

Research Article

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The limit Riemann solutions to nonisentropic Chaplygin Euler equations

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Abstract: We mainly consider the limit behaviors of the Riemann solutions to Chaplygin Euler equations for nonisentropic fluids. The formation of delta shock wave and the appearance of vacuum state are found as parameter ε tends to a certain value. Different from the isentropic fluids, the weight of delta shock wave is determined by variance density ρ and internal energy H . Meanwhile, involving the entropy inequality, the uniqueness of delta shock wave is obtained.

Keywords: nonisentropic fluids, Riemann solutions, delta shock, vacuum state, entropy inequality

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1 Introduction

One-dimensional compressible Euler equations for nonisentropic fluids can be written as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho, s))_x = 0, \\ (\rho u^2/2 + \rho e)_t + ((\rho u^2/2 + \rho e + P(\rho, s))u)_x = 0, \end{cases} \quad (1.1)$$

where the variables ρ , u , s , P , e stand for the density, velocity, specific entropy, pressure and specific energy, respectively, and $P(\varepsilon, \rho) = \varepsilon p$ satisfies $\lim_{\varepsilon \rightarrow 0} P(\rho, \varepsilon) = 0$. P and e are the functions of ρ and s , and fulfill the thermodynamical constraint

$$de = Tds - Pd\frac{1}{\rho}, \quad (1.2)$$

where $T = T(\rho, s)$ represents the temperature. The equation of state with Chaplygin gas can be expressed as

$$p = -\frac{1}{\rho}, \quad (1.3)$$

which was introduced by Chaplygin [1] in 1904. In some theories of cosmology, Chaplygin gas explains the acceleration and the dark energy of the universe, and the formation of delta shock wave may be used to illustrate the different periods of evolution of the universe. As for the related results, one can see [2–6].

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In 2005, Brenier [7] considered the Riemann problem of the isentropic Chaplygin gas Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 - \frac{1}{\rho} \right)_x = 0, \end{cases} \quad (1.4)$$

and obtained the concentration solutions when the initial value belongs to a certain region in the phase plane. In 2010, Guo et al. [8] put away this restriction to system (1.4) and received the global solutions including the delta shock. Wang and Zhang [9] investigated the Riemann problem with delta initial data and obtained four kinds of the global generalized solutions. In 2014, Nedeljkov [10] studied higher order shadow waves and delta shock blow up in the Chaplygin gas and found that a double shadow wave interacted with an outgoing wave and formed a singled weighted shadow wave, which is in general called delta shock wave. Meanwhile, Nedeljkov proved that this delta shock has a variable strength and variable speed. For more detailed knowledge of delta shock, interested readers can refer to [11–17].

As the pressure vanishes, equations (1.4) converge to the transport equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \quad (1.5)$$

which are also called the pressureless Euler equations and can be used to describe the motion of free particles sticking under collision in [18–20]. Equations (1.5) have been extensively studied since 1994 such as in [21–23]. In 2016, Shen [24] considered the Riemann problem for the Chaplygin gas equations with a source term. Furthermore, Guo et al. [25] studied the vanishing pressure limits of Riemann solutions and analyzed the phenomena of concentration and cavitation to the Chaplygin gas equations with a source term. As for the pressure vanishing limits of the isentropic Euler equations, let us refer to [26–31] for more details.

Kraiko [32] studied system (1.1) with $P(\rho, s) = 0$ in 1979. In order to construct a solution for any initial data, they needed the discontinuities which are different from classical waves that carry mass, impulse and energy. In 2012, Cheng [33] solved the Riemann problem for (1.1) with $P(\rho, s) = 0$ and found two kinds of solutions containing vacuum state and delta shock with Dirac delta function in both the density and the internal energy. We replace internal energy ρe by H , therefore, system (1.1) can be transformed into the following equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P)_x = 0, \\ (\rho u^2/2 + H)_t + ((\rho u^2/2 + H + P)u)_x = 0, \end{cases} \quad (1.6)$$

where H denoted the internal energy and $H \geq 0$. Pang [34] considered the system of (1.6) for Chaplygin gas equations with the following initial data

$$(\rho, u, H)(x, 0) = \begin{cases} (\rho_-, u_-, H_-), & x < 0, \\ (\rho_+, u_+, H_+), & x > 0, \end{cases} \quad (1.7)$$

where $\rho_{\pm} > 0$, $u_{\pm} > 0$ and $H_{\pm} > 0$ are different constants. For more detailed information on the nonisentropic Euler equations, interested readers can refer to [35–38].

In this article, we mainly focus our attention to the vanishing pressure limits of Riemann solutions for system (1.6)–(1.7), when the pressure vanishes, equation (1.6) can be translated into (1.5), and an additional conservation law

$$(\rho u^2/2 + H)_t + (\rho u^3/2 + Hu)_x = 0. \quad (1.8)$$

As pressure vanishes, we identify and analyze the formation of delta shock waves and vacuum states in the Riemann solutions. Furthermore, in the sense of distributions, entropy inequality corresponding to equation (1.8) will be verified

$$(\rho u^2)_t + (\rho u^3)_x \geq 0. \quad (1.9)$$

The remainder of this article can be organized as follows: in Sections 2 and 3, we review the Riemann solutions to (1.5) and (1.6), respectively. In Section 4, we consider the vanishing pressure limits of Riemann solutions to (1.6) and (1.7). In Section 5, we give some discussions.

2 Riemann problem for (1.5)

In this section, we review some results on Riemann solution to system (1.5) with initial data

$$(\rho, u)(x, 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \quad (2.1)$$

where $\rho_{\pm} > 0$, the details can be referred to in [23].

For the case $u_- < u_+$, we know that the Riemann solutions of (1.5) contain two-contact discontinuities J_1 , J_2 and a vacuum state between two-contact discontinuities, and J_1 , J_2 satisfy

$$J_1 : u = u_-, \quad J_2 : u = u_+. \quad (2.2)$$

For the case $u_- = u_+$, the Riemann solution include a contact discontinuity J that connects (ρ_-, u_-) to (ρ_+, u_+) , and J satisfies

$$J : u = u_- = u_+. \quad (2.3)$$

While for the case $u_- > u_+$, the superposition of S and J leads to the singularity for ρ on the line $x = x(t)t$ as a weighted Dirac delta function, which was named as the so-called delta shock wave. Thus, the delta shock wave solution to the Riemann problem (1.6) and (1.7) should be constructed when $u_- > u_+$. Then, let us recollect the definition of delta shock wave in [13,22].

Definition 2.1. For arbitrary $\psi(x, t) \in C_0^\infty(R + R_+)$, the two-dimensional weighted Dirac delta function $\beta(s)\delta_\Gamma$ with the support on a parameterized smooth curve $\Gamma = \{(x(s), t(s)) : a < s < b\}$ is defined by

$$\langle \beta(s)\delta_\Gamma, \psi(x, t) \rangle = \int \beta(s)\psi(x(s), t(s))ds.$$

By virtue of the above definition, the Riemann solution of (1.6) and (1.7) contains a delta shock wave. It can be briefly expressed by

$$(\rho_-, u_-) + \delta S + (\rho_+, u_+), \quad (2.4)$$

namely,

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (\omega(t)\delta(x - x(t)), u_\delta), & x = x(t), \\ (\rho_+, u_+), & x(t) < x, \end{cases} \quad (2.5)$$

where $w(t)$ and $\sigma(t)$ denote the weight and velocity of delta shock wave, respectively.

While for the case $u_- > u_+$, singularity must happen. We use a delta shock wave to construct the Riemann solution. The details can be found in [22]. The location, weight and velocity of the delta shock are given by computing generalized Rankine-Hugoniot relations, which are

$$x(t) = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}}t, \quad \omega(t) = \frac{t}{1 + \sigma^2}(\sigma[\rho] - [\rho u]), \quad \sigma = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}}. \quad (2.6)$$

In addition, the delta shock wave satisfies the generalized entropy condition

$$u_+ < \sigma < u_-, \quad (2.7)$$

which means that all characteristics on both sides of the δ -shock wave curve are incoming. Furthermore, the uniqueness of delta shock wave can be obtained.

