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Research Article

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On an equivalence between regular ordered Γ-semigroups and regular ordered semigroups

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Abstract: In this paper, we develop a technique which enables us to obtain several results from the theory of Γ-semigroups as logical implications of their semigroup theoretical analogues.

Keywords: ordered Γ-semigroup, regular ordered Γ-semigroup, ordered semigroup, regular ordered semigroup, quasi-ideal

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1 Introduction and preliminaries

The theory of Γ-semigroups has been around for more than three decades and counts hundreds of research papers and many PhD theses. Along with Γ-semigroups, other structures such as ordered Γ-semigroups and fuzzy Γ-semigroups have been studied in recent years. The majority of the results proved so far are Γ-analogues of the well-known results of ordinary semigroups which their authors pretend to be genuine generalizations of their semigroup counterparts. It should be noted that there is a striking similarity between the proofs of the original semigroup theorems and their Γ-semigroup analogues. It is this similarity that is causing a growing concern among Γ-skeptics that many of the results in Γ-semigroup theory are logically equivalent with their counterparts in ordinary semigroups. But so far there has been no evidence that this concern is mathematically based. The aim of this paper is to develop a technique whose purpose is to demonstrate the equivalence for a pair of analogue results from the two theories. This technique is a refinement of that developed in [1] and has the advantage that it works for regular Γ-semigroups endowed with a partial order. More specifically, given an ordered Γ-semigroup $(S, \Gamma, \leq_S)$, we construct an ordered semigroup $(\Omega_{\gamma_0}, \cdot, \leq_{\Omega_{\gamma_0}})$ and prove that S is regular if and only if Ω_{γ_0} is regular. This shows that regularity in the theory of Γ-semigroups can be interpreted as the usual regularity of semigroups. We go on further to prove that two characterizations of regularity, one for ordered Γ-semigroups and the other for ordered semigroups are logically equivalent. The characterization of the regularity of ordered Γ-semigroups is Theorem 8(iii) of [2] and also Theorem 3 of [3], which states that an ordered Γ-semigroup $(S, \Gamma, \leq_S)$ is regular if and only if one-sided ideals of $(S, \Gamma, \leq_S)$ are idempotent, and for every right ideal R and every left ideal L of $(S, \Gamma, \leq_S)$, $(R\Gamma L]$ is a quasi ideal of $(S, \Gamma, \leq_S)$. On the other hand, the characterization of the regularity of ordered semigroups is Theorem 3.1(iii) of [4], which states that an ordered semigroup $(S, \cdot, \leq_S)$ is regular if and only if, one-sided ideals of $(S, \cdot, \leq_S)$ are idempotent, and for every right ideal R and every left ideal L of

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$(S, \Gamma, \leq_S)$, $(R\Gamma L]$ is a quasi ideal of $(S, \cdot, \leq_S)$. Proving that the above analogue theorems are equivalent gives points to the idea that producing Γ-analogues of known results from the semigroup theory brings nothing new to the theory as pretended, but simply replicates those results in a new setting.

In what follows, we give a few basic notions that will be used throughout the paper. Let S and Γ be two nonempty sets. Any map from $S \times \Gamma \times S$ to S will be called a Γ-multiplication in S and is denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. In 1986, Sen and Saha [5,6] introduced the concept of a Γ-semigroup S as an ordered pair $(S, (\cdot)_\Gamma)$, where S and Γ are nonempty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on S, which satisfies the following property:

$$\forall\ (a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

Here we give some necessary definitions from ordered semigroup and ordered Γ-semigroup theory. An ordered semigroup S is a semigroup $(S, \cdot)$ together with an order relation $\leq$ such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $c \in S$. An ordered semigroup S is called regular if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalently, if $a \in (aSa]$ for every $a \in S$ or if $A \subseteq (ASA]$ for any subset A of S. A nonempty subset Q of an ordered semigroup S is called a quasi-ideal of S if (1) $(Q] = Q$ and (2) $(QS] \cap (SQ] \subseteq Q$.

An ordered Γ-semigroup (shortly po-Γ-semigroup) defined by Sen and Seth in [7] is a Γ-semigroup together with an order relation $\leq$ such that $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $c \in S$ and all $\gamma \in \Gamma$. A nonempty subset A of a *po*-Γ-semigroup S is called a right (resp. left) ideal of S if (1) $A\Gamma S \subseteq A$ resp. $(S\Gamma A \subseteq A)$ and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. A right (left) ideal A can be obviously written as $(A] = A$. An ordered Γ-semigroup S is called regular if for every $a \in S$ there exist $x \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $a \leq a\gamma_1 x\gamma_2 a$. A *po*-Γ-semigroup S is regular if and only if $a \in (a\Gamma S\Gamma a]$ for all $a \in S$, equivalently, if $A \subseteq (A\Gamma S\Gamma A]$ for all $A \subseteq S$. A nonempty subset Q of an ordered Γ-semigroup S is called a quasi-ideal of S if (1) $(Q] = Q$ and (2) $(Q\Gamma S] \cap (S\Gamma Q] \subseteq Q$.

2 Construction of Ω_{γ_0}

Given an ordered Γ-semigroup $(S, \Gamma, \leq_S)$, we define an ordered semigroup $(\Omega_{\gamma_0}, \cdot, \leq_{\Omega_{\gamma_0}})$. To define Ω_{γ_0} we use the fact that we can always define a multiplication $\bullet$ on any nonempty set Γ in such a way that $(\Gamma;\bullet)$ becomes a group. This in fact is equivalent to the axiom of choice. Also, we use the concept of the free product of two semigroups. Material related to this concept can be found in [8, pp. 258–261]. Furthermore, let $(F;\cdot)$ be the free semigroup on S. Its elements are finite strings $(x_1,\ldots,x_n)$, where each $x_i \in S$ and the product $\cdot$ is the concatenation of words. Now we define Ω_{γ_0} as the quotient semigroup of the free product $F*\Gamma$ of $(F;\cdot)$ with $(\Gamma, \bullet)$ by the congruence generated from the set of relations

$$((x, y), x\gamma_0 y), ((x, \gamma, y), x\gamma y)$$

for all $x, y \in S$, $\gamma \in \Gamma$ and with $\gamma_0 \in \Gamma$ a fixed element. The following is Lemma 2.1 of [9]. We have included here for convenience. Readers unfamiliar with rewriting systems can find anything necessary to understand the proof in the monograph [10].

