (cf. [2, Sections 17.1, 17.2 and 17.3]). For a finitely generated tensor norm α, the *adjoint ideal* $[\mathcal{A}^{\mathrm{adj}}, \|\cdot\|_{\mathcal{A}^{\mathrm{adj}}}]$ is the maximal Banach operator ideal associated with the *adjoint tensor norm* $\alpha^* := (\alpha')^t = (\alpha^t)'$.

Lemma 2.1. [2, Theorem 17.5] *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm α. Then for all Banach spaces X and Y, $\mathcal{A}(X, Y^*)$ is isometric to $(X \otimes_{\alpha'} Y)^*$ and $\mathcal{A}(X, Y)$ is isometrically imbedded in $(X \otimes_{\alpha'} Y^*)^*$ by the natural dual actions.*

Let α be a finitely generated tensor norm. According to [2, Proposition 21.7(4)], a Banach space X has the α-AP if and only if for every Banach space Y, the natural map

$$J_\alpha : Y^* \hat{\otimes}_\alpha X \to Y^* \hat{\otimes}_\varepsilon X \hookrightarrow \mathcal{L}(Y, X)$$

is injective.

Theorem 2.2. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm α. Then the following statements are equivalent for a Banach space X.*
(a) X has the α-AP.
(b) For every Banach space Y, $\mathcal{F}(X, Y)$ is dense in $\mathcal{A}^{\mathrm{adj}}(X, Y)$ with the weak* topology on $\big(X \hat{\otimes}_{\alpha^t} Y^*\big)^*$.
(c) For every Banach space Y, $\mathcal{F}(X, Y^*)$ is dense in $\mathcal{A}^{\mathrm{adj}}(X, Y^*)$ with the weak* topology on $\big(X \hat{\otimes}_{\alpha^t} Y\big)^*$.

Proof. (a) ⇒ (b): Let Y be a Banach space. Since $[\mathcal{A}^{\mathrm{adj}}, \|\cdot\|_{\mathcal{A}^{\mathrm{adj}}}]$ is associated with α^* and $(\alpha^*)' = ((\alpha^t)')' = \alpha^t$, by Lemma 2.1, $\mathcal{A}^{\mathrm{adj}}(X, Y)$ is isometrically imbedded in $\big(X \hat{\otimes}_{\alpha^t} Y^*\big)^*$. Let $T \in \mathcal{A}^{\mathrm{adj}}(X, Y)$. Suppose that $T \notin \overline{\mathcal{F}(X, Y)}^{\mathrm{weak}^*}$. Then by the separation theorem, there exists a $u \in X \hat{\otimes}_{\alpha^t} Y^*$ such that for every $S \in \mathcal{F}(X, Y)$,

$$\langle S, u\rangle = 0 \text{ but } \langle T, u\rangle \neq 0,$$

where $\langle\cdot,\cdot\rangle$ is the dual action on $\big(X \hat{\otimes}_{\alpha^t} Y^*\big)^*$. We will show that $u = 0$ in $X \hat{\otimes}_{\alpha^t} Y^*$, which is a contradiction. Let

$$J_{\alpha^t} : X \hat{\otimes}_{\alpha^t} Y^* \to X \hat{\otimes}_\varepsilon Y^* \hookrightarrow \mathcal{L}(X^*, Y^*)$$

be the natural map. To show that $J_{\alpha^t} u = 0$ in $X \hat{\otimes}_\varepsilon Y^*$, let $x^* \in X^*$ and $y \in Y$. For every $v = \sum_{k=1}^m x_k \otimes y_k^* \in X \otimes Y^*$,

$$\langle x^* \otimes y, v\rangle = \sum_{k=1}^m x^*(x_k) y_k^*(y) = ((J_{\alpha^t} v)x^*)(y).$$

Let (u_n) be a sequence in $X \otimes Y^*$ such that $\lim_{n\to\infty} \alpha^t(u_n - u) = 0$. Then

$$\lim_{n\to\infty} \langle x^* \otimes y, u_n\rangle = \langle x^* \otimes y, u\rangle.$$

Since

$$|((J_{\alpha^t} u_n)x^*)(y) - ((J_{\alpha^t} u)x^*)(y)| \leq \|x^*\|\|y\|\varepsilon(J_{\alpha^t}(u_n - u); X, Y^*) \leq \|x^*\|\|y\|\alpha^t(u_n - u) \to 0$$

as $n \to \infty$, and for every n, $\langle x^* \otimes y, u_n\rangle = ((J_{\alpha^t} u_n)x^*)(y)$,

$$0 = \langle x^* \otimes y, u\rangle = ((J_{\alpha^t} u)x^*)(y).$$

Thus, $J_{\alpha^t} u = 0$ in $X \hat{\otimes}_\varepsilon Y^*$.

The aforementioned argument also shows that

$$x^*((J_\alpha u^t)y) = ((J_{\alpha^t} u)x^*)(y)$$

for every $x^* \in X^*$ and $y \in Y$, where $J_\alpha : Y^* \hat{\otimes}_\alpha X \to Y^* \hat{\otimes}_\varepsilon X \hookrightarrow \mathcal{L}(Y, X)$ is the natural map. Consequently, $J_\alpha u^t = 0$ in $Y^* \hat{\otimes}_\varepsilon X$. Since X has the α-AP, $u^t = 0$ in $Y^* \hat{\otimes}_\alpha X$ and so $u = 0$ in $X \hat{\otimes}_{\alpha^t} Y^*$.

