Research Article

Ankit Gupta, Abdulkareem Saleh Hamarsheh, Ratna Dev Sarma, and Reny George*

A study of uniformities on the space of uniformly continuous mappings

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Abstract: New families of uniformities are introduced on $UC(X, Y)$, the class of uniformly continuous mappings between X and Y, where $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces. Admissibility and splittingness are introduced and investigated for such uniformities. Net theory is developed to provide characterizations of admissibility and splittingness of these spaces. It is shown that the point-entourage uniform space is splitting while the entourage-entourage uniform space is admissible.

Keywords: uniform space, function space, splittingness, admissibility, uniformly continuous mappings

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1 Introduction

The function space $C(X, Y)$, where X, Y are topological spaces, can be equipped with various interesting topologies. Properties of these topologies *vis-a-vis* that of X and Y have been an active area of research in recent years. In [1], it is shown that several fundamental properties hold for a hyperspace convergence τ on $C(X, \$)$ at X if and only if they hold for $\tau^{\Uparrow}$ on $C(X, \mathbb{R})$ at origin (where $\$$ is the Sierpinski topology and $\tau^{\Uparrow}$ is the convergence on $C(X, \mathbb{R})$ determined by τ). In [2], function space topologies are introduced and investigated for the space of continuous multifunctions between topological spaces. In [3], some conditions are discussed under which the compact-open, Isbell or natural topologies on the set of continuous real-valued functions on a space may coincide. Properties of c-compact-open topology on the $C(X)$ such as metrizability, separability, and second countability have been discussed in [4]. Function space topologies over the generalized topological spaces, defined by Császár, are introduced and studied in [5]. Their dual topologies have been investigated in [6]. Similarly, the space $C(X)$, the space of continuous mappings from X to $\mathbb{R}$, where X is a completely regular or a Tychonoff space, has also been studied by several researchers in recent years [7,8]. It is well known that every metric space has a uniformity induced by its metric, but not every uniform space is metrizable. Similarly, every uniformity induces a topology. But not every topological space is uniformizable. The metric topology of a space can be derived purely from the properties of the induced uniform space via its uniform topology. In this sense, uniformities are positioned between metric spaces and topological structures. Hence, it is natural to look for similar studies for uniform spaces also. However, not much literature is available so far regarding function space uniformities over uniform spaces. In most of these studies, the topologies and not the uniformities of the underlying spaces are considered. For example, in [9], a quasi-uniformity is formed over a topological space X. In [10], fuzzy topology on X is considered for

*** Corresponding author: Reny George,** Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, Alkharj 11942, Saudi Arabia, e-mail: renygeorge02@yahoo.com
Ankit Gupta: Department of Mathematics, Bharati College, University of Delhi, Delhi 110058, India
Abdulkareem Saleh Hamarsheh: Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, Alkharj 11942, Saudi Arabia
Ratna Dev Sarma: Department of Mathematics, Rajdhani College, University of Delhi, Delhi 110015, India

the space $Cf(X, Y)$. Similar is the case in [11,12]. None of these studies relate to uniformities over uniform spaces, *per se*.

In the present paper, we provide a study of the possible uniform structures on the space of uniformly continuous mappings between uniform spaces. We have verified the existence of two such families, namely, entourage-entourage uniformities and point-entourage uniformities, for the space of uniformly continuous mappings. Our present study is centered around developing the well-known topological concepts of function spaces such as admissibility and splittingness for the function space uniformities over uniform structures. Unlike in [13–15], net theory has been used as a tool in our study. For this purpose, we have introduced the concept of pairwise Cauchy nets for uniformities. This has helped us develop a net theoretic characterization for uniform continuity between uniformities. All the concepts introduced and studied in Section 3 are new, although similar concepts do exist for function space topologies between topological spaces. It is found that a uniformity on $UC(Y, Z)$ is splitting if and only if every pair of nets in $UC(Y, Z)$ is pairwise Cauchy whenever it is continuously Cauchy. On the other hand, a uniformity is admissible if and only if every pair of nets in $UC(Y, Z)$ is continuously Cauchy whenever it is pairwise Cauchy. While the point-entourage uniformity is splitting, entourage-entourage uniformity is found to be admissible. Several examples are provided to explain the theory developed in the paper. The successful application of net theory in the entire investigation testifies that like in topology, net theory is an effective tool for uniformities too. We have concluded the present work with some open questions for future work.

2 Preliminaries

A uniform space is a non-empty set with a uniform structure on it. A uniform structure (or a uniformity) on a set X is a collection of subsets of $X \times X$ satisfying certain conditions. More precisely, we have the following definition.

Definition 2.1. [16,17] *A uniform structure or uniformity* on a non-empty set X is a family $\mathcal{U}$ of subsets of $X \times X$ satisfying the following properties:

(2.1.1) if $U \in \mathcal{U}$, then $\Delta X \subseteq U$;
where $\Delta X = \{(x, x) \in X \times X$ for all $x \in X\}$;
(2.1.2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,
where U^{-1} is called the *inverse relation* of U and is defined as:

$$U^{-1} = \{(x, y) \in X \times X | (y, x) \in U\};$$

(2.1.3) if $U \in \mathcal{U}$, then there exists some $V \in \mathcal{U}$ such that $V \circ V \subseteq U$,
where the composition $U \circ V = \{(x, z) \in X \times X \mid$ for some $y \in X, (x, y) \in V$, and $(y, z) \in U\}$;
(2.1.4) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;
(2.1.5) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a *uniform space* and the members of $\mathcal{U}$ are called *entourages*.

Remark 2.1. There are two more approaches to define uniformity on a set. One of them [18] uses a certain specification of a system of coverings on X. The other is via a system of pseudo-metrics. The one we have provided here is originally due to Weil [19]. This definition centers around the idea of closeness of points of X. In metric spaces, the metric defines the closeness between points. However, in topological spaces, we can only talk of a point being arbitrarily close to a set (i.e., being in the closure of the set). In a uniform space, as defined above, the closeness of points x and y is equivalent to the ordered pair (x, y) belonging to some entourage.

Definition 2.2. [20] A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is called a *base* for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$.

In view of (2.1.5), a base is enough to specify the corresponding uniformity unambiguously as the uniformity consists of just the supersets of members of $\mathcal{B}$. Also, every uniformity possesses a base.

Definition 2.3. [20] A subfamily $\mathcal{S}$ of a uniformity $\mathcal{U}$ is called subbase for $\mathcal{U}$ if the family of finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{U}$.

The finite intersection of the members of a subbase generates a base. A uniformity is obtained by taking the collection of the supersets of the members of its base.

Remark 2.2. The aforementioned definitions of base and subbase are similar to those of a topology. In fact, if we replace uniformity $\mathcal{U}$ by a topology τ in Definitions 2.2 and 2.3, we get the definitions of base and subbase of topology τ on X. As in topology, these definitions help us to restrict our study to a smaller collection of subsets.

The conditions under which a collection of subsets of $X \times X$ becomes a base (respectively, a subbase) of a uniformity on X are provided in the following theorems.

Theorem 2.4. [20] *A non-empty family $\mathcal{U}$ of subsets of $X \times X$ is a base for some uniformity for X if and only if the aforementioned conditions* (2.1.1)–(2.1.4) *hold.*

Theorem 2.5. [20] *A non-empty family $\mathcal{U}$ of subsets of $X \times X$ is a subbase for some uniformity for X if and only if the aforementioned conditions* (2.1.1)–(2.1.3) *hold.*

In particular, the union of any collection of uniformities for X forms a subbase for a uniformity for X. The fact that a subbase (respectively, a base) uniquely defines a uniformity is being utilized in this paper for defining new uniformities.

