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Research Article

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The edge-regular complete maps

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Abstract: A map is called edge-regular if it is edge-transitive but not arc-transitive. In this paper, we show that a complete graph K_n has an orientable edge-regular embedding if and only if $n = p^d > 3$ with p an odd prime such that $p^d \equiv 3 \pmod 4$. Furthermore, K_{p^d} has $\frac{p^d-3}{4d}\phi\left(\frac{p^d-1}{2}\right)$ non-isomorphic orientable edge-regular embeddings.

Keywords: edge-regular maps, arc-transitive maps, complete graphs

MSC 2020: 05C25, 20B05, 20B15, 20B30

1 Introduction

An *orientable map* is a 2-cell embedding of a finite graph in an orientable surface, and thus a map is an incident triple of vertex set V, edge set E, and face set F, denoted by $\mathcal{M} = (V, E, F)$. The finite graph is called the *underlying graph* of the map, and the surface is called the *supporting surface* of the map. In this paper, we focus on the orientable supporting surface.

An automorphism of a map $\mathcal{M} = (V, E, F)$ is a permutation of $V \cup E \cup F$ that preserves the incidence relation between vertices, edges, and faces. A map $\mathcal{M}$ is called *edge-transitive* (or an *edge-transitive embedding* of its underlying graph) or *arc-transitive* if the automorphism group $\mathrm{Aut}\mathcal{M}$ is transitive on the edge set or the arc set, respectively. Sometimes, the arc-transitive map is called a rotary map or a symmetrical map. For convenience, a map $\mathcal{M}$ is called *edge-regular* if $\mathcal{M}$ is edge-transitive but not arc-transitive.

The purpose of this paper is to give a classification of orientable edge-regular maps with underlying graphs being complete graphs. For convenience, a map with underlying graph being a complete graph is called a *complete map*.

In general, a *flag* of a map is an incident triple of vertex, edge, and face. A map $\mathcal{M}$ is called *flag-transitive* if $\mathrm{Aut}\mathcal{M}$ is transitive on the flag set. Since the action of $\mathrm{Aut}\mathcal{M}$ on the flag set is always semi-regular, a flag-transitive map is flag-regular, and it is simply called *regular*. It is easy to see that if a subgroup $G \le \mathrm{Aut}\mathcal{M}$ is transitive on the edge set of $\mathcal{M}$, then the index of G in $\mathrm{Aut}\mathcal{M}$ is not more than four. Thus, an edge-transitive map is highly symmetrical.

The literature studying "symmetrical" maps is rich, refer to [1]. Recent investigation began with Biggs' study in [2,3] of arc-transitive complete maps. Among the arc-transitive embeddings of K_n constructed by Biggs, the unique embeddings for $n = 2$, 3, and 4 are flag-regular, others with $n \ge 5$ a prime power are all arc-regular. In the past four decades, plenty of results about symmetrical maps have been obtained, see [4–10] and references therein. In particular, arc-transitive complete maps are classified in [3,7,11]; vertex-transitive complete maps are characterized in [12]. Very recently, some special families of edge-transitive

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maps with underlying complete bipartite graphs are classified in [13–16]. In this paper, we study the edge-regular complete maps. The main result of the paper is stated in the following theorem.

Theorem 1.1. *A complete graph K_n has an orientable edge-regular embedding if and only if $n = p^d$, where p is an odd prime and d is a positive integer such that $p^d \equiv 3 \pmod 4$. Furthermore, K_{p^d} has $\frac{p^d-3}{4d}\,\phi\left(\frac{p^d-1}{2}\right)$ non-isomorphic edge-regular embeddings, where ϕ is Euler' s function, and each of the maps has full automorphism group isomorphic to $\mathbb{Z}_p^d : \mathbb{Z}_{\frac{p^d-1}{2}} < \mathrm{AGL}(1, p^d)$.*

Remarks on Theorem 1.1.