(b) ⇒ (c): Let Y be a Banach space. By Lemma 2.1, $\mathcal{A}^{\mathrm{adj}}(X, Y^*)$ is isometric to $\big(X \widehat{\otimes}_{\alpha^t} Y\big)^*$. Let $T \in \mathcal{A}^{\mathrm{adj}}(X, Y^*)$. Then by (b),

$$T \in \overline{\mathcal{F}(X, Y^*)}^{\mathrm{weak}^*} \ \text{ on } \ \big(X \widehat{\otimes}_{\alpha^t} Y^{**}\big)^*.$$

Since the canonical imbedding from $X \widehat{\otimes}_{\alpha^t} Y$ to $X \widehat{\otimes}_{\alpha^t} Y^{**}$ is an isometry,

$$T \in \overline{\mathcal{F}(X, Y^*)}^{\mathrm{weak}^*} \ \text{ on } \ \big(X \widehat{\otimes}_{\alpha^t} Y\big)^*.$$

(c) ⇒ (a): Let Y be a Banach space. We show that the natural map

$$J_\alpha : Y \widehat{\otimes}_\alpha X \to Y \widehat{\otimes}_\varepsilon X \hookrightarrow \mathcal{L}(Y^*, X)$$

is injective. Assume that $J_\alpha u = 0$ in $Y \widehat{\otimes}_\varepsilon X$. To show that $u = 0$ in $Y \widehat{\otimes}_\alpha X$, we will show that $u^t = 0$ in $X \widehat{\otimes}_{\alpha^t} Y$, that is, $\langle T, u^t\rangle = 0$ for every $T \in \mathcal{A}^{\mathrm{adj}}(X, Y^*)$. Let $T \in \mathcal{A}^{\mathrm{adj}}(X, Y^*)$ be fixed. Since $J_\alpha u = 0$ in $Y \widehat{\otimes}_\varepsilon X$, for every $x^* \in X^*$ and $y^* \in Y^*$,

$$y^*((J_{\alpha^t} u^t)x^*) = x^*((J_\alpha u)y^*) = 0,$$

where $J_{\alpha^t} : X \widehat{\otimes}_{\alpha^t} Y \to X \widehat{\otimes}_\varepsilon Y \hookrightarrow \mathcal{L}(X^*, Y)$ is the natural map. As in the proof of (a) ⇒ (b), we see that

$$\langle x^* \otimes y^*, u^t\rangle = y^*((J_{\alpha^t} u^t)x^*) = 0$$

for every $x^* \in X^*$ and $y^* \in Y^*$, and so

$$\langle S, u^t\rangle = 0$$

for every $S \in \mathcal{F}(X, Y^*)$. Since $T \in \overline{\mathcal{F}(X, Y^*)}^{\mathrm{weak}^*}$ on $\big(X \widehat{\otimes}_{\alpha^t} Y\big)^*$, $\langle T, u^t\rangle = 0$. □

Let $1 \le p, q \le \infty$ with $1/p + 1/q \le 1$ and let $1 \le r \le \infty$ with $1/p + 1/q + 1/r^* = 1$, where $1/r + 1/r^* = 1$. A linear map $T : X \to Y$ is called (p, q)*-dominated* if there exists a $C > 0$ such that

$$\|(y_n^*(Tx_n))_n\|_r \le C\, \|(x_n)_n\|_p^w\, \|(y_n^*)_n\|_q^w$$

for every $(x_n)_n \in \ell_p^w(X)$ and $(y_n^*)_n \in \ell_q^w(Y^*)$. We denote by $\mathcal{D}_{p,q}(X, Y)$ the collection of all (p, q)-dominated operators from X to Y and for $T \in \mathcal{D}_{p,q}(X, Y)$, let $\|T\|_{\mathcal{D}_{p,q}}$ be the infimum C satisfying all such inequalities. Then $[\mathcal{D}_{p,q}, \|\cdot\|_{\mathcal{D}_{p,q}}]$ is a Banach operator ideal (cf. [2, Section 19]). $\mathcal{P}_p := \mathcal{D}_{p,\infty}$ is well known as the *ideal of absolutely p-summing operators* (cf. [2,8–10]) and $\mathcal{D}_p := \mathcal{D}_{p,p^*}$ is the *ideal of p-dominated operators*. For $1/p + 1/q \ge 1$, let $[\mathcal{L}_{p,q}, \|\cdot\|_{\mathcal{L}_{p,q}}]$ be the maximal Banach operator ideal associated with the tensor norm $\alpha_{p,q}$. $\mathcal{L}_{p,q}$ is well known as the *ideal of* (p, q)*-factorable operators*. Then

$$[\mathcal{L}_{p,q}, \|\cdot\|_{\mathcal{L}_{p,q}}]^{\mathrm{adj}} = [\mathcal{D}_{p^*,q^*}, \|\cdot\|_{\mathcal{D}_{p^*,q^*}}]$$

(see [2, Section 17.12] and [9, Section 17.4]).