3 Riemann problem for (1.6)–(1.7) for the Chaplygin gas

From thermodynamical constraint (1.2), we derive

$$Tds = d\left(e - \frac{\varepsilon}{2\rho^2}\right),$$

thus, there is a function $f(s)$ satisfying

$$T = f'(s), \quad e = \frac{\varepsilon}{2\rho^2} + f(s).$$

Due to the value of e is positive, which means that the function $g(x) = \frac{1}{2}x^2 + f(s) \geq 0$ when $x \in (0, +\infty)$, so $f(s) > 0$, namely, $e - \frac{\varepsilon}{2\rho^2} \geq 0$. Then, the physically relevant region can be expressed as

$$\aleph = \left\{(\rho, u, H) \mid \rho > 0, \quad H \geq \frac{\varepsilon}{2\rho}, \quad u \in \mathbb{R}\right\}.$$

In this section, we review results on the Riemann problem of (1.6) for the Chaplygin gas, see [34] for the details. Equations (1.6) have three eigenvalues

$$\lambda_1 = u - \frac{\sqrt{\varepsilon}}{\rho}, \quad \lambda_2 = u, \quad \lambda_3 = u + \frac{\sqrt{\varepsilon}}{\rho}, \quad (3.1)$$

with corresponding right eigenvectors

$$\vec{r}_1 = \left(-\frac{\rho^2}{\sqrt{\varepsilon}}, 1, \sqrt{\varepsilon} - \frac{1}{\sqrt{\varepsilon}}H\rho\right)^T, \quad \vec{r}_2 = (0, 0, 1)^T, \quad \vec{r}_3 = \left(\frac{\rho^2}{\sqrt{\varepsilon}}, 1, -\sqrt{\varepsilon} + \frac{1}{\sqrt{\varepsilon}}H\rho\right)^T.$$

Direct calculation yields $\nabla\lambda_i \cdot \vec{r}_i = 0$, for $i = 1, 2, 3$, which indicates that all the characteristic fields are contact discontinuities.

For any given constant state (ρ_-, u_-, H_-) in the phase plane, we can derive three families of contact discontinuities

$$J_1^\varepsilon(\rho_-, u_-, H_-) : \begin{cases} \sigma_1^\varepsilon = u - \frac{\sqrt{\varepsilon}}{\rho} = u_- - \frac{\sqrt{\varepsilon}}{\rho}, \\ (2H\rho - \varepsilon)\rho_-^2 = (2H_-\rho - \varepsilon)\rho^2, \end{cases} \quad (3.2)$$

$$J_2^\varepsilon(\rho_-, u_-, H_-) : \begin{cases} \sigma_2^\varepsilon = u = u_-, \quad \rho = \rho_-, \\ H \neq H_-, \end{cases} \quad (3.3)$$

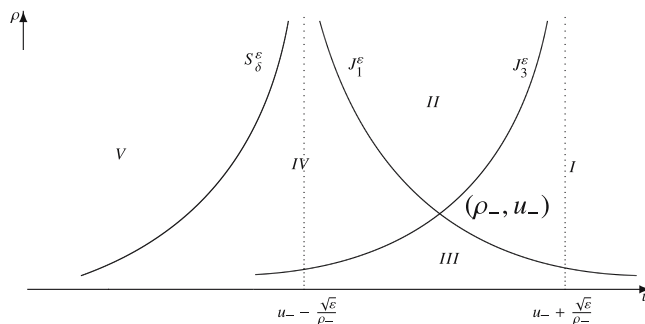
$$J_3^\varepsilon(\rho_-, u_-, H_-) : \begin{cases} \sigma_3^\varepsilon = u + \frac{\sqrt{\varepsilon}}{\rho} = u_- + \frac{\sqrt{\varepsilon}}{\rho}, \\ (2H\rho - \varepsilon)\rho_-^2 = (2H_-\rho - \varepsilon)\rho^2. \end{cases} \quad (3.4)$$

On the physical correlation region, that is $(\rho_-, u_-, H_-) \in \aleph$, from given state (ρ_-, u_-, H_-) , we can draw the one-contact discontinuity curve J_1^ε that satisfies (3.2) and the three-contact discontinuity curve J_3^ε that satisfies (3.4). And from the point $(\rho_-, u_- - \frac{2\sqrt{\varepsilon}}{\rho_-}, H_-)$ draw three-contact discontinuity curve S_δ^ε that satisfies (3.4). In fact, this curve S_δ^ε consists of some states that can be connected to the states (ρ_-, u_-, H_-) on the right by a δS .

We project these curves onto the (ρ, u) -plane. J_1^ε has two asymptotes $u = u_- - \frac{\sqrt{\varepsilon}}{\rho}$ and $\rho = 0$, J_3^ε has two asymptotes $u = u_- + \frac{\sqrt{\varepsilon}}{\rho}$ and $\rho = 0$, and S_δ^ε satisfies

$$u + \frac{\sqrt{\varepsilon}}{\rho} = u_- - \frac{\sqrt{\varepsilon}}{\rho}, \quad (3.5)$$

which has two asymptotic lines $u = u_- - \frac{\sqrt{\varepsilon}}{\rho_-}$ and $\rho = 0$. Thus, the phase plane can be divided into five regions.



When the projection of (ρ_+, u_+, H_+) belongs to $I(\rho_-, u_-) \cup II(\rho_-, u_-) \cup III(\rho_-, u_-) \cup IV(\rho_-, u_-)$ in the (ρ, u) -plane, the Riemann solution can be briefly expressed by

$$(\rho_-, u_-, H_-) + J_1^\varepsilon + (\rho_{*1}^\varepsilon, u_{*1}^\varepsilon, H_{*1}^\varepsilon) + J_2^\varepsilon + (\rho_{*2}^\varepsilon, u_{*2}^\varepsilon, H_{*2}^\varepsilon) + J_3^\varepsilon + (\rho_+, u_+, H_+), \quad (3.6)$$

where $(\rho_{*1}^\varepsilon, u_{*1}^\varepsilon, H_{*1}^\varepsilon)$ and $(\rho_{*2}^\varepsilon, u_{*2}^\varepsilon, H_{*2}^\varepsilon)$ are the intermediate states.

For the projection of (ρ_+, u_+, H_+) belongs to $V(\rho_-, u_-)$ in the (ρ, u) -plane, the Riemann solution can be given by

$$(\rho_-, u_-, H_-) + \delta S + (\rho_+, u_+, H_+). \quad (3.7)$$

The details can be referred to in [34]. The delta shock wave holds the generalized Rankine-Hugoniot conditions

$$\begin{cases} \frac{dx(t, \varepsilon)}{dt} = u_\delta(t, \varepsilon), \\ \frac{d\omega(t, \varepsilon)}{dt} = u_\delta(t, \varepsilon)[\rho] - [\rho u], \\ \frac{d(\omega(t, \varepsilon)u_\delta(t))}{dt} = u_\delta(t, \varepsilon)[\rho u] - [\rho u^2 + P], \\ \frac{d(\omega(t)u_\delta^2(t)/2 + h(t))}{dt} = u_\delta(t, \varepsilon)\left[\frac{\rho u^2}{2} + H\right] - \left[\left(\frac{\rho u^2}{2} + H + P\right)u\right], \\ (x, \omega, u_\delta, h)(0) = (0, 0, 0, 0), \end{cases} \quad (3.8)$$

where $\omega(t, \varepsilon)$ and $u_\delta(t, \varepsilon)$ are weight and velocity of delta shock wave, respectively.

It can be derived from (3.8) that

$$\begin{cases} x(t, \varepsilon) = \frac{1}{[\rho]}([\rho u]t + \omega(t, \varepsilon)), \\ u_\delta(t, \varepsilon) = \frac{1}{[\rho]}([\rho u] + \omega'(t, \varepsilon)), \\ \omega(t, \varepsilon) = \sqrt{\rho_+ \rho_- \left(\left(u_- + \frac{\sqrt{\varepsilon}}{\rho_-} \right) - \left(u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} \right) \right) \left(\left(u_- - \frac{\sqrt{\varepsilon}}{\rho_-} \right) - \left(u_+ - \frac{\sqrt{\varepsilon}}{\rho_+} \right) \right)} t, \\ h(t, \varepsilon) = -\omega(t, \varepsilon)u_\delta(t, \varepsilon)^2/2 + x(t, \varepsilon)[\rho u^2/2 + H] - [(\rho u^2/2 + H + P)u]t, \end{cases} \quad (3.9)$$

for $[\rho] = \rho_+ - \rho_- \neq 0$, and

$$\begin{cases} x(t, \varepsilon) = \frac{(u_- + u_+)t}{2}, \\ u_\delta(t, \varepsilon) = \frac{u_- + u_+}{2}, \\ \omega(t, \varepsilon) = \rho_-(u_- - u_+)t, \\ h(t, \varepsilon) = -\omega(t, \varepsilon)u_\delta^2(t)/2 + x(t, \varepsilon)[\rho u^2/2 + H] - [(\rho u^2/2 + H + P)u]t, \end{cases} \quad (3.10)$$

for $[\rho] = \rho_+ - \rho_- = 0$.