Remark 2.3. The aforementioned two results provide us simplified methods to check whether a given collection of subsets of $X \times X$ qualifies to generate a uniformity on X.

In fact, Theorem 2.5 is used in this paper in Lemmas 3.4 and 3.5 to establish the existence of the point-entourage uniformity and the entourage-entourage uniformity on $UC(X, Y)$ and $UC(Y, Z)$, respectively.

Definition 2.6. [20] Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces. A mapping $f : X \to Y$ is called *uniformly continuous* if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $f_2[U] \subset V$, where $f_2 : X \times X \to Y \times Y$ is a map corresponding to f defined as $f_2(x, x') = (f(x), f(x'))$ for $(x, x') \in X \times X$.

In other words, $f : X \to Y$ is uniformly continuous if for each $V = V_1 \times V_2 \in \mathcal{V}$, there exists $U = U_1 \times U_2 \in \mathcal{U}$ such that $f(U_1) \times f(U_2) \subseteq V_1 \times V_2$.

The collection of all uniformly continuous functions from X to Y is denoted by $UC(X, Y)$.

3 The main results

The development of this section is as follows. In Section 3.1, we first define pairwise Cauchy nets and then use them to characterize uniform continuity. In Section 3.2, we establish the existence of uniformities in $UC(X, Y)$. Two such uniformities on $UC(X, Y)$ are point-entourage and entourage-entourage uniformities, respectively. Next we define admissibility and splittingness for such uniformities on $UC(X, Y)$. Net theory has been extensively used to provide alternative characterizations for these notions. Finally, we prove that point-entourage uniformity on $UC(X, Y)$ is splitting, while entourage-entourage uniformity is admissible.

Here, it may be mentioned that $\mathcal{I}$-Cauchy nets and $\mathcal{I}$-convergence using ideal of the directed sets were introduced in [21] for uniform spaces. However, convergence of $\mathcal{I}$-nets was defined there using open sets of the corresponding topology. In our paper, we are using nets without any such restrictions. The convergence defined here is purely in terms of uniformity. We have not come across any results in the literature resembling the net-theoretic characterization of uniform continuity provided here. Splittingness and admissibility have been studied by several authors for the function space topologies [13,17,21]. In [16], it has been proved that point-open topology is splitting and open-open topology is admissible. In this section, we extend these notions to uniformities on the space of uniformly continuous mappings. We provide characterizations of these notions using net theory. We also provide examples of splitting and admissible uniformities. The successful development of net theory and its effective applications here have established that like in topology, and net theory is an useful tool for studying uniform structures.

3.1 Uniformly continuous mappings and net theory

Definition 3.1. Let $(X, \mathcal{U})$ be a uniform space. Two nets $\{x_n\}_{n\in D_1}$ and $\{y_m\}_{m\in D_2}$ in $(X, \mathcal{U})$, where D_i are directed sets, are called pairwise Cauchy if $\{(x_n, y_m)\}_{(n,m)\in D_1\times D_2}$ is eventually contained in each entourage $U \in \mathcal{U}$, that is, for each $U \in \mathcal{U}$, $(x_n, y_m) \in U$ for all $n \geq n_0$ and $m \geq m_0$ for some $(n_0, m_0) \in D_1 \times D_2$.

For brevity, we simply say that $\{(x_n, y_m)\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy to indicate that $\{x_n\}_{n\in D_1}$ and $\{y_m\}_{m\in D_2}$ are pairwise Cauchy nets with respect to the uniformity concerned.

Now we provide a characterization for uniformly continuous mappings.

Proposition 3.2. *Let $(X, \mathcal{U})$ be a uniform space and $\{(x_n, y_m)\}_{(n,m)\in D_1\times D_2}$ be a pair of nets. Then $\{(x_n, y_m)\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy if and only if the pair of nets $\{(y_m, x_n)\}_{(m,n)\in D_2\times D_1}$ is pairwise Cauchy.*

Proof. Let $\{(x_n, y_m)\}_{(n,m)\in D1\times D_2}$ be a pairwise Cauchy nets. Let $U \in \mathcal{U}$ be any entourage. Then there exists $U^{-1} \in \mathcal{U}$. Since the pair of nets $\{(x_n, y_m)\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy, $(x_n, y_m) \in U^{-1}$ eventually. Thus, $(y_m, x_n) \in U$ eventually. Hence, the pair of nets $\{(y_m, x_n)\}_{(m,n)\in D_2\times D_1}$ is pairwise Cauchy.

Converse is true obviously. □

Here, it may be mentioned that pairwise Cauchy nets remain pairwise Cauchy if finitely many members of the pair are replaced by other elements. In other words, the results related to pairwise Cauchy nets will remain valid if the pair of nets is eventually Cauchy.

In our next theorem, we provide an equivalent criterion for uniform continuity.

Proposition 3.3. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces. Then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous if and only if the image of every pairwise Cauchy nets in X is again pairwise Cauchy in Y.*

Proof. Let f be uniformly continuous and $\{(x_n, y_m)\}_{(n,m)\in D_1\times D_2}$ be pairwise Cauchy nets in X. Let $V \in \mathcal{V}$ be any entourage. Since f is uniformly continuous, there exists $U \in \mathcal{U}$ such that $f_2[U] \subset V$. As $(x_n, y_m) \in U$ eventually, we have $f_2(x_n, y_m) \in f_2[U] \subset V$, eventually. That is, $(f(x_n), f(y_m)) \in V$ eventually. Hence, $\{(f(x_n), f(y_m))\}_{n\times m\in D_1\times D_2}$, the image of $\{(x_n, y_m)\}_{(n,m)\in D_1\times D_2}$ is also pairwise Cauchy.

Conversely, let the image of every pairwise Cauchy nets be pairwise Cauchy. Let if possible, f be not uniformly continuous. Then there exists an entourage $V \in \mathcal{V}$ such that there is no entourage $U \in \mathcal{U}$ with $f_2[U] \subset V$. Hence for each $U \in \mathcal{U}$, we have $f_2[U] \not\subset V$. Thus, for each entourage $U \in \mathcal{U}$, there exists a pair $(x_u, y_u) \in U$ such that $f_2(x_u, y_u) = (f(x_u), f(y_u)) \notin V$. Now the collection of all entourages $U \in \mathcal{U}$ forms a directed set under the relation $\geq$, which is defined by "$U \geq V$ implies $U \subset V$." Now we show that $\{(x_u, y_u)\}_{u\in\mathcal{U}}$ is pairwise Cauchy in X but its image is not pairwise Cauchy in Y.

Let $U_0 \in \mathcal{U}$. For $U \geq U_0$, that is, $U \subset U_0$, we have $(x_u, y_u) \in U \subset U_0$. Hence, $(x_u, y_u) \in U_0$ for all $U \geq U_0$. Thus, $\{(x_u, y_u)\}_{u \in \mathcal{U}}$ is pairwise Cauchy. Now consider entourage $V \in \mathcal{V}$, we have $(f(x_u), f(y_u)) \notin V$, for each $u \in \mathcal{U}$. Hence, the image of $\{(x_u, y_u)\}_{u \in \mathcal{U}}$ is not pairwise Cauchy. Thus, we got a contradiction. Therefore, f is uniformly continuous. □

3.2 Uniformity over uniform spaces

We now define a uniformity on $UC(X, Y)$ in the following way:
Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces. For $V \in \mathcal{V}$ and $x \in X$, we define:

$$(x, V) = \{(f, g) \in UC(X, Y) \times UC(X, Y) | (f(x), g(x)) \in V\}.$$

Let $\mathcal{S}_{p,\mathcal{V}} = \{(x, V) | x \in X, V \in V\}$.