(1) The main results of Theorem 1.1 were obtained by James [17] and Jones [18], both of which used the methods of group theories. However, the research approach adopted in the present paper is more concise and easier.
(2) A map $\mathcal{M}$ is called a *Cayley map* if Aut$\mathcal{M}$ contains a subgroup G which is regular on the vertices. Furthermore, if $G \lhd \mathrm{Aut}\mathcal{M}$, then $\mathcal{M}$ is called a *normal map*. Thus, all the maps in Theorem 1.1 are normal maps.
(3) A Cayley map $\mathcal{M}$ is called *balanced* if each generated element s and its inverse element s^{-1} of s are placed on the antipodal points, see [10]. It is known that an arc-transitive Cayley map $\mathcal{M}$ is balanced if and only if $\mathcal{M}$ is a normal map. Thus, the maps in Theorem 1.1 are not necessary balanced.

We end this section in the following corollary, the result can also be deduced from [19, Corollary 2].

Corollary 1.2. *A normal edge-regular Cayley map is not necessarily a balanced map.*

2 Construction and enumeration

Let p be an odd prime and d a positive integer. Let $F = \mathrm{GF}(p^d)$ be a field of order p^d. Let F^+ and $F^\times$ be the additive group and the multiplicative group of F, respectively. Then $F^+ \cong \mathbb{Z}_p^d$ and $F^\times \cong \mathbb{Z}_{p^d-1}$. Furthermore, $F^\times$ naturally acts on F^+ by multiplication. This defines a group

$$X = F^+ : F^\times = \mathrm{AGL}(1, p^d),$$

which is an affine group. Let $k = p^d - 1$. Since p is odd, k is even.

Assume that $p^d \equiv 3 \pmod 4$. Then $\frac{k}{2}$ is odd, and d is odd too. Let $F^\times = \langle \xi \rangle$, and let $\rho = \xi^2$ and $\sigma = \xi^{\frac{k}{2}}$. Then $\langle \xi \rangle = \langle \rho \rangle \times \langle \sigma \rangle \cong \mathbb{Z}_{\frac{k}{2}} \times \mathbb{Z}_2$. Let

$$G = F^+ : \langle \rho \rangle \cong \mathbb{Z}_p^d : \mathbb{Z}_{\frac{k}{2}}.$$

We now construct edge-regular embeddings of the complete graph $\Gamma = K_{p^d}$. Let $\mathbf{0}$ be the identity of F^+. Then $\Gamma = \mathrm{Cay}(F^+, F^+ \setminus \{\mathbf{0}\})$, and in the case where $p^d \equiv 3 \pmod 4$, G is edge-transitive but not arc-transitive on Γ. Noting that the group $\langle \rho \rangle$ partitions $\Gamma(\mathbf{0}) = F^+ \setminus \{\mathbf{0}\}$ into two orbits, say Δ_0 and Δ_1. We make labelings for the elements of Δ_0 and Δ_1 as follows.

Let $v \in \Delta_0$ be a non-identity element of F^+, and set

$\beta_0 = v$ and $\beta_{2i+2} = \beta_{2i}^{\rho}$, where $0 \le i < \frac{k}{2} - 1$;

$\beta_1 \in F^+ \setminus \{\mathbf{0}, \beta_0, \beta_2, \ldots, \beta_{k-2}\}$ and $\beta_{2i+1} = \beta_{2i-1}^{\rho}$, where $1 \le i < \frac{k}{2}$.

Then the orbits $\Delta_0 = \beta_0^{\langle \rho \rangle}$ and $\Delta_1 = \beta_1^{\langle \rho \rangle}$.

For a vertex α, a cyclic permutation of the neighbor set $\Gamma(\alpha)$ of α is called a *rotation* at α and denoted by R_α. A *rotation system* $R(\Gamma)$ of a graph Γ is the product of rotations at all vertices, that is, $R(\Gamma) = \prod_{\alpha \in V} R_\alpha$. Then each rotation system $R(\Gamma)$ defines an orientable embedding of Γ, refer to [1, pp. 104–108].

Construction 2.1
Use the aforementioned notations.