In addition, it is easy to see that the delta shock wave satisfies the generalized entropy condition

$$u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} \leq u_\delta(t, \varepsilon) \leq u_- - \frac{\sqrt{\varepsilon}}{\rho_-}, \quad (3.11)$$

which ensures the uniqueness of Riemann solutions.

4 Limits of Riemann solutions to (1.6)–(1.7)

In this section, we concentrate on the limit behavior of Riemann solutions to equations (1.6)–(1.7), and the formation of delta shock and the vacuum phenomenon are considered in the case $u_- > u_+$ and the case $u_- < u_+$.

4.1 Limits of Riemann solutions in the case $u_- > u_+$

Lemma 4.1. Assume $u_- > u_+$, and then there exist two constant values $\varepsilon_1, \varepsilon_2$, $\varepsilon_1 > \varepsilon_2 > 0$, such that the projection of (ρ_+, u_+, H_+) belongs to $IV(\rho_-, u_-)$ when $\varepsilon_2 < \varepsilon < \varepsilon_1$, and belongs to $V(\rho_-, u_-)$ when $0 < \varepsilon < \varepsilon_2$.

Proof. Suppose $u_- > u_+$, the states (ρ_+, u_+, H_+) connect with (ρ_-, u_-, H_-) by contact discontinuities that satisfy

$$u_+ = u_- - \frac{\sqrt{\varepsilon}}{\rho_-} + \frac{\sqrt{\varepsilon}}{\rho_+}, \quad \rho_+ > \rho_-, \quad (4.1)$$

$$u_+ = u_- + \frac{\sqrt{\varepsilon}}{\rho_-} - \frac{\sqrt{\varepsilon}}{\rho_+}, \quad \rho_+ < \rho_-. \quad (4.2)$$

If $\rho_- \neq \rho_+$, the projection pertains to $IV(\rho_-, u_-)$ or $V(\rho_-, u_-)$, we have

$$\varepsilon_1 = \left(\frac{(u_- - u_+)\rho_+\rho_-}{\rho_- - \rho_+} \right)^2, \quad (4.3)$$

that is, the projection belongs to $IV(\rho_-, u_-)$ or $V(\rho_-, u_-)$ when $0 < \varepsilon < \varepsilon_1$.

If the projection pertains to $V(\rho_-, u_-)$, we have

$$\varepsilon_2 = \left(\frac{(u_- - u_+)\rho_+\rho_-}{\rho_- + \rho_+} \right)^2, \quad (4.4)$$

that is, projection is located in $IV(\rho_-, u_-)$ when $\varepsilon_2 < \varepsilon < \varepsilon_1$, and projection belongs to $V(\rho_-, u_-)$ when $0 < \varepsilon < \varepsilon_2$.

If $\rho_- = \rho_+$, the conclusion is clearly valid. \square

From Lemma 4.1, we know there is no delta shock wave when $\varepsilon > \varepsilon_2$. We find that the curves of two-contact discontinuities become steeper when ε decreases, that is, when ε decreases, the projection of (ρ_+, u_+, H_+) must belong to $IV(\rho_-, u_-)$ or $V(\rho_-, u_-)$.

First, we consider the situation $\varepsilon_2 < \varepsilon < \varepsilon_1$, namely, the projection of (ρ_+, u_+, H_+) pertains to $IV(\rho_-, u_-)$. In this situation, the Riemann solution to (1.6)–(1.7) is

$$(\rho_-, u_-, H_-) + J_1^\varepsilon + (\rho_{*1}^\varepsilon, u_{*1}^\varepsilon, H_{*1}^\varepsilon) + J_2^\varepsilon + (\rho_{*2}^\varepsilon, u_{*2}^\varepsilon, H_{*2}^\varepsilon) + J_3^\varepsilon + (\rho_+, u_+, H_+), \quad (4.5)$$

the intermediate states satisfy the following formulae

$$J_1^\varepsilon : \begin{cases} u_{*1}^\varepsilon - u_- = \sqrt{\varepsilon} \left(\frac{1}{\rho_{*1}^\varepsilon} - \frac{1}{\rho_-} \right), & \rho_{*1}^\varepsilon > \rho_-, \\ (2H_{*1}^\varepsilon \rho_{*1}^\varepsilon - \varepsilon) \rho_-^2 = (2H_- \rho_- - \varepsilon) (\rho_{*1}^\varepsilon)^2, \end{cases} \quad (4.6)$$

$$J_2^\varepsilon : \begin{cases} \rho_{*1}^\varepsilon = \rho_{*2}^\varepsilon, & u_{*1}^\varepsilon = u_{*2}^\varepsilon, \\ H_{*1}^\varepsilon \neq H_{*2}^\varepsilon, \end{cases} \quad (4.7)$$

$$J_3^\varepsilon : \begin{cases} u_+ - u_{*2}^\varepsilon = \sqrt{\varepsilon} \left(\frac{1}{\rho_{*2}^\varepsilon} - \frac{1}{\rho_+} \right), & \rho_{*2}^\varepsilon > \rho_+, \\ (2H_+ \rho_+ - \varepsilon) (\rho_{*2}^\varepsilon)^2 = (2H_{*2}^\varepsilon \rho_{*2}^\varepsilon - \varepsilon) \rho_+^2, \end{cases} \quad (4.8)$$

where $(\rho_{*1}^\varepsilon, u_{*1}^\varepsilon, H_{*1}^\varepsilon)$ and $(\rho_{*2}^\varepsilon, u_{*2}^\varepsilon, H_{*2}^\varepsilon)$ are the intermediate states.

Lemma 4.2. *The intermediate density ρ_*^ε becomes unbounded as $\varepsilon \rightarrow \varepsilon_2$, that is,*

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \rho_*^\varepsilon = \infty, \quad (4.9)$$

where $\rho_*^\varepsilon := \rho_{*1}^\varepsilon = \rho_{*2}^\varepsilon$, the intermediate internal energy $H_{*1}^\varepsilon, H_{*2}^\varepsilon$ become unbounded, i.e.,

$$\lim_{\varepsilon \rightarrow \varepsilon_2} H_{*1}^\varepsilon = \infty, \quad \lim_{\varepsilon \rightarrow \varepsilon_2} H_{*2}^\varepsilon = \infty. \quad (4.10)$$

Proof. From (4.6) to (4.8), it is easy to calculate that

$$\rho_*^\varepsilon := \rho_{*1}^\varepsilon = \rho_{*2}^\varepsilon, \quad u_*^\varepsilon := u_{*1}^\varepsilon = u_{*2}^\varepsilon, \quad (4.11)$$

$$u_*^\varepsilon = \frac{1}{2} \left(u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} \right) + \frac{1}{2} \left(u_- - \frac{\sqrt{\varepsilon}}{\rho_-} \right), \quad (4.12)$$

$$\frac{\sqrt{\varepsilon}}{\rho_*^\varepsilon} = \frac{1}{2} \left(u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} \right) - \frac{1}{2} \left(u_- - \frac{\sqrt{\varepsilon}}{\rho_-} \right). \quad (4.13)$$

Therefore,

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \frac{\sqrt{\varepsilon}}{\rho_*^\varepsilon} = \lim_{\varepsilon \rightarrow \varepsilon_2} \frac{1}{2} \left(u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} \right) - \frac{1}{2} \left(u_- - \frac{\sqrt{\varepsilon}}{\rho_-} \right) = 0, \quad (4.14)$$

which implies that $\lim_{\varepsilon \rightarrow \varepsilon_2} \rho_*^\varepsilon = \infty$.