Lemma 2.1. *Every element of Ω_{γ_0} can be represented by an irreducible word which has the form (γ, x, γ'), (γ, x), (x, γ), γ or x, where $x \in S$ and $\gamma, \gamma' \in \Gamma$.*

Proof. First, we have to prove that the reduction system arising from the given presentation is Noetherian and confluent, and therefore any element of Ω_{γ_0} is given by a unique irreducible word from $S \cup \Gamma$. Second, we have to prove that the irreducible words have one of these five forms. So if ω is a word of the form $\omega = (u, x, \gamma, y, v)$ for $\gamma \in \Gamma$, $x, y \in S$ and u, v possibly empty words, then ω reduces to $\omega' = (u, x\gamma y, v)$. And if $\omega = (u, x, y, v)$, then it reduces to $\omega' = (u, x\gamma_0 y, v)$. In this way, we obtain a reduction system which is length reducing and therefore it is Noetherian. To prove that this system is confluent, from Newman's lemma, it is sufficient to prove that it is locally confluent. For this, we need to see only the overlapping pairs.

1. $(x, y, z) \to (x\gamma_0 y, z)$ and $(x, y, z) \to (x, y\gamma_0 z)$ which both reduce to $(x\gamma_0 y\gamma_0 z)$;
2. $(x, \gamma, y, z) \to (x\gamma y, z)$ and $(x, \gamma, y, z) \to (x, \gamma, y\gamma_0 z)$ which both reduce to $(x\gamma y\gamma_0 z)$;
3. $(x, y, \gamma, z) \to (x\gamma_0 y, \gamma, z)$ and $(x, y, \gamma, z) \to (x, y\gamma z)$ which both reduce to $(x\gamma_0 y\gamma z)$;
4. $(x, \gamma, y, \gamma', z) \to (x\gamma y, \gamma', z)$ and $(x, \gamma, y, \gamma', z) \to (x, \gamma, y\gamma' z)$ which both reduce to $(x\gamma y\gamma' z)$;
5. $(\gamma_1, \gamma_2, \gamma_3) \to (\gamma_1 \bullet \gamma_2, \gamma_3)$ and $(\gamma_1, \gamma_2, \gamma_3) \to (\gamma_1, \gamma_2 \bullet \gamma_3)$ which both reduce to $\gamma_1 \bullet \gamma_2 \bullet \gamma_3$.

To complete the proof, we need to show that the irreducible word representing the element of Ω_{γ_0} has one of the five forms stated. If the word which has neither a prefix nor a suffix made entirely of letters from Γ, then it reduces to an element of S by performing the appropriate reductions. If the word has the form $(\alpha, \omega, \alpha')$, (α, ω) or (ω, α'), where ω is a word which has neither a prefix nor a suffix made entirely of letters from Γ, and α, α' have only letters from Γ, then it reduces to an element of one of the first three forms. □

Definition 2.1. We define an order relation $\le_{\Omega_{\gamma_0}}$ in terms of $\le_S$ as follows:
(1) For every $x, y \in S$, we let $x \le_{\Omega_{\gamma_0}} y \Leftrightarrow x \le_S y$;
(2) For every $x, y \in S$, and $\gamma \in \Gamma$ we let $\gamma x \le_{\Omega_{\gamma_0}} \gamma y \Leftrightarrow x \le_S y$;
(3) For every $x, y \in S$, and $\gamma \in \Gamma$ we let $x\gamma \le_{\Omega_{\gamma_0}} y\gamma \Leftrightarrow x \le_S y$;
(4) For every $x, y \in S$, and $\gamma \in \Gamma$ we let, $\gamma x\gamma' \le_{\Omega_{\gamma_0}} \gamma y\gamma' \Leftrightarrow x \le_S y$;
(5) The restriction of the relation in Γ is taken to be the equality.

Using the fact that $\le_S$ is an order relation in the Γ-semigroup S, we can prove that $\le_{\Omega_{\gamma_0}}$ is an order relation in the semigroup Ω_{γ_0}. It is obvious that $\le_{\Omega_{\gamma_0}}$ is reflexive, and very easy to see that it is antisymmetric. We check for convenience the transitivity.

(1) If $x, y, z \in S$ are such that $x \le_{\Omega_{\gamma_0}} y$ and $y \le_{\Omega_{\gamma_0}} z$, then by the definition, $x \le_S y$ and $y \le_S z$, hence $x \le_S z$ because $\le_S$ is transitive, and then $x \le_{\Omega_{\gamma_0}} z$.
(2) If $x, y, z \in S$ and $\gamma \in \Gamma$ are such that $\gamma x \le_{\Omega_{\gamma_0}} \gamma y$ and $\gamma y \le_{\Omega_{\gamma_0}} \gamma z$, then $x \le_S y$ and $y \le_S z$, hence $x \le_S z$ and as a result $\gamma x \le_{\Omega_{\gamma_0}} \gamma z$.
(3) If $x, y, z \in S$ and $\gamma \in \Gamma$ are such that $x\gamma \le_{\Omega_{\gamma_0}} y\gamma$ and $y\gamma \le_{\Omega_{\gamma_0}} z\gamma$, then similarly with above $x \le_S y$ and $y \le_S z$, hence $x \le_S z$, consequently $x\gamma \le_{\Omega_{\gamma_0}} z\gamma$.
(4) If $x, y, z \in S$ and $\gamma \in \Gamma$ are such that $\gamma x\gamma' \le_{\Omega_{\gamma_0}} \gamma y\gamma'$ and $\gamma y\gamma' \le_{\Omega_{\gamma_0}} \gamma z\gamma'$, then $x \le_S y$ and $y \le_S z$, hence $x \le_S z$, consequently $\gamma x\gamma' \le_{\Omega_{\gamma_0}} \gamma z\gamma'$.
(5) If $\alpha, \beta, \gamma \in \Gamma$ are such that $\alpha \le_{\Omega_{\gamma_0}} \beta$ and $\beta \le_{\Omega_{\gamma_0}} \gamma$, then $\alpha = \beta = \gamma$.