(1) Label the end points of the arcs emitting from the vertex $\mathbf{0} \in F^+$ as

$$\beta_0, \beta_1, \beta_2, \beta_3, \ldots, \beta_{k-1}.$$

(2) For each vertex $x \in F^+$, label the end points of the arcs emitting from x as

$$\beta_0 x, \beta_1 x, \beta_2 x, \beta_3 x, \ldots, \beta_{k-1} x.$$

(3) Define a rotation of the end points of arcs emitting from x by

$$R_x = (\beta_0 x, \beta_1 x, \beta_2 x, \beta_3 x, \ldots, \beta_{k-1} x),$$

where $x \in V = F^+$, and let $R(\Gamma) = \prod_{x \in F^+} R_x$.

(4) Observing that the rotation system $R(\Gamma)$ is uniquely determined by β_0, β_1, and ρ, we denote the orientable map defined by the rotation system as follows:

$$\mathcal{M}(\beta_0, \beta_1, \rho).$$

We next study the maps $\mathcal{M}(\beta_0, \beta_1, \rho)$. The first lemma determines the relation between β_i's and their inverses.

Lemma 2.2. *There exist positive integers* ℓ *and* m *such that* $\beta_1 = (\beta_0^{-1})^{\rho^\ell} = \beta_{2\ell}^{-1}$ *and* $\beta_0^{-1} = \beta_{2m+1}$, *where* $m = \frac{p^d - 1}{2} - \ell$.

Proof. Assume that β_i and β_i^{-1} lie in the same orbit of $\langle\rho\rangle$, where $0 \le i < p^d - 1$. Then $\beta_i^{-1} = \beta_I^{\rho^t}$ with $1 \le t < \frac{p^d - 1}{2}$, and $(\beta_i^{\rho^t})^{\rho^t} = (\beta_i^{-1})^{\rho^t} = (\beta_i^{\rho^t})^{-1} = (\beta_i^{-1})^{-1} = \beta_i$. Thus, $(\rho^t)^2 = 1$. A contradiction occurred since $|\rho| = \frac{p^d - 1}{2}$ is odd. So β_i and β_i^{-1} lie in different orbits of $\langle\rho\rangle$, where $0 \le i < p^d - 1$. Hence, β_0^{-1} lies in the orbit of β_1 under $\langle\rho\rangle$. It follows that there exists a positive integer ℓ such that $\beta_1 = (\beta_0^{-1})^{\rho^\ell} = (\beta_0^{\rho^\ell})^{-1} = \beta_{2\ell}^{-1}$.

Furthermore, $\beta_0^{-1} = \beta_1^{\rho^{-\ell}} = \beta_1^{\rho^{\frac{p^d-1}{2} - \ell}} = \beta_{2\left(\frac{p^d-1}{2} - \ell\right)+1}$. Thus, letting $m = \frac{p^d - 1}{2} - \ell$, we have $\beta_0^{-1} = \beta_{2m+1}$. □

For a vertex $v \in V\Gamma$, noting that the vertex rotations R_v can be regarded as permutations not only of the set $\Gamma(v)$ but also of the generating set S. So Cayley maps have another equivalent definitions, see [19]. A map with underlying graph being Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is a *Cayley map* if the induced local cyclic permutations of S are all the same. The next lemma shows the edge-transitivity of $\mathcal{M}(\beta_0, \beta_1, \rho)$.

Lemma 2.3. *Let* $\mathcal{M} = \mathcal{M}(\beta_0, \beta_1, \rho)$ *and* $G = F^+ : \langle\rho\rangle$, *as defined above. Then the following statements hold.*

(*i*) $G \le \mathrm{Aut}\mathcal{M}$, *and* G *is transitive on the edges of* $\mathcal{M}$;

(*ii*) $\mathcal{M}$ *is a Cayley map of* F^+.