Theorem 2.2 applied to the tensor norm $\alpha_{p,q}$ covers [7, Theorem 1].

Corollary 2.3. *Let* $1 \le p, q \le \infty$ *with* $1/p + 1/q \ge 1$. *The following statements are equivalent for a Banach space X.*

(*a*) *X has the* $\alpha_{p,q}$*-AP.*

(*b*) *For every Banach space Y,* $\mathcal{F}(X, Y)$ *is dense in* $\mathcal{D}_{p^*,q^*}(X, Y)$ *with the weak** *topology on* $\big(X \widehat{\otimes}_{\alpha_{q,p}} Y^*\big)^*$.

(*c*) *For every Banach space Y,* $\mathcal{F}(X, Y^*)$ *is dense in* $\mathcal{D}_{p^*,q^*}(X, Y^*)$ *with the weak** *topology on* $\big(X \widehat{\otimes}_{\alpha_{q,p}} Y\big)^*$.

Recall that a Banach space X has the AP if and only if X has the π-AP. Then the most special case of Corollary 2.3 is the following.

Corollary 2.4. *The following statements are equivalent for a Banach space X*
(*a*) *X has the AP.*
(*b*) *For every Banach space Y, $\mathcal{F}(X, Y)$ is dense in $\mathcal{L}(X, Y)$ with the weak* topology on $(X \widehat{\otimes}_\pi Y^*)^*$.*
(*c*) *For every Banach space Y, $\mathcal{F}(X, Y^*)$ is dense in $\mathcal{L}(X, Y^*)$ with the weak* topology on $(X \widehat{\otimes}_\pi Y)^*$.*

Proof. It is well known that π is associated with the ideal $\mathcal{I}$ of integral operators and $\mathcal{I}^{\mathrm{adj}} = \mathcal{L}$ holds isometrically (cf. [2]). Since $\pi^t = \pi$, we have the conclusion. □

Theorem 2.5. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm α. Then a Banach space X has the α^t-AP if and only if for every Banach space Y, $\mathcal{F}(Y, X^*)$ is dense in $\mathcal{A}^{\mathrm{adj}}(Y, X^*)$ with the weak* topology on $\big(Y \widehat{\otimes}_{\alpha^t} X\big)^*$.*

Proof. Assume that X has the α^t-AP. Let Y be a Banach space. By Lemma 2.1, $\mathcal{A}^{\mathrm{adj}}(Y, X^*)$ is isometric to $\big(Y \widehat{\otimes}_{\alpha^t} X\big)^*$. Let $T \in \mathcal{A}^{\mathrm{adj}}(Y, X^*)$. Suppose that $T \notin \overline{\mathcal{F}(Y, X^*)}^{\mathrm{weak}^*}$. Then there exists a $u \in Y \widehat{\otimes}_{\alpha^t} X$ such that for every $S \in \mathcal{F}(Y, X^*)$,

$$\langle S, u\rangle = 0 \ \text{ but } \ \langle T, u\rangle \neq 0.$$

Then as in the proof of Theorem 2.2, we can show that the natural map $J_{\alpha^t} : Y \widehat{\otimes}_{\alpha^t} X \to Y \widehat{\otimes}_\varepsilon X$ is not injective. This contradicts the assumption that X has the α^t-AP.

To show the converse, let Y be a Banach space. We want to show that the natural map

$$J_{\alpha^t} : Y \widehat{\otimes}_{\alpha^t} X \to Y \widehat{\otimes}_\varepsilon X \hookrightarrow \mathcal{L}(Y^*, X)$$

is injective. Assume that $J_{\alpha^t} u = 0$ in $Y \widehat{\otimes}_\varepsilon X$. Let $T \in \mathcal{A}^{\mathrm{adj}}(Y, X^*)$. Since $J_{\alpha^t} u = 0$ in $Y \widehat{\otimes}_\varepsilon X$, we see that for every $y^* \in Y^*$ and $x^* \in X^*$,

$$\langle y^* \otimes x^*, u\rangle = x^*((J_{\alpha^t} u)y^*) = 0.$$

Thus, $\langle S, u\rangle = 0$ for every $S \in \mathcal{F}(Y, X^*)$. Since $T \in \overline{\mathcal{F}(Y, X^*)}^{\mathrm{weak}^*}$, $\langle T, u\rangle = 0$. Hence, $u = 0$ in $Y \widehat{\otimes}_{\alpha^t} X$. □

Corollary 2.6. *Let $1 \le p, q \le \infty$ with $1/p + 1/q \ge 1$. Then a Banach space X has the $\alpha_{q,p}$-AP if and only if for every Banach space Y, $\mathcal{F}(Y, X^*)$ is dense in $\mathcal{D}_{p^*,q^*}(Y, X^*)$ with the weak* topology on $\big(Y \widehat{\otimes}_{\alpha_{q,p}} X\big)^*$.*

3 The bounded α-approximation property

Let α be a tensor norm and let X and Y be Banach spaces. Recall from [2, 12.4] that for every $u \in X \otimes Y$, let

$$\overrightarrow{\alpha}(u; X, Y) := \inf\{\alpha(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty\}$$

and

$$\overleftarrow{\alpha}(u; X, Y) := \sup\{\alpha((q_K^X \otimes q_L^Y)(u); X/K, Y/L) : \dim X/K, \dim Y/L < \infty\}.$$

It follows that $\overleftarrow{\alpha} \le \alpha \le \overrightarrow{\alpha}$. A tensor norm α is called *totally accessible* if $\overleftarrow{\alpha} = \overrightarrow{\alpha}$.