Lemma 3.4. *$\mathcal{S}_{p,\mathcal{V}}$ forms a subbase for a uniformity over $UC(X, Y)$.*

Proof. By Theorem 2.5, it is enough to show that $\mathcal{S}_{p,\mathcal{V}}$ satisfies conditions (2.1.1)–(2.1.3). We proceed as follows:

1. $\Delta = \{(f, f) | f \in UC(X, Y)\} \subset (x, V)$.
   This follows from the definition of $\mathcal{S}_{p,\mathcal{V}}$.
2. For every $(x, V) \in \mathcal{S}_{p,\mathcal{V}}$, $(x, V)^{-1} \in \mathcal{S}_{p,\mathcal{V}}$.
   Since $V \in \mathcal{V}$, $V^{-1} \in \mathcal{V}$. We claim that $(x, V)^{-1} = (x, V^{-1})$.
   Let $(f, g) \in (x, V)^{-1}$, then $(g, f) \in (x, V)$. Thus, we have $(g(x), f(x)) \in V$. Hence, $(f(x), g(x)) \in V^{-1}$. Therefore, $(f, g) \in (x, V^{-1})$ and hence $(x, V)^{-1} \subset (x, V^{-1})$. On the same line, one can prove that $(x, V^{-1}) \subset (x, V)^{-1}$. Hence, $(x, V^{-1}) = (x, V)^{-1}$.
3. For every $(x, V) \in \mathcal{S}_{p,\mathcal{V}}$, there exists some $A \in \mathcal{S}_{p,\mathcal{V}}$ such that $A \circ A \subset (x, V)$.
   Let $(x, V) \in \mathcal{S}_{p,\mathcal{V}}$. For $V \in \mathcal{V}$ there exists $V' \in \mathcal{V}$ such that $V' \circ V' \subset V$. Now, we claim that for $(x, V') \in \mathcal{S}_{p,\mathcal{V}}$ we have $(x, V') \circ (x, V') \subset (x, V)$.
   Let $(f, h) \in (x, V') \circ (x, V')$. Then there exists $g \in UC(X, Y)$ such that $(f, g), (g, h) \in (x, V')$, that is, $(f(x), g(x)) \subset V'$ and $(g(x), h(x)) \subset V'$. Thus, we have $(f(x), g(x)) \circ (g(x), h(x)) \subset V' \circ V' \subset V$. Hence, $(f(x), h(x)) \in V$ which implies $(f, h) \in (x, V)$. Thus, $(x, V') \circ (x, V') \subseteq (x, V) \in \mathcal{S}_{p,\mathcal{V}}$. Therefore, $(f, h) \in (x, V)$.

Hence, $\mathcal{S}_{p,\mathcal{V}}$ forms a subbase for a uniformity on $UC(X, Y)$. □

Uniformity generated by this subbase is called the *point-entourage uniformity* for $UC(X, Y)$ and is denoted by $\mathfrak{U}_{p,\mathcal{V}}$.

Example 3.1. Let $X = \mathbb{Z}$ be the set of integers. The p-adic uniform structure on $\mathbb{Z}$, for a given prime number p, is the uniformity $\mathcal{V}$ generated by the subsets $\mathbb{Z}_n$ of $\mathbb{Z} \times \mathbb{Z}$, for $n = 1, 2, 3, \ldots$, where $\mathbb{Z}_n$ is defined as:

$$\mathbb{Z}_n = \{(k, m) | \ k \equiv m \bmod p^n\}.$$

Consider the family of subsets

$$U_\varepsilon = \{(x, y) | \ |x - y| < \varepsilon\}$$

of $\mathbb{R} \times \mathbb{R}$ for $\varepsilon > 0$. The uniform structure generated by the subsets U_ε for $\varepsilon > 0$ is called the *Euclidean uniformity* of $\mathbb{R}$. Specifically, a subset D of $\mathbb{R} \times \mathbb{R}$ is an entourage if $U_\varepsilon \subset D$ for some $\varepsilon > 0$.

Now, we consider, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(x, \mathbb{Z}_n) = \{(f, g) \in UC(X, Y) \times UC(X, Y) | \ (f(x), g(x)) \in \mathbb{Z}_n\}.$$

Let $\mathcal{S}_{p,\mathcal{V}} = \{(x, \mathbb{Z}_n) | \ x \in \mathbb{R}, \mathbb{Z}_n \in \mathcal{V}\}$. It can be easily verified that $\mathcal{S}_{p,\mathcal{V}}$ satisfies (2.1.1) to (2.1.3) of Definition 2.1. Thus, $\mathcal{S}_{p,\mathcal{V}}$ forms a subbase for a uniformity over $UC(\mathbb{R}, \mathbb{Z})$ which is a *point-entourage uniformity* for $UC(\mathbb{R}, \mathbb{Z})$. Here, structure of the entourage in the uniformity generated by the subbase $\mathcal{S}_{p,\mathcal{V}}$ is the collection of the pair of uniformly continuous functions from $\mathbb{R}$ to $\mathbb{Z}$, (f, g) such that $f(x) - g(x)$ is divisible by p^n, for some given $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Now, let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Let $V \in \mathcal{V}$ be any symmetric entourage, that is, $V = V^{-1}$. For $U \in \mathcal{U}$, we define:

$$(V, U) = \{(f, g) \in UC(Y, Z) \times UC(Y, Z) | \ (f(V_1), g(V_2)) \subseteq U\} \cup \{(f, f) | \ f \in UC(Y, Z)\},$$

where $V = V_1 \times V_2$.

Consider $\mathcal{S}_{\mathcal{V},\mathcal{U}} = \{(V, U) | \ V \in \mathcal{V}, U \in \mathcal{U}, V \text{ is symmetric}\}$.

Lemma 3.5. *$\mathcal{S}_{\mathcal{V},\mathcal{U}}$ forms a subbase for a uniformity over $UC(Y, Z)$.*

Proof. By Theorem 2.5, it is enough to show that $\mathcal{S}_{\mathcal{V},\mathcal{U}}$ satisfies conditions (2.1.1)–(2.1.3). We proceed as follows:

1. $\Delta = \{(f, f) | \ f \in UC(Y, Z)\} \subset (V, U)$.
   This follows from the definition of $\mathcal{S}_{\mathcal{V},\mathcal{U}}$.
2. For every $(V, U) \in \mathcal{S}_{\mathcal{V},\mathcal{U}}$, $(V, U)^{-1} \in \mathcal{S}_{\mathcal{V},\mathcal{U}}$.
   Let $(f, g) \in (V, U)$, which implies $(f(V_1), g(V_2)) \subseteq U$, where $V = V_1 \times V_2$. Since $V \in \mathcal{V}$ and $U \in \mathcal{U}$, V^{-1} and U^{-1} belong to $\mathcal{V}$ and $\mathcal{U}$, respectively.
   We claim that $(V, U)^{-1} = (V^{-1}, U^{-1})$.
   Let $(f, g) \in (V, U)^{-1}$, then $(g, f) \in (V, U)$. Thus, we have $(g(V_1), f(V_2)) \subseteq U$. Hence, $(f(V_2), g(V_1)) \subseteq U^{-1}$. Therefore, $(f, g) \in (V^{-1}, U^{-1})$ and hence $(V, U)^{-1} \subset (V^{-1}, U^{-1})$. On the same line, one can prove that $(V^{-1}, U^{-1}) \subset (V, U)^{-1}$. Hence, $(V^{-1}, U^{-1}) = (V, U)^{-1}$.
3. For every $(V, U) \in \mathcal{S}_{\mathcal{V},\mathcal{U}}$, there exists some $A \in \mathcal{S}_{\mathcal{V},\mathcal{U}}$ such that $A \circ A \subset (V, U)$.
   Let $(V, U) \in \mathcal{S}_{\mathcal{V},\mathcal{U}}$. For $U \in \mathcal{U}$ there exists $U' \in \mathcal{U}$ such that $U' \circ U' \subset U$ and similarly, there exists $U'' \in \mathcal{U}$ such that $U'' \circ U'' \subseteq U'$ and hence $U'' \circ U'' \circ U'' \circ U'' \subset U$. Now, we claim that for $(V, U'') \in \mathcal{S}_{\mathcal{V},\mathcal{U}}$ we have $(V, U'') \circ (V, U'') \subset (V, U)$.
   Let $(f, h) \in (V, U'') \circ (V, U'')$. Then there exists $g \in UC(Y, Z)$ such that $(f, g), (g, h) \in (V, U'')$, that is, $(f(V_1), g(V_2)) \subset U''$ and $(g(V_1), h(V_2)) \subset U''$, where $V = V_1 \times V_2$. Since $(g, g) \in (V, U'')$, where V is a symmetric entourage, we have $(g(V_1), g(V_2)) \subset U''$ and hence $(g(V_2), g(V_1)) \subset U''$.
   Thus, we have $(f(V_1), g(V_2)) \circ (g(V_2), g(V_1)) \circ (g(V_1), g(V_1)) \circ (g(V_1), h(V_2)) \subset U'' \circ U'' \circ U'' \circ U'' \subset U$. Hence, $(f(V_1), h(V_2)) \in U$ which implies $(f, h) \in (V, U)$. Thus, $(V, U'') \circ (V, U'') \subseteq (V, U)$.

Hence, $\mathcal{S}_{\mathcal{V},\mathcal{U}}$ forms a subbase for a uniformity on $UC(Y, Z)$. □

The uniform space generated by the aforementioned subbase is called *the entourage-entourage uniformity* and it is denoted by $\mathfrak{U}_{\mathcal{V},\mathcal{U}}$.

Example 3.2. Now, we again consider the set of integers $\mathbb{Z}$, with p-adic uniformity $\mathcal{V}$ and the set of real numbers $\mathbb{R}$ with Euclidean uniformity $\mathcal{U}$, defined in Example 3.1. We for given $\varepsilon > 0$ and $n \in \mathbb{N}$ define:

$$(U_\varepsilon, \mathbb{Z}_n) = \{(f, g) \in UC(\mathbb{R}, \mathbb{Z}) \times UC(\mathbb{R}, \mathbb{Z}) | \ (f(V_1), g(V_2)) \subseteq \mathbb{Z}_n\} \cup \{(f, f) | \ f \in UC(\mathbb{R}, \mathbb{Z})\},$$

where $U_\varepsilon = V_1 \times V_2$.

Consider $\mathcal{S}_{\mathcal{U},\mathcal{V}} = \{(U_\varepsilon, \mathbb{Z}_n) | \ U_\varepsilon \in U, \mathbb{Z}_n \in \mathcal{V}\}$.

It is easy to verify that $\mathcal{S}_{\mathcal{U},\mathcal{V}}$ satisfies (2.1.1) to (2.1.3) of Definition 2.1. Hence, $\mathcal{S}_{\mathcal{U},\mathcal{V}}$ forms a subbase for a uniformity over $UC(\mathbb{R}, \mathbb{Z})$. This is an example of an *entourage-entourage uniformity* and it is denoted by $\mathfrak{U}_{\mathcal{U},\mathcal{V}}$. Here, the structure of entourage in entourage-entourage uniformity over $UC(\mathbb{R}, \mathbb{Z})$ is the collection of pair of all uniformly continuous functions (f, g) from $\mathbb{R}$ to $\mathbb{Z}$ such that for given $\varepsilon > 0$, there always exists a natural number $n \in \mathbb{N}$ such that $f(x) - g(y)$ is divisible by p^n whenever $|x - y| < \varepsilon$.

The above discussion clearly indicates that several uniformities do exist on $UC(X, Y)$.

Let $\mathcal{A}$ be a uniformity on $UC(X, Y)$, then the pair $(UC(X, Y), \mathcal{A})$ is called a *uniform space over uniformly continuous mappings* or *uniform space over uniform continuity*.

Now we introduce the notions of admissibility and splittingness for the uniform spaces over uniform continuity. Admissibility and splittingness are two very important notions in the topology of function spaces. They were introduced by Arens and Dugundji [16] and have been studied by several authors thereafter. In recent years, Georgiou, Iliadis, and others [13–15,17] have significantly contributed to the study of these notions. In the following, we proceed to extend the notions of splittingness and admissibility to the domain of uniformities.

Definition 3.6. Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces and let $(X, \mathcal{W})$ be another uniform space. Then for a map $g : X \times Y \to Z$, we define $g^\star : X \to UC(Y, Z)$ by $g^\star(x)(y) = g(x, y)$.

The mappings g and $g^\star$ related in this way are called *associated maps*.

Definition 3.7. Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces. A uniformity $\mathcal{A}$ on $UC(Y, Z)$ is called
1. *admissible* if for each uniform space $(X, \mathcal{W})$, uniform continuity of $g^\star : X \to UC(Y, Z)$ implies uniform continuity of the associated map $g : X \times Y \to Z$;
2. *splitting* if for each uniform space $(X, \mathcal{W})$, uniform continuity of $g : X \times Y \to Z$ implies uniform continuity of $g^\star : X \to UC(Y, Z)$, where $g^\star$ is the associated map of g.

Now, we prove that the point-entourage uniformity defined in Example 3.1 over $UC(\mathbb{R}, \mathbb{Z})$ is splitting.

Example 3.3. Let $Y = \mathbb{R}$, the set of real numbers with Euclidean uniformity $\mathcal{U}$ and $Z = \mathbb{Z}$, the set of all integers with p-adic uniformity $\mathcal{V}$ be two uniform spaces. Let $UC(\mathbb{R}, \mathbb{Z})$ be the space of all uniform continuous functions from Y to Z with point-entourage uniformity $\mathfrak{U}_{p,\mathcal{V}}$, defined in Example 3.1. Let $(X, \mathcal{W})$ be any uniform space such that the map $g : X \times \mathbb{R} \to \mathbb{Z}$ is uniformly continuous. We have to show that the associated map $g^\star : X \to UC(\mathbb{R}, \mathbb{Z})$ is uniformly continuous, where $g^\star$ is defined as $g^\star(x)(y) = g(x, y)$.

Let $(x, \mathbb{Z}_n)$ be any entourage in $UC(\mathbb{R}, \mathbb{Z})$. Since, the map g is uniformly continuous, there exists an entourage V of $X \times \mathbb{R}$ such that $g_2[V] \subseteq \mathbb{Z}_n$, where $V = U' \times U_\varepsilon$ for some $\varepsilon > 0$ and $U' \in \mathcal{W}$. We have $g_2(U' \times U_\varepsilon) \subseteq \mathbb{Z}_n$, that is, $(g(a, x), g(b, y)) \in \mathbb{Z}_n$ for all $(a, b) \in U'$ and $(x, y) \in U_\varepsilon$. That is, $g(a, x) \equiv g(b, y) \bmod p^n$ for all $(a, b) \in U'$ and $(x, y) \in U_\varepsilon$. That is, $g^\star(a)(x) \equiv g^\star(b)(y) \bmod p^n$ for all $(a, b) \in U'$ and $(x, y) \in U_\varepsilon$. Since $U_\varepsilon \in \mathcal{U}$ is an entourage, $(x, x) \in U_\varepsilon$ for all $x \in \mathbb{R}$. Thus, we have $g^\star(a)(x) \equiv g^\star(b)(x) \bmod p^n$, which implies $(g^\star(a), g^\star(b)) \in (x, \mathbb{Z}_n)$ for all $(a, b) \in U'$. Hence, we have $g_2^\star[U'] \subseteq (x, \mathbb{Z}_n)$. The associated map $g^\star$ is uniformly continuous, therefore, the point-entourage uniformity over $UC(Y, Z)$ is splitting.