Proof. By the definition of the rotation system $\prod_{x \in F^+} R_x$, each element of F^+ is an automorphism of $\mathcal{M}$. Since F^+ acts regularly on the vertices of $\mathcal{M}$, $\mathcal{M}$ is a Cayley map of F^+, and the underlying graph of $\mathcal{M}$ is a complete graph of order p^d. Furthermore, for each element $x \in F^+$, the conjugation ρ^x is such that, for $0 \le i \le k - 1$,

$$(\beta_i x)^{\rho^x} = (\beta_i x)^{x^{-1}\rho x} = \beta_i^{\rho x} = \beta_{i+2}^{x} = \beta_{i+2} x,$$

reading the subscripts modulo k. Thus, ρ is an automorphism of the map $\mathcal{M}$, refer to [1, pp. 109–110]. Since G is transitive on the edges of the underlying graph Γ, we have that the map $\mathcal{M}$ is G-edge-transitive. □

The Cayley map $\mathcal{M}(\beta_0, \beta_1, \rho)$ is balanced if s and $s^{-1} = -s$ are placed on the antipodal points for all elements $s \in F^+ \backslash \{\mathbf{0}\}$. Let ρ_0 be the unique involution of $\mathrm{GL}(1, p^d)$. Then

$$\rho_0 : x \mapsto -x, \quad \text{for all } x \in F^+,$$

which is an automorphism of $\mathcal{M}$.

Lemma 2.4. *A map $\mathcal{M}(\beta_0, \beta_1, \rho)$ is balanced if and only if $\beta_0^{-1} = \beta_{\frac{p^d-1}{2}}$ and $\beta_1^{-1} = \beta_{\frac{p^d+1}{2}}$.*

Proof. Assume that $\mathcal{M}(\beta_0, \beta_1, \rho)$ is balanced. Then for every i with $0 \le i < \frac{p^d-1}{2}$, the vertex β_i^{-1} is placed at the antipodal position of the vertex β_i. Thus, $\beta_0^{-1} = \beta_{\frac{p^d-1}{2}}$ and $\beta_1^{-1} = \beta_{\frac{p^d+1}{2}}$.

Conversely, assume that $\beta_0^{-1} = \beta_{\frac{p^d-1}{2}}$ and $\beta_1^{-1} = \beta_{\frac{p^d+1}{2}}$. Then we have

$$\beta_{2i}^{-1} = (\beta_0^{-1})^{\rho^i} = \left(\beta_{\frac{p^d-1}{2}}\right)^{\rho^i} = \beta_{\frac{p^d-1}{2}+2i},$$

$$\beta_{2i+1}^{-1} = (\beta_1^{-1})^{\rho^i} = \left(\beta_{\frac{p^d+1}{2}}\right)^{\rho^i} = \beta_{\frac{p^d-1}{2}+2i+1}.$$

So $\beta_j^{-1} = \beta_{\frac{p^d-1}{2}+j}$ is at the antipodal position of β_j, for all j with $0 \le j < \frac{p^d-1}{2}$, and therefore, $\mathcal{M}(\beta_0, \beta_1, \rho)$ is balanced. □

The *mirror-image* $\overline{\mathcal{M}}$ of an orientable map $\mathcal{M}$ is given by reversing the cyclic order of the edges at each vertex. An orientable map $\mathcal{M}$ is called *reflexible* if it has an automorphism that reverses the orientation of the supporting surface, equivalently, $\mathcal{M} \cong \overline{\mathcal{M}}$, see [20, Chapter 16].

Lemma 2.5. *The orientable map $\mathcal{M}(\beta_0, \beta_1, \rho)$ is not reflexible, and the automorphism group* $\mathrm{Aut}\mathcal{M}$ *is a subgroup of* $\mathrm{AGL}(1, p^d)$.

Proof. Let $\mathcal{M} = \mathcal{M}(\beta_0, \beta_1, \rho)$, and let $A = \mathrm{Aut}\mathcal{M}$. Let A^+ be the group of elements of A which preserves the orientation of the supporting surface of $\mathcal{M}$. Then G is of index at most 2 in A^+, and A^+ is of index at most 2 in A. Thus, $G \lhd A^+ \lhd A$, and the index $|A : G| = 1$, 2, or 4. Since G is of odd order, it follows that G is a characteristic subgroup of A. As G is primitive on the vertex set V, so is A. Hence, $C_A(G) \le G$, and $A \le \mathrm{Aut}(G) = \mathrm{A\Gamma L}(1, p^d) = (F^+ : F^\times){:}\mathbb{Z}_d$. By Frattini's argument, $A_\alpha \le F^\times : \mathbb{Z}_d$ for any $\alpha \in V = F^+$. Since $A_\alpha \le D_{2(p^d-1)}$ and d is odd, it follows that $A_\alpha \le F^\times \cong \mathbb{Z}_{p^d-1}$ and $A \le \mathrm{AGL}(1, p^d)$.