Next we prove that the compatibility of $\le_S$ in S implies that of $\le_{\Omega_{\gamma_0}}$ in Ω_{γ_0}. We obtain the proof only for relations of type (4) of Definition 2.1 since the proofs for the other types are analogous. So let $\gamma x\gamma' \le_{\Omega_{\gamma_0}} \gamma y\gamma'$, and want to prove that the inequality is preserved after multiplying both sides of the above on the left (resp. on the right) by one of the following elements: $\alpha \in \Gamma$, $z \in S$, $\alpha z \in \Gamma S$, $z\beta \in S\Gamma$, $\alpha z\beta \in \Gamma S\Gamma$. Since the proofs for the compatibility on the right are symmetric to those on the left, we obtain them only for the left multiplication.

(1) $\alpha(\gamma x\gamma') \le_{\Omega_{\gamma_0}} \alpha(\gamma y\gamma') \Leftrightarrow (\alpha \bullet \gamma)x\gamma' \le_{\Omega_{\gamma_0}} (\alpha \bullet \gamma)x\gamma' \Leftrightarrow x \le_S y \Leftrightarrow \gamma x\gamma' \le_{\Omega_{\gamma_0}} \gamma y\gamma'$.
(2) $z(\gamma x\gamma') \le_{\Omega_{\gamma_0}} z(\gamma y\gamma') \Leftrightarrow (z\gamma x)\gamma' \le_{\Omega_{\gamma_0}} (z\gamma y)\gamma' \Leftrightarrow z\gamma x \le_S z\gamma y$, where the latter is true since $x \le_S y$.
(3) $(\alpha z)(\gamma x\gamma') \le_{\Omega_{\gamma_0}} (\alpha z)(\gamma y\gamma') \Leftrightarrow \alpha(z\gamma x)\gamma' \le_{\Omega_{\gamma_0}} \alpha(z\gamma y)\gamma' \Leftrightarrow z\gamma x \le_{\Omega_{\gamma_0}} z\gamma y$, where the latter is true since $x \le_S y$.
(4) $(z\beta)(\gamma x\gamma') \le_{\Omega_{\gamma_0}} (z\beta)(\gamma y\gamma') \Leftrightarrow z(\beta \bullet \gamma)x\gamma' \le_{\Omega_{\gamma_0}} z(\beta \bullet \gamma)y\gamma' \Leftrightarrow z(\beta \bullet \gamma)x \le_{\Omega_{\gamma_0}} z(\beta \bullet \gamma)y$, where the last inequality is true since $x \le_S y$.
(5) $(\alpha z\beta)(\gamma x\gamma') \le_{\Omega_{\gamma_0}} (\alpha z\beta)(\gamma y\gamma') \Leftrightarrow \alpha(z(\beta \bullet \gamma)x)\gamma' \le_{\Omega_{\gamma_0}} \alpha(z(\beta \bullet \gamma)y)\gamma' \Leftrightarrow z(\beta \bullet \gamma)x \le_{\Omega_{\gamma_0}} z(\beta \bullet \gamma)y$, where the last inequality is true since $x \le_S y$.

Therefore, $\le_{\Omega_{\gamma_0}}$ is compatible with the multiplication of Ω_{γ_0}. Summarizing, we have the following.

Proposition 2.1. *The triple* $(\Omega_{\gamma_0}, \cdot, \le_{\Omega_{\gamma_0}})$ *is an ordered semigroup.*

Since in Section 3 we deal with ordered ideals in both structures, $(S, \Gamma, \leq_S)$ and $(\Omega_{\gamma_0}, \cdot, \leq_{\Omega_{\gamma_0}})$, we will not use the standard notation $(X]$ to indicate the ordered ideal, but we introduce a new one as in the following definition.

Definition 2.2. For $C \subseteq S$, we define $L_S(C) = \{x \in S : \exists c \in C \text{ such that } x \leq_S c\}$, and for every $D \subseteq \Omega_{\gamma_0}$, $L_{\Omega_{\gamma_0}}(D) = \{w \in \Omega_{\gamma_0} : \exists d \in D \text{ such that } w \leq_{\Omega_{\gamma_0}} d\}$.

We remark by passing the following.

Lemma 2.2. *$L_{\Omega_{\gamma_0}}(C) = L_S(C)$ for $C \subseteq S$.*

Proof. First, we prove that $L_S(C) \subseteq L_{\Omega_{\gamma_0}}(C)$. If $y \in L_S(C)$, then $y \leq_S c$ for $c \in C$. Since $y, c \in S$, then $y \leq_S c$ by Definition 2.1 is equivalent to $y \leq_{\Omega_{\gamma_0}} c$, hence $y \in L_{\Omega_{\gamma_0}}(C)$.

Conversely, if $\omega \in L_{\Omega_{\gamma_0}}(C)$, then $w \leq_{\Omega_{\gamma_0}} c$ for $c \in C$ and by Definition 2.1 we must have that $w \in S$, and that $\omega \leq_S c$, proving that $\omega \in L_S$. □

The following lemma gives a relationship between the principal ordered ideal in S generated by some $x \in S$ and the principal ordered ideal in Ω_{γ_0} generated by the same element x.

