Research Article

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Hyers-Ulam-Rassias stability of (m, n)-Jordan derivations

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Abstract: In this paper, we study the Hyers-Ulam-Rassias stability of (m, n)-Jordan derivations. As applications, we characterize (m, n)-Jordan derivations on C^*-algebras and some non-self-adjoint operator algebras.

Keywords: Hyers-Ulam-Rassias stability, (m,n)-Jordan derivation, C^*-algebra

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1 Introduction

In 1940, S. Ulam [1] posed a problem about group homomorphisms. *Suppose that G_1 is a group, (G_2, d) is a metric group, and ε is a positive number. Does there exist a positive number δ, such that if a mapping f from G_1 into G_2 satisfies the inequality*

$$\mathrm{d}(f(xy), f(x)f(y)) \leqslant \delta$$

for each x, y in G_1, then there exists a homomorphism h from G_1 into G_2 such that

$$\mathrm{d}(f(x), h(x)) \leqslant \varepsilon$$

for every x in G_1? If this problem has a solution, we say that the homomorphisms from G_1 into G_2 are *stable*.

In 1941, D. Hyers [2] answers the question of Ulam's problem for Banach spaces. *Suppose that X_1 is a normed space and X_2 is a Banach space. If f is a mapping from X_1 into X_2, and there exists a positive number ε such that*

$$\|f(x + y) - f(x) - f(y)\| < \varepsilon$$

for each x and y in X_1, then there exists a unique additive mapping h from X_1 into X_2 such that

$$\|f(x) - h(x)\| < \varepsilon$$

for every x in X_1. This stability phenomenon is called the Hyers-Ulam stability of the additive functional equation $h(x + y) = h(x) + h(y)$.

In 1950, T. Aoki [3] generalized Hyers's theorem. *Suppose that X_1 and X_2 are two Banach spaces. If f is a mapping from X_1 into X_2, and there exists a positive number ε and $0 \leqslant p < 1$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p)$$

for each x and y in X_1, then there exists a positive number θ and a unique linear mapping h from X_1 into X_2 such that

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$$\|f(x) - h(x)\| \leqslant \theta\|x\|^p$$

for every x in X_1.

In 1978, Th. Rassias [4] introduced unbounded Cauchy difference and proved the stability of linear mappings between Banach spaces. *Suppose that X_1 and X_2 are two Banach spaces. If f is a mapping from X_1 into X_2, and there exist positive constants ε and $0 \leqslant p < 1$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(\|x\|^p + \|y\|^p)$$

for each x and y in X_1, then there exists a unique additive mapping h from X_1 into X_2 such that

$$\|f(x) - h(x)\| \leqslant \frac{2\varepsilon}{2 - 2^p}\|x\|^p$$

for every x in X_1. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for every x in X_1, then h is a linear mapping, where $\mathbb{R}$ denotes the set of the real numbers. In 1991, Z. Gajda [5] proved that the result in [4] is also true when $p > 1$, and Z. Gajda [5] also gave an example to show that the Rassias's stability result is not valid for $p = 1$. This phenomenon is called the *Hyers-Ulam-Rassias stability*.

On the other hand, J. Rassias [6–8] generalized Hyers's stability result by presenting a weaker condition involving a product of different powers of norms. *Suppose that X_1 is a normed space and X_2 is a Banach space. If f is a mapping from X_1 into X_2 and there exist positive constants $\varepsilon \geqslant 0$, p_1 and p_2 in $\mathbb{R}$ with $p = p_1 + p_2 \neq 1$, such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon\|x\|^{p_1}\|y\|^{p_2}$$

for each x and y in X_1, then there exists a unique additive mapping h from X_1 into X_2 such that

$$\|f(x) - h(x)\| \leqslant \frac{\varepsilon}{|2 - 2^p|}\|x\|^p$$

for every x in X_1. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for every x in X_1, then h is a linear mapping.

In this paper, we suppose that $\mathcal{A}$ is an algebra over the field of complex numbers $\mathbb{C}$, and all linear mappings are $\mathbb{C}$-linear mappings.

Let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *derivation* if

$$\delta(xy) = x\delta(y) + \delta(x)y$$

for each x, y in $\mathcal{A}$; and δ is called a *Jordan derivation* if

$$\delta(x^2) = x\delta(x) + \delta(x)x$$

for every x in $\mathcal{A}$. In 1996, K. Jun and D. Park [9] considered the stability of derivations from a Banach algebra $C^n[0,1]$ into a finite dimensional Banach $C^n[0,1]$-bimodule. In 2004, C. Park [10] gave a characterization of the stability of derivations from a Banach algebra into its Banach bimodule.

