Research Article

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Ramanujan's function $k(\tau) = r(\tau)r^2(2\tau)$ and its modularity

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Abstract: We study the modularity of Ramanujan's function $k(\tau) = r(\tau)r^2(2\tau)$, where $r(\tau)$ is the *Rogers-Ramanujan continued fraction*. We first find the modular equation of $k(\tau)$ of "an" level, and we obtain some symmetry relations and some congruence relations which are satisfied by the modular equations; these relations are quite useful for reduction of the computation cost for finding the modular equations. We also show that for some τ in an imaginary quadratic field, the value $k(\tau)$ generates the ray class field over an imaginary quadratic field modulo 10; this is because the function k is a generator of the field of the modular function on $\Gamma_1(10)$. Furthermore, we suggest a rather optimal way of evaluating the singular values of $k(\tau)$ using the modular equations in the following two ways: one is that if $j(\tau)$ is the elliptic modular function, then one can explicitly evaluate the value $k(\tau)$, and the other is that once the value $k(\tau)$ is given, we can obtain the value $k(r\tau)$ for any positive rational number r immediately.

Keywords: Ramanujan's function k, modular function, class field theory

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1 Introduction

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$, $\mathfrak{H}$ be the complex upper half plane, and $\mathfrak{H}^* := \mathbb{Q} \cup \{\infty\} \cup \mathfrak{H}$. We consider the compact Riemann surface $\Gamma\backslash\mathfrak{H}^*$ and the field $\mathcal{K}(\Gamma)$ of meromorphic functions which are invariant under Γ. If the congruence subgroup Γ has genus zero, there is a function $f(\tau)$ which satisfies $\mathcal{K}(\Gamma) = \mathbb{C}(f(\tau))$. For a positive integer N, we call the function $j^{(N)}(\tau)$ a *Hauptmodul of level N* if it is a generator for $\mathcal{K}(\Gamma_0(N))$ which has a simple pole at the cusp at infinity. Clearly, it is unique up to a constant. Note that the genus of $\Gamma_0(N)$ is zero only for $N = 1, 2, \ldots, 10, 12, 13, 16, 18$, and 25, and Table 1 describes Hauptmoduln $j^{(N)}(\tau)$ in terms of η-quotients for the levels $N \geq 2$ [1,2], where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$ with $q = e^{2\pi i\tau}$.

Table 1: Hauptmoduln of levels 2,…, 10, 12, 13, 16, 18, and 25

N	2	3	4	5	6	7	8
$j^{(N)}(\tau)$	$\frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}}$	$\frac{\eta(\tau)^{12}}{\eta(3\tau)^{12}}$	$\frac{\eta(\tau)^{8}}{\eta(4\tau)^{8}}$	$\frac{\eta(\tau)^{6}}{\eta(5\tau)^{6}}$	$\frac{\eta(2\tau)^3\eta(3\tau)^9}{\eta(\tau)^3\eta(6\tau)^9}$	$\frac{\eta(\tau)^{4}}{\eta(7\tau)^{4}}$	$\frac{\eta(\tau)^4\eta(4\tau)^2}{\eta(2\tau)^2\eta(8\tau)^4}$
N	9	10	12	13	16	18	25
$j^{(N)}(\tau)$	$\frac{\eta(\tau)^{3}}{\eta(9\tau)^{3}}$	$\frac{\eta(2\tau)\eta(5\tau)^5}{\eta(\tau)\eta(10\tau)^5}$	$\frac{\eta(4\tau)^4\eta(6\tau)^2}{\eta(2\tau)^2\eta(12\tau)^4}$	$\frac{\eta(\tau)^{2}}{\eta(13\tau)^{2}}$	$\frac{\eta(\tau)^2\eta(8\tau)}{\eta(2\tau)\eta(16\tau)^2}$	$\frac{\eta(6\tau)\eta(9\tau)^3}{\eta(3\tau)\eta(18\tau)^3}$	$\frac{\eta(\tau)}{\eta(25\tau)}$

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We consider the following equation:

$$\frac{1}{X_N} - X_N = j^{(N)}(\tau) + c_N \tag{1.1}$$

with a solution (X_N, c_N), where X_N is the modular function and c_N is a constant. Then the solutions $(r^5(\tau), 11)$, $(r_{13}(\tau), 3)$ and $(r(5\tau), 1)$ are found for $N = 5, 13$, and 25, respectively. In detail,

$$\frac{1}{r^5(\tau)} - r^5(\tau) = j^{(5)}(\tau) + 11, \tag{1.2}$$

$$\frac{1}{r_{13}(\tau)} - r_{13}(\tau) = j^{(13)}(\tau) + 3, \tag{1.3}$$

and

$$\frac{1}{r(5\tau)} - r(5\tau) = j^{(25)}(\tau) + 1, \tag{1.4}$$

where $r(\tau)$ is the *Rogers-Ramanujan continued fraction*

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} = q^{\frac{1}{5}} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{5}\right)},$$

and $r_{13}(\tau)$ is *the level 13 analogue of the Rogers-Ramanujan continued fraction*

$$r_{13}(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{13}\right)}$$

with the Legendre symbol $\left(\frac{n}{p}\right)$. Identities (1.2) and (1.4) were stated by Ramanujan [3, pp. 85 and 267] and proved by Watson [4], and Identity (1.3) was proved by Cooper and Ye [5] and Lee and Park [6].