Before proceeding further, we mention few basic results which will be used later.

Proposition 3.8. *Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$, and $(Z, \mathcal{W})$ be uniform spaces and let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $g : (Y, \mathcal{V}) \to (Z, \mathcal{W})$ be two uniformly continuous maps. Then $g \circ f : (X, \mathcal{U}) \to (Z, \mathcal{W})$ is again uniformly continuous.*

Proposition 3.9. *Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$, and $(Z, \mathcal{W})$ be uniform spaces and let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be uniformly continuous. Then $F : X \times Z \to Y \times Z$, defined by $F(x, z) = (f(x), z)$ is also uniformly continuous.*

Proof. Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be any uniformly continuous function. We have to show $F : X \times Z \to Y \times Z$, defined by $F(x, z) = (f(x), z)$ is uniformly continuous. Let $A = V \times W$ be any entourage in the uniformity of $Y \times Z$, where $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Since f is uniformly continuous, there exists an entourage $U \in \mathcal{U}$ such that $f_2[U] \subset V$. Thus, $f_2(U) \times W \subset V \times W$, which implies $F_2[U \times W] \subset V \times W$. Hence, F is uniformly continuous. ◻

The notion of evaluation maps of topology can be extended for uniformities also to obtain the following characterization of admissibility.

Theorem 3.10. *Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces. Then a uniformity $\mathcal{A}$ on $UC(Y, Z)$ is admissible if and only if the evaluation mapping $e : UC(Y, Z) \times Y \to Z$ defined by $e(f, y) = f(y)$ is uniformly continuous.*

Proof. Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces and uniformity $\mathcal{A}$ on $UC(Y, Z)$ be admissible, that is, uniform continuity of the map $g^\star : X \to UC(Y, Z)$ implies the uniform continuity of the associated map $g : X \times Y \to Z$ for each uniform space $(X, \mathcal{W})$. We take $X = UC(Y, Z)$, then the identity map $g^\star : UC(Y, Z) \to UC(Y, Z)$, where $g^\star(f) = f$ for all $f \in UC(Y, Z)$ is uniformly continuous. Then by the given hypothesis, the associated map $g : UC(Y, Z) \times Y \to Z$ is also uniformly continuous. Consider, $g(f, y) = g^\star(f)(y) = f(y) = e(f, y)$. Thus, we have $g \equiv e$. Hence, the evaluation map $e : UC(Y, Z) \times Y \to Z$ is uniformly continuous.

Conversely, let $g^\star : X \to UC(Y, Z)$ be uniformly continuous. We define a map $h : X \times Y \to UC(Y, Z) \times Y$ defined by $h(x, y) = (g^\star(x), y)$. In the light of Proposition 3.9, the map h is uniformly continuous. Since the given evaluation map $e : UC(Y, Z) \times Y \to Z$ is uniformly continuous. Thus, the composition map $e \circ h : X \times Y \to Z$ is also uniformly continuous and $e \circ h \equiv g$ because $e \circ h(x, y) = e[h(x, y)] = e(g^\star(x), y) = g^\star(x)(y) = g(x, y)$. Hence, the associated map g is uniformly continuous. This completes the proof. ☐

The uniform space $UC(\mathbb{R}, \mathbb{Z})$ is admissible under entourage-entourage uniformity, defined in Example 3.2.

Example 3.4. Let $Y = \mathbb{R}$, the set of all real numbers with the Euclidean uniformity $\mathcal{U}$ and $Z = \mathbb{Z}$, the set of all integers with p-adic uniformity $\mathcal{V}$. We have to show that the evaluation mapping $e : UC(\mathbb{R}, \mathbb{Z}) \times \mathbb{R} \to \mathbb{Z}$, defined as $e(f, y) = f(y)$, is uniformly continuous under the entourage-entourage uniformity over $UC(\mathbb{R}, \mathbb{Z})$ defined in Example 3.2. Let $\mathbb{Z}_n$ be any entourage in $(\mathbb{Z}, \mathcal{V})$.

Consider an entourage $V_1 = (U_\varepsilon, \mathbb{Z}_n)$ and $V_2 = U_\varepsilon$, for some $\varepsilon > 0$. Then $e_2(V_1 \times V_2) = (e(f, x), e(g, y)) = (f(x), g(y))$. We have $(f, g) \in (U_\varepsilon, \mathbb{Z}_n)$, thus $(f(a), g(b)) \in \mathbb{Z}_n$, that is, $f(a) \equiv f(b) \bmod p^n$, for all $(a, b) \in \mathbb{Z}_n$. Therefore, $e_2(V_1 \times V_2) \subseteq \mathbb{Z}_n$. Thus, the evaluation map is uniformly continuous and hence $UC(\mathbb{R}, \mathbb{Z})$ under entourage-entourage uniformity is admissible.

Before coming to the main results of this paper, we provide a small discussion on directed sets.

Let D_1 and D_2 be two directed sets. We define a uniformity $\mathcal{U}_0$ on $\Delta = D_1 \cup D_2$, generated by $\{U_{n_0,m_0} \mid (n_0, m_0) \in D_1 \times D_2\}$, where $U_{n_0,m_0} = \delta \cup \{(n, m) \mid (n, m) \geq (n_0, m_0)\} \cup \{(m, n) \mid (m, n) \geq (m_0, n_0)$: $(n_0, m_0) \in D_1 \times D_2\}$ and $\delta = \{(n, n) \mid n \in \Delta\}$, where "$\geq$" being defined component-wise.

Lemma 3.11. *Let $(Y, \mathcal{U})$ be a uniform space and $\{(y_n, y'_m\}_{(n,m)\in D_1\times D_2}$ be a pair of nets in Y. Then $\{(y_n, y'_m\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy if and only if the function $s : \Delta \to Y$ defined by $s(n) = y_n$ for $n \in D_1$, $s(m) = y'_m$ for $m \in D_2$, $n \neq m$ is uniformly continuous under $\mathcal{U}_0$ defined above on $\Delta = D_1 \cup D_2$ and $\mathcal{U}$ on Y. In case, $D_1 \cap D_2 \neq \emptyset$, that is, $n = m$ for some $n \in D_1$ and $m \in D_2$, take $s(n) = s(m) = y_n$ in the above definition.*

Proof. Let $\{(y_n, y'_m)\}_{(n,m)\in D_1\times D_2}$ be any pairwise Cauchy nets in Y. Let $U \in \mathcal{U}$ be any entourage, then $(y_n, y'_m) \in U$ eventually, that is, there exists $(n_0, m_0) \in D_1 \times D_2$ such that $(y_n, y'_m) \in U$ for all $(n, m) \geq (n_0, m_0)$. That is, there exists $U_{n_0,m} \in \mathcal{U}_0$ such that $s_2(U_{n_0,m_0}) \subset U$. Hence, s is uniformly continuous.