In 1990, M. Brešar and J. Vukman [11] introduced the concepts of left derivations and Jordan left derivations. Let $\mathcal{M}$ be a left $\mathcal{A}$-module. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *left derivation* if

$$\delta(xy) = x\delta(y) + y\delta(x)$$

for each x, y in $\mathcal{A}$; and δ is called a *Jordan left derivation* if

$$\delta(x^2) = 2x\delta(x)$$

for every x in $\mathcal{A}$. In 2008, Y. Jung [12] characterized the stability of left derivations.

In 2008, J. Vukman [13] introduced the concept of (m, n)-Jordan derivations. Let $\mathcal{M}$ be an $\mathcal{A}$-bimodule, $m \geqslant 0$ and $n \geqslant 0$ be two fixed integers with $m + n \neq 0$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called an (m, n)*-Jordan derivation* if

$$(m+n)\delta(x^2) = 2mx\delta(x) + 2n\delta(x)x$$

for every x in $\mathcal{A}$. By simple calculation, it is easy to show that δ is an (m, n)-Jordan derivation if and only if

$$(m+n)\delta(xy+yx) = 2mx\delta(y) + 2n\delta(x)y + 2my\delta(x) + 2n\delta(y)x$$

for each x and y in $\mathcal{A}$. It is clear that the notions of Jordan derivations and Jordan left derivations are particular cases of (m, n)-Jordan derivations, obtained when $m = n = 1$ and when $m = 1$ and $n = 0$, respectively.

This paper is organized as follows. In Section 2, we study the Hyers-Ulam-Rassias stability of (m, n)-Jordan derivations.

In Section 3, we give the applications on C^*-algebras and some non-self-adjoint operator algebras.

2 Stability of (m, n)-Jordan derivations

In this section, we denote $\mathbb{T}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. To prove the main theorem, we need the following results.

Lemma 2.1. [14] *Suppose that X and Y are two linear spaces. If f is an additive mapping from X into Y such that $f(\lambda x) = \lambda f(x)$ for every x in X and every λ in $\mathbb{T}^1$, then f is a linear mapping.*

In [15], P. Găvruţa generalized the results in [4] and [6] with the admissible control function as follows.

Lemma 2.2. [15] *Suppose that $(G, +)$ is an abelian group and X is a Banach space. Let φ be a mapping from $G \times G$ into $[0, \infty)$ such that*

$$\tilde{\varphi}(x, y) := 2^{-1}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty$$

for each x and y in G. If f is a mapping from G into X such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varphi(x, y)$$

for each x and y in G, then there exists a unique additive mapping h from G into X such that

$$\|f(x) - h(x)\| \leqslant \tilde{\varphi}(x, x)$$

for every x in G.

Let $\mathcal{A}$ be a normed algebra, $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule, and m, n be two fixed non-negative integers with $m + n \neq 0$. For the sake of convenience, we use the same symbol $\|\cdot\|$ to represent the norms on $\mathcal{A}$ and $\mathcal{M}$. A mapping φ from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ is said to have *property* $\mathbb{P}$ if

$$\tilde{\varphi}(x, y) := 2^{-1}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty \tag{2.1}$$

for each x, y in $\mathcal{A}$. A mapping f from $\mathcal{A}$ into $\mathcal{M}$ is said to have *property* $\mathbb{Q}$ if f satisfies the following two inequalities:

$$\|f(\lambda x + y) - \lambda f(x) - f(y)\| \leqslant \varphi(x, y) \tag{2.2}$$

and

$$\|(m+n)f(xy+yx) - 2mxf(y) - 2nf(x)y - 2myf(x) - 2nf(y)x\| \leqslant \varphi(x, y) \tag{2.3}$$

for each x, y in $\mathcal{A}$ and every λ in $\mathbb{T}^1$. Depending on f, we define a mapping δ_f from $\mathcal{A}$ into $\mathcal{M}$ such that

$$\delta_f(x) = \lim_{k\to\infty} 2^{-k} f(2^k x) \tag{2.4}$$

for every x in $\mathcal{A}$. The definition of δ_f will be used in most of the theorems and corollaries of the paper.

Theorem 2.3. *Let $\mathcal{A}$ be a normed algebra, $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule, and m, n be two fixed non-negative integers with $m + n \neq 0$. Suppose that φ is a mapping from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\mathbb{P}$ and f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. Then there exists a unique (m, n)-Jordan derivation δ_f from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$\|f(x) - \delta_f(x)\| \leqslant \tilde{\varphi}(x, x)$$

for every x in $\mathcal{A}$.

Proof. First we prove that there exists a unique linear mapping δ_f from $\mathcal{A}$ into $\mathcal{M}$ such that

$$\|f(x) - \delta_f(x)\| \leqslant \tilde{\varphi}(x, x)$$

for every x in $\mathcal{A}$. Let $\lambda = 1$ in (2.2), it implies that

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varphi(x, y) \tag{2.5}$$

for each x and y in $\mathcal{A}$.