Using (4.6)₂, we derive

$$\frac{2H_{*1}^\varepsilon}{\rho_*^\varepsilon} - \frac{\varepsilon}{(\rho_*^\varepsilon)^2} = \frac{2H_-}{\rho_-} - \frac{\varepsilon}{\rho_-^2}, \quad (4.15)$$

thus

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \frac{2H_{*1}^\varepsilon}{\rho_*^\varepsilon} = \frac{2H_-}{\rho_-} - \frac{\varepsilon_2}{\rho_-^2}, \quad (4.16)$$

namely,

$$\lim_{\varepsilon \rightarrow \varepsilon_2} H_{*1}^\varepsilon = \infty. \quad (4.17)$$

By using (4.8)₂, we obtain the same conclusion

$$\lim_{\varepsilon \rightarrow \varepsilon_2} H_{*2}^\varepsilon = \infty. \quad (4.18)$$

□

Lemma 4.3. *Let*

$$u_\delta = \frac{\rho_+ u_+ + \rho_- u_-}{\rho_+ + \rho_-}, \quad (4.19)$$

then

$$\lim_{\varepsilon \rightarrow \varepsilon_2} u_*^\varepsilon = \lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_1^\varepsilon = \lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_2^\varepsilon = \lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_3^\varepsilon = u_\delta. \quad (4.20)$$

Proof. Involving the first term of (3.2)–(3.4), the following equations are obtained

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_1^\varepsilon = \lim_{\varepsilon \rightarrow \varepsilon_2} \left(u_- - \frac{\sqrt{\varepsilon}}{\rho_-} \right) = u_- - \frac{\rho_+(u_- - u_+)}{\rho_+ + \rho_-} = \frac{\rho_+ u_+ + \rho_- u_-}{\rho_+ + \rho_-} = u_\delta, \quad (4.21)$$

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_3^\varepsilon = \lim_{\varepsilon \rightarrow \varepsilon_2} \left(u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} \right) = u_+ + \frac{\rho_-(u_- - u_+)}{\rho_+ + \rho_-} = \frac{\rho_+ u_+ + \rho_- u_-}{\rho_+ + \rho_-} = u_\delta, \quad (4.22)$$

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \sigma_2^\varepsilon = \lim_{\varepsilon \rightarrow \varepsilon_2} u_*^\varepsilon = u_\delta. \quad (4.23)$$

□

Lemma 4.4.

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \rho_*^\varepsilon (\sigma_2^\varepsilon - \sigma_1^\varepsilon) = \frac{(u_- - u_+) \rho_+ \rho_-}{\rho_+ + \rho_-}, \quad (4.24)$$

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \rho_*^\varepsilon (\sigma_3^\varepsilon - \sigma_2^\varepsilon) = \frac{(u_- - u_+) \rho_+ \rho_-}{\rho_+ + \rho_-}. \quad (4.25)$$

Proof. The expressions of σ_i ($i = 1, 2, 3$) are employed again, and the following discussions will be presented

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \rho_*^\varepsilon (\sigma_2^\varepsilon - \sigma_1^\varepsilon) = \lim_{\varepsilon \rightarrow \varepsilon_2} \rho_*^\varepsilon \left(u_*^\varepsilon - u_*^\varepsilon + \frac{\sqrt{\varepsilon}}{\rho_*^\varepsilon} \right) = \lim_{\varepsilon \rightarrow \varepsilon_2} \sqrt{\varepsilon} = \frac{(u_- - u_+) \rho_+ \rho_-}{\rho_+ + \rho_-}, \quad (4.26)$$

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \rho_*^\varepsilon (\sigma_3^\varepsilon - \sigma_2^\varepsilon) = \lim_{\varepsilon \rightarrow \varepsilon_2} \rho_*^\varepsilon \left(u_*^\varepsilon + \frac{\sqrt{\varepsilon}}{\rho_*^\varepsilon} - u_*^\varepsilon \right) = \lim_{\varepsilon \rightarrow \varepsilon_2} \sqrt{\varepsilon} = \frac{(u_- - u_+) \rho_+ \rho_-}{\rho_+ + \rho_-}. \quad (4.27)$$

□

Lemma 4.5.

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \left(\frac{\rho_*^\varepsilon (u_*^\varepsilon)^2}{2} + H_*^\varepsilon \right) (\sigma_3^\varepsilon - \sigma_1^\varepsilon) = \left(u_\delta \left[\frac{\rho u^2}{2} + H \right] - \left[\left(\frac{\rho u^2}{2} + H - \frac{\varepsilon_2}{\rho} \right) u \right] \right), \quad (4.28)$$

where

$$H_*^\varepsilon = \begin{cases} H_{*1}^\varepsilon, & \sigma_1^\varepsilon < \xi < \sigma_2^\varepsilon, \\ H_{*2}^\varepsilon, & \sigma_2^\varepsilon < \xi < \sigma_3^\varepsilon. \end{cases}$$

Proof. Using the Rankine-Hugoniot conditions of (1.6), we obtain the following forms:

$$\begin{cases} \sigma_1^\varepsilon \left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 1}^\varepsilon - \frac{\rho u_-^2}{2} - H_- \right) = \left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 1}^\varepsilon - \frac{\varepsilon}{\rho_\star^\varepsilon} \right) u_\star^\varepsilon - \left(\frac{\rho u_-^2}{2} + H_- - \frac{\varepsilon}{\rho} \right) u_-, \\ \sigma_2^\varepsilon \left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 2}^\varepsilon - \frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} - H_{\star 1}^\varepsilon \right) = \left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 2}^\varepsilon - \frac{\varepsilon}{\rho_\star^\varepsilon} \right) u_\star^\varepsilon - \left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 1}^\varepsilon - \frac{\varepsilon}{\rho_\star^\varepsilon} \right) u_\star^\varepsilon, \\ \sigma_3^\varepsilon \left(\frac{\rho_+ u_+^2}{2} + H_+ - \frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} - H_{\star 2}^\varepsilon \right) = - \left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 2}^\varepsilon - \frac{\varepsilon}{\rho_\star^\varepsilon} \right) u_\star^\varepsilon + \left(\frac{\rho_+ u_+^2}{2} + H_+ - \frac{\varepsilon}{\rho_+} \right) u_+. \end{cases} \quad (4.29)$$

When $\varepsilon \rightarrow \varepsilon_2$, utilizing (4.29) and taking limits, we have

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \left(\left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 1}^\varepsilon \right) (\sigma_2^\varepsilon - \sigma_1^\varepsilon) + \left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 2}^\varepsilon \right) (\sigma_3^\varepsilon - \sigma_2^\varepsilon) \right) = u_\delta \left[\frac{\rho u^2}{2} + H \right] - \left[\left(\frac{\rho u^2}{2} + H - \frac{\varepsilon_2}{\rho} \right) u \right], \quad (4.30)$$

which implies that

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \left(\frac{\rho_\star^\varepsilon (u_\star^\varepsilon)^2}{2} + H_{\star 2}^\varepsilon \right) (\sigma_3^\varepsilon - \sigma_1^\varepsilon) = \left(u_\delta \left[\frac{\rho u^2}{2} + H \right] - \left[\left(\frac{\rho u^2}{2} + H - \frac{\varepsilon_2}{\rho} \right) u \right] \right).$$

The proof is complete. \square

Theorem 4.6. When $u_- > u_+$, the Riemann solution tends to a delta shock wave as $\varepsilon \rightarrow \varepsilon_2$. The limit functions ρ , ρu and H are the sums of a step function and a δ -measure with weights

$$\frac{t}{\sqrt{1+u_\delta^2}} (u_\delta [\rho] - [\rho u]), \quad \frac{t}{\sqrt{1+u_\delta^2}} \left(u_\delta [\rho u] - \left[\rho u^2 - \frac{\varepsilon_2}{\rho} \right] \right), \quad \frac{t}{\sqrt{1+u_\delta^2}} \left(u_\delta \left[\frac{\rho u^2}{2} + H \right] - \left[\left(\frac{\rho u^2}{2} + H - \frac{\varepsilon_2}{\rho} \right) u \right] \right),$$

where $u_\delta = \frac{\rho_+ u_+ + \rho_- u_-}{\rho_+ + \rho_-}$.

Proof.