Lemma 2.3. *Let $x \in S$ by an arbitrary element. The following hold true.*

(*i*) *The principal left ordered ideal of Ω_{γ_0} generated by x is the set $(x)_l^{\leq_{\Omega_{\gamma_0}}} = (x)_l^{\leq_S} \cup \Gamma(x)_l^{\leq_S}$, where $(x)_l^{\leq_S} = L_S(x \cup S\Gamma x)$ is the left ordered ideal of S generated by x and $\Gamma(x)_l^{\leq_S} = \{\gamma y | \gamma \in \Gamma, y \in L_S(S\Gamma x \cup x)\}$.*

(*ii*) *The principal right ordered ideal of Ω_{γ_0} generated by x is the set $(x)_r^{\leq_{\Omega_{\gamma_0}}} = (x)_r^{\leq_S} \cup (x)_r^{\leq_S}\Gamma$, where $(x)_r^{\leq_S} = L_S(x\Gamma S \cup x)$ is the right ordered ideal of S generated by x and $(x)_r^{\leq_S}\ \Gamma = \{y\gamma | \gamma \in \Gamma, y \in L_S(x\Gamma S \cup x)\}$.*

Proof. We obtain the proof for (*i*) since the proof for (*ii*) is dual to that of (*i*). So we have to prove that $A \in (x)_l^{\leq_{\Omega_{\gamma_0}}}$ if and only if $A \in (x)_l^{\leq_S} \cup \Gamma(x)_l^{\leq_S}$. Indeed, if $A \in (x)_l^{\leq_{\Omega_{\gamma_0}}}$, then $A \leq_{\Omega_{\gamma_0}} B$, where $B \in L_{\Omega_{\gamma_0}}(\Omega_{\gamma_0} x \cup x)$. If $B = x$, we have $A \leq_{\gamma_0} x$ and by Definition 2.1 $A \leq_S x$, so $A \in (x)_l^{\leq_S}$. If $B \in \Omega_{\gamma_0} x$, then B may have these forms:

(1) $B = (\alpha y \beta)x = \alpha(y\beta x)$. In this case, $A \leq_{\Omega_{\gamma_0}} \alpha(y\beta x)$ which forces $A = \alpha z$, where $z \leq_S y\beta x$, hence $z \in (x)_l^{\leq_S}$ and then $A \in \Gamma(x)_l^{\leq_S}$.

(2) $B = (\alpha y)x = \alpha(y\gamma_0 x)$. In this case, $A \leq_{\Omega_{\gamma_0}} \alpha(y\gamma_0 x)$. One can see that in the same way as above, $A \in \Gamma(x)_l^{\leq_S}$.

(3) $B = (y\alpha)x = y\alpha x$. In this case, $A \leq_{\Omega_{\gamma_0}} y\alpha x$ and by definition 2.1 we have that $A \leq_S y\alpha x$, therefore, $A \in (x)_l^{\leq_S}$.

(4) $B = (\alpha)x = \alpha x$. In this case, $A \leq_{\Omega_{\gamma_0}} \alpha x$, then $A = \alpha z$, where $z \leq_S x$, hence $z \in (x)_l^{\leq_S}$ and then $A \in \Gamma(x)_l^{\leq_S}$.

Conversely, if $A \in (x)_l^{\leq_S} \cup \Gamma(x)_l^{\leq_S}$, then either $A \leq_S B$ where $B \in (x)_l^{\leq_S}$ or $A \leq_{\Omega_{\gamma_0}} B$, where $B = \alpha C$ with $\alpha \in \Gamma$ and $C \in (x)_l^{\leq_S}$. In the first case, it follows at once that $A \in (x)_l^{\leq_{\Omega_{\gamma_0}}}$. In the second case, when $B = \alpha C$ and $C \in (x)_l^{\leq_S}$, the inequality $A \leq_{\Omega_{\gamma_0}} \alpha C$ implies that $A = \alpha A'$ with $A' \leq_S C$, and as a consequence $A' \in (x)_l^{\leq_S}$. Thus, $A \in \Gamma(x)_l^{\leq_S}$. □

3 Regularity in ordered Γ-semigroups as a consequence of regularity in ordered semigroups

The following proposition shows that the regularity of an ordered Γ-semigroup can be completely characterized as the regularity of an ordered semigroup.

Proposition 3.1. *S is a regular ordered Γ-semigroup if and only if Ω_{γ_0} is a regular ordered semigroup.*

Proof. If S is a regular ordered Γ-semigroup, then for all $a \in S$, $\exists x \in S$ and $\gamma_1, \gamma_2 \in \Gamma$, such that $a \leq_S a\gamma_1 x\gamma_2 a$. To prove Ω_{γ_0} is a regular ordered semigroup, we have to prove that every element of Ω_{γ_0} have an ordered inverse in Ω_{γ_0}. By Lemma 2.1, we have that the elements of Ω_{γ_0} can be represented by an irreducible word which has only five forms. We prove regularity for elements of each of these five forms. So let first $\alpha_1 a\alpha_2 \in \Omega_{\gamma_0}$. To find its ordered inverse we take $a \leq_S a\gamma_1 x\gamma_2 a$ and then by Definition 2.1 we have $\alpha_1 a\alpha_2 \leq_{\Omega_{\gamma_0}} \alpha_1(a\gamma_1 x\gamma_2 a)\alpha_2 = (\alpha_1 a\alpha_2)(\alpha_2^{-1}\gamma_1 x\gamma_2\alpha_1^{-1})(\alpha_1 a\alpha_2)$ which tells us that $\alpha_1 a\alpha_2$ is regular in Ω_{γ_0} and $\alpha_2^{-1}\gamma_1 x\gamma_2\alpha_1^{-1} \in \Omega_{\gamma_0}$ is its ordered inverse. Second, for showing that $\alpha a \in \Omega_{\gamma_0}$ is regular, we take $a \leq_S a\gamma_1 x\gamma_2 a$ and then by Definition 2.1 we can write $\alpha a \leq_{\Omega_{\gamma_0}} \alpha(a\gamma_1 x\gamma_2 a) = (\alpha a)(\gamma_1 x\gamma_2\alpha^{-1})(\alpha a)$. This tells us that αa is regular and as its inverse we can take $\gamma_1 x\gamma_2\alpha^{-1} \in \Omega_{\gamma_0}$. In the same way, one may prove that $a\alpha \in \Omega_{\gamma_0}$ is regular with inverse $\alpha^{-1}\gamma_1 x\gamma_2 \in \Omega_{\gamma_0}$. Furthermore, we prove that $a \in \Omega_{\gamma_0}$ is regular with inverse $\gamma_1 x\gamma_2 \in \Omega_{\gamma_0}$, since by Definition 2.1, $a \leq_S a\gamma_1 x\gamma_2 a$ and then $a \leq_{\Omega_{\gamma_0}} a\gamma_1 x\gamma_2 a$. And finally, $\gamma \in \Omega_{\gamma_0}$ is regular since $\gamma \leq_{\Omega_{\gamma_0}} \gamma\gamma^{-1}\gamma$. Hence, we showed that Ω_{γ_0} is a regular ordered semigroup. Conversely, if Ω_{γ_0} is regular ordered semigroup, then every $a \in S$ has an inverse in Ω_{γ_0}. To show that S is a regular ordered Γ-semigroup, we show that very $a \in S$ has an inverse in S. For this, we distinguish between the five following forms. First, if the inverse of a in Ω_{γ_0} has the form $\alpha x\beta \in \Omega_{\gamma_0}$, for $x \in S$, then $a \leq_{\Omega_{\gamma_0}} a\alpha x\beta a$ by Definition 2.1 implies that $a \leq_S a\alpha x\beta a$, showing that a is regular in S. Second, if the inverse of a in Ω_{γ_0} has the form αx, then $a \leq_{\Omega_{\gamma_0}} a(\alpha x)a = a\alpha x\gamma_0 a$ and then Definition 2.1 implies that $a \leq_S a\alpha x\gamma_0 a$, which means that a is regular in S. Third, the inverse of a in Ω_{γ_0} has the form $x\alpha$, this case is similar to the second case. Fourth, the inverse of a in S is x, then $a \leq_{\Omega_{\gamma_0}} axa = a\gamma_0 x\gamma_0 a$ and by Definition 2.1 $a \leq_S a\gamma_0 x\gamma_0 a$, which means that a is regular in S. Finally, if the inverse have the form $\alpha \in \Gamma$, then $a \leq_{\Omega_{\gamma_0}} a\gamma a \leq_{\Omega_{\gamma_0}} a\gamma a\gamma a$ and by Definition 2.1 we have that $a \leq_S a\gamma a\gamma a$ and a is regular in the ordered Γ semigroup S. ◻