Define a mapping δ_f from $\mathcal{A}$ into $\mathcal{M}$ as in (2.4). By the proof of Lemma 2.2 in [15] and (2.5), and also that the sequence $\{2^{-k}f(2^k x)\}$ is convergent, we know that δ_f is a unique additive mapping from $\mathcal{A}$ into $\mathcal{M}$ such that

$$\|f(x) - \delta_f(x)\| \leqslant \tilde{\varphi}(x, x)$$

for every x in $\mathcal{A}$. Replacing x, y by $2^k x$, $2^k y$ in (2.2), respectively. It follows that

$$\|f(2^k(\lambda x + y)) - \lambda f(2^k x) - f(2^k y)\| \leqslant \varphi(2^k x, 2^k y) \tag{2.6}$$

for each x, y in $\mathcal{A}$, and every λ in $\mathbb{T}^1$. Multiplying 2^{-k} from the left of (2.6), we can obtain the following inequality

$$\|2^{-k}f(2^k(\lambda x + y)) - 2^{-k}\lambda f(2^k x) - 2^{-k}f(2^k y)\| \leqslant 2^{-k}\varphi(2^k x, 2^k y) \tag{2.7}$$

for each x, y in $\mathcal{A}$, and every λ in $\mathbb{T}^1$. Taking the limit in (2.7) as $k \to \infty$ and by (2.4), we have that

$$\delta_f(\lambda x + y) = \lambda\delta_f(x) + \delta_f(y)$$

for each x, y in $\mathcal{A}$, and every λ in $\mathbb{T}^1$. By Lemma 2.1, we know that δ_f is a linear mapping.

Next we show that δ_f is an (m, n)-Jordan derivation. Replacing x, y by $2^k x$, $2^k y$ in (2.3), respectively, we can obtain that

$$\begin{aligned} \|(m + n)f(2^{2k}(xy + yx)) - 2m2^k xf(2^k y) - 2nf(2^k x)2^k y \\ - 2m2^k yf(2^k x) - 2nf(2^k y)2^k x\| \leqslant \varphi(2^k x, 2^k y) \end{aligned} \tag{2.8}$$

for each x and y in $\mathcal{A}$. Multiplying 2^{-2k} from the left of (2.8), we have that

$$\begin{aligned} \|(m + n)2^{-2k}f(2^{2k}(xy + yx)) - 2m2^{-k}xf(2^k y) - 2nf(2^k x)2^{-k}y \\ - 2m2^{-k}yf(2^k x) - 2nf(2^k y)2^{-k}x\| \leqslant 2^{-2k}\varphi(2^k x, 2^k y) \end{aligned} \tag{2.9}$$

for each x and y in $\mathcal{A}$. By the convergence of (2.1), it follows that $\lim_{k\to\infty} 2^{-2k}\varphi(2^k x, 2^k y) = 0$. Taking the limit as $k \to \infty$ in (2.9) and by (2.4), we have that

$$(m + n)\delta_f(xy + yx) = 2mx\delta_f(y) + 2n\delta_f(x)y + 2my\delta_f(x) + 2n\delta_f(y)x$$

for each x and y in $\mathcal{A}$. It means that δ_f is an (m, n)-Jordan derivation. □

Corollary 2.4. *Let $\mathcal{A}$ be a normed algebra, $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule, and $m = 1$, $n = 1$. Suppose that φ is a mapping from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\mathbb{P}$ and f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. Then there exists a unique Jordan derivation δ_f from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$\|f(x) - \delta_f(x)\| \leqslant \tilde{\varphi}(x, x)$$

for every x in $\mathcal{A}$.

In the following, we assume that $\mathcal{A}$ is a unital normed algebra with a unit element e and $\mathcal{M}$ is a unital Banach $\mathcal{A}$-bimodule, that is,

$$em = me = m$$

for every m in $\mathcal{M}$. A mapping φ from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ is said to have the *property* $\hat{\mathbb{P}}$ if φ satisfies the property $\mathbb{P}$ and the following equation:

$$\lim_{k\to\infty} 2^{-k}\varphi(2^k e, x) = 0 \tag{2.10}$$

for each x in $\mathcal{A}$.

Lemma 2.5. *Let $\mathcal{A}$ be a unital normed algebra, $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule, and m, n be two fixed non-negative integers with $m + n \neq 0$. Suppose that φ is a mapping from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\hat{\mathbb{P}}$ and f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. Then $f(\lambda x) = \lambda f(x)$ for every x in $\mathcal{A}$ and λ in $\mathbb{C}$.*