On the other hand, Cooper [7, Theorem 3.5] showed that $(X_{10}, c_{10}) = (k(\tau), 0)$ is a solution to the following equation:

$$\frac{1}{X_{10}} - X_{10} = j^{(10)}(\tau) + c_{10} \tag{1.5}$$

by using the following identity:

$$k(\tau) = r(\tau) r^2(2\tau). \tag{1.6}$$

In fact, Ramanujan used the function $k(\tau)$ as a parameter for expressing $r^5(\tau)$ and $r^5(2\tau)$ in [8, p. 326]:

$$\text{if } r^5(\tau) = k\left(\frac{1-k}{1+k}\right)^2, \quad \text{then } r^5(2\tau) = k^2\left(\frac{1+k}{1-k}\right).$$

Therefore, it is clear that there is a significant meaning of the function $k(\tau)$, but there has been no investigation yet for $k(\tau)$ as a modular function. It is thus certainly worthy of studying the modularity of the function $k(\tau)$. This is one of the motivations for this paper: we study the modularity of Ramanujan's function $k(\tau) = r(\tau)r^2(2\tau)$.

The field of modular functions on $\Gamma(5)$ and $\Gamma_0(13)$ is generated by $r(\tau)$ and $1/r_{13}(\tau) - r_{13}(\tau)$, respectively. Koo and Shin found all the generators of $\Gamma_1(N)$ whose genus is zero [9, p. 161], and they chose $1/k(\tau)$ as a generator of $\Gamma_1(10)$. Therefore, the modular function $k(\tau)$ is a generator of the function field $\mathcal{K}(\Gamma_1(10))$. Furthermore, using (1.5), we see that $1/k(\tau) - k(\tau)$ is a generator of $\mathcal{K}(\Gamma_0(10))$. One can get the relation between $k(\tau)$ and $k(n\tau)$ for any positive integer n using the previous facts as above. For $\tau' = \tau + 1/2$, $2\tau + 1/2$, 3τ, and 5τ, the relations of this type were found by Xia and Yao [10]. These relations are called *modular equations* which satisfy Kronecker's congruence.

In this paper, we investigate the modularity of Ramanujan's function $k(\tau) = r(\tau)r^2(2\tau)$. We first find the modular equation of $k(\tau)$ of "any" level, and we obtain some symmetry relations and some congruence relations which are satisfied by the modular equations; these relations are quite useful for reduction of the computation cost for finding the modular equations (Theorem 1.1). We also show that for some τ in an imaginary quadratic field, the value $k(\tau)$ generates the ray class field over an imaginary quadratic field modulo 10; this is because the function k is a generator of the field of the modular function on $\Gamma_1(10)$ (Theorem 1.2). Furthermore, we suggest a rather optimal way of evaluating the singular values of $k(\tau)$ using the modular equations in the following two ways: one is that if $j(\tau)$ is the elliptic modular function, then one can explicitly evaluate the value $k(\tau)$ (Theorem 1.3), and the other is that once the value $k(\tau)$ is given, we can obtain the value $k(r\tau)$ for any positive rational number r immediately (Theorem 1.4). For any congruence subgroup Γ such that $\Gamma_1(10) \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, any function $f \in \mathcal{K}(\Gamma)$ can be written as the rational function of $k(\tau)$ since $k(\tau)$ is the generator of $\mathcal{K}(\Gamma_1(10))$. For example, combining the first formula in [11, Corollary 3.40] with (1.5), we get the same formula in Theorem 1.3. In Theorem 1.3, we focus on finding the explicit relation of $k(\tau)$ with the modular j-function directly. If one is interested in expressing $j(d\tau)$ in terms of $k(\tau)$ for $d = 1, 2, 5, 10$, then one can use the result in [11, Theorems 10.8 and 10.13] in order to write $\eta(d\tau)$ and $Q(q^d)$ in terms of $k(\tau)$ since $j(\tau) = Q^3(q)/\eta^{24}(\tau)$, where $Q(q) = 1/240 + \sum_{m=1}^{\infty} m^3q^m/(1-q^m)$.