1. For $\xi = \frac{x}{t}$, the Riemann solutions are denoted by

$$(\rho^\varepsilon(\xi), u^\varepsilon(\xi)) = \begin{cases} (\rho_-, u_-), & \xi < \sigma_1^\varepsilon, \\ (\rho_\star^\varepsilon, u_\star^\varepsilon), & \sigma_1^\varepsilon < \xi < \sigma_3^\varepsilon, \\ (\rho_+, u_+), & \xi > \sigma_3^\varepsilon, \end{cases} \quad \text{and} \quad H^\varepsilon(\xi) = \begin{cases} H_-, & \xi < \sigma_1^\varepsilon, \\ H_{\star 1}^\varepsilon, & \sigma_1^\varepsilon < \xi < \sigma_2^\varepsilon, \\ H_{\star 2}^\varepsilon, & \sigma_2^\varepsilon < \xi < \sigma_3^\varepsilon, \\ H_+, & \xi > \sigma_3^\varepsilon, \end{cases}$$

which satisfies the following weak formulae:

$$\langle -\xi \rho_\xi^\varepsilon + (\rho^\varepsilon u^\varepsilon)_\xi, \psi \rangle = \int_{-\infty}^{+\infty} (-\xi \rho_\xi^\varepsilon + (\rho^\varepsilon u^\varepsilon)_\xi) \psi d\xi = 0, \quad (4.31)$$

$$\left\langle -\xi (\rho^\varepsilon u^\varepsilon)_\xi + \left(\rho^\varepsilon (u^\varepsilon)^2 - \frac{\varepsilon}{\rho^\varepsilon} \right)_\xi, \psi \right\rangle = \int_{-\infty}^{+\infty} \left(-\xi (\rho^\varepsilon u^\varepsilon)_\xi + \left(\rho^\varepsilon (u^\varepsilon)^2 - \frac{\varepsilon}{\rho^\varepsilon} \right)_\xi \right) \psi d\xi = 0, \quad (4.32)$$

$$\begin{aligned} & \left\langle -\xi \left(\frac{\rho^\varepsilon (u^\varepsilon)^2}{2} + H^\varepsilon \right)_\xi + \left(\left(\frac{\rho^\varepsilon (u^\varepsilon)^2}{2} + H^\varepsilon - \frac{\varepsilon}{\rho^\varepsilon} \right) u^\varepsilon \right)_\xi, \psi \right\rangle \\ &= \int_{-\infty}^{+\infty} \left(-\xi \left(\frac{\rho^\varepsilon (u^\varepsilon)^2}{2} + H^\varepsilon \right)_\xi + \left(\frac{\rho^\varepsilon (u^\varepsilon)^3}{2} + H^\varepsilon u^\varepsilon - \frac{\varepsilon u^\varepsilon}{\rho^\varepsilon} \right)_\xi \right) \psi d\xi = 0, \end{aligned} \quad (4.33)$$

for arbitrary $\psi \in C_0^\infty(\mathbb{R})$.

2. From (4.31), one can obtain

$$\int_{-\infty}^{+\infty} (-\xi \rho_{\xi}^{\varepsilon} + (\rho^{\varepsilon} u^{\varepsilon})_{\xi}) \psi d\xi = I_1 + I_2, \quad (4.34)$$

where

$$I_1 = \int_{-\infty}^{+\infty} \rho^{\varepsilon} \psi d\xi, \quad I_2 = \int_{-\infty}^{+\infty} \rho^{\varepsilon} (\xi - u^{\varepsilon}) \psi_{\xi} d\xi.$$

We decompose I_2 as

$$I_2 = \left(\int_{-\infty}^{\sigma_1^{\varepsilon}} + \int_{\sigma_1^{\varepsilon}}^{\sigma_3^{\varepsilon}} + \int_{\sigma_3^{\varepsilon}}^{+\infty} \right) \rho^{\varepsilon} (\xi - u^{\varepsilon}) \psi_{\xi} d\xi.$$

The total of the first and last terms is

$$\begin{aligned} & \int_{-\infty}^{\sigma_1^{\varepsilon}} \rho_{-} (\xi - u_{-}) \psi_{\xi} d\xi + \int_{\sigma_3^{\varepsilon}}^{+\infty} \rho_{+} (\xi - u_{+}) \psi_{\xi} d\xi \\ &= \rho_{+} u_{+} \psi(\sigma_3^{\varepsilon}) - \rho_{-} u_{-} \psi(\sigma_1^{\varepsilon}) + \rho_{-} \sigma_1^{\varepsilon} \psi(\sigma_1^{\varepsilon}) - \rho_{+} \sigma_3^{\varepsilon} \psi(\sigma_3^{\varepsilon}) - \int_{-\infty}^{\sigma_1^{\varepsilon}} \rho_{-} \psi d\xi - \int_{\sigma_3^{\varepsilon}}^{+\infty} \rho_{+} \psi d\xi, \end{aligned}$$

when $\varepsilon \rightarrow \varepsilon_2$, it converges to the following equality:

$$[\rho u] \psi - u_{\delta} [\rho] \psi - \int_{-\infty}^{u_{\delta}} \rho_{-} \psi d\xi - \int_{u_{\delta}}^{+\infty} \rho_{+} \psi d\xi = ([\rho u] - u_{\delta} [\rho]) \psi(u_{\delta}) - \int_{-\infty}^{+\infty} \rho_0 (\xi - u_{\delta}) \psi(\xi) d\xi, \quad (4.35)$$

where $\rho_0 (\xi - u_{\delta}) = \rho_{-} + [\rho] \mathcal{F}(\xi)$ and $\mathcal{F}(\xi)$ is a Heaviside function.

In addition,

$$\int_{\sigma_1^{\varepsilon}}^{\sigma_3^{\varepsilon}} \rho_{\star}^{\varepsilon} (\xi - u_{\star}^{\varepsilon}) \psi_{\xi} d\xi = \rho_{\star}^{\varepsilon} (\sigma_3^{\varepsilon} - \sigma_1^{\varepsilon}) \left(\frac{\sigma_3^{\varepsilon} \psi(\sigma_3^{\varepsilon}) - \sigma_1^{\varepsilon} \psi(\sigma_1^{\varepsilon})}{\sigma_3^{\varepsilon} - \sigma_1^{\varepsilon}} - u_{\star}^{\varepsilon} \frac{\psi(\sigma_3^{\varepsilon}) - \psi(\sigma_1^{\varepsilon})}{\sigma_3^{\varepsilon} - \sigma_1^{\varepsilon}} - \frac{1}{\sigma_3^{\varepsilon} - \sigma_1^{\varepsilon}} \int_{\sigma_1^{\varepsilon}}^{\sigma_3^{\varepsilon}} \psi d\xi \right), \quad (4.36)$$

when $\varepsilon \rightarrow \varepsilon_2$, it leads to

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \int_{\sigma_1^{\varepsilon}}^{\sigma_3^{\varepsilon}} \rho_{\star}^{\varepsilon} (\xi - u_{\star}^{\varepsilon}) \psi_{\xi} d\xi = \rho_{\star}^{\varepsilon} (\sigma_3^{\varepsilon} - \sigma_1^{\varepsilon}) ((u_{\delta} \psi)' - u_{\delta} \psi' - \psi) = 0. \quad (4.37)$$

Above all, from (4.34), it yields

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \int_{-\infty}^{\infty} \rho^{\varepsilon} \psi - \rho_0 (\xi - u_{\delta}) \psi(\xi) d\xi = (u_{\delta} [\rho] - [\rho u]) \psi(u_{\delta}). \quad (4.38)$$

3. We deduce the limit of the momentum $m^{\varepsilon} = \rho^{\varepsilon} u^{\varepsilon}$ from momentum equation (4.32), that is to say,

$$\int_{-\infty}^{\infty} \left(-\xi (\rho^{\varepsilon} u^{\varepsilon})_{\xi} + \left(\rho^{\varepsilon} (u^{\varepsilon})^2 - \frac{\varepsilon}{\rho^{\varepsilon}} \right) \right)_{\xi} \psi d\xi = \int_{-\infty}^{\infty} \left(\rho^{\varepsilon} u^{\varepsilon} (\xi - u^{\varepsilon}) + \frac{\varepsilon}{\rho^{\varepsilon}} \right) \psi_{\xi} d\xi + \int_{-\infty}^{\infty} \rho^{\varepsilon} u^{\varepsilon} \psi d\xi = 0. \quad (4.39)$$