Proof. Define a mapping δ_f from $\mathcal{A}$ into $\mathcal{M}$ as in (2.4). By Theorem 2.3, we know that δ_f is a unique (m, n)-Jordan derivation from $\mathcal{A}$ into $\mathcal{M}$ such that

$$\|f(x) - \delta_f(x)\| \leqslant 2^{-1}\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x, 2^i x) \tag{2.11}$$

for every x in $\mathcal{A}$. Let e be a unit element of $\mathcal{A}$ and λ be in $\mathbb{C}$, we have the following inequality:

$$\begin{aligned}&\|(m+n)\delta_f((2^k e)(\lambda x)) - \lambda[2m(2^{k-1}e)f(x) + 2nf(2^{k-1}e)x + 2mxf(2^{k-1}e) + 2nf(x)(2^{k-1}e)]\|\\&\quad\leqslant |\lambda|\|(m+n)f(2^k ex) - 2m2^{k-1}ef(x) - 2nf(2^{k-1}e)x\\&\qquad - 2mxf(2^{k-1}e) - 2nf(x)2^{k-1}e\| + |\lambda|(m+n)\|\delta_f(2^k ex) - f(2^k ex)\|\end{aligned} \tag{2.12}$$

for every x in $\mathcal{A}$ and every k in $\mathbb{N}$. By (2.3), (2.11), and (2.12), it follows that

$$\begin{aligned}&\|(m+n)\delta_f((2^k e)(\lambda x)) - \lambda[2m(2^{k-1}e)f(x) + 2nf(2^{k-1}e)x + 2mxf(2^{k-1}e) + 2nf(x)(2^{k-1}e)]\|\\&\quad\leqslant |\lambda|\varphi(2^{k-1}e, x) + |\lambda|(m+n)2^{-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^{i+k}ex, 2^{i+k}ex)\end{aligned} \tag{2.13}$$

for every x in $\mathcal{A}$ and every k in $\mathbb{N}$. By (2.11) and (2.13), we can obtain the following inequality:

$$\begin{aligned}&\|(m+n)f((2^k e)(\lambda x)) - \lambda[2m(2^{k-1}e)f(x) + 2nf(2^{k-1}e)x + 2mxf(2^{k-1}e) + 2nf(x)(2^{k-1}e)]\|\\&\quad\leqslant \|(m+n)\delta_f((2^k e)(\lambda x)) - \lambda[2m(2^{k-1}e)f(x) + 2nf(2^{k-1}e)x\\&\qquad + 2mxf(2^{k-1}e) + 2nf(x)(2^{k-1}e)]\| + (m+n)\|f((2^k e)(\lambda x)) - \delta_f((2^k e)(\lambda x))\|\\&\quad\leqslant |\lambda|\varphi(2^{k-1}e, x) + |\lambda|(m+n)2^{-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^{i+k}ex, 2^{i+k}ex)\\&\qquad + (m+n)2^{-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^{i+k}e\lambda x, 2^{i+k}e\lambda x)\end{aligned} \tag{2.14}$$

for every x in $\mathcal{A}$ and every k in $\mathbb{N}$. Since

$$\begin{aligned}&\|2m2^{k-1}(f(\lambda x) - \lambda f(x))\|\\&\quad= \|2m2^{k-1}e(f(\lambda x) - \lambda f(x))\|\\&\quad\leqslant \|2m2^{k-1}ef(\lambda x) + 2nf(2^{k-1}e)\lambda x + 2m\lambda xf(2^{k-1}e) + 2nf(\lambda x)2^{k-1}e - (m+n)f((2^k e)(\lambda x))\|\\&\qquad + \|(m+n)f((2^k e)(\lambda x)) - 2m\lambda 2^{k-1}ef(x) - 2n\lambda f(2^{k-1}e)x - 2m\lambda xf(2^{k-1}e) - 2n\lambda f(x)2^{k-1}e\|\\&\qquad + \|2n2^{k-1}(f(\lambda x) - \lambda f(x))\|\end{aligned} \tag{2.15}$$

for every x in $\mathcal{A}$ and every k in $\mathbb{N}$. By (2.3), (2.14), and (2.15), we can obtain the following inequality:

$$
\begin{aligned}
&(2m+2n)2^{k-1}\|f(\lambda x)-\lambda f(x)\| \\
&\quad \leqslant \|2m2^{k-1}ef(\lambda x)+2nf(2^{k-1}e)\lambda x+2m\lambda xf(2^{k-1}e)+2nf(\lambda x)2^{k-1}e-(m+n)f((2^ke)(\lambda x))\| \\
&\qquad + \|(m+n)f((2^ke)(\lambda x))-2m\lambda 2^{k-1}ef(x)-2n\lambda f(2^{k-1}e)x-2m\lambda xf(2^{k-1}e)-2n\lambda f(x)2^{k-1}e\| \\
&\quad \leqslant \varphi(2^{k-1}e,\lambda x)+|\lambda|\varphi(2^{k-1}e,x)+|\lambda|(m+n)2^{-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^{i+k}ex,2^{i+k}ex)+(m+n)2^{-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^{i+k}e\lambda x,2^{i+k}e\lambda x)
\end{aligned}
$$

for every x in $\mathcal{A}$ and every k in $\mathbb{N}$. This means that

$$
\begin{aligned}
&(m+n)\|f(\lambda x)-\lambda f(x)\| \\
&\quad \leqslant 2^{-k}\Bigg[\varphi(2^{k-1}e,\lambda x)+|\lambda|\varphi(2^{k-1}e,x)+|\lambda|(m+n)2^{-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^{i+k}ex,2^{i+k}ex) \\
&\qquad + (m+n)2^{-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^{i+k}e\lambda x,2^{i+k}e\lambda x)\Bigg]
\end{aligned}
\tag{2.16}
$$

for every x in $\mathcal{A}$ and every k in $\mathbb{N}$.