Now we state four theorems, which are our main results in detail:

Theorem 1.1. *Let $k(\tau)$ be a solution to* (1.1) *for $N = 10$ and $c_{10} = 0$.*

(1) *One can explicitly obtain the modular equation of $k(\tau)$ of level n for any positive integer n.*
(2) *For every positive integer n with $(n, 10) = 1$, the modular equation $F_n(X, Y)$ of $k(\tau)$ has the following symmetry:*

$$\begin{cases} F_n(X,Y) = F_n(Y,X) & \text{if } n \equiv \pm 1 \pmod{10}, \\ F_n(X,Y) = Y^d F_n(-Y^{-1}, X) & \text{if } n \equiv \pm 3 \pmod{10}, \end{cases}$$

where $d = n\prod_{p\mid n}(1+p^{-1})$.

(3) *For any odd prime $p \neq 5$, the modular equation $F_p(X, Y)$ of $k(\tau)$ of level p is congruent to*

$$\begin{cases} (X^p - Y)(X - Y^p) \mod p\mathbb{Z}[X,Y] & \text{if } p \equiv \pm 1 \pmod{10}, \\ (X^p - Y)(XY^p + 1) \mod p\mathbb{Z}[X,Y] & \text{if } p \equiv \pm 3 \pmod{10}. \end{cases}$$

Using the modularity, one can construct some ray class fields over an imaginary quadratic field. Theorem 1.2 shows that $k(\tau)$ generates the ray class field modulo 10 over an imaginary quadratic field.

Theorem 1.2. *Let K be an imaginary quadratic field with discriminant d_K and $\tau \in K \cap \mathfrak{H}$ be a root of the primitive equation $ax^2 + bx + c = 0$ such that $b^2 - 4ac = d_K$ and $(a, 10) = 1$, where $a, b, c \in \mathbb{Z}$. Then $K(k(\tau))$ is the ray class field modulo* 10 *over K.*

It is mentioned that the value $k(\tau)$ is a unit for an imaginary quadratic quantity $\tau \in \mathfrak{H}$ [9, Theorem 6.7]. We prove that this value $k(\tau)$ can be written in the following two ways. Theorem 1.3 shows that we can evaluate the value $k(\tau)$ by using its relation with $j(\tau)$, and the other way is given in Theorem 1.4.

Theorem 1.3. *Let k be the modular function $k(\tau)$. Then $j(\tau) = G_1(1/k - k)^3/G_2(1/k - k)$, where*

$$G_1(X) = X^6 + 236X^5 + 1{,}440X^4 + 1{,}920X^3 + 3{,}840X^2 + 256X + 256,$$
$$G_2(X) = X^2(X+1)^5(X-4)^{10}$$

and $j(\tau)$ is a generator of $\mathcal{K}(SL_2(\mathbb{Z}))$ with q-expansion $q^{-1} + 744 + 196{,}884q + O(q^2)$.

Theorem 1.4. *When $k(\tau)$ is expressed in terms of radicals, we can express $k(r\tau)$ in terms of radicals for any positive rational number r.*

We present some examples (Examples 4.4 and 4.5) for Theorems 1.3 and 1.4, respectively, in Section 4.

This paper is organized as follows. We briefly mention some definitions and lemmas which are used to prove our theorems in Section 2. In Section 3, we focus on the modular equations of $k(\tau)$. We use the fact that $k(\tau)$ generates the field of modular functions on $\Gamma_1(10)$, and we find the modular equations which satisfy Kronecker's congruence (Theorem 1.1). It is possible to reduce the computation cost for finding modular equations by using useful properties for the coefficients $C_{i,j}$'s of modular equations given in Theorem 1.1 (2) and Proposition 3.5. Modular equations of levels 2, 3, 4, 5, 7, 8, 11, and 19 are found, and they are presented in Appendix A. The proofs of Theorems 1.2–1.4 are given in Section 4. First of all, the value $k(\tau)$ generates the ray class field modulo 10 over an imaginary quadratic field (Theorem 1.2). Some relation between $k(\tau)$ and $j(\tau)$ is obtained (Theorem 1.3), and it is used to get the value $k(\tau)$ in terms of radicals (Theorem 1.4). Using the modular equations, we find an explicit method for evaluating the value $k(\tau)$ in terms of radicals. Some calculations in this paper are performed by the MAPLE program.