The first term on the right of equation (4.39) can be rewritten as

$$\int_{-\infty}^{\infty} \left(\rho^\varepsilon u^\varepsilon (\xi - u^\varepsilon) + \frac{\varepsilon}{\rho^\varepsilon} \right) \psi_\xi d\xi = \left(\int_{-\infty}^{\sigma_1^\varepsilon} + \int_{\sigma_1^\varepsilon}^{\sigma_3^\varepsilon} + \int_{\sigma_3^\varepsilon}^{\infty} \right) \left(\rho^\varepsilon u^\varepsilon (\xi - u^\varepsilon) + \frac{\varepsilon}{\rho^\varepsilon} \right) \psi_\xi d\xi. \quad (4.40)$$

The sum of the first and last terms of equation (4.40) is

$$\begin{aligned} & \int_{-\infty}^{\sigma_1^\varepsilon} \left(\rho_- u_- (\xi - u_-) + \frac{\varepsilon}{\rho_-} \right) \psi_\xi d\xi + \int_{\sigma_3^\varepsilon}^{\infty} \left(\rho_+ u_+ (\xi - u_+) + \frac{\varepsilon}{\rho_+} \right) \psi_\xi d\xi \\ &= \rho_- u_- \sigma_1^\varepsilon \psi(\sigma_1^\varepsilon) - \rho_+ u_+ \sigma_3^\varepsilon \psi(\sigma_3^\varepsilon) + \left(\rho_+ u_+^2 - \frac{\varepsilon}{\rho_+} \right) \psi(\sigma_3^\varepsilon) - \left(\rho_- u_-^2 - \frac{\varepsilon}{\rho_-} \right) \psi(\sigma_1^\varepsilon) - \int_{-\infty}^{\sigma_1^\varepsilon} \rho_- u_- \psi d\xi - \int_{\sigma_3^\varepsilon}^{\infty} \rho_+ u_+ \psi d\xi. \end{aligned} \quad (4.41)$$

Letting $\varepsilon \rightarrow \varepsilon_2$, we derive

$$\begin{aligned} & \lim_{\varepsilon \rightarrow \varepsilon_2} \int_{-\infty}^{\sigma_1^\varepsilon} \left(\rho_- u_- (\xi - u_-) + \frac{\varepsilon}{\rho_-} \right) \psi_\xi d\xi + \int_{\sigma_3^\varepsilon}^{\infty} \left(\rho_+ u_+ (\xi - u_+) + \frac{\varepsilon}{\rho_+} \right) \psi_\xi d\xi \\ &= -u_\delta [\rho u] \psi + \left[\rho u^2 - \frac{\varepsilon_2}{\rho} \right] \psi - \int_{-\infty}^{\infty} m_0(\xi - u_\delta) \psi(\xi) d\xi, \end{aligned} \quad (4.42)$$

where $m_0(\xi) = \rho_- u_- + [\rho u] \mathcal{F}(\xi)$ and $\mathcal{F}(\xi)$ is a Heaviside function.

$$\begin{aligned} & \int_{\sigma_1^\varepsilon}^{\sigma_3^\varepsilon} \left(\rho_*^\varepsilon u_*^\varepsilon (\xi - u_*^\varepsilon) + \frac{\varepsilon}{\rho_*^\varepsilon} \right) \psi_\xi d\xi \\ &= \left(\rho_*^\varepsilon u_*^\varepsilon (\xi - u_*^\varepsilon) + \frac{\varepsilon}{\rho_*^\varepsilon} \right) \psi \Big|_{\sigma_1^\varepsilon}^{\sigma_3^\varepsilon} - \int_{\sigma_1^\varepsilon}^{\sigma_3^\varepsilon} \rho_*^\varepsilon u_*^\varepsilon \psi d\xi \\ &= \rho_*^\varepsilon u_*^\varepsilon (\sigma_3^\varepsilon - \sigma_1^\varepsilon) \left(\frac{\sigma_3^\varepsilon \psi(\sigma_3^\varepsilon) - \sigma_1^\varepsilon \psi(\sigma_1^\varepsilon)}{\sigma_3^\varepsilon - \sigma_1^\varepsilon} - u_*^\varepsilon \frac{\psi(\sigma_3^\varepsilon) - \psi(\sigma_1^\varepsilon)}{\sigma_3^\varepsilon - \sigma_1^\varepsilon} - \frac{\int_{\sigma_1^\varepsilon}^{\sigma_3^\varepsilon} \rho_* u_* \psi d\xi}{\sigma_3^\varepsilon - \sigma_1^\varepsilon} \right) + \frac{\varepsilon}{\rho_*^\varepsilon} (\psi(\sigma_3^\varepsilon) - \psi(\sigma_1^\varepsilon)), \end{aligned} \quad (4.43)$$

it converges to

$$\rho_*^\varepsilon u_*^\varepsilon (\sigma_3^\varepsilon - \sigma_1^\varepsilon) ((u_\delta \psi(u_\delta))' - u_\delta \psi' - \psi) = 0, \quad (4.44)$$

as $\varepsilon \rightarrow \varepsilon_2$.

Above all, from (4.39), we obtain

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \int_{-\infty}^{\infty} \rho^\varepsilon u^\varepsilon \psi(\xi) - m_0(\xi - u_\delta) \psi(\xi) d\xi = \left(u_\delta [\rho u] - \left[\rho u^2 - \frac{\varepsilon_2}{\rho} \right] \right) \psi(\sigma). \quad (4.45)$$

4. Next, let us consider the conservation of energy (4.33), we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(-\xi \left(\frac{\rho^\varepsilon (u^\varepsilon)^2}{2} + H^\varepsilon \right)_\xi + \left(\frac{\rho^\varepsilon (u^\varepsilon)^3}{2} + H^\varepsilon u^\varepsilon - \frac{\varepsilon u^\varepsilon}{\rho^\varepsilon} \right)_\xi \right) \psi d\xi \\ &= \int_{-\infty}^{\infty} \left(\frac{\rho^\varepsilon (u^\varepsilon)^2}{2} + H^\varepsilon \right) (\xi - u^\varepsilon) \psi_\xi + \frac{\varepsilon u^\varepsilon}{\rho^\varepsilon} \psi_\xi d\xi + \int_{-\infty}^{\infty} \left(\frac{\rho^\varepsilon (u^\varepsilon)^2}{2} + H^\varepsilon \right) \psi d\xi. \end{aligned} \quad (4.46)$$

The first integral in (4.46) can be decomposed into

$$\left(\int_{-\infty}^{\sigma_1^\varepsilon} + \int_{\sigma_1^\varepsilon}^{\sigma_3^\varepsilon} + \int_{\sigma_3^\varepsilon}^{\infty} \right) \left(\left(\frac{\rho^\varepsilon(u^\varepsilon)^2}{2} + H^\varepsilon \right) (\xi - u^\varepsilon) + \frac{\varepsilon u^\varepsilon}{\rho} \right) \psi_\xi d\xi. \quad (4.47)$$

The sum of the first and the last terms in (4.47) is

$$\begin{aligned} & \left(\int_{-\infty}^{\sigma_1^\varepsilon} + \int_{\sigma_3^\varepsilon}^{\infty} \right) \left(\left(\frac{\rho^\varepsilon(u^\varepsilon)^2}{2} + H^\varepsilon \right) (\xi - u^\varepsilon) + \frac{\varepsilon u^\varepsilon}{\rho} \right) \psi_\xi d\xi \\ &= \left(\frac{\rho u_-^2}{2} + H_- \right) \sigma_1^\varepsilon \psi(\sigma_1^\varepsilon) - \left(\frac{\rho u_+^2}{2} + H_+ \right) \sigma_3^\varepsilon \psi(\sigma_3^\varepsilon) - \left(\frac{\rho u_-^3}{2} + H_- u_- - \frac{\varepsilon u_-}{\rho} \right) \psi(\sigma_1^\varepsilon) \\ & \quad + \left(\frac{\rho u_+^3}{2} + H_+ u_+ - \frac{\varepsilon u_+}{\rho} \right) \psi(\sigma_3^\varepsilon) - \int_{-\infty}^{\sigma_1^\varepsilon} \left(\frac{\rho u_-^2}{2} + H_- \right) \psi d\xi - \int_{\sigma_3^\varepsilon}^{\infty} \left(\frac{\rho u_+^2}{2} + H_+ \right) \psi d\xi, \end{aligned} \quad (4.48)$$

which converges to

$$\left(-u_\delta \left[\frac{\rho u^2}{2} + H \right] + \left[\left(\frac{\rho u^2}{2} + H - \frac{\varepsilon_2}{\rho} \right) u \right] \right) \psi(u_\delta) - \int_{-\infty}^{\infty} \varepsilon_0(\xi - u_\delta) \psi(\xi) d\xi \quad (4.49)$$

as $\varepsilon \rightarrow \varepsilon_2$, with $\varepsilon_0(\xi) = \frac{\rho u_-^2}{2} + H_- + \left[\frac{\rho u^2}{2} + H \right] \mathcal{F}(\xi)$ and $\mathcal{F}(\xi)$ is a Heaviside function.