If $\mathcal{M}$ is reflexible, then there exists an involution $\tau \in A \backslash A^+$ which reverses ρ. However, note that d is odd, then this is not possible. So $\mathcal{M}$ is not reflexible. □

An orientable map is called *chiral* if it is arc-regular, and each of its automorphisms preserves the orientation of the supporting surface. However, chirality does not always require arc-regularity, sometimes chiral simply means "not allowing reflections." The next lemma determines the chirality of the maps $\mathcal{M}(\beta_0, \beta_1, \rho)$.

Lemma 2.6. *Let $\mathcal{M} = \mathcal{M}(\beta_0, \beta_1, \rho)$, a map as constructed in Construction 2.1. Then either*

(*i*) *$\mathcal{M}$ is chiral (arc-regular), $\beta_1 = \beta_0^{\xi}$, and* $\mathrm{Aut}\mathcal{M} = \mathrm{AGL}(1, p^d)$; *or*

(*ii*) *$\mathcal{M}$ is edge-regular, and* $\mathrm{Aut}\mathcal{M} = G$.

Proof. Assume that $\mathcal{M}$ is arc-transitive. Then $\mathrm{Aut}\mathcal{M} = \mathrm{AGL}(1, p^d) = F^+ : F^\times = F^+ : \langle\xi\rangle$ by Lemma 2.5. Since β_1 and β_0 lie in different orbits of $\langle\rho\rangle = \langle\xi^2\rangle$, there exists an odd integer ℓ with $1 \le \ell < p^d - 1$ such that $\beta_1 = \beta_0^{\xi^\ell}$. Thus, $\beta_0^{\xi^2} = \beta_0^{\rho} = \beta_2 = \beta_1^{\xi^\ell} = \beta_0^{\xi^{2\ell}}$. Since $\langle\xi\rangle$ is regular on $F^+\backslash\{\mathbf{0}\}$, it follows that $\ell = 1$, and so $\beta_1 = \beta_0^{\xi}$.

Assume that $\mathcal{M}$ is not arc-transitive. Then by Lemma 2.5, $\mathrm{Aut}\mathcal{M} = G$, which is regular on the edge set of $\mathcal{M}$. □

Noting that the rotation $R_x = (\beta_0 x, \beta_1 x, \beta_2 x, \beta_3 x, \ldots, \beta_{k-1} x)$ for all $x \in V = F^+$ is a circular permutation, namely,

$$(\beta_0 x, \beta_1 x, \beta_2 x, \beta_3 x, \ldots, \beta_{k-1} x) = R_x = (\beta_i x, \beta_{i+1} x, \beta_{i+2} x, \beta_{i+3} x, \ldots, \beta_{i+k-1} x),$$

where $0 \le i \le k-1$ and the subscripts are modulo k, we have

$$\mathcal{M}(\beta_0, \beta_1, \rho) = \mathcal{M}(\beta_i, \beta_{i+1}, \rho).$$

Therefore, to enumerate different edge-transitive embeddings of K_{p^d}, we may fix the first element β_0.

The following lemma gives a counting formula of the maps $\mathcal{M}(\beta_0, \beta_1, \rho)$.