2 Preliminaries

Ramanujan's function $k(\tau)$ can be written as an infinite product

$$k(\tau) = q\prod_{n=1}^{\infty}\frac{(1-q^{10n-9})(1-q^{10n-8})(1-q^{10n-2})(1-q^{10n-1})}{(1-q^{10n-7})(1-q^{10n-6})(1-q^{10n-4})(1-q^{10n-3})}$$

by (1.6) and

$$r(\tau) = q^{1/5}\prod_{n=1}^{\infty}\frac{(1-q^{5n-4})(1-q^{5n-1})}{(1-q^{5n-3})(1-q^{5n-2})}.$$

Hence, in [9, Table 2, p. 161] the generator of $\mathcal{K}(\Gamma_1(10))$ is $G_{10}(\tau) = 1/k(\tau)$, and we get the following:

Proposition 2.1. *The field $\mathcal{K}(\Gamma_1(10))$ is generated by $k(\tau)$, that is,*

$$\mathcal{K}(\Gamma_1(10)) = \mathbb{C}(k(\tau)).$$

We assume that two modular functions $f_1(\tau)$ and $f_2(\tau)$ satisfy the relation $F(f_1(\tau), f_2(\tau)) = 0$, where $F(X, Y)$ is a two-variable polynomial with complex coefficients. The following lemma which is proved by Ishida and Ishii [12] provides some information on the coefficients of $F(X, Y)$.

Lemma 2.2. *For any congruence subgroup Γ', let $f_1(\tau)$ and $f_2(\tau)$ be nonconstants such that $\mathbb{C}(f_1(\tau), f_2(\tau)) = \mathcal{K}(\Gamma')$ with the total degree D_k of poles of $f_k(\tau)$ for $k = 1, 2$. Let*

$$F(X, Y) = \sum_{\substack{0\le i\le D_2\\ 0\le j\le D_1}} C_{i,j}X^iY^j \in \mathbb{C}[X, Y]$$

be such that $F(f_1(\tau), f_2(\tau)) = 0$. Let $S_{\Gamma'}$ be a set of all the inequivalent cusps of Γ', and for $k = 1, 2$,

$$S_{k,0} = \{s \in S_{\Gamma'} : f_k(\tau) \text{ has zeros at } s\},$$

$$S_{k,\infty} = \{s \in S_{\Gamma'} : f_k(\tau) \text{ has poles at } s\},$$

$$a = -\sum_{s\in S_{1,\infty}\cap S_{2,0}} \operatorname{ord}_s f_1(\tau) \quad\text{and}\quad b = \sum_{s\in S_{1,0}\cap S_{2,0}} \operatorname{ord}_s f_1(\tau).$$

We assume that a (resp. b) is zero if $S_{1,\infty}\cap S_{2,0}$ (resp. $S_{1,0}\cap S_{2,0}$) is empty. Then we obtain the following assertions:

(1) $C_{D_2,a} \neq 0$. If further $S_{1,\infty} \subset S_{2,\infty}\cup S_{2,0}$, then $C_{D_2,j} = 0$ for any $j \neq a$.

(2) $C_{0,b} \neq 0$. If further $S_{1,0} \subset S_{2,\infty}\cup S_{2,0}$, then $C_{0,j} = 0$ for any $j \neq b$.

If we interchange the roles of f_1 and f_2, then we obtain further properties similar to (1) and (2).

3 Modular equations: Proof of Theorem 1.1

Let $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ for positive integers N and m. We denote the subgroup Δ by

$$\Delta := \{\overline{\pm(1+Nk)} \in (\mathbb{Z}/mN\mathbb{Z})^\times : k = 0, \ldots, m-1\}.$$

We understand that $\pm 1/0 = \infty$. Then we use the following lemmas to obtain useful properties.

Lemma 3.1. *Let $a, c, a', c' \in \mathbb{Z}$ be such that $(a, c) = (a', c') = 1$. Then with the notation Δ as above, a/c and a'/c' are equivalent under $\Gamma_1(N) \cap \Gamma_0(mN)$ if and only if there exist $\bar{s} \in \Delta \subset (\mathbb{Z}/mN\mathbb{Z})^\times$ and $n \in \mathbb{Z}$ such that $(a', c') \equiv (\bar{s}^{-1}a + nc, \bar{s}c) \pmod{mN}$.*

Proof. See [13, Lemma 1]. □

Let $\mathcal{K}(\Gamma)_{\mathbb{Q}}$ be the subfield of $\mathcal{K}(\Gamma)$ which consists of all modular functions whose Fourier coefficients are in $\mathbb{Q}$.

Lemma 3.2. *Let n be a positive integer. Then we have*

$$\mathbb{Q}(k(\tau), k(n\tau)) = \mathcal{K}(\Gamma_1(10) \cap \Gamma_0(10n))_{\mathbb{Q}}.$$

Proof. For a convenience, let $\Gamma := \Gamma_1(10) \cap \Gamma_0(10n)$. Since $\mathbb{Q}(k(\tau)) = \mathcal{K}(\Gamma_1(10))_{\mathbb{Q}}$, we know that for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, $k(\alpha\tau) = k(\tau)$ if and only if $\alpha \in \mathbb{Q}^\times \cdot \Gamma_1(10)$. Let $\beta = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$. Note that

$$\Gamma_1(10) \cap \beta^{-1}\Gamma_1(10)\beta = \Gamma_1(10) \cap \Gamma_0(10n) = \Gamma.$$

Hence, we get $k(\tau), k(n\tau) \in \mathcal{K}(\Gamma)_{\mathbb{Q}}$. Now we show that $\mathbb{Q}(k(\tau), k(n\tau))$ contains $\mathcal{K}(\Gamma)_{\mathbb{Q}}$. We choose $M_i \in \Gamma_1(10)$ satisfying

$$\Gamma_1(10) = \cup_i \Gamma M_i, \tag{3.1}$$

which is a disjoint union.