For the second term in (4.47), we get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow \varepsilon_2} \int_{\sigma_1^\varepsilon}^{\sigma_3^\varepsilon} \left(\left(\frac{\rho_\star^\varepsilon(u_\star^\varepsilon)^2}{2} + H_\star^\varepsilon \right) (\xi - u_\star^\varepsilon) + \frac{\varepsilon u_\star^\varepsilon}{\rho_\star^\varepsilon} \right) \psi_\xi d\xi \\ &= \lim_{\varepsilon \rightarrow \varepsilon_2} \left(\frac{\rho_\star^\varepsilon(u_\star^\varepsilon)^2}{2} + H_\star^\varepsilon \right) (\sigma_3^\varepsilon - \sigma_1^\varepsilon) \left(\frac{\sigma_3^\varepsilon \psi(\sigma_3^\varepsilon) - \sigma_1^\varepsilon \psi(\sigma_1^\varepsilon)}{\sigma_3^\varepsilon - \sigma_1^\varepsilon} - u_\star^\varepsilon \frac{\psi(\sigma_3^\varepsilon) - \psi(\sigma_1^\varepsilon)}{\sigma_3^\varepsilon - \sigma_1^\varepsilon} - \frac{\psi(\sigma_3^\varepsilon - \sigma_1^\varepsilon)}{\sigma_3^\varepsilon - \sigma_1^\varepsilon} \right) \\ & \quad + \frac{\varepsilon u_\star^\varepsilon}{\rho_\star^\varepsilon} (\psi(\sigma_3^\varepsilon) - \psi(\sigma_1^\varepsilon)) = \left(\frac{\rho_\star^\varepsilon(u_\star^\varepsilon)^2}{2} + H_\star^\varepsilon \right) (\sigma_3^\varepsilon - \sigma_1^\varepsilon) ((u_\delta \psi)' - u_\delta \psi' - \psi) + 0 = 0, \end{aligned} \quad (4.50)$$

where

$$H_\star^\varepsilon = \begin{cases} H_{\star_1}^\varepsilon, & \sigma_1^\varepsilon < \xi < \sigma_2^\varepsilon, \\ H_{\star_2}^\varepsilon, & \sigma_2^\varepsilon < \xi < \sigma_3^\varepsilon. \end{cases} \quad (4.51)$$

From (4.46), we obtain that

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \int_{-\infty}^{\infty} \left(\frac{\rho^\varepsilon(u^\varepsilon)^2}{2} + H^\varepsilon \right) \psi(\xi) + \varepsilon_0(\xi - u_\delta) \psi(\xi) d\xi = u_\delta \left[\frac{\rho u^2}{2} + H \right] \psi + \left[\left(\frac{\rho u^2}{2} + H - \frac{\varepsilon_2}{\rho} \right) u \right] \psi. \quad (4.52)$$

- By considering the time dependence of weights of δ -measures, the limits of density, momentum and energy are obtained.

For any $\psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$, letting $\tilde{\psi}(\xi, t) = \psi(\xi t, t)$, we obtain

$$\langle \rho^\varepsilon, \psi \rangle = \lim_{\varepsilon \rightarrow \varepsilon_2} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho^\varepsilon \left(\frac{x}{t} \right) \psi(x, t) d(\xi t) dt = \lim_{\varepsilon \rightarrow \varepsilon_2} \int_0^{+\infty} t \int_{-\infty}^{+\infty} \rho^\varepsilon(\xi) \tilde{\psi}(\xi, t) d\xi dt, \quad (4.53)$$

as $(\rho^\varepsilon, u^\varepsilon, H^\varepsilon)$ is a self-similar solution.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \rho^\varepsilon(\xi) \tilde{\psi}(\xi, t) d\xi &= \int_{-\infty}^{+\infty} \rho_0(\xi - u_\delta) \tilde{\psi}(\xi, t) d\xi + (\sigma[\rho] - [\rho u]) \tilde{\psi}(u_\delta, t) \\
&= t^{-1} \int_{-\infty}^{+\infty} \rho_0(\xi - u_\delta) \psi(x, t) dx + (\sigma[\rho] - [\rho u]) \psi(u_\delta t, t), \\
\langle \rho^\varepsilon, \psi \rangle &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0(\xi - u_\delta t) \psi(x, t) dx dt + \int_0^{+\infty} t(u_\delta[\rho] - [\rho u]) \psi(u_\delta t, t) dt,
\end{aligned} \tag{4.54}$$

$$\rho(x, t) = \rho_0(x, t) + \omega_0(t) \delta_s, \tag{4.55}$$

and

$$\langle \omega_0(t) \delta_s, \psi \rangle = \int_0^{+\infty} \omega_0(t) \sqrt{x'(t)^2 + 1} \psi(x, t) dt, \tag{4.56}$$

utilizing (4.54)–(4.56), we have

$$\begin{aligned}
\langle \rho, \psi \rangle &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0(x, t) \psi(x, t) dx dt + \int_0^{+\infty} \omega_0(t) \sqrt{x'(t)^2 + 1} \psi(x, t) dt \\
&= \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0(\xi - \sigma t) \psi(x, t) dx dt + \int_0^{+\infty} \omega_0(t) \psi(x, t) \sqrt{u_\delta^2 + 1} \psi(x, t) dt,
\end{aligned} \tag{4.57}$$

therefore,

$$\omega_0(t) \sqrt{u_\delta^2 + 1} = t(u_\delta[\rho] - [\rho u]), \tag{4.58}$$

namely,

$$\omega_0(t) = \frac{t}{\sqrt{1 + u_\delta^2}} (u_\delta[\rho] - [\rho u]). \tag{4.59}$$

Similarly, it can be shown that

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \int_{-\infty}^{\infty} \int_0^{\infty} \rho_*^\varepsilon u_*^\varepsilon \psi(x, t) dx dt = \int_{-\infty}^{\infty} \int_0^{\infty} m_0(x - u_\delta t) \psi(x, t) dx dt + \int_0^{\infty} \omega_1(t) \psi(u_\delta t, t) \sqrt{u_\delta^2 + 1} dt, \tag{4.60}$$

with

$$\omega_1(t) = \frac{t}{\sqrt{1 + u_\delta^2}} \left(u_\delta[\rho u] - \left[\rho u^2 - \frac{\varepsilon_2}{\rho} \right] \right). \tag{4.61}$$

$$\lim_{\varepsilon \rightarrow \varepsilon_2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\frac{\rho_*^\varepsilon (u_*^\varepsilon)^2}{2} + H_*^\varepsilon \right) \psi(x, t) dx dt = \int_{-\infty}^{\infty} \int_0^{\infty} \varepsilon_0(x - \sigma t) \psi(x, t) dx dt + \int_0^{\infty} \omega_2(t) \psi(u_\delta t, t) \sqrt{u_\delta^2 + 1} dt, \tag{4.62}$$

with

$$\omega_2(t) = \frac{t}{\sqrt{1 + u_\delta^2}} \left(u_\delta \left[\frac{\rho u^2}{2} + H \right] - \left[\left(\frac{\rho u^2}{2} + H - \frac{\varepsilon_2}{\rho} \right) u \right] \right). \tag{4.63}$$

When $0 < \varepsilon < \varepsilon_2$, the projection of the state $(\rho_\varepsilon, u_\varepsilon, H_\varepsilon)$ belongs to $V(\rho, u_-)$ onto the (ρ, u) -plane. From [34], when $\varepsilon \rightarrow 0$, we know that the limit solutions of (1.6) satisfy the following generalized Rankine-Hugoniot condition:

$$\begin{cases} \frac{dx(t)}{dt} = \sigma(t), \\ \frac{d\omega(t)}{dt} = \sigma(t)[\rho] - [\rho u], \\ \frac{d(\omega(t)\sigma(t))}{dt} = \sigma(t)[\rho u] - [\rho u^2], \\ \frac{d(\omega(t)\sigma^2(t)/2 + h(t))}{dt} = \sigma(t)\left[\frac{\rho u^2}{2} + H\right] - \left[\left(\frac{\rho u^2}{2} + H\right)u\right], \end{cases} \quad (4.64)$$

from (4.64), it is easy to calculate that the solutions can be expressed as

$$\begin{cases} x(t) = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}}t, \\ \sigma(t) = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}}, \\ \omega(t) = \sqrt{\rho_+\rho_-}(u_- - u_+)t, \\ h(t) = -\omega(t)\sigma^2/2 + x(t)[\rho u^2/2 + H] - [(\rho u^2/2 + H)u]t, \end{cases} \quad (4.65)$$

where σ satisfies the entropy condition.