Lemma 2.7. *A complete graph K_{p^d} with $p^d \equiv 3 \pmod 4$ has precisely $\frac{p^d-3}{4d}\phi\left(\frac{p^d-1}{2}\right)$ non-isomorphic orientable edge-regular embeddings, and $\frac{1}{d}\phi\left(\frac{p^d-1}{2}\right)$ non-isomorphic orientable chiral embeddings.*

Proof. As argued above, we may fix β_0. Then there are precisely $\frac{p^d-1}{2}$ choices for β_1, and there are precisely $\phi\left(\frac{p^d-1}{2}\right)$ choices for ρ, where ϕ is Euler's function. Thus, there are exactly $\frac{p^d-1}{2}\phi\left(\frac{p^d-1}{2}\right)$ different triples for (β_0, β_1, ρ), so the number of different maps of the form $\mathcal{M}(\beta_0, \beta_1, \rho)$ is equal to $\frac{p^d-1}{2}\phi\left(\frac{p^d-1}{2}\right)$.

By Lemma 2.6, a map $\mathcal{M}(\beta_0, \beta_1, \rho)$ is arc-transitive if and only if $\beta_1 = \beta_0^{\xi}$, and if and only if it is arc-regular. Thus, the number of different arc-transitive embeddings of the form $\mathcal{M}(\beta_0, \beta_0^{\xi}, \rho)$ of K_{p^d} equals the number of choices of ξ with $|\xi| = p^d - 1$, which is equal to $\phi\left(\frac{p^d-1}{2}\right)$ since $\frac{p^d-1}{2}$ is odd and $\phi(p^d-1) = \phi\left(\frac{p^d-1}{2}\right) \times \phi(2) = \phi\left(\frac{p^d-1}{2}\right)$. The others are all edge-regular, and the number of them equals $\frac{p^d-1}{2}\phi\left(\frac{p^d-1}{2}\right) - \phi\left(\frac{p^d-1}{2}\right) = \frac{p^d-3}{2}\phi\left(\frac{p^d-1}{2}\right)$.

Now, we determine isomorphism classes of the $\frac{p^d-1}{2}\phi\left(\frac{p^d-1}{2}\right)$ different maps described above. Let

$$\mathcal{S} = \left\{\mathcal{M}(\beta_0, \beta_1, \rho^i) \mid \beta_1 \in F^+ \backslash \{\mathbf{0}, \beta_0, \beta_2, \ldots, \beta_{k-2}\},\ 1 \le i < \frac{p^d-1}{2} \text{ and } \left(i, \frac{p^d-1}{2}\right) = 1\right\}.$$

Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{S}$ which are isomorphic, and let φ be an isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$. Then φ induces an isomorphism from $\mathrm{Aut}\mathcal{M}_1$ to $\mathrm{Aut}\mathcal{M}_2$.

Case 1. First, assume that $\mathcal{M}_1$ is edge-regular. Then so is $\mathcal{M}_2$. By Lemma 2.6, we have

$$\mathrm{Aut}\mathcal{M}_1 = \mathrm{Aut}\mathcal{M}_2 = F^+ : \langle\rho\rangle = G.$$

Hence, $G^{\varphi} = (\mathrm{Aut}\mathcal{M}_1)^{\varphi} = \mathrm{Aut}\mathcal{M}_2 = G$, namely, φ is an automorphism of G. Since F^+ is a normal Sylow p-subgroup of G, it follows that F^+ is a characteristic subgroup of G, and φ normalizes $F^+ \cong \mathbb{Z}_p^d$. Thus, $\varphi \in \mathrm{Aut}(\mathbb{Z}_p^d) = \mathrm{GL}(d, p)$, and $\varphi \in N_{\mathrm{GL}(d,p)}(\langle\rho\rangle)$. We may choose φ to normalize $\langle\rho\rangle$. By [21, Theorem 7.3 (a) of Chapter 2],

$$\varphi \in N_{\mathrm{GL}(d,p)}(\langle\rho\rangle) = \Gamma\mathrm{L}(1, p^d) = \mathbb{Z}_{p^d-1} : \mathbb{Z}_d = \langle\xi\rangle : \langle\tau\rangle,$$

where $\tau : \xi \mapsto \xi^p$ is the Frobenius automorphism of $\mathrm{GF}(p^d)$.