Let $f(\tau)$ be $k(n\tau) = (k \circ \beta)(\tau)$. For distinct indices i and j, we assume that $f \circ M_i = f \circ M_j$. Then $k \circ \beta \circ M_i = k \circ \beta \circ M_j$; so, $k \circ \beta M_i M_j^{-1} \beta^{-1} = k$. This means that $\beta M_i M_j^{-1}\beta^{-1} \in \mathbb{Q}^\times \cdot \Gamma_1(10)$ and $M_i M_j^{-1} \in \beta^{-1}\Gamma_1(10)\beta$. Since $M_i M_j^{-1} \in \Gamma_1(10)$, $M_i M_j^{-1} \in \Gamma_1(10) \cap \beta^{-1}\Gamma_1(10)\beta = \Gamma$, which is a contradiction to (3.1). Therefore, all functions $f \circ M_i$ are distinct with distinct indices and $\mathbb{C}(k(\tau), k(n\tau)) = \mathcal{K}(\Gamma_1(10) \cap \Gamma_0(10n))_{\mathbb{Q}}$. □

The following lemma tells us the behavior of $k(\tau)$.

Lemma 3.3. *Let $a, c, a', c' \in \mathbb{Z}$. Then the functions $k(\tau)$ and $k(n\tau)$ have the following properties:*
(1) *$k(\tau)$ has a pole at $a/c \in \mathbb{Q} \cup \{\infty\}$ with $(a, c) = 1$ if and only if $(a, c) = 1$, $a \equiv \pm 3 \pmod{10}$, and $c \equiv 0 \pmod{10}$.*
(2) *$k(n\tau)$ has a pole at $a'/c' \in \mathbb{Q} \cup \{\infty\}$ if and only if there exist $a, c \in \mathbb{Z}$ such that $a/c = na'/c'$, $(a, c) = 1$, $a \equiv \pm 3 \pmod{10}$, and $c \equiv 0 \pmod{10}$.*
(3) *$k(\tau)$ has a zero at $a/c \in \mathbb{Q} \cup \{\infty\}$ with $(a, c) = 1$ if and only if $(a, c) = 1$, $a \equiv \pm 1 \pmod{10}$, and $c \equiv 0 \pmod{10}$.*
(4) *$k(n\tau)$ has a zero at $a'/c' \in \mathbb{Q} \cup \{\infty\}$ if and only if there exist $a, c \in \mathbb{Z}$ such that $a/c = na'/c'$, $(a, c) = 1$, $a \equiv \pm 1 \pmod{10}$, and $c \equiv 0 \pmod{10}$.*

Proof. First we prove that $k(\tau)$ has a simple zero at ∞ and a simple pole at $3/10$. From Table 1, $j^{(10)}(\tau)$ has a pole at x if and only if $k(\tau)$ has a pole or zero at x. Since $j^{(10)}(\tau)$ has a pole at ∞, if $\mathrm{ord}_x k(\tau)$ is nonzero, then x is equivalent to ∞ under $\Gamma_0(10)$. Among the elements of $S_{\Gamma_1(10)}$, ∞ and $3/10$ are equivalent under $\Gamma_0(10)$ by Lemma 3.1. It is easy to see that $\mathrm{ord}_\infty k(\tau) = 1$, thus we know that $\mathrm{ord}_{3/10} k(\tau) \le 0$. Since $\sum_{x \in \Gamma_1(10)\backslash\mathfrak{H}^*} \mathrm{ord}_x k(\tau) = 0$, $\mathrm{ord}_{3/10} k(\tau) = -1$.

By Lemma 3.1, $k(\tau)$ has a simple pole at a/c if and only if $(a,c) \equiv \pm(3,0) \pmod{10}$. Similarly, $k(\tau)$ has a simple zero at a/c if and only if $(a,c) \equiv \pm(1,0) \pmod{10}$. Hence, we proved (1) and (3).

We easily obtain (2) and (4) by (1) and (3). ☐

Hereafter, we fix the sets $S_{\Gamma_0(10)}$ and $S_{\Gamma_1(10)}$ of inequivalent cusps of $\Gamma_0(10)$ and $\Gamma_1(10)$, respectively, as follows:

$$S_{\Gamma_0(10)} := \left\{\infty, 0, \frac{1}{2}, \frac{1}{5}\right\}$$

and

$$S_{\Gamma_1(10)} := \left\{\infty, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \frac{3}{5}, \frac{3}{10}\right\}.$$

Before proving Theorem 1.1 we check the action of $\Gamma_0(10)$ on the function $k(\tau)$ in the following lemma.