Finally, we show that the law of conservation of energy (1.6)₃ actually produces the entropy inequality (1.9) of the transport equation (1.5) and obtains the entropy consistency.

Theorem 4.7. *There are limit functions (ρ, u) which are a measure solution of the transport equation (1.5), and these meet*

$$(\rho u^2)_t + (\rho u^3)_x \geq 0, \quad (4.66)$$

in the sense of distributions.

Proof. Due to $\left(\frac{\rho^\varepsilon(u^\varepsilon)^2}{2}\right)_t + \left(\frac{\rho^\varepsilon(u^\varepsilon)^3}{2} + H^\varepsilon u^\varepsilon\right)_x = 0$ in the sense of distributions,

$$(\rho^\varepsilon(u^\varepsilon)^2)_t + (\rho^\varepsilon(u^\varepsilon)^3)_x = -2(H_t^\varepsilon + (H^\varepsilon u^\varepsilon)_x). \quad (4.67)$$

Furthermore, for each given positive test function $\psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$

$$\langle (\rho^\varepsilon(u^\varepsilon)^2)_t + (\rho^\varepsilon(u^\varepsilon)^3)_x, \psi \rangle = 2\langle H_t^\varepsilon, \psi_t \rangle + 2\langle H^\varepsilon u^\varepsilon, \psi_x \rangle. \quad (4.68)$$

We note that

$$\begin{aligned} \lim_{\varepsilon \rightarrow \varepsilon_2} \int_0^\infty \int_{-\infty}^\infty (H^\varepsilon \psi_t + H^\varepsilon u^\varepsilon \psi_x) dx dt &= \lim_{\varepsilon \rightarrow \varepsilon_2} \int_0^\infty (\sigma_1^\varepsilon(H_{*1}^\varepsilon - H_-) - (H_{*1}^\varepsilon u_{*1}^\varepsilon - H_- u_-)) \psi(\sigma_1^\varepsilon t, t) dt \\ &+ \lim_{\varepsilon \rightarrow \varepsilon_2} \int_0^\infty (\sigma_2^\varepsilon(H_{*2}^\varepsilon - H_{*1}^\varepsilon) - (H_{*2}^\varepsilon u_{*1}^\varepsilon - H_{*1}^\varepsilon u_{*1}^\varepsilon)) \psi(\sigma_2^\varepsilon t, t) dt \\ &+ \lim_{\varepsilon \rightarrow \varepsilon_2} \int_0^\infty (\sigma_3^\varepsilon(H_+ - H_{*2}^\varepsilon) - (H_+ u_+ - H_{*2}^\varepsilon u_{*1}^\varepsilon)) \psi(\sigma_3^\varepsilon t, t) dt \\ &= \int_0^\infty (u_\delta[H] - [Hu]) \psi(u_\delta t, t) dt = \int_0^\infty (u_- - u_+) (\alpha H_+ + (1 - \alpha) H_-) \psi(u_\delta t, t) dt \geq 0, \end{aligned} \quad (4.69)$$

for $\alpha = \frac{\rho_-}{\rho_- + \rho_+} \in (0, 1)$

$$u_\delta = \frac{\rho_+ u_+ + \rho_- u_-}{\rho_+ + \rho_-}.$$

Through the aforementioned discussions, we verify the consistency of entropy. \square

4.2 Limits of Riemann solutions in the case $u_- < u_+$

In this section, we show the phenomenon of cavitation of Riemann solutions for (1.6)–(1.7), in the case $u_- < u_+$ as the pressure decreases.

Lemma 4.8. *Suppose that $u_- < u_+$, then there exists a ε_3 , when $0 < \varepsilon < \varepsilon_3$, the projection of (ρ, u, H) onto the (ρ, u) -plane belongs to $I(\rho_-, u_-)$.*

Proof. If $\rho_+ = \rho_-$, then the conclusion is obviously true. Next, we discuss the situation $\rho_+ \neq \rho_-$. Assume that $u_- < u_+$, the states (ρ_+, u_+, H_+) connect with (ρ_-, u_-, H_-) by contact discontinuities that satisfy

$$u_+ = u_- - \frac{\sqrt{\varepsilon}}{\rho_-} + \frac{\sqrt{\varepsilon}}{\rho_+}, \quad \rho_+ < \rho_-, \quad (4.70)$$

$$u_+ = u_- + \frac{\sqrt{\varepsilon}}{\rho_-} - \frac{\sqrt{\varepsilon}}{\rho_+}, \quad \rho_+ > \rho_-. \quad (4.71)$$

The projection of (ρ_+, u_+, H_+) belongs to $I(\rho_-, u_-)$, we have

$$\varepsilon_3 = \left(\frac{(u_+ - u_-)\rho_+\rho_-}{\rho_- - \rho_+} \right)^2, \quad (4.72)$$

it is easy to see that the projection of (ρ_+, u_+, H_+) onto the (ρ, u) -plane belongs to $I(\rho_-, u_-)$ as $\varepsilon < \varepsilon_3$. \square

Lemma 4.9. *When $u_- < u_+$, the cavitation occurs as $\varepsilon \rightarrow 0$. Namely,*

$$\lim_{\varepsilon \rightarrow 0} \rho_{*1}^\varepsilon = \lim_{\varepsilon \rightarrow 0} \rho_{*2}^\varepsilon = 0. \quad (4.73)$$

Proof. From the first equation of (4.6)–(4.8), respectively, we derive

$$u_{*1} - \frac{\sqrt{\varepsilon}}{\rho_{*1}} = u_- - \frac{\sqrt{\varepsilon}}{\rho_-}, \quad (4.74)$$

$$\rho_{*1}^\varepsilon = \rho_{*2}^\varepsilon, \quad u_{*1}^\varepsilon = u_{*2}^\varepsilon, \quad (4.75)$$

$$u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} = u_{*2} + \frac{\sqrt{\varepsilon}}{\rho_{*2}}. \quad (4.76)$$

Hence, one can easily see

$$\lim_{\varepsilon \rightarrow 0} \frac{\sqrt{\varepsilon}}{\rho_{*1}} = \lim_{\varepsilon \rightarrow 0} \frac{\sqrt{\varepsilon}}{\rho_{*2}} = \lim_{\varepsilon \rightarrow 0} \frac{u_+ + \frac{\sqrt{\varepsilon}}{\rho_+} - \left(u_- - \frac{\sqrt{\varepsilon}}{\rho_-} \right)}{2} = \frac{u_- - u_+}{2}, \quad (4.77)$$

thus, we have

$$\lim_{\varepsilon \rightarrow 0} \rho_{*1}^\varepsilon = \lim_{\varepsilon \rightarrow 0} \rho_{*2}^\varepsilon = 0. \quad (4.78)$$

\square

5 Discussion

We have considered the limit behavior of Riemann solutions to Chaplygin Euler equations for nonisentropic fluids, when $u_- > u_+$ and $u_- < u_+$, and we studied the formation of delta shock wave and the appearance of vacuum state for equations (1.6)–(1.7), respectively. When $u_- > u_+$ and the parameter ε tends to ε_2 , the weight of delta shock wave for (1.6)–(1.7) is analyzed. When $u_- < u_+$ and $\varepsilon \rightarrow 0$, we show the phenomenon of cavitation for (1.6)–(1.7).

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