Conversely, let $\{(y_n, y'_m)\}_{(n,m)\in D_1\times D_2}$ be any pair of nets in Y and s be a uniformly continuous mapping. Let $V \in \mathcal{U}$ be any entourage. Then there exists an entourage $U_{n_0,m_0} \in \mathcal{U}_0$ such that $s_2(U_{n_0,m_0}) \subset V$. Thus, $s_2((n, m)) \in V$ for all $(n, m) \geq (n_0, m_0)$. Hence, $\{(y_n, y'_m\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy. ☐

In the next pair of theorems, we provide some characterizations of splittingness and admissibility of the uniform spaces over uniformly continuous mappings. We introduce the notion of continuously Cauchy nets for this purpose. The need for such notion has arisen purely out of uniformity structure of the space and has no topological or metric counterpart.

Definition 3.12. Let $\{(f_n, g_m)\}_{(n,m)\in D_1\times D_2}$ be a pair of nets in $UC(Y, Z)$. Then $\{(f_n, g_m)\}_{(n,m)\in D_1\times D_2}$ is said to be *continuously Cauchy* if for each pairwise Cauchy net $\{(y_k, y'_l)\}_{(k,l)\in D_3\times D_4}$ in Y, $\{(f_n(y_k), g_m(y'_l))\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is pairwise Cauchy in Z.

Lemma 3.13. *Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces and let $\{(y_n, y'_m)\}_{(n,m)\in D_1\times D_2}$ and $\{(z_k, z'_l)\}_{(k,l)\in D_3\times D_4}$ be two pairwise Cauchy nets in Y and Z, respectively. Then $\{(y_n, z_k), (y'_m, z'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is pairwise Cauchy in $Y \times Z$ with respect to the product uniformity $\mathcal{U} \times \mathcal{V}$ and vice versa.*

Proof. Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces and let $\{(y_n, y'_m)\}_{(n,m)\in D_1\times D_2}$ and $\{(z_k, z'_l)\}_{(k,l)\in D_3\times D_4}$ be two pairwise Cauchy nets in Y and Z, respectively. Therefore, for each $U \in \mathcal{U}$ and $V \in \mathcal{V}$, we have $(y_n, y'_m) \in U$ and $(z_k, z'_l) \in V$ eventually. Hence, $\{(y_n, z_k), (y'_m, z'_l)\} \in U \times V$ eventually. Thus, $\{(y_n, z_k), (y'_m, z'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is pairwise Cauchy in $Y \times Z$. The converse can be proved in similar manner. □

In the remaining part of this section, we use net theory to provide further investigations about admissibility and splittingness for uniformities on $UC(Y, Z)$. In the first two theorems, we provide net-theoretic characterization for splittingness and admissibility, respectively. In the topological parlance, Arens and Dugundji were the ones to introduce the concept of continuous convergence. They have provided characterizations for splittingness and admissibility for topological function spaces, by using the concept of continuous convergence. Here, we extend the same for uniformity and use the concept of pairwise Cauchy nets to arrive at our results. Theorems 3.16 and 3.17 provide examples of splittingness and admissibility families of uniform space, respectively, on $UC(Y, Z)$.

Theorem 3.14. *Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces. A uniformity $\mathcal{A}$ on $UC(Y, Z)$ is splitting if and only if each pair of nets $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ in $UC(Y, Z)$ is pairwise Cauchy whenever it is continuously Cauchy.*

Proof. Let $(X, \mathcal{W})$ be any uniform space such that $g : X \times Y \to Z$ be uniformly continuous. We have to show that the associated map $g^\star : X \to UC(Y, Z)$ is uniformly continuous. Let $\{(x_n, x'_m)\}_{(n,m)\in D_1\times D_2}$ be any pairwise Cauchy nets in X. We have to show that $\{(g^\star(x_n), g^\star(x'_m))\}_{(n,m)\in D_1\times D_2}$ is again pairwise Cauchy in $UC(Y, Z)$. Let $\{(y_k, y'_l)\}_{(k,l)\in D_3\times D_4}$ be any pairwise Cauchy net in Y. Then $\{(x_n, y_k), (x'_m, y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is a pairwise Cauchy net in $X \times Y$. Since $g : X \times Y \to Z$ is uniformly continuous, $\{g(x_n, y_k), g(x'_m, y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is a pairwise Cauchy net in Z. Let us define $g^\star(x_n) = f_n$ and $g^\star(x'_m) = f'_m$. Then $\{g(x_n, y_k), g(x'_m, y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ $= \{g^\star(x_n)(y_k), g^\star(x'_m)(y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4} = \{f_n(y_k), f'_m(y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is pairwise Cauchy in Z. Therefore, the pair of nets $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ is continuously Cauchy. By the hypothesis, the pair of nets $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy in $UC(Y, Z)$ and hence $\{(g^\star(x_n), g^\star(x'_m))\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy. Therefore, $g^\star$ is uniformly continuous and hence $(UC(Y, Z), \mathcal{A})$ is splitting.

In the above proof, let if possible, $x_n = x'_m = x_k$ (say) for some $n \in D_1$ and $m \in D_2$, then we should proceed as follows.

We then define the map $g^\star : X \to UC(Y, Z)$ as $g^\star(x_n) = f_n$, $g^\star(x'_m) = f'_m$ for $x_n \neq x_m$ and $g^\star(x_k) = f_k$ whenever $x_n = x'_m = x_k$. Then $\{g(x_n, y_k), g(x'_m, y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4} = \{g^\star(x_n)(y_k), g^\star(x'_m)(y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4} =$ $\{f_n(y_k), f'_m(y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$, where $f'_m(y'_l) = f_k(y'_l)$, whenever $x_n = x'_m = x_k$, is pairwise Cauchy net in Z as they are eventually pairwise Cauchy. Therefore, $\{f_n(y_k), f'_m(y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is again pairwise Cauchy. Hence, the pair of nets $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ is continuously Cauchy. By the hypothesis, the pair of nets $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy in $UC(Y, Z)$ and hence $\{(g^\star(x_n), g^\star(x'_m))\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy. Therefore, $g^\star$ is uniformly continuous and hence $(UC(Y, Z), \mathcal{A})$ is splitting.

The case where $x_n = x'_m$ for infinitely many indices, may be treated in a similar manner.

Conversely, let $(UC(Y, Z), \mathcal{A})$ be splitting and $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ be any pair of nets in $UC(Y, Z)$ which is continuously Cauchy. We have to show that the pair $(f_n, f'_m)_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy in $(UC(Y, Z), \mathcal{A})$. Let $\mathcal{U}_0$ be the uniformity generated on Δ, where $\Delta = D_1 \cup D_2$. Then the only non-trivial pair of nets in Δ is $(n, m)_{(n,m)\in D_1\times D_2}$, which is pairwise Cauchy. Let $\mathcal{S}$ be any pairwise Cauchy net in $\Delta \times Y$. Then $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are nets in $\Delta \times Y$, where $\mathcal{S}_1 = \{n, y_k\}_{(n,k)\in D_1\times D_3}$ and $\mathcal{S}_2 = \{m, y'_l\}_{(m,l)\in D_2\times D_4}$ Then $\{n, m\}_{(n,m)\in D_1\times D_2}$ and $\{y_k, y_l\}_{(k,l)\in D_3\times D_4}$. Then we define a map $g : \Delta \times Y \to Z$ as $g(n) = f_n(y)$ and $g(m, y) = f'_m(y)$. Thus, $g_2(\mathcal{S}) = (g(n, y_k), g(m, y'_l))_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$. That is, $g_2(\mathcal{S}) = (f_n(y_k), f'_m(y'_l))_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$. Since the pair $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ is given to be a continuously Cauchy pair, $\{(g(n, y_k), g(m, y'_l))\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is pairwise Cauchy. Hence, the map g is uniformly continuous. As $\mathcal{A}$ is splitting, this implies $g^\star$ is uniformly continuous. Since $\{(n, m)\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy, we have $\{(g^\star(n), g^\star(m))\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy in $UC(Y, Z)$. Now consider, $g^\star(n)(y) = g(n, y) = f_n(y)$ and $g^\star(m)(y) = g(m, y) = f'_m(y)$. That is, $g^\star(n) = f_n$ and $g^\star(m) = f'_m$. Hence, $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy in $(UC(Y, Z), \mathcal{A})$.