Lemma 3.4. *Let $\gamma \in \Gamma_0(10) - \Gamma_1(10)$. Then $k \circ \gamma(\tau) = -1/k(\tau)$.*

Proof. Note that $\mathcal{K}(\gamma^{-1}\Gamma_1(10)\gamma) = \mathcal{K}(\Gamma_1(10))$. Since

$$\mathbb{C}((k \circ \gamma)(\tau)) = \mathcal{K}(\gamma^{-1}\Gamma_1(10)\gamma) = \mathcal{K}(\Gamma_1(10)) = \mathbb{C}(k(\tau)),$$

there are four constants a, b, c, and d satisfying that

$$(k \circ \gamma)(\tau) = \frac{ak(\tau)+b}{ck(\tau)+d} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \pm I.$$

By Lemma 3.3 (1), $k(\tau)$ has a pole at 3/10 and the q-expansion of $(k \circ \gamma)(\tau) = q^{-1} + \cdots$. This means that we may assume that $a = 0$, $c = 1$, and $b \neq 0$. Then

$$\begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix}^2 = \begin{pmatrix} b & bd \\ d & d^2+b \end{pmatrix} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so we have $d = 0$ and $b = \pm 1$. Hence, $k \circ \gamma = \pm 1/k$. Suppose that $k \circ \gamma = 1/k$. Using $\gamma \in \Gamma_0(10)$ and (1.5),

$$j^{(10)}(\tau) = (j^{(10)} \circ \gamma)(\tau) = \left(\left(\frac{1}{k} - k\right) \circ \gamma\right)(\tau) = k(\tau) - \frac{1}{k(\tau)} = -j^{(10)}(\tau);$$

this is a contradiction to the fact that $j^{(10)}(\tau)$ is a generator of the field of modular functions on $\Gamma_0(10)$. Therefore, $(k \circ \gamma)(\tau) = -1/k(\tau)$. ☐

Until the proof of Theorem 1.1 ends, assume that n is a positive integer relatively prime to 10. We take $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \pmod{10}$ for any integer a relatively prime to 10. Then by [14, Proposition 3.36] we have a disjoint union such as

$$\Gamma_1(10)\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Gamma_1(10) = \bigcup_{0<a|n} \bigcup_{\substack{0 \le b < n/a \\ (a,b,n/a)=1}} \Gamma_1(10)\sigma_a\begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix}.$$

Let $d = n\prod_{p|n}(1+p^{-1})$ for prime p dividing n. Then $[\Gamma_1(10)\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Gamma_1(10) : \Gamma_1(10)] = d$. It is clear that σ_a depends on $a \pmod{10}$, and so we may take σ_a as follows:

$$\sigma_{\pm 1} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{\pm 3} = \pm\begin{pmatrix} -3 & -10 \\ 10 & 33 \end{pmatrix}.$$

Then $k \circ \sigma_{\pm 1} = k$ and $k \circ \sigma_{\pm 3} = -1/k$ by Lemma 3.4.

We denote $\alpha_{a,b} := \sigma_a \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix}$ for integers a and b such that a is a positive divisor of n, $0 \le b < n/a$, and $(a, b, n/a) = 1$. Consider the polynomial $\Psi_n(X, \tau)$ having an indeterminate X

$$\Psi_n(X, k(\tau)) := \prod_{0<a|n} \prod_{\substack{0\le b<n/a \\ (a,b,n/a)=1}} (X - (k \circ \alpha_{a,b})(\tau)).$$

We just write $\Psi_n(X, k(\tau))$ instead of $\Psi_n(X, \tau)$ because $\Psi_n(X, \tau) \in \mathbb{C}(k(\tau))[X]$ from the fact that all the coefficients of $\Phi_n(X, \tau)$ are elementary symmetric functions of $k \circ \alpha_{a,b}$ (i.e., they are invariant under the action of $\Gamma_1(10)$).

Let $S_{m,0}$ be the set of cusps which $k(m\tau)$ has a zero and $S_{m,\infty}$ is the set of cusps which $k(m\tau)$. We denote by r_n the nonnegative integer

$$r_n := -\sum_{s\in S_{1,\infty}\cap S_{n,0}} \mathrm{ord}_s k(\tau),$$

where $S_{m,0}$ is the set of cusps which $k(m\tau)$ has a zero and $S_{m,\infty}$ is the set of cusps which $k(m\tau)$. Here we consider $k(\tau)$ as a function on $\mathcal{K}(\Gamma_1(10) \cap \Gamma_0(10n))$ and $r_n = 0$ if the set $S_{n,0} \cap S_{1,\infty}$ is empty. Then define the polynomial

$$F_n(X, k(\tau)) = k(\tau)^{r_n} \Psi_n(X, k(\tau)),$$

that is, $F_n(X, Y) = Y^{r_n} \Psi_n(X, Y)$.