From [2, Proposition 21.7(2)], a Banach space X has the λ-bounded α-AP if and only if for every Banach space Y,

$$\alpha(u; Y, X) \le \lambda \overleftarrow{\alpha}(u; Y, X)$$

for every $u \in Y \otimes X$. Since $\overleftarrow{\alpha^t} = (\overleftarrow{\alpha})^t$, it follows that a Banach space X has the λ-bounded α^t-AP if and only if for every Banach space Y,

$$\alpha(u; X, Y) \le \lambda \overleftarrow{\alpha}(u; X, Y)$$

for every $u \in X \otimes Y$.

Lemma 3.1. [2, Theorem 15.5] *For all Banach spaces X and Y, and a tensor norm α, the natural maps*

$$I_{\overleftarrow{\alpha}} : X \otimes_{\overleftarrow{\alpha}} Y \to \big(X^* \otimes_{\alpha'} Y^*\big)^*,$$
$$I_{\overleftarrow{\alpha}} : X^* \otimes_{\overleftarrow{\alpha}} Y \to \big(X \otimes_{\alpha'} Y^*\big)^*$$

are isometries.

The following lemma is a reformulation of [2, Lemma 16.2].

Lemma 3.2. *Let α be a tensor norm and let X and Y be Banach spaces. Let $\lambda \geq 1$. Then $\alpha \leq \lambda\overleftarrow{\alpha}$ on $X \otimes Y$ if and only if for every $\phi \in B_{(X\otimes_\alpha Y)^*}$, there exists a net $(T_\eta)_\eta$ in $\lambda B_{X^*\otimes_{\alpha'} Y^*}$ such that for every $x \in X$ and $y \in Y$,*

$$\lim_\eta (T_\eta x)(y) = \langle \phi, x \otimes y\rangle.$$

Proof. Suppose that $\alpha \leq \lambda\overleftarrow{\alpha}$ on $X \otimes Y$. Let $\phi \in B_{(X\otimes_\alpha Y)^*} \subset (X \otimes_{\overleftarrow{\alpha}} Y)^*$. By Lemma 3.1, we can choose a Hahn-Banach extension $\hat{\phi} \in (X^* \otimes_{\alpha'} Y^*)^{**}$ of ϕ. By Goldstine's theorem, there exists a net $(T_\eta)_\eta$ in $X^* \otimes Y^*$ with $\alpha'(T_\eta; X^*, Y^*) \leq \|\hat{\phi}\|_{(X^*\otimes_{\alpha'} Y^*)^{**}}$ such that

$$\lim_\eta \langle f, T_\eta\rangle = \langle \hat{\phi}, f\rangle$$

for every $f \in (X^* \otimes_{\alpha'} Y^*)^*$. Thus, for every $x \in X$ and $y \in Y$,

$$\lim_\eta (T_\eta x)(y) = \lim_\eta \langle x \otimes y, T_\eta\rangle = \langle \hat{\phi}, x \otimes y\rangle = \langle \phi, x \otimes y\rangle.$$

Also, since

$$\|\hat{\phi}\|_{(X^*\otimes_{\alpha'} Y^*)^{**}} = \|\phi\|_{\left(X\otimes_{\overleftarrow{\alpha}} Y\right)^*} \leq \lambda\ \|\phi\|_{(X\otimes_\alpha Y)^*} \leq \lambda,$$

the net (T_η) is in $\lambda B_{X^*\otimes_{\alpha'} Y^*}$.

To show the converse, let $u = \sum_{k=1}^m x_k \otimes y_k \in X \otimes Y$. Then there exists $\phi \in B_{(X\otimes_\alpha Y)^*}$ such that $\alpha(u; X, Y) = \langle \phi, u\rangle$. By assumption, there exists a net (T_η) in $\lambda B_{X^*\otimes_{\alpha'} Y^*}$ such that

$$\lim_\eta \langle u, T_\eta\rangle = \lim_\eta \sum_{k=1}^m (T_\eta x_k)(y_k) = \langle \phi, u\rangle.$$

Hence,

$$\alpha(u; X, Y) = \langle \phi, u\rangle \leq \lambda \sup\{|\langle u, v\rangle| : v \in B_{X^*\otimes_{\alpha'} Y^*}\} = \lambda\ \|u\|_{(X^*\otimes_{\alpha'} Y^*)^*} = \lambda\overleftarrow{\alpha}(u; X, Y).$$ □

Lemma 3.3. [2, Proposition 21.8] *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm α and let $\lambda \geq 1$. Let X and Y be Banach spaces. Then $\alpha \leq \lambda\overleftarrow{\alpha}$ on $Y^* \otimes X$ if and only if for every $T \in B_{\mathcal{A}^{\mathrm{adj}}(X, Y^{**})}$, there exists a net $(T_\eta)_\eta$ in $\lambda B_{X^*\otimes_\alpha Y}$ such that for every $x \in X$ and $y^* \in Y^*$,*

$$\lim_\eta y^*(T_\eta x) = (Tx)(y^*).$$

We denote the *strong operator topology* and the *weak operator topology* on $\mathcal{L}$, respectively, by τ_{so} and τ_{wo}. For a net $(T_\alpha)_\alpha$ in $\mathcal{L}(X, Y^*)$, we say that $T_\alpha \to 0$ in the weak* *operator topology* if

$$\lim_\eta (T_\alpha x)(y) \to 0$$

for every $x \in X$ and $y \in Y$. We denote the weak* operator topology by $\tau_{\mathrm{w^*o}}$.