Before we prove our main theorem we need this technical lemma which is the analogue of implication $(iii) \Rightarrow (ii)$ of Theorem 9.4 of [11].

Lemma 3.1. *If $(S, \Gamma, \leq_S)$ is an ordered Γ-semigroup such that one-sided ideals of $(S, \Gamma, \leq_S)$ are idempotent, and for every right ideal R and every left ideal L of $(S, \Gamma, \leq_S)$, $L_S(R\Gamma L)$ is a quasi ideal of $(S, \Gamma, \leq_S)$, then for every $a \in S$, $(a)_r^{\leq_S} \cap (a)_l^{\leq_S} = L_S((a)_r^{\leq_S}\Gamma(a)_l^{\leq_S})$.*

Proof. Observe first that

$$(a)_r^{\leq_S} = L_S(a \cup a\Gamma S) = L_S(a \cup a\Gamma S)\Gamma L_S(a \cup a\Gamma S) \subseteq L_S(a\Gamma S) \subseteq (a)_r^{\leq_S},$$

from which we derive that $(a)_r^{\leq_S} = L_S(a\Gamma S)$. In a similar fashion, one can get that $(a)_l^{\leq_S} = L_S(S\Gamma a)$. Furthermore, since

$$L_S((a)_r^{\leq_S}\Gamma(a)_l^{\leq_S}) = L_S(L_S(a\Gamma S)\Gamma L_S(S\Gamma a))$$

is a quasi ideal, we have

$$L_S(L_S(L_S(a\Gamma S)\Gamma L_S(S\Gamma a))\Gamma S) \cap L_S(S\Gamma L_S(L_S(a\Gamma S)\Gamma L_S(S\Gamma a))) \subseteq L_S(L_S(a\Gamma S)\Gamma L_S(S\Gamma a)).$$

From this and the previous assumptions, we obtain

$$\begin{aligned}(a)_r^{\leq_S} \cap (a)_l^{\leq_S} &= L_S(a\Gamma S) \cap L_S(S\Gamma a)\\ &= (L_S(a\Gamma S)\Gamma L_S(a\Gamma S)\Gamma L_S(a\Gamma S)) \cap (L_S(S\Gamma a)\Gamma L_S(S\Gamma a)\Gamma L_S(S\Gamma a))\\ &\subseteq L_S(L_S(L_S(a\Gamma S)\Gamma L_S(S\Gamma a))\Gamma S) \cap L_S(S\Gamma L_S(L_S(a\Gamma S)\Gamma L_S(S\Gamma a)))\\ &\subseteq L_S(L_S(a\Gamma S)\Gamma L_S(S\Gamma a))\\ &= L_S((a)_r^{\leq_S}\Gamma(a)_l^{\leq_S}),\end{aligned}$$

proving thus the nonobvious inclusion $(a)_r^{\leq_S} \cap (a)_l^{\leq_S} \subseteq L_S((a)_r^{\leq_S}\Gamma(a)_l^{\leq_S})$. ◻

Theorem 3.1. *The following are logically equivalent.*

(i) *An ordered* Γ*-semigroup* $(S, \Gamma, \leq_S)$ *is regular if and only if, one-sided ideals of* $(S, \Gamma, \leq_S)$ *are idempotent, and for every right ideal R and every left ideal L of* $(S, \Gamma, \leq_S)$, $L_S(R\Gamma L)$ *is a quasi ideal of* $(S, \Gamma, \leq_S)$.

(ii) *An ordered semigroup* $(S, \cdot, \leq_S)$ *is regular if and only if, one-sided ideals of* $(S, \cdot, \leq_S)$ *are idempotent, and for every right ideal R and every left ideal L of* $(S, \Gamma, \leq_S)$, $L_S(R\Gamma L)$ *is a quasi ideal of* $(S, \cdot, \leq_S)$.

Proof. (i) ⇒ (ii) is trivial since any regular ordered semigroup can be regarded as an regular ordered Γ-semigroup, where Γ is a singleton. Also, one-sided ideals and quasi ideals in ordered semigroups are the same as those in ordered Γ-semigroups when Γ is a singleton.