Taking the limit in (2.16) as $k \to \infty$, by the convergence of (2.1) and the property $\hat{\mathbb{P}}$, we have that

$$(m+n)(f(\lambda x)-\lambda f(x))=0$$

for every x in $\mathcal{A}$ and λ in $\mathbb{C}$. Since $m+n\neq 0$, $f(\lambda x)=\lambda f(x)$ for every x in $\mathcal{A}$ and λ in $\mathbb{C}$. □

The following theorem is the main result in this section.

Theorem 2.6. *Let $\mathcal{A}$ be a unital normed algebra, $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule, and m, n be two fixed non-negative integers with $m+n\neq 0$. Suppose that φ is a mapping from $\mathcal{A}\times\mathcal{A}$ into $[0,\infty)$ satisfying the property $\hat{\mathbb{P}}$ and f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. Then f is an (m,n)-Jordan derivation.*

Proof. Define a mapping δ_f from $\mathcal{A}$ into $\mathcal{M}$ as in (2.4). By Theorem 2.3, we know that δ_f is a unique (m,n)-Jordan derivation from $\mathcal{A}$ into $\mathcal{M}$ such that

$$\|f(x)-\delta_f(x)\| \leqslant 2^{-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^ix,2^ix)$$

for every x in $\mathcal{A}$. By Lemma 2.4, it follows that $f(\lambda x)=\lambda f(x)$ for every x in $\mathcal{A}$ and λ in $\mathbb{C}$. Hence, we have the following inequality:

$$\|f(x)-\delta_f(x)\|=\|2^{-k}f(2^kx)-2^{-k}\delta_f(2^kx)\|\leqslant 2^{-k-1}\sum_{i=0}^{\infty}2^{-i}\varphi(2^{i+k}x,2^{i+k}x) \tag{2.17}$$

for every x in $\mathcal{A}$ and every k in $\mathbb{N}$. Taking the limit in (2.17) as $k\to\infty$ and by the convergence of (2.1), we have that $f(x)=\delta_f(x)$ for every x in $\mathcal{A}$. It means that f is an (m,n)-Jordan derivation. □

By Th. Rassias [4], we have the following result.

Corollary 2.7. *Let $\mathcal{A}$ be a unital normed algebra, $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule, and m, n be two fixed non-negative integers with $m+n\neq 0$. Suppose that f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$\|f(\lambda x+y)-\lambda f(x)-f(y)\|\leqslant\theta(\|x\|^p+\|y\|^p)$$

and

$$\|(m+n)f(xy+yx)-2mxf(y)-2nf(x)y-2myf(x)-2nf(y)x\|\leqslant\theta(\|x\|^p+\|y\|^p)$$

for each x, y in $\mathcal{A}$ and every λ in $\mathbb{T}^1$, where $\theta\geqslant 0$ and $0<p<1$. Then f is an (m,n)-Jordan derivation.

Proof. Suppose that $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is defined by

$$\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$$

for each x, y in $\mathcal{A}$, with $\theta \geq 0$ and $0 < p < 1$. In the following we show that φ satisfies the property $\hat{\mathbb{P}}$. Since $0 < p < 1$, it follows that

$$\tilde{\varphi}(x, y) := 2^{-1}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) = 2^{-1}\sum_{k=0}^{\infty} 2^{-k}\theta(\|2^k x\|^p + \|2^k y\|^p) = 2^{-1}\sum_{k=0}^{\infty} 2^{k(p-1)}\theta(\|x\|^p + \|y\|^p) < \infty \quad (2.18)$$

and

$$\lim_{k\to\infty} 2^{-k}\varphi(2^k e, x) = \lim_{k\to\infty} 2^{-k}\theta(\|2^k e\|^p + \|x\|^p) = \lim_{k\to\infty} \theta(2^{k(p-1)} + 2^{-k}\|x\|^p) = 0. \quad (2.19)$$

Thus by (2.18), (2.19), and Theorem 2.5, we can conclude that f is an (m, n)-Jordan derivation. □

By J. Rassias [6], we have the following result.