Since $\rho = \xi^2$ is an automorphism of $\mathcal{M} \in \mathcal{S}$, it fixes all maps in $\mathcal{S}$. Let $\mathcal{S}_1$ be the subset of $\mathcal{S}$ which consists of edge-regular maps. So $\rho = \xi^2$ fixes each element of $\mathcal{S}_1$. It follows that $\langle\xi\rangle : \langle\tau\rangle / \langle\xi^2\rangle \cong \langle\sigma\rangle \times \langle\tau\rangle$, where $\sigma = \xi^{\frac{p^d-1}{2}}$. Thus, $N_{\mathrm{GL}(d,p)}(\langle\rho\rangle) = \langle\xi\rangle : \langle\tau\rangle$ acting on $\mathcal{S}_1$ is isomorphic to $\langle\sigma\rangle \times \langle\tau\rangle$. As $\mathrm{Aut}\mathcal{M} = G$, the group $\langle\sigma\rangle \times \langle\tau\rangle$ acts semiregularly on $\mathcal{S}_1$. Obviously, two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic if and only if they lie in the same orbit of $\langle\sigma\rangle \times \langle\tau\rangle$ acting on $\mathcal{S}_1$. Since $|\mathcal{S}_1| = \frac{p^d-3}{2}\phi\left(\frac{p^d-1}{2}\right)$, it follows that there are exactly $\frac{1}{2d} \cdot \frac{p^d-3}{2}\phi\left(\frac{p^d-1}{2}\right)$ non-isomorphic edge-regular maps in $\mathcal{S}$.

Case 2. Assume now that $\mathcal{M}_1$ and $\mathcal{M}_2$ are arc-regular. By Lemma 2.6, we have

$$\mathrm{Aut}\mathcal{M}_1 = \mathrm{Aut}\mathcal{M}_2 = F^+ : \langle\xi\rangle = G.\langle\sigma\rangle,$$

where $\sigma = \xi^{\frac{p^d-1}{2}}$. So $(G.\langle\sigma\rangle)^\varphi = (\mathrm{Aut}\mathcal{M}_1)^\varphi = \mathrm{Aut}\mathcal{M}_2 = G.\langle\sigma\rangle$, namely, φ is an automorphism of G. Arguing as in the previous case shows that $\varphi \in \Gamma\mathrm{L}(1, p^d) = \mathbb{Z}_{p^d-1} : \mathbb{Z}_d = \langle\xi\rangle : \langle\tau\rangle$, where $\tau : \xi \mapsto \xi^p$. Since for any $\mathcal{M}(\beta_0, \beta_1, \rho^i) \in \mathcal{S}$,

$$\mathcal{M}(\beta_0, \beta_1, \rho^i)^\xi = \mathcal{M}(\beta_0^\xi, \beta_1^\xi, (\rho^i)^\xi) = \mathcal{M}(\beta_1, \beta_2, \rho^i),$$
$$\mathcal{M}(\beta_0, \beta_1, \rho^i)^\tau = \mathcal{M}(\beta_0^\tau, \beta_1^\tau, (\rho^i)^\tau) = \mathcal{M}(\beta_1^p, \beta_2^p, \rho^{ip}),$$

and $\mathcal{M}(\beta_1, \beta_2, \rho^i) \in \mathcal{S}$ and $\mathcal{M}(\beta_1^p, \beta_2^p, \rho^{ip}) \in \mathcal{S}$, it follows that each element of $\langle\xi\rangle : \langle\tau\rangle$ is an isomorphism among maps in $\mathcal{S}$. Let $\mathcal{S}_2$ be the subset of $\mathcal{S}$ consisting of arc-regular maps. As ξ is an automorphism of each map in $\mathcal{S}_2$, the induced action of $\langle\xi\rangle : \langle\tau\rangle$ is isomorphic to $\langle\tau\rangle \cong \mathbb{Z}_d$ and $\langle\tau\rangle$ is semiregular on $\mathcal{S}_2$. It follows that there are precisely $\frac{1}{d}\,\phi\left(\frac{p^d-1}{2}\right)$ non-isomorphic arc-regular maps in $\mathcal{S}$. □

3 Proof of the main theorem

Let $\Gamma = (V, E) = K_n$ be a complete graph of order n. Then the edges of Γ are precisely the 2-subsets of the vertex set V, and the arcs of Γ are the ordered pairs of distinct vertices. Recall that a permutation group $G \le \mathrm{Sym}(V)$ is said to be 2-*homogeneous* on V if G is transitive on the set of all 2-subsets of V.