If $D_1 \cap D_2$ in non-empty and finite, then in the above discussion, we take $g(n, y) = f_n(y)$ and $g(m, z) = f'_m(z)$, whenever $y \neq z \in Y$. Furthermore, if we have, $n = m$ and $y = z$ for some $y, z \in Y$, then we define $g(n, y) = g(m, z) = f_n(y)$. Then in the above proof, $g_2(\mathcal{S}) = (f_n(y_k), f'_m(y'_l))_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$. Since the pair $\{(f_n, f'_m)\}_{(n,m)\in D_1\times D_2}$ is continuously Cauchy, $g_2(\mathcal{S}) = (f_n(y_k), f'_m(y'_l))_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is pairwise Cauchy. Hence, the proof.

If $D_1 \cap D_2$ in non-empty and infinite and $y = z$, then the images of $\mathcal{S}_1$ and $\mathcal{S}_2$ under g will coincide in infinitely many places. Thus, proof becomes a trivial case of the above discussion for the converse part. ☐

Example 3.5. Let $(\mathbb{R}, \mathcal{U})$ and $(\mathbb{Z}, \mathcal{V})$ be uniform spaces as stated in Example 3.1. It has been shown that the point-entourage uniform space $(UC(\mathbb{R}, \mathbb{Z}), \mathfrak{U}_{p,\mathcal{V}})$ is splitting (see Example 3.3). We now show that this uniformity satisfies the conditions stated in the above theorem.

For this, we have to show that if a pair of nets $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ in $UC(\mathbb{R}, \mathbb{Z}) \times UC(\mathbb{R}, \mathbb{Z})$ is continuously Cauchy, then $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy under $(UC(\mathbb{R}, \mathbb{Z}), \mathfrak{U}_{p,\mathcal{V}})$. Let $(y, \mathbb{Z}_n) \in \mathfrak{U}_{p,\mathcal{V}}$, for some $y \in \mathbb{R}$ and for some $n \in \mathbb{N}$, be any entourage in the point-entourage uniformity. Then, consider $(y_l, y_k)_{(l,k)\in D_3\times D_4} = (y, y)$, which is constant and hence is pairwise Cauchy net. Since $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is assumed to be continuously Cauchy, $(f_n(y), f_m(y))_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy net in $\mathbb{Z}$. Hence, for $\mathbb{Z}_n \in \mathcal{V}$, we have $(f_n(y), f_m(y)) \in \mathbb{Z}_n$ eventually. Therefore, $(f_n, f_m) \in (y, \mathbb{Z}_n)$ eventually. Hence, the pair of nets $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy net in $(UC(Y, Z), \mathfrak{U}_{p,\mathcal{V}})$. Thus, the point-entourage uniform space $(UC(Y, Z), \mathfrak{U}_{p,\sqsubset})$ satisfies the conditions of Theorem 3.14.

Theorem 3.15. *Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces. A uniformity $\mathcal{A}$ on $UC(Y, Z)$ is admissible if and only if each pair of nets $(f_n, f'_m)_{(n,m)\in D_1\times D_2}$ in $UC(Y, Z)$ is continuously Cauchy under $\mathcal{A}$ if $(f_n, f'_m)_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy.*

Proof. Let $(UC(Y, Z), \mathcal{A})$ be admissible and $(f_n, g_m)_{(n,m)\in D_1\times D_2}$ be a pair of nets in $UC(Y, Z)$, which is pairwise Cauchy. We have to show that the pair $(f_n, g_m)_{(n,m)\in D_1\times D_2}$ is continuously Cauchy in $(UC(Y, Z), \mathcal{A})$. Let $\mathcal{T}_0$ be the uniformity generated on Δ, where $\Delta = D_1 \cup D_2$. The only non-trivial pair of nets in Δ, which is pairwise Cauchy, is $\{(n, m)\}_{(n,m)\in D_1\times D_2}$. Then define a map $g^\star : \Delta \to UC(Y, Z)$ by $g^\star(n) = f_n$ and $g^\star(m) = g_m$. Consider $(g^\star(n), g^\star(m))_{(n,m)\in D_1\times D_2} = (f_n, g_m)_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy. Hence, the map $g^\star$ is uniformly continuous. Since $(UC(Y, Z), \mathcal{A})$ is given to be admissible, the associated map $g : \Delta \times Y \to Z$ is also uniformly continuous. Let $(y_k, y'_l)_{(k,l)\in D_3\times D_4}$ be any pairwise net in Y. Therefore, $\{(n, y_k), (m, y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is again a pairwise Cauchy net in $\Delta \times Y$. Since the map g is uniformly continuous, $\{g(n, y_k), g(m, y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4}$ is pairwise Cauchy.

That is, $\{g^\star(n)(y_k), g^\star(m)(y'_l)\}_{(n,m,k,l)\in D_1\times D_2\times D_3\times D_4} = (f_n(y_k), g_m(y'_l))$ is pairwise Cauchy. Hence, the pair $(f_n, g_m)_{(n,m)\in D_1\times D_2}$ is continuously Cauchy.

Conversely, let $g^\star : X \to UC(Y,Z)$ be uniformly continuous. We have to show that the associated map g is uniformly continuous. Let $\{(x_n, y_m), (x'_n, y'_m)\}_{(n,m)\in D_1\times D_2}$ be a pairwise Cauchy net in $X \times Y$. Then $\{(x_n, x'_m\}_{(n,m)\in D_1\times D_2}$ and $\{(y_n, y'_m)\}_{(n,m)\in D_1\times D_2}$ are pairwise Cauchy net in X and Y, respectively. Since $\{(x_n, x'_m)\}_{(n,m)\in D_1\times D_2}$ is a pairwise Cauchy net in X and $g^\star$ is uniformly continuous, $\{g^\star(x_n), g^\star(x'_m)\}_{(n,m)\in D_1\times D_2}$ is also a pairwise Cauchy net in $UC(Y,Z)$, that is, $\{(f_n, g_m)\}_{(n,m)\in D_1\times D_2}$ is a pairwise Cauchy in $UC(Y,Z)$, where $f_n = g^\star(x_n)$ and $g_m = g^\star(x'_m)$, respectively. Then, by the given hypothesis, the pair $\{(f_n, g_m)\}_{(n,m)\in D_1\times D_2}$ is continuously Cauchy. Hence, for the pairwise Cauchy net $\{(y_n, y'_m)\}_{(n,m)\in D_1\times D_2}$ in Y, we have $\{(f_n(y_n), g_m(y'_m))\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy in Z, that is, $\{g(x_n, y_n), g(x'_m, y'_m)\}_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy. Hence, g is uniformly continuous. Therefore, $(UC(Y,Z), \mathcal{A})$ is admissible. □

Remark 3.1. In the above proof, if $x_n = x'_m$, for some $n \in D_1$, $m \in D_2$, we may proceed in a similar way as in Theorem 3.14. Similarly, in the converse part, if $D_1 \cap D_2 \neq \varnothing$, we proceed as in Theorem 3.14.

Example 3.6. In Example 3.4, it has shown that the uniform space $UC(\mathbb{R}, \mathbb{Z})$ is admissible under the entourage-entourage uniformity defined as in Example 3.2. Now, we show that this uniformity satisfies the conditions laid down in the above theorem.