Corollary 2.8. *Let $\mathcal{A}$ be a unital normed algebra, $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule, and m, n be two fixed non-negative integers with $m + n \neq 0$. Suppose that f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$\|f(\lambda x + y) - \lambda f(x) - f(y)\| \leqslant \theta(\|x\|^{p_1}\|y\|^{p_2})$$

and

$$\|(m + n)f(xy + yx) - 2mxf(y) - 2nf(x)y - 2myf(x) - 2nf(y)x\| \leqslant \theta(\|x\|^{p_1}\|y\|^{p_2})$$

for each x, y in A and every λ in $\mathbb{T}^1$, where $\theta \geqslant 0$, and $p_1, p_2 \in \mathbb{R}$ with $p = p_1 + p_2 < 1$. Then f is an (m, n)-Jordan derivation.

Proof. Since $p = p_1 + p_2 < 1$, without loss of generality, we can assume that $p_1 < 1$. Suppose that $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is defined by

$$\varphi(x, y) = \theta(\|x\|^{p_1}\|y\|^{p_2})$$

for each x, y in A. In the following we show that φ satisfies the property $\hat{\mathbb{P}}$. Since $p = p_1 + p_2 < 1$ and $p_1 < 1$, it follows that

$$\tilde{\varphi}(x, y) := 2^{-1}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) = 2^{-1}\sum_{k=0}^{\infty} 2^{-k}\theta(\|2^k x\|^{p_1}\|2^k y\|^{p_2}) = 2^{-1}\sum_{k=0}^{\infty} 2^{k(p_1+p_2-1)}\theta(\|x\|^{p_1}\|y\|^{p_2}) < \infty \quad (2.20)$$

and

$$\lim_{k\to\infty} 2^{-k}\varphi(2^k e, x) = \lim_{k\to\infty} 2^{-k}\theta(\|2^k e\|^{p_1}\|x\|^{p_2}) = \lim_{k\to\infty} 2^{k(p_1-1)}\theta\|x\|^{p_2} = 0. \quad (2.21)$$

Thus by (2.20), (2.21) and Theorem 2.5, we can deduce that f is an (m, n)-Jordan derivation. □

Corollary 2.9. *Let $\mathcal{A}$ be a unital normed algebra, $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule, and $m = 1, n = 1$. Suppose that φ is a mapping from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\hat{\mathbb{P}}$ and f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. Then f is an Jordan derivation.*

Remark 1. Suppose that φ is a mapping from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\mathbb{P}$ and f_1, f_2 are two mappings from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. It is obvious that the mapping $|\mu|\varphi$ is also a mapping from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\mathbb{P}$, μf, and $f_1 + f_2$ are two mappings from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. Moreover, we have that

$$\mu\delta_f = \delta_{\mu f} \quad \text{and} \quad \delta_{f_1} + \delta_{f_2} = \delta_{f_1+f_2}$$

are (m, n)-Jordan derivations. Let $V = \{\delta_f \mid f : \mathcal{A} \to \mathcal{M}$ with the property $\mathbb{Q}\}$ and $V_0 = \{\delta_f = 0 \mid f : \mathcal{A} \to \mathcal{M}$ with the property $\mathbb{Q}\}$, it follows that $W = V/V_0$ is a linear space and it is interesting to consider the structure of V_0 and W.

Remark 2. We should notice that if $\mathcal{A}$ is an algebra and a mapping φ from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\mathbb{P}$:

$$\tilde{\varphi}(x, y) := 2^{-1} \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty$$

for each x and y in $\mathcal{A}$, then we cannot deduce the following equation

$$\lim_{k\to\infty} 2^{-k} \varphi(2^k e, x) = 0$$

for each x in $\mathcal{A}$. Indeed, suppose that $\mathcal{A} = \mathbb{R}$ and define a two-variable non-negative function φ by

$$\varphi(x, y) = \frac{|xy|}{y^2 + 1}$$

for each x and y in $\mathbb{R}$. It is clear that $\tilde{\varphi}(x, y)$ is a convergent series for each x, y in $\mathbb{R}$, but $\lim_{k\to\infty} 2^{-k}\varphi(2^k, x) \neq 0$ when $x \neq 0$. It means that (2.10) is not a consequence of (2.1).

Remark 3. In 2010, J. Vukman [16] gave the definition of (m, n)-Jordan centralizer. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule, $m \geqslant 0$ and $n \geqslant 0$ be two fixed integers with $m + n \neq 0$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called an *(m, n)-Jordan centralizer* if

$$(m + n)\delta(x^2) = mx\delta(x) + n\delta(x)x$$

for every x in $\mathcal{A}$. It is clear that δ is an (m, n)-Jordan centralizer if and only if

$$(m + n)\delta(xy + yx) = mx\delta(y) + n\delta(x)y + my\delta(x) + n\delta(y)x$$

for each x and y in $\mathcal{A}$.

Similarly, via the same technique used in the proof of Theorem 2.3, we can characterize the stability of (m, n)-Jordan centralizers.

3 Some applications

In [17], G. An and J. He proved that every (m, n)-Jordan derivation from a C^*-algebra $\mathcal{A}$ into its Banach $\mathcal{A}$-bimodule is zero. Thus, by Theorem 2.5, we have the following result.