(ii) ⇒ (i). Assume first that the ordered Γ-semigroup S is regular. Then by Proposition 3.1, Ω_{γ_0} is a regular ordered semigroup. Theorem 3.2 of [4] implies that for every right ideal R and every left ideal L of Ω_{γ_0}, $R \cap L = L_{\Omega_{\gamma_0}}(RL)$, $L_{\Omega_{\gamma_0}}(RL)$ is a quasi-ideal of Ω_{γ_0}, and also every right and left ideal of the semigroup Ω_{γ_0} is idempotent. Let A now be an ordered right ideal of S and consider the subset $R = L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ of Ω_{γ_0}. We show that $R = L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ is a right ideal of the ordered semigroup Ω_{γ_0}. To this end, we have to prove that it satisfies the two conditions of right ideals: (1) $L_{\Omega_{\gamma_0}}(A \cup A\Gamma)\Omega_{\gamma_0} \subseteq L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$, (2) if $a \in L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ and $b \leq_{\Omega_{\gamma_0}} a$, then $b \in L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$. To show the first we have to show that for every $b \in L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ and every $C \in \Omega_{\gamma_0}$, $bC \in L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$. Since $b \in L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$, either $b \leq_S a$ where $a \in A$, or $b \leq_{\Omega_{\gamma_0}} a\gamma$, in which case $b = a'\gamma$ and $a' \leq_S a$. In the first case, when $a, b \in S$ and $b \leq_S a$, it follows immediately that $bC \leq_{\Omega_{\gamma_0}} aC$ whatever the value of C is, since $\leq_{\Omega_{\gamma_0}}$ is a compatible relation. But still we have to prove that $aC \in L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ and this depends on the value of C. Since

$$C \in S \cup \Gamma S \cup S\Gamma \cup \Gamma S\Gamma \cup \Gamma,$$

then

$$aC \in aS \cup a\Gamma S \cup aS\Gamma \cup a\Gamma S\Gamma \cup a\Gamma \subseteq a\Gamma S \cup a\Gamma S\Gamma \cup a\ \Gamma \subseteq L_{\Omega_{\gamma_0}}(A \cup A\Gamma),$$

where the last inclusion comes from the fact that A is an ordered right ideal of $(S, \Gamma, \leq_S)$. It remains to prove that the same holds true in the second case when $b = a'\gamma$ with $a' \in S$ and $a' \leq_S a$, and $\gamma \in \Gamma$. Depending on the value of C we have to prove that $a\gamma C \in L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$. Indeed,

$$a\gamma C \in (a\gamma)S \cup (a\gamma)\Gamma S \cup (a\gamma)S\Gamma \cup (a\gamma)\,\Gamma S\Gamma \cup (a\gamma)\Gamma \subseteq a\Gamma S \cup a\Gamma S\Gamma \cup a\Gamma \subseteq L_{\Omega_{\gamma_0}}(A \cup A\Gamma).$$

All the above verifications prove the first condition, while the second condition is obvious. So $R = L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ is a right ideal of the ordered semigroup Ω_{γ_0} and from [4] it follows that $R = L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ is an idempotent. Passing now from the ordered semigroup Ω_{γ_0} to the ordered Γ semigroup S, we show that if every right ideal of the ordered semigroup Ω_{γ_0} is idempotent, then so is every right ideal of the ordered Γ-semigroup S. Let A be a right ideal of the ordered Γ semigroup S, we have to prove A is an idempotent in S, that is, $L_S(A\Gamma A) = A$. Since A is a right ideal, $L_S(A\Gamma A) \subseteq L_S(A\Gamma S) \subseteq A$. To prove the converse, we utilize the fact that $L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ is an idempotent in Ω_{γ_0}, thus

$$A \subseteq L_{\Omega_{\gamma_0}}(A \cup A\Gamma) = \left(L_{\Omega_{\gamma_0}}\left(A \cup A\Gamma\right)\right)^2 \subseteq L_{\Omega_{\gamma_0}}(AA \cup AA\Gamma \cup A\Gamma A \cup A\Gamma A\Gamma) \subseteq L_{\Omega_{\gamma_0}}(A\Gamma A \cup A\Gamma).$$

This implies that every $a \in A$ is lower with respect to $\leq_{\Omega_{\gamma_0}}$ than some element of A or some element of $A\Gamma$. The second case is impossible from the way we have defined $\leq_{\Omega_{\gamma_0}}$, so it remains that there is some $\gamma \in \Gamma$, and $a', a'' \in A$ such that $a \leq_{\Omega_{\gamma_0}} a'\gamma a''$. But this is the same as to say that $a \leq_S a'\gamma a''$, so $a \in L_S(A\Gamma A)$, and as a result $A \subseteq L_S(A\Gamma A)$. One can show that left ideals of S too are idempotent by first proving in a similar fashion to above that for every left ideal B of S, the set $L_{\Omega_{\gamma_0}}(B \cup \Gamma B)$ is a left ideal of Ω_{γ_0}. Finally, if A is a right ideal and B a left ideal of the ordered Γ semigroup S, we have to prove that $L_S(A\Gamma B)$ is a quasi ideal of $(S, \Gamma, \leq_S)$, which means that:

$$L_S(L_S(A\Gamma B)\Gamma S \cap S\Gamma L_S(A\Gamma B)) \subseteq L_S(A\Gamma B) \tag{1}$$

and

$$L_S(L_S(A\Gamma B)) = L_S(A\Gamma B). \tag{2}$$

From [4] we have that for the right ideal $L_{\Omega_{\gamma_0}}(A \cup A\Gamma)$ and for the left ideal $L_{\Omega_{\gamma_0}}(B \cup \Gamma B)$ of Ω_{γ_0}, the set $L_{\Omega_{\gamma_0}}(L_{\Omega_{\gamma_0}}(A \cup A\Gamma)\cdot L_{\Omega_{\gamma_0}}(B \cup \Gamma B)) = L_{\Omega_{\gamma_0}}((A \cup A\Gamma)\cdot(B \cup \Gamma B))$ is a quasi ideal of Ω_{γ_0}. To prove the first condition (1), we see that