For this, we show that if a pair of nets $(f_n, f_m)_{(n,m)\in D_1\times D_2} \in UC(\mathbb{R}, \mathbb{Z})$ is pairwise Cauchy, then $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is continuously Cauchy.

Let $(y_l, y_k)_{(l,k)\in D_3\times D_4}$ be a pair of Cauchy nets in $(\mathbb{R}, \mathcal{U})$. Thus, for any given $\varepsilon > 0$, $(y_l, y_k) \in U_\varepsilon$ eventually. Since the pair of nets $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is also a Cauchy pair, $(f_n, f_m) \in (U_\varepsilon, \mathbb{Z}_n)$ eventually. Hence, $(f_n(V_1), f_m(V_2)) \in \mathbb{Z}_n$ eventually, where $U_\varepsilon = V_1 \times V_2$. Therefore, we have $(f_n(y_l), f_m(y_k)) \in \mathbb{Z}_n$ eventually. Hence, $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is continuously Cauchy. Thus, entourage-entourage uniform space $(UC(Y,Z), \mathfrak{U}_{\mathcal{V},\mathcal{U}})$ satisfies the condition laid down in Theorem 3.15.

In our next pair of theorems, we provide the existence of some uniform spaces over $UC(Y,Z)$, which satisfy the conditions of splittingness and admissibility, respectively. Using the results obtained so far, we show that every point-entourage uniform space is splitting, whereas every entourage-entourage uniform space is admissible.

Theorem 3.16. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Then the point-entourage uniform space $(UC(Y,Z), \mathfrak{U}_{p,\mathcal{U}})$ is splitting.*

Proof. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. We have to prove that the point-entourage uniform space $(UC(Y,Z), \mathfrak{U}_{p,\mathcal{U}})$ is splitting. For this, we have to show that if a pair of nets $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ in $UC(Y,Z) \times UC(Y,Z)$ is continuously Cauchy, then $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy under $(UC(Y,Z), \mathfrak{U}_{p,\mathcal{U}})$.

Let $(y, U) \in \mathfrak{U}_{p,\mathcal{U}}$ be any entourage in point-entourage uniformity. Consider $(y_l, y_k)_{(l,k)\in D_3\times D_4} = (y, y)$ is the pairwise constant Cauchy net. Since $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is continuously Cauchy, $(f_n(y), f_m(y))_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy net in Z. Hence for $U \in \mathcal{U}$, we have $(f_n(y), f_m(y)) \in U$ eventually. Therefore, $(f_n, f_m) \in (y, U)$ eventually. Hence, the pair of nets $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is pairwise Cauchy net in $(UC(Y,Z), \mathfrak{U}_{p,\mathcal{U}})$. Thus, the point-entourage uniform space $(UC(Y,Z), \mathfrak{U}_{p,\mathcal{U}})$ is splitting. □

In the next theorem, we show that the entourage-entourage uniform space $(UC(Y,Z), \mathfrak{U}_{\mathcal{V},\mathcal{U}})$ is admissible.

Theorem 3.17. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Then the entourage-entourage uniform space $(UC(Y,Z), \mathfrak{U}_{\mathcal{V},\mathcal{U}})$ is admissible.*

Proof. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. We have to prove that the entourage-entourage uniform space $(UC(Y,Z), \mathfrak{U}_{\mathcal{V},\mathcal{U}})$ is admissible. For this, we have to show that if a pair of nets $(f_n, f_m)_{(n,m)\in D_1\times D_2} \in UC(Y,Z)$ is pairwise Cauchy provided $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is continuously Cauchy.

Let $(y_l, y_k)_{(l,k)\in D_3\times D_4}$ be a net of Cauchy pair in $(Y, \mathcal{V})$. Thus, $(y_l, y_k) \in V$ eventually for all $V \in \mathcal{V}$. Since the pair of nets $(f_n, f_m)_{(n,m)\in D_1\times D_2}$ is also a Cauchy pair, $(f_n, f_m) \in (V, U)$ eventually. Hence, $(f_n(V_1), f_m(V_2)) \subseteq U$ eventually, where $V = V_1 \times V_2$. Therefore, we have $(f_n(y_l), f_m(y_k)) \in U$ eventually. Thus, entourage-entourage uniform space $(UC(Y, Z), \mathfrak{U}_{\mathcal{V},\mathcal{U}})$ is admissible. □

Does there exist any uniformity on $UC(Y, Z)$ which is both admissible and splitting? The answer is yes. In the following, we provide an example to show this fact. This example also highlights applications of net-theoretic characterization of admissibility obtained in Theorem 3.15.

Let X be a non-empty set. We call a uniformity $\mathcal{I}$ on X an *indiscrete uniformity* if it has only one entourage, that is, $X \times X$. In an indiscrete uniform space, every pair of nets is a Cauchy pair.

Example 3.7. Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces, where $\mathcal{V}$ is the indiscrete uniformity over Z. The point-entourage uniformity, $\mathfrak{U}_{p,\mathcal{V}}$ generated over the class of all uniform continuous function $UC(Y, Z)$ is splitting in view of Theorem 3.14.

Now, let $(f_n, f'_m)_{(n,m)\in D_1\times D_2}$ be pairwise Cauchy in $UC(Y, Z)$. We show that it is continuously Cauchy. Let $(y_k, y'_l)_{(k,l)\in D_3\times D_4}$ be pairwise Cauchy nets in Y. As $\mathcal{V}$ is indiscrete uniformity, the pair of nets $(f_n(y_k), f'_m(y'_l))$ is pairwise Cauchy. Therefore, (f_n, f'_m) is continuously Cauchy. Hence, $UC(Y, Z)$ is admissible, in view of Theorem 3.15.

Remark 3.2. Consider the point-entourage uniformity over the space of all uniform functions $UC(\mathbb{R}, \mathbb{R})$, with Euclidean uniformity. Then the convergence of the sequence of functions in the topology generated by point-entourage uniformity coincides with the point-wise convergence of the sequence of the function.

4 Conclusion

This study establishes that the uniformly continuous mappings between uniform spaces do possess interesting uniform structures. We have also shown that net theory can successfully be used in the realm of uniform spaces. The authors have not come across similar work in the literature so far in the domain of uniform spaces. The present work is expected to encourage researchers working in the field of metric spaces and topologies as uniform spaces lie between metric spaces and topological spaces. Also, one can develop the concept of dual uniformity in the line of the studies carried out in [6].

To begin with, let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces, $\mathfrak{U}$ be a uniformity on $UC(Y, Z)$. For $\mathcal{H} \subseteq UC(Y, Z) \times UC(Y, Z)$, $U \in \mathcal{U}$, $V = V_1 \times V_2 \in \mathcal{V}$, we define

$$(\mathcal{H}, U) = \{(f_2^{-1}(U), g_2^{-1}(U))|\ (f, g) \in \mathcal{H}\},$$

$$\mathcal{S}(\mathfrak{T}) = \{(\mathcal{H}, U)|\ H \in \mathfrak{T}, U \in \mathcal{U}\}.$$

Does $\mathcal{S}(\mathfrak{T})$ form a subbase for a uniformity on $\mathcal{U}_Z(Y)$, where

$$\mathcal{U}_Z(Y) = \{f_2^{-1}(U)|\ f \in UC(Y, Z), U \in \mathcal{U}\}?$$

If yes, do properties of this uniformity depend on that of $\mathfrak{U}$ on $UC(Y, Z)$ and *vice versa*? How to define splittingness and admissibility for such spaces and how they are related to those of $\mathfrak{U}$ on $UC(Y, Z)$?

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