Corollary 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{M}$ be a unital Banach $\mathcal{A}$-bimodule, and m, n be two fixed non-negative integers with $m + n \neq 0$. Suppose that φ is a mapping from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\hat{\mathbb{P}}$ and f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. Then $f \equiv 0$.*

Let $\mathcal{H}$ be a complex Hilbert space, and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. By a *subspace lattice* on $\mathcal{H}$, we mean a collection $\mathcal{L}$ of subspaces of $\mathcal{H}$ with (0) and $\mathcal{H}$ in $\mathcal{L}$ such that, for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$. For a Hilbert space $\mathcal{H}$, we disregard the distinction between a closed subspace and the orthogonal projection onto it. Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$, $\mathcal{L}$ is called a *commutative subspace lattice* (*CSL*) if it consists of mutually commuting projections.

Let $\mathcal{B}$ be a von Neumann algebra on $\mathcal{H}$, and $\mathcal{L} \subseteq \mathcal{B}$ be a CSL on $\mathcal{H}$. Then $\mathcal{A} = \mathcal{B} \cap \mathrm{Alg}\mathcal{L}$ is said to be a *CSL subalgebra of the von Neumann algebra $\mathcal{B}$.*

In [17], G. An and J. He proved that if $\mathcal{B}$ is a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathcal{L} \subseteq \mathcal{B}$ is a CSL on $\mathcal{H}$, then every (m, n)-Jordan derivation from $\mathcal{B} \cap \mathrm{Alg}\mathcal{L}$ into $B(\mathcal{H})$ is zero. Thus by Theorem 2.5, we have the following result.

Corollary 3.2. *Let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, $\mathcal{L} \subseteq \mathcal{B}$ be a CSL on $\mathcal{H}$, and m, n be two fixed non-negative integers with $m + n \neq 0$. Suppose that φ is a mapping from $(\mathcal{B} \cap \mathrm{Alg}\mathcal{L}) \times (\mathcal{B} \cap \mathrm{Alg}\mathcal{L})$ into $[0, \infty)$ satisfying property $\hat{\mathbb{P}}$ and f is a mapping from $\mathcal{B} \cap \mathrm{Alg}\mathcal{L}$ into $B(\mathcal{H})$ satisfying the property $\mathbb{Q}$. Then $f \equiv 0$.*

Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Suppose that $\mathcal{J}$ is an ideal of $\mathcal{A}$, we say that $\mathcal{J}$ is a *right separating set* (resp. *left separating set*) of $\mathcal{M}$ if for every m in $\mathcal{M}$, $\mathcal{J}m = \{0\}$ implies $m = 0$ (resp. $m\mathcal{J} = \{0\}$ implies $m = 0$). We denote by $\mathfrak{J}(\mathcal{A})$ the subalgebra of $\mathcal{A}$ generated algebraically by all idempotents in $\mathcal{A}$.

Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$, define $\mathcal{P}_{\mathcal{L}} = \{E \in \mathcal{L} : E_- \not\supseteq E\}$, where $E_- = \vee\{F \in \mathcal{L} : F \not\supseteq E\}$ and let $E_+ = \cap\{F \in \mathcal{L} : F \not\subseteq E\}$. A subspace $\mathcal{L}$ is called a *completely distributive* if $L = \vee\{E \in \mathcal{L} : E_- \not\supseteq L\}$ for every $L \in \mathcal{L}$; $\mathcal{L}$ is called a *$\mathcal{P}$-subspace lattice* if $\vee\{E : E \in \mathcal{P}_{\mathcal{L}}\} = \mathcal{H}$ or $\cap\{E_- : E \in \mathcal{P}_{\mathcal{L}}\} = (0)$. For some properties of completely distributive subspace lattices and $\mathcal{P}$-subspace lattices, see [19,18]. A totally ordered subspace lattice $\mathcal{N}$ is called a *nest*.

By [18,20], we know that if $\mathcal{A}$ and $\mathcal{M}$ satisfy one of the following conditions:

(1) $\mathcal{A} = \mathcal{B} \cap \mathrm{Alg}\mathcal{N}$ and $\mathcal{M} = \mathcal{B}$, where $\mathcal{N}$ is a nest in a factor von Neumann algebra $\mathcal{B}$;
(2) $\mathcal{A} = \mathrm{Alg}\mathcal{L}$ with $(0)_+ \neq (0)$ or $\mathcal{H}_- \neq \mathcal{H}$, $\mathcal{M} = B(\mathcal{H})$;
(3) $\mathcal{A} = \mathrm{Alg}\mathcal{L}$ with $\vee\{E : E \in \mathcal{P}_{\mathcal{L}}\} = \mathcal{H}$ or $\cap\{E_- : E \in \mathcal{P}_{\mathcal{L}}\} = (0)$, $\mathcal{M} = B(\mathcal{H})$;
(4) $\mathcal{A} = \mathrm{Alg}\mathcal{L}$ and $\mathcal{M}$ is a dual normal Banach $\mathcal{A}$-bimodule, where $\mathcal{L}$ is a completely distributive subspace lattice on a Hilbert space $\mathcal{H}$;

then $\mathcal{M}$ has a right or a left separating set $\mathcal{J}$ with $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$.