$$\begin{aligned} &L_S(L_S(A\Gamma B)\Gamma S \cap S\Gamma L_S(A\Gamma B)) \\ &\subseteq L_{\Omega_{\gamma_0}}((L_{\Omega_{\gamma_0}}(A \cup A\Gamma)\cdot L_{\Omega_{\gamma_0}}(B \cup \Gamma B))\cdot \Omega_{\gamma_0} \cap \Omega_{\gamma_0} \cdot L_{\Omega_{\gamma_0}}(L_{\Omega_{\gamma_0}}(A \cup A\Gamma)\cdot L_{\Omega_{\gamma_0}}(B \cup \Gamma B)) \\ &\subseteq L_{\Omega_{\gamma_0}}((L_{\Omega_{\gamma_0}}(A \cup A\Gamma)\cdot L_{\Omega_{\gamma_0}}(B \cup \Gamma B)) \\ &= L_{\Omega_{\gamma_0}}((A \cup A\Gamma)\cdot(B \cup \Gamma B)) = L_{\Omega_{\gamma_0}}(A\Gamma B) = L_S(A\Gamma B), \end{aligned}$$

where the last equality follows from Lemma 2.2. The second condition (2) is obviously true since $L_S(A\Gamma B)$ is an ordered ideal.

Conversely, we assume that every right and left ideal of S is an idempotent, and for every right ideal A of S, and every left ideal B of S, the set $L_S(A\Gamma B)$ is a quasi ideal of S, and want to prove that S is regular. The strategy is to prove that under the given conditions, Ω_{γ_0} is a regular ordered semigroup, and then from Proposition 3.1 we obtain straightaway that S is a regular Γ-semigroup. To prove the regularity of Ω_{γ_0}, it is enough to prove that all right ideals R and all left ideals L of Ω_{γ_0} are idempotent, and $L_{\Omega_{\gamma_0}}(R \cdot L)$ is a quasi ideal of Ω_{γ_0}. Let R be a right ideal of Ω_{γ_0} and want to prove that $RR = R$. The inclusion $RR \subseteq R$ is trivial. To prove the reverse inclusion $R \subseteq RR$, we need to prove that every $x \in R$ is of the form $x = x_1x_2$, where $x_1, x_2 \in R$. There are several possibilities for $x \in R$.

(i) $x \in S$. Then, from Lemma 2.3 we have that $(x)_r^{\leq_{\Omega_{\gamma_0}}} = (x)_r^{\leq_S} \cup (x)_r^{\leq_S}\Gamma$. Recalling that $(x)_r^{\leq_S}$ is idempotent from the assumption, so $(x)_r^{\leq_S} = (x)_r^{\leq_S}\Gamma(x)_r^{\leq_S}$, hence $x = x_1\gamma x_2$, where $x_1, x_2 \in (x)_r^{\leq_S}$ and $\gamma \in \Gamma$. Now $x_1\gamma \in (x)_r^{\leq_S}\Gamma$ and $x_2 \in (x)_r^{\leq_S}$, consequently

$$x = (x_1\gamma)\cdot x_2 \in (x)_r^{\leq_{\Omega_{\gamma_0}}}\cdot (x)_r^{\leq_{\Omega_{\gamma_0}}} \subseteq R \cdot R.$$

(ii) Let the element of R be of the form $x\alpha$ with $x \in S$ and $\alpha \in \Gamma$. Since R is a right ideal of Ω_{γ_0}, then

$$x\Gamma = x(\alpha\bullet\Gamma) = (x\alpha)\Gamma \subseteq (x\alpha)\Omega_{\gamma_0} \subseteq R\Omega_{\gamma_0} \subseteq R,$$

and then we also obtain that

$$x\Gamma S = x\Gamma\Gamma S \subseteq R\Gamma S \subseteq R\Omega_{\gamma_0} \subseteq R.$$

Now we prove that the element x above is necessarily in R. For this, we use again the fact that $(x)_r^{\leq_S}$ is idempotent. It follows from this assumption that $x = x'\gamma x''$, where either $x' \leq_S x$ or $x' \leq_S x\beta s$ with $s \in S$. In the first case,

$$x = x'\gamma x'' \leq_S x\gamma x'' \in x\Gamma S \subseteq R,$$

from which it follows that $x \in R$. Similarly, in the second case we see that

$$x = x'\gamma x'' \leq_S x\beta s\gamma x'' \in x\Gamma S \subseteq R,$$

and then $x \in R$. From the proof of (i) above we have that $x = (x_1\gamma)\cdot x_2$, where $x_1\gamma \in R$ and $x_2 \in R$. Then $x\alpha = (x_1\gamma)\cdot(x_2\alpha)$, where again $x_1\gamma \in R$ and $x_2\alpha \in R$. Thus, we proved that $x\alpha$ is expressed as a product of two elements from R as desired.

(iii) The element of R is of the form αx with $\alpha \in \Gamma$ and $x \in S$. In this case, we have to show first the equality $(\alpha x)_r^{\leq_{\Omega_{\gamma_0}}} = \alpha(x)_r^{\leq_{\Omega_{\gamma_0}}}$. Indeed,

$$\begin{aligned} \xi \in (\alpha x)_r^{\leq_{\Omega_{\gamma_0}}} &\Leftrightarrow \xi \in L_{\Omega_{\gamma_0}}(\alpha x \cup \alpha x\Omega_{\gamma_0}) \\ &\Leftrightarrow \xi = \alpha x', \quad \text{where } x' \in L_{\Omega_{\gamma_0}}(x \cup x\Omega_{\gamma_0}) \\ &\Leftrightarrow \xi \in \alpha(x)_r^{\leq_{\Omega_{\gamma_0}}}, \end{aligned}$$

which proves that $(\alpha x)_r^{\leq_{\Omega_{\gamma_0}}} = \alpha(x)_r^{\leq_{\Omega_{\gamma_0}}}$. From Lemma 2.3, we derive that

$$(\alpha x)_r^{\leq_{\Omega_{\gamma_0}}} = \alpha((x)_r^{\leq_S} \cup (x)_r^{\leq_S}\Gamma).$$