Now we are ready to prove our first theorem.

Proof of Theorem 1.1. Let $F_n(X, Y)$ be the polynomial defined as before. Then it is easy to see that $F_n(X, Y) \in \mathbb{Z}[X, Y]$, $\deg_X F_n(X, Y) = n\prod_{p|n}(1 + p^{-1})$, and $F_n(X, Y)$ is irreducible as a polynomial in X (resp. Y) over $\mathbb{C}(Y)$ (resp. $\mathbb{C}(X)$); this is obtained in a similar way to that of [12, Theorem 10], so we omit the proof.

(1) First, we show that the modular equations of $k(\tau)$ exist for all levels. Let $\mathbb{C}(f_1(\tau), f_2(\tau))$ be the field of modular functions on a certain congruence subgroup Γ for nonconstants $f_1(\tau)$ and $f_2(\tau)$. The degree $[\mathbb{C}(f_1(\tau), f_2(\tau)) : \mathbb{C}(f_1(\tau))]$ (resp. $[\mathbb{C}(f_1(\tau), f_2(\tau)) : \mathbb{C}(f_2(\tau))]$) of field of extension is the total degree d_1 (resp. d_2) of poles $f_1(\tau)$ (resp. $f_2(\tau)$) in Riemann surface $\Gamma\backslash\mathfrak{H}^*$.

Assume that $f_1(\tau) = k(\tau)$ and $f_2(\tau) = k(n\tau)$ for any positive integer n. By Lemma 3.2, $\mathbb{C}(f_1(\tau), f_2(\tau)) = \mathcal{K}(\Gamma_1(10) \cap \Gamma_0(10n))$. From Lemma 2.2, one can have a polynomial $F(X, Y)$ such that $F(k(\tau), k(n\tau)) = 0$, $\deg_X F(X, Y)$ is the total degree of poles $k(n\tau)$, and $\deg_Y F(X, Y)$ is the total degree of poles of $k(\tau)$. So we obtain the modular equation $F(X, Y)$ of $k(\tau)$ of level n.

(2) Assume that $n \equiv \pm1 \pmod{10}$. As before, $F_n(X, k(\tau))$ is an irreducible polynomial in X over $\mathbb{C}(k(\tau))$ with $F_n(k(\tau/n), k(\tau)) = 0$. Using $\Psi_n(k(n\tau), k(\tau)) = 0$, it is easy to see that $k(\tau/n)$ is a root of $F_n(k(\tau), X) \in \mathbb{Z}[X, k(\tau)]$. Hence, there is a polynomial $G(X, k(\tau)) \in \mathbb{Z}[X, k(\tau)]$ such that $F_n(k(\tau), X) = G(X, k(\tau))F_n(X, k(\tau))$. By interchanging the places of X and $k(\tau)$, we have that $F_n(X, k(\tau)) = G(k(\tau), X)F_n(k(\tau), X)$ and $F_n(k(\tau), X) = G(X, k(\tau))G(k(\tau), X)F_n(k(\tau), X)$. So $G(X, k(\tau))$ should be ±1. Suppose that $G := G(X, k(\tau)) = -1$. Then $F_n(X, k(\tau)) + F_n(k(\tau), X) = 0$. When $X = k(\tau)$, $F_n(k(\tau), k(\tau)) = 0$ and $k(\tau)$ is a root of $F_n(X, k(\tau))$. Since $X - k(\tau)$ divides $F_n(X, k(\tau))$, $F_n(X, k(\tau))$ is not irreducible by considering $d > 1$; this is a contradiction to the irreducibility of $F_n(X, Y)$. Hence, we get

$$F_n(X, Y) = F_n(Y, X).$$

Now assume that $n \equiv \pm3 \pmod{10}$. We already know that $\Psi_n(-1/k(n\tau), k(\tau)) = 0$ and $\Psi_n(-1/k(\tau), k(\tau/n)) = 0$. Then $k(\tau)^d F_n(-1/k(\tau), X) \in \mathbb{Z}[X, k(\tau)]$ also has a root $k(\tau/n)$. Hence, $k(\tau)^d F_n(-1/k(\tau), X)$ is written as follows:

$$k(\tau)^d F_n\left(-\frac{1}{k(\tau)}, X\right) = G(X, k(\tau))F_n(X, k(\tau))$$

for some $G(X, k(\tau)) \in \mathbb{Z}[X, k(\tau)]$ by using the fact that $F_n(X, k(\tau))$ is an irreducible polynomial with root $k(\tau/n)$.