Theorem 3.4. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm α* and let $\lambda \geq 1$. *Then the following statements are equivalent for a Banach space X.*

(*a*) *X has the* λ*-bounded* α*-AP.*
(*b*) *For every Banach space* Y *and every* $T \in \mathcal{A}^{\mathrm{adj}}(X, Y)$,

$$T \in \overline{\left\{S \in \mathcal{F}\left(X, Y\right) : \alpha^*\left(S; X^*, Y\right) \le \lambda \|T\|_{\mathcal{A}^{\mathrm{adj}}}\right\}}^{\tau_{\mathrm{so}}}.$$

(*c*) *For every Banach space* Y *and every* $T \in \mathcal{A}^{\mathrm{adj}}(X, Y^*)$,

$$T \in \overline{\left\{S \in \mathcal{F}\left(X, Y^*\right) : \alpha^*\left(S; X^*, Y^*\right) \le \lambda \|T\|_{\mathcal{A}^{\mathrm{adj}}}\right\}}^{\tau_{\mathrm{w^*o}}}.$$

Proof.
(b) ⇒ (c) is trivial.

(a) ⇒ (b): This proof is essentially due to [11, Theorem 4.1]. Let Y be a Banach space and let $T \in \mathcal{A}^{\mathrm{adj}}(X, Y)$. Consider $i_Y T \in \mathcal{A}^{\mathrm{adj}}(X, Y^{**})$, where $i_Y : Y \to Y^{**}$ is the canonical isometry. Since X has the λ-bounded α-AP, by Lemma 3.3, there exists a net $(T_\eta)_\eta$ in $\mathcal{F}(X, Y)$ with $\alpha^*(T_\eta; X^*, Y) \le \lambda$ such that for every $x \in X$ and $y^* \in Y^*$,

$$\lim_\eta y^*(\|i_Y T\|_{\mathcal{A}^{\mathrm{adj}}} T_\eta x) = (i_Y Tx)(y^*) = y^*(Tx).$$

Since $\alpha^*(\|i_Y T\|_{\mathcal{A}^{\mathrm{adj}}} T_\eta; X^*, Y) \le \lambda \; \|T\|_{\mathcal{A}^{\mathrm{adj}}}$,

$$T \in \overline{\left\{S \in \mathcal{F}\left(X, Y\right) : \alpha^*\left(S; X^*, Y\right) \le \lambda \|T\|_{\mathcal{A}^{\mathrm{adj}}}\right\}}^{\tau_{\mathrm{wo}}} = \overline{\left\{S \in \mathcal{F}\left(X, Y\right) : \alpha^*\left(S; X^*, Y\right) \le \lambda \|T\|_{\mathcal{A}^{\mathrm{adj}}}\right\}}^{\tau_{\mathrm{so}}}.$$

(c) ⇒ (a): Let Y be a Banach space. Since $\overleftarrow{\alpha^t} = (\overleftarrow{\alpha})^t$, $\alpha \le \lambda \overleftarrow{\alpha}$ on $Y \otimes X$ if and only if $\alpha^t \le \lambda \overleftarrow{\alpha^t}$ on $X \otimes Y$. So, in order to show that X has the λ-bounded α-AP, we will show that $\alpha^t \le \lambda \overleftarrow{\alpha^t}$ on $X \otimes Y$ using Lemma 3.2.

Now, let $\phi \in B_{(X \otimes_{\alpha^t} Y)^*}$. By Lemma 2.1, we can choose the representation $T_\phi \in \mathcal{A}^{\mathrm{adj}}(X, Y^*)$ of ϕ with $\|T_\phi\|_{\mathcal{A}^{\mathrm{adj}}} \le 1$. Then by (c), there exists a net $(S_\eta)_\eta$ in $\lambda B_{X^* \otimes_{\alpha^*} Y^*}$ such that for every $x \in X$ and $y \in Y$,

$$\lim_\eta (S_\eta x)(y) = (T_\phi x)(y) = \langle \phi, x \otimes y \rangle. \qquad \square$$

Hence by Lemma 3.2, we complete the proof.