Lemma 3.1. *Let $\Gamma = (V, E) = K_n$ and $G \le \mathrm{Aut}\Gamma$. Assume that G is edge-transitive but not arc-transitive on Γ, then the following statements hold:*
(i) $n = p^d \equiv 3 \pmod 4$, *where p is an odd prime and d is a positive integer;*
(ii) $G \cong \mathbb{Z}_p^d : \mathbb{Z}_{\frac{p^d-1}{2}} < \mathrm{AGL}(1, p^d)$.

Proof. Since G is edge-transitive, but not arc-transitive, we have that $G \le \mathrm{S}_n$ is a 2-homogeneous, but not 2-transitive permutation group on V. Noting that G_α is a cyclic group or a dihedral group. By Kantor's classification (see [22, Theorem 9.4B]), G has a unique minimal normal subgroup $N \cong \mathbb{Z}_p^d$ for some odd prime p and positive integer d such that $p^d \equiv 3 \pmod 4$, and $G_\alpha = \mathbb{Z}_{\frac{p^d-1}{2}}$, where α is a vertex. Thus, $G = N : G_\alpha \cong \mathbb{Z}_p^d : \mathbb{Z}_{\frac{p^d-1}{2}} < \mathrm{AGL}(1, p^d)$. □

Now we are ready to prove our main theorem.

Proof of Theorem 1.1. Let $\mathcal{M} = (V, E, F)$ be a 2-cell embedding of $\Gamma = K_n$, and let $G = \mathrm{Aut}\mathcal{M}$. Then, for a vertex $\alpha \in V$, the stabilizer G_α is a cyclic or dihedral group. Assume that G is edge-transitive but not arc-transitive on $\mathcal{M}$. We need to prove that $\mathcal{M}$ is a map as given in Construction 2.1.

By Lemma 3.1, there exist an odd prime p and a positive integer d such that $G = N : G_\alpha = \mathbb{Z}_p^d : \mathbb{Z}_{\frac{p^d-1}{2}}$, where $n = p^d \equiv 3 \pmod 4$, and N is regular on the vertex set V. Thus, we may identify the vertex set V with N. Furthermore, we identify N with the additive group F^+ of the field $F = \mathrm{GF}(p^d)$. Let the multiplicative group $F^\times$ of F be generated by ξ, namely, $F^\times = \langle\xi\rangle \cong \mathbb{Z}_{\frac{p^d-1}{2}}$. Let $\rho = \xi^2$, and let α be the vertex of $\mathcal{M}$ corresponding to $\mathbf{0} \in F^+$. Then $G_\alpha = \langle\rho\rangle \cong \mathbb{Z}_{\frac{p^d-1}{2}}$.

The action of $G_\alpha = \langle\rho\rangle$ on $V\backslash\{\alpha\}$ is semiregular and divides $V\backslash\{\alpha\}$ into two orbits. Since ρ fixes the vertex $\mathbf{0}$ and preserves the supporting surface, the rotation of the end points of the arcs emanating from $\mathbf{0}$ has the form

$$(\beta_0, \beta_1, \beta_2, \beta_3, \ldots, \beta_{k-2}, \beta_{k-1}),$$

where $k = p^d - 1$ such that

$$\beta_i^\rho = \beta_{i+2}, \text{ for } 0 \le i \le k-1,$$

reading the subscripts $i + 2$ modulo k. Thus, the two orbits of $\langle \rho \rangle$ acting on $\Gamma(\alpha)$ are $\{\beta_0, \beta_2, \ldots, \beta_{k-2}\}$ and $\{\beta_1, \beta_3, \ldots, \beta_{k-1}\}$. Therefore, the rotation system for $\mathcal{M}$ is the same as the one constructed in Construction 2.1, and so the map $\mathcal{M}$ is as constructed in Construction 2.1. □

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