Now the right ideal $(x)_r^{\leq_S}$ is idempotent, which means that $(x)_r^{\leq_S} = (x)_r^{\leq_S}\Gamma(x)_r^{\leq_S}$, hence

$$(\alpha x)_r^{\leq_{\Omega_{\gamma_0}}} = \alpha((x)_r^{\leq_S}\Gamma(x)_r^{\leq_S} \cup (x)_r^{\leq_S}\Gamma). \tag{3}$$

It follows from (3) that $\alpha x = \alpha x_1\gamma x_2$, where $x_1, x_2 \in (x)_r^{\leq_S}$. Now if we rewrite this term as $\alpha x = (\alpha x_1)(\gamma\alpha^{-1})(\alpha x_2)$, we see that

$$\begin{aligned}\alpha x &= (\alpha x_1)(\gamma\alpha^{-1})(\alpha x_2)\\ &\in \alpha((x)_r^{\leq_S} \cup (x)_r^{\leq_S}\Gamma)\alpha((x)_r^{\leq_S} \cup (x)_r^{\leq_S}\Gamma)\\ &= (\alpha x)_r^{\leq_{\Omega_{\gamma_0}}}\cdot(\alpha x)_r^{\leq_{\Omega_{\gamma_0}}},\end{aligned}$$

which proves the claim.

(iv) The element of R is of the form $\alpha x\beta$. We first note that

$$\begin{aligned}(\alpha x\beta)_r^{\leq_{\Omega_{\gamma_0}}} &= L_{\Omega_{\gamma_0}}(\alpha x\beta \cup \alpha x\beta(\Gamma \cup S \cup \Gamma S \cup S\Gamma \cup \Gamma S\Gamma))\\ &= L_{\Omega_{\gamma_0}}(\alpha x\Gamma \cup \alpha x\Gamma S \cup \alpha x\Gamma S\Gamma).\end{aligned}$$

Since $(x)_r^{\leq_S}$ is idempotent, $x = x_1\gamma x_2$ where $x_1, x_2 \in (x)_r^{\leq_S}$. It follows that

$$\alpha x_1\gamma\alpha^{-1} \in L_{\Omega_{\gamma_0}}(\alpha x\Gamma \cup \alpha x\Gamma S \cup \alpha x\Gamma S\Gamma)$$

and similarly,

$$\alpha x_2\beta \in L_{\Omega_{\gamma_0}}(\alpha x\,\Gamma\cup \alpha x\Gamma S \cup \alpha x\Gamma S\Gamma),$$

hence

$$\alpha x\beta = (\alpha x_1\gamma\alpha^{-1})(\alpha x_2\beta) \in \left(L_{\Omega_{\gamma_0}}\left(\alpha x\,\Gamma\cup \alpha x\Gamma S \cup \alpha x\Gamma S\Gamma\right)\right)^2 = \left(\left(\alpha x\beta\right)_r^{\leq_{\Omega_{\gamma_0}}}\right)^2 \subseteq R^2.$$

This shows that $\alpha x\beta$ is expressed as a product of two elements of R.

(v) The final case is when the element of R is some $\gamma \in \Gamma$. Observe that

$$\Gamma = \gamma\,\Gamma \subseteq R\Omega_{\gamma_0} \subseteq R.$$

Now letting 1 be the unit element of $(\Gamma, \bullet)$ we have that

$$\gamma = \gamma \cdot 1 \in RR,$$

and once again, $\gamma \in R$ is expressed as a product of two elements in R, namely, γ and 1.

Recollecting, we have proved that any right ideal R of Ω_{γ_0} is idempotent. Similarly, we can show that any left ideal L is idempotent in Ω_{γ_0}. Now if R is a right ideal and L a left ideal of the ordered semigroup Ω_{γ_0}, we have to prove that $L_{\Omega_{\gamma_0}}(RL)$ is a quasi ideal of Ω_{γ_0}. This would follow immediately if we prove that $R \cap L \subseteq L_{\Omega_{\gamma_0}}(RL)$, since on one hand, $R \cap L$ is a quasi-ideal, and on the other hand, $L_{\Omega_{\gamma_0}}(RL) \subseteq R \cap L$. Let $\alpha x\beta \in R \cap L$ where $\alpha, \beta \in \Gamma$ are operators from Γ, and $x \in S$ such that $\alpha x\beta \leq_{\Omega_{\gamma_0}} \alpha a\beta$ where $\alpha a\beta \in R \cap L$ and $a \in S$. From Definition 2.1, we have that $x \leq_S a$. Consequently,

$$x \in (a)_r^{\leq_S} \cap (a)_l^{\leq_S} = L_S((a)_r^{\leq_S}\Gamma(a)_l^{\leq_S}),$$

where the equality follows from Lemma 3.1. Then there are $\xi_1, \xi_2, \xi_3 \in \Gamma$ and $s, t \in S$ such that $x \leq_S a\xi_1 s\xi_2 t\xi_3 a$. Furthermore, we have that

$$\alpha x\beta \leq_{\Omega_{\gamma_0}} (\alpha a\beta)(\beta^{-1}\xi_1)((s\xi_2 t)(\xi_3\alpha^{-1})(\alpha a\beta)) \in L_{\Omega_{\gamma_0}}(RL)$$

since $(\beta^{-1}\xi_1)((s\xi_2 t)(\xi_3\alpha^{-1})(\alpha a\beta)) \in L$. The remaining cases for an element from $R \cap L$ include elements of the form αx, $x\beta$ or simply x, where $\alpha, \beta \in \Gamma$ and $x \in S$. These cases are dealt similarly as above. □

The following is straightforward.

Corollary 3.1. *Any of the characterizations of the regularity of an ordered* Γ*-semigroup given in* Theorem 8 *of* [2] *is logically equivalent to its corresponding characterization of the regularity of an ordered semigroup given in* Theorem 3.1 of [4].

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