Corollary 3.5. *Let* $1 \le p, q \le \infty$ *with* $1/p + 1/q \ge 1$ *and let* $\lambda \ge 1$. *The following statements are equivalent for a Banach space X.*
(*a*) *X has the* λ*-bounded* $\alpha_{p,q}$*-AP.*
(*b*) *For every Banach space* Y *and every* $T \in \mathcal{D}_{p^*,q^*}(X, Y)$,

$$T \in \overline{\{S \in \mathcal{F}(X, Y) : \|S\|_{\mathcal{D}_{p^*,q^*}} \le \lambda \; \|T\|_{\mathcal{D}_{p^*,q^*}}\}}^{\tau_{\mathrm{so}}}.$$

(*c*) *For every Banach space* Y *and every* $T \in \mathcal{D}_{p^*,q^*}(X, Y^*)$,

$$T \in \overline{\{S \in \mathcal{F}(X, Y^*) : \|S\|_{\mathcal{D}_{p^*,q^*}} \le \lambda \|T\|_{\mathcal{D}_{p^*,q^*}}\}}^{\tau_{\mathrm{w^*o}}}.$$

Proof. If $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is the maximal Banach operator ideal associated with a totally accessible finitely generated tensor norm α, then by Lemmas 2.1 and 3.1, $\alpha = \|\cdot\|_{\mathcal{A}}$ on $\mathcal{F}$. Since $\alpha^*_{p,q}$ is totally accessible (cf. [2, Theorem 21.5]), by Theorem 3.4, we have the conclusion. The equivalence (a) ⇔ (b) is also a consequence of [11, Theorem 4.1]. □

For the following result, we will need [11, Corollary 2.14], which can be reformulated as follows.

Lemma 3.6. *Let* $1 \le p \le \infty$ *and let* $\lambda \ge 1$. *The following statements are equivalent for a Banach space X.*
(*a*) *For every Banach space* Y *and every* $T \in \mathcal{P}_p(X, Y)$,

$$T \in \overline{\{S \in \mathcal{F}(X, Y) : \|S\|_{\mathcal{P}_p} \le \lambda \; \|T\|_{\mathcal{P}_p}\}}^{\tau_{\mathrm{so}}}.$$

(*b*) *For every Banach space* Y *and every* $T \in \mathcal{P}_p(X, Y)$,

$$\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X, X) : \|TS\|_{\mathcal{P}_p} \le \lambda \; \|T\|_{\mathcal{P}_p}\}}^{\tau_{\mathrm{so}}}.$$

According to [11, Definition 1.2], for a Banach operator ideal $\mathcal{A}$, a Banach space X is said to have the *weak* λ-BAP *for* $\mathcal{A}$ if for every Banach space Y and every $T \in \mathcal{A}(X, Y)$,

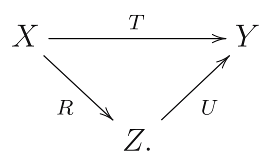
$$\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X, X) : \|TS\|_{\mathcal{A}} \le \lambda \, \|T\|_{\mathcal{A}}\}}^{\tau_{so}}.$$

Theorem 3.7. *Let $1 \le p, q \le \infty$ with $1/p + 1/q \ge 1$ and let $\lambda \ge 1$. If a Banach space X has the λ-bounded g_p-AP, then X has the weak λ-BAP for $\mathcal{D}_{p^*,q^*}$.*

Proof. By Corollary 3.5 and Lemma 3.6, if X has the λ-bounded g_p-AP, then for every Banach space Z and every $T \in \mathcal{P}_{p^*}(X, Z)$,

$$\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X, X) : \|TS\|_{\mathcal{P}_{p^*}} \le \lambda \, \|T\|_{\mathcal{P}_{p^*}}\}}^{\tau_{so}}.$$

Now, let Y be a Banach space and let $T \in \mathcal{D}_{p^*,q*}(X, Y)$. Let $\delta > 0$. Then by Kwapień's factorization theorem (cf. [2, Theorem 19.3]), there exist a Banach space Z, $R \in \mathcal{P}_{p^*}(X, Z)$ and $U^* \in \mathcal{P}_{q^*}(Y^*, Z^*)$ with $\|U^*\|_{\mathcal{P}_{q^*}} \|R\|_{\mathcal{P}_{p^*}} \le (1 + \delta) \|T\|_{\mathcal{D}_{p^*,q^*}}$ such that the following diagram is commutative.



By the aforementioned statement, for every finite $x_1, \dots, x_m \in X$ and every $\varepsilon > 0$, there exists an $S \in \mathcal{F}(X, X)$ with $\|RS\|_{\mathcal{P}_{p^*}} \le \lambda \|R\|_{\mathcal{P}_{p^*}}$ such that

$$\|Sx_i - x_i\| \le \varepsilon$$

for every $i = 1, \dots, m$. Since

$$\|TS\|_{\mathcal{D}_{p^*,q^*}} \le \|U^*\|_{\mathcal{P}_{q^*}} \|RS\|_{\mathcal{P}_{p^*}} \le (1 + \delta)\lambda \, \|T\|_{\mathcal{D}_{p^*,q^*}},$$

we have shown that for every $\delta > 0$,

$$\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X, X) : \|TS\|_{\mathcal{D}_{p^*,q^*}} \le (1 + \delta)\lambda \, \|T\|_{\mathcal{D}_{p^*,q^*}}\}}^{\tau_{so}}.$$

Let $x_1, \dots, x_m \in X$ and let $\varepsilon > 0$. Choose a $\delta > 0$ so that

$$(\delta\lambda/(1 + \delta)\lambda) \max_{1 \le k \le m} \|x_k\| \le \varepsilon/2.$$