In [21], G. An and J. Li showed that if $\mathcal{A}$ is a unital algebra and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule with a right (left) separating set generated algebraically by idempotents in $\mathcal{A}$, then every (m, n)-Jordan derivation from a $\mathcal{A}$ into $\mathcal{M}$ is zero. By Theorem 2.5, we have the following result.

Corollary 3.3. *Let $\mathcal{A}$ be a unital algebra, $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule, with a right (left) separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$, and m, n be two fixed non-negative integers with $m + n \neq 0$. Suppose that φ is a mapping from $\mathcal{A} \times \mathcal{A}$ into $[0, \infty)$ satisfying the property $\hat{\mathbb{P}}$ and f is a mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the property $\mathbb{Q}$. Then $f \equiv 0$.*

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References

[1] S. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, 1964.
[2] D. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27** (1941), no. 4, 222–224, DOI: https://doi.org/10.1093/jahist/jav119.
[3] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), no. 1, 64–66, DOI: https://doi.org/10.2969/jmsj/00210064.
[4] Th. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300, DOI: https://doi.org/10.1090/S0002-9939-1978-0507327-1.
[5] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Math. Sci. **14** (1991), no. 3, 431–434, DOI: https://doi.org/10.1155/S016117129100056X.

[6] J. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. **46** (1982), no. 1, 126–130, DOI: https://doi.org/10.1016/0022-1236(82)90048-9.
[7] J. Rassias, *On approximately of approximately linear mappings by linear mappings*, Bull. Sci. Math. **108** (1984), no. 4, 445–446.
[8] J. Rassias, *Solution of a problem of Ulam*, J. Approx. Theory **57** (1989), no. 3, 268–273, DOI: https://doi.org/10.1016/0021-9045(89)90041-5.
[9] K. Jun and D. Park, *Almost derivations on the Banach algebra* $C^n[0,1]$, Bull. Korean Math. Soc. **33** (1996), no. 3, 359–366.
[10] C. Park, *Linear derivations on Banach algebras*, Nonlinear Funct. Anal. Appl. **9** (2004), no. 3, 359–368.
[11] M. Brešar and J. Vukman, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 7–16, DOI: https://doi.org/10.1090/S0002-9939-1990-1028284-3.
[12] Y. Jung, *On the generalized Hyers-Ulam stability of module left derivations*, J. Math. Anal. Appl. **339** (2008), no. 1, 108–114, DOI: https://doi.org/10.1016/j.jmaa.2007.07.003.
[13] J. Vukman, *On* (m,n)*-Jordan derivations and commutativity of prime rings*, Demonstr. Math. **41** (2008), no. 4, 773–778, DOI: https://doi.org/10.1515/dema-2008-0405.
[14] C. Park, *Homomorphisms between Poisson* JC^**-algebras*, Bull. Braz. Math. Soc. **36** (2005), no. 1, 79–97, DOI: https://doi.org/10.1007/s00574-005-0029-z.
[15] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), no. 3, 431–436, DOI: https://doi.org/10.1006/jmaa.1994.1211.
[16] J. Vukman, *On* (m,n)*-Jordan centralizers in rings and algebras*, Glas. Mat. Ser. III **45** (2010), no. 1, 43–53, DOI: https://doi.org/10.3336/gm.45.1.04.
[17] G. An and J. He, *Characterizations of* (m,n)*-Jordan derivations on some algebras*, Acta Math. Sin. (Engl. Ser.) **35** (2019), no. 3, 378–390, DOI: https://doi.org/10.1007/s10114-018-7495-x.
[18] Y. Chen and J. Li., *Mappings on some reflexive algebras characterized by action on zero products or Jordan zero products*, Studia Math. **206** (2011), no. 2, 121–134, DOI: https://doi.org/10.4064/sm206-2-2.
[19] W. Longstaff, *Strongly reflexive lattices*, J. London. Math. Soc. **11** (1975), no. 2, 491–498, DOI: https://doi.org/10.1112/jlms/s2-11.4.491.
[20] D. Hadwin and J. Li., *Local derivations and local automorphisms on some algebras*, J. Operator Theory **60** (2008), no. 1, 29–44, DOI: https://doi.org/10.1016/j.jmaa.2003.10.015.
[21] G. An and J. Li, *Characterizations of* (m,n)*-Jordan derivations on operator algebras*, Acta Math. Sinica (Chin. Ser.) **60** (2017), no. 1, 173–184, DOI: https://doi.org/10.1007/s10114-018-7495-x.