Then, there exists an $S \in \{S \in \mathcal{F}(X, X) : \|TS\|_{\mathcal{D}_{p^*,q^*}} \le (1 + \delta)\lambda\|T\|_{\mathcal{D}_{p^*,q^*}}\}$ such that for every $i = 1, \dots, m$, $\|Sx_i - x_i\| \le \varepsilon/2$. Consider

$$(\lambda/(1 + \delta)\lambda)S \in \{S \in \mathcal{F}(X, X) : \|TS\|_{\mathcal{D}_{p^*,q^*}} \le \lambda \, \|T\|_{\mathcal{D}_{p^*,q^*}}\}.$$

Then for every $i = 1, \dots, m$,

$$\left\| \frac{\lambda}{(1 + \delta)\lambda} Sx_i - x_i \right\| \le \frac{\lambda}{(1 + \delta)\lambda} \|Sx_i - x_i\| + \frac{\delta\lambda}{(1 + \delta)\lambda} \max_{1 \le k \le m} \|x_k\| \le \varepsilon.$$

Hence, $\mathrm{id}_X \in \overline{\{S \in \mathcal{F}(X, X) : \|TS\|_{\mathcal{D}_{p^*,q^*}} \le \lambda\|T\|_{\mathcal{D}_{p^*,q^*}}\}}^{\tau_{so}}$. □

In [7, Proposition 2], it was shown that if a Banach space X has the g_p-AP, then X has the $\alpha_{p,q}$-AP. From Theorem 3.7 and Corollary 3.5, we have:

Corollary 3.8. *Let $1 \le p, q \le \infty$ with $1/p + 1/q \ge 1$ and let $\lambda \ge 1$. If a Banach space X has the λ-bounded g_p-AP, then X has the λ-bounded $\alpha_{p,q}$-AP.*

Theorem 3.9. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm α and let $\lambda \geq 1$. Then a Banach space X has the λ-bounded α^t-AP if and only if for every Banach space Y and every $T \in \mathcal{A}^{\mathrm{adj}}(Y, X^*)$,*

$$T \in \overline{\{S \in \mathcal{F}(Y, X^*) : \alpha^*(S; Y^*, X^*) \leq \lambda \|T\|_{\mathcal{A}^{\mathrm{adj}}}\}}^{\tau_{w^*o}}.$$

Proof. Let

$$i : X \otimes_\alpha Y \to Y \otimes_{\alpha^t} X$$

be the isometry defined by $i(u) = u^t$.

Suppose that X has the λ-bounded α^t-AP. Let Y be a Banach space and let $T \in \mathcal{A}^{\mathrm{adj}}(Y, X^*)$. By Lemma 2.1, we can choose the representation $\phi_T \in (Y \otimes_{\alpha^t} X)^*$ of T. Consider $\phi_T i \in (X \otimes_\alpha Y)^*$. Since X has the λ-bounded α^t-AP, $\alpha \leq \lambda \overleftarrow{\alpha}$ on $X \otimes Y$. Then by Lemma 3.2, there exists a net $(u_\eta)_\eta$ in $\lambda B_{X^* \otimes_{\alpha'} Y^*}$ such that for every $x \in X$ and $y \in Y$,

$$\lim_\eta \langle x \otimes y, u_\eta \rangle = \left\langle \frac{\phi_T i}{\|\phi_T i\|_{(X \otimes_\alpha Y)^*}}, x \otimes y \right\rangle.$$

Let us consider the net $(\|\phi_T i\|_{(X \otimes_\alpha Y)^*} u_\eta^t)_\eta$ in $Y^* \otimes X^* = \mathcal{F}(Y, X^*)$. Then

$$\alpha^*(\|\phi_T i\|_{(X \otimes_\alpha Y)^*} u_\eta^t; Y^*, X^*) = \|\phi_T i\|_{(X \otimes_\alpha Y)^*} \alpha'(u_\eta; X^*, Y^*) \leq \lambda \, \|T\|_{\mathcal{A}^{\mathrm{adj}}}$$

and for every $y \in Y$ and $x \in X$,

$$\lim_\eta (\|\phi_T i\|_{(X \otimes_\alpha Y)^*} u_\eta^t y)(x) = \lim_\eta \, \|\phi_T i\|_{(X \otimes_\alpha Y)^*} \langle x \otimes y, u_\eta \rangle = \langle \phi_T i, x \otimes y \rangle = (Ty)(x).$$

Hence, $T \in \overline{\{S \in \mathcal{F}(Y, X^*) : \alpha^*(S; Y^*, X^*) \leq \lambda \, \|T\|_{\mathcal{A}^{\mathrm{adj}}}\}}^{\tau_{w^*o}}$.

To show the converse, we also use Lemma 3.2. Let Y be a Banach space and let $\phi \in B_{(X \otimes_\alpha Y)^*}$. Consider $\phi i^{-1} \in B_{(Y \otimes_{\alpha^t} X)^*}$. By Lemma 2.1, we can choose the representation $T_{\phi i^{-1}} \in \mathcal{A}^{\mathrm{adj}}(Y, X^*)$ of ϕi^{-1} with $\|T_{\phi i^{-1}}\|_{\mathcal{A}^{\mathrm{adj}}} \leq 1$. By assumption, there exists a net $(S_\eta)_\eta$ in $\lambda B_{Y^* \otimes_{\alpha^*} X^*}$ such that for every $y \in Y$ and $x \in X$,

$$\lim_\eta (S_\eta y)(x) = (T_{\phi i^{-1}} y)(x).$$

Consider the net $(S_\eta^t)_\eta$ in $X^* \otimes Y^*$. Then $\alpha'(S_\eta^t; X^*, Y^*) = \alpha^*(S_\eta; Y^*, X^*) \leq \lambda$ and for every $x \in X$ and $y \in Y$,

$$\lim_\eta (S_\eta^t x)(y) = \lim_\eta (S_\eta y)(x) = (T_{\phi i^{-1}} y)(x) = \langle \phi i^{-1}, y \otimes x \rangle = \langle \phi, x \otimes y \rangle.$$

Thus by Lemma 3.2, $\alpha \leq \lambda \overleftarrow{\alpha}$ on $X \otimes Y$. Hence, X has the λ-bounded α^t-AP. □

From Theorem 3.9, we have:

Corollary 3.10. *Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \geq 1$ and let $\lambda \geq 1$. A Banach space X has the λ-bounded $\alpha_{q,p}$-AP if and only if for every Banach space Y and every $T \in \mathcal{D}_{p^*,q^*}(Y, X^*)$,*

$$T \in \overline{\{S \in \mathcal{F}(Y, X^*) : \|S\|_{\mathcal{D}_{p^*,q^*}} \leq \lambda \, \|T\|_{\mathcal{D}_{p^*,q^*}}\}}^{\tau_{w^*o}}.$$

4 Open problems

The following question is a well-known problem (cf. [2, Section 21.12]).

Problem 1
Is the tensor norm w_p $(1 < p < \infty, p \neq 2)$ totally accessible?

Since a finitely generated tensor norm α is totally accessible if and only if every Banach space has the 1-bounded α-AP, the problem can be reformulated as follows.

Problem 1
Does every Banach space have the 1-bounded w_p-AP ($1 < p < \infty, p \neq 2$)?

According to Corollaries 3.5 and 3.10, a Banach space X has the 1-bounded w_p-AP if and only if for every Banach space Y and every $T \in \mathcal{D}_{p^*}(X, Y)$,

$$T \in \overline{\{S \in \mathcal{F}(X, Y) : \|S\|_{\mathcal{D}_{p^*}} \leq \|T\|_{\mathcal{D}_{p^*}}\}}^{\tau_{so}}$$

if and only if for every Banach space Y and every $T \in \mathcal{D}_p(Y, X^*)$,

$$T \in \overline{\{S \in \mathcal{F}(Y, X^*) : \|S\|_{\mathcal{D}_p} \leq \|T\|_{\mathcal{D}_p}\}}^{\tau_{w^*o}}.$$

Therefore, the problem can be reformulated as follows.

Problem 1
Let $1 < p < \infty, p \neq 2$. For all Banach spaces X, Y and every $T \in \mathcal{D}_{p^*}(X, Y)$,

$$T \in \overline{\{S \in \mathcal{F}(X, Y) : \|S\|_{\mathcal{D}_{p^*}} \leq \|T\|_{\mathcal{D}_{p^*}}\}}^{\tau_{so}}?$$

Or, for every $T \in \mathcal{D}_p(Y, X^*)$,

$$T \in \overline{\{S \in \mathcal{F}(Y, X^*) : \|S\|_{\mathcal{D}_p} \leq \|T\|_{\mathcal{D}_p}\}}^{\tau_{w^*o}}?$$

Reinov [12] constructed Banach spaces failing to have the g_p-AP and the d_p-AP ($1 \leq p \leq \infty, p \neq 2$). From [7, Proposition 2], if a Banach space has the g_p-AP, then it has the w_p-AP. It is not known whether every Banach space has the w_p-AP ($1 < p < \infty, p \neq 2$). According to Corollaries 2.3 and 2.6, a Banach space X has the w_p-AP if and only if $\mathcal{F}(X, Y)$ is dense in $\mathcal{D}_{p^*}(X, Y)$ with the *weak** topology on $\left(X \hat{\otimes}_{w_{p^*}} Y^*\right)^*$ for every Banach space Y if and only if $\mathcal{F}(Y, X^*)$ is dense in $\mathcal{D}_p(Y, X^*)$ with the *weak** topology on $\left(Y \hat{\otimes}_{w_p} X\right)^*$ for every Banach space Y. We ask:

Problem 2
Let $1 < p < \infty, p \neq 2$. For all Banach spaces X and Y, is the space $\mathcal{F}(X, Y)$ dense in $\mathcal{D}_{p^*}(X, Y)$ with the weak* topology on $\left(X \hat{\otimes}_{w_{p^*}} Y^*\right)^*$?

Or, is the space $\mathcal{F}(Y, X^*)$ dense in $\mathcal{D}_p(Y, X^*)$ with the *weak** topology on $\left(Y \hat{\otimes}_{w_p} X\right)^*$?

Acknowledgments: The author would like to thank the referees for valuable comments. This work was supported by the National Research Foundation of Korea (NRF-2018R1D1A1B07043566).

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