Then

$$\deg_X F_n(X, Y) + \deg_X G(X, Y) = \deg_X Y^d F_n\left(-\frac{1}{Y}, X\right) = \deg_Y F_n(X, Y)$$

and

$$\deg_Y F_n(X, Y) + \deg_Y G(X, Y) = \deg_Y Y^d F_n\left(-\frac{1}{Y}, X\right) = \deg_X F_n(X, Y),$$

where the second identity is obtained because $Y^d F_n(-1/Y, X)$ is written as

$$\frac{1}{C_{d_n,r}}((-1)^{d_n} C_{d_n,r} X^r Y^{d-d_n} + C_{0,s} X^s Y^d + (-1)^{r'} C_{r',d_1} X^{d_1} Y^{d-r'} + (-1)^{s'} C_{s',0} Y^{d-s'} + \text{(lower degree terms)}),$$

where d_m is the total degrees of pole of $k(m\tau)$, $r := r_n$,

$$r' := -\sum_{s \in S_{n,\infty} \cap S_{1,0}} \operatorname{ord}_s k(n\tau), \quad s := \sum_{s \in S_{1,0} \cap S_{n,0}} \operatorname{ord}_s k(\tau), \quad \text{and} \quad s' := \sum_{s \in S_{1,0} \cap S_{n,0}} \operatorname{ord}_s k(n\tau).$$

Hence, $\deg_X G(X, Y) + \deg_Y G(X, Y) = 0$ and $G := G(X, Y)$ is constant. Moreover, we obtain that $d = \deg_X F_n(X, Y) = \deg_Y F_n(X, Y)$. By using that $F_n(X, Y)$ is a primitive polynomial, $G = \pm 1$. Since $F_n(-1/Y, X) = G \cdot Y^{-d} F_n(X, Y)$,

$$F_n(X, Y) = G(-X)^d F_n\left(Y, -\frac{1}{X}\right) = G(-X)^d (Y^d(-X)^{-r} + \cdots) = G(-1)^{d-r} X^{d-r} Y^d + \cdots.$$

Noting that $\Psi_n(X, k(\tau))$ can be written as the following product:

$$\prod_{\substack{0<a|n \\ a \equiv \pm 1 \pmod{10}}} \prod_{\substack{0 \le b < n/a \\ (a,b,n/a)=1}} \left(X - \zeta_n^{ab} q^{a^2/n} + \cdots\right) \prod_{\substack{0<a|n \\ a \equiv \pm 3 \pmod{10}}} \prod_{\substack{0 \le b < n/a \\ (a,b,n/a)=1}} \left(X + \zeta_n^{-ab} q^{-a^2/n} + \cdots\right),$$

we can get that the coefficient of $X^{d-r} Y^d$ in $F_n(X, Y)$ is given by

$$\prod_{\substack{0<a|n \\ a \equiv \pm 3 \pmod{10}}} \prod_{\substack{0 \le b < n/a \\ (a,b,n/a)=1}} \zeta_n^{ab}. \tag{3.2}$$

Hence, (3.2) should be $(-1)^{d-r} G$. Denote by $\Pi\Pi$ the double product

$$\prod_{\substack{0<a|n \\ a \equiv \pm 3 \pmod{10}}} \prod_{\substack{0 \le b < n/a \\ (a,b,n/a)=1}} \cdot$$

By [15, Lemma 6.7], in (3.2) $\prod\prod \zeta_n^{ab} = 1$. Since $\prod\prod(-1) = (-1)^r$, we have $G = (-1)^d$. Hence,

$$F_n(X, Y) = Y^d F_n\left(-\frac{1}{Y}, X\right)$$

because n is odd and d is even.

(3) Now we focus on the congruence properties which the modular equation of $k(\tau)$ satisfies. Let p be an odd prime not 5. We denote $x \equiv y \pmod{\alpha}$ for $x, y \in R$ if $x - y \in \alpha R$.
Write $k(\tau) = \sum_{n=1}^{\infty} c_m a^m$ with integers c_m. Since

$$(k \circ \alpha_{1,b})(\tau) = \left(k \circ \sigma_1 \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}\right)(\tau) = k\left(\frac{\tau + b}{p}\right) = \sum_{m \ge 1} c_m \zeta_p^{bm} q^{\frac{m}{p}} \equiv \sum_{m \ge 1} c_m q^{\frac{m}{p}} = (k \circ \alpha_{1,0})(\tau) \pmod{1 - \zeta_p},$$

$$\prod_{b=0}^{p-1}(X-(k\circ\alpha_{1,b})(\tau))\equiv(X-(k\circ\alpha_{1,0})(\tau))^p\pmod{1-\zeta_p}$$
$$\equiv(X^p-(k\circ\alpha_{1,0})(\tau)^p)\pmod{1-\zeta_p}$$
$$\equiv(X^p-k(\tau))\pmod{1-\zeta_p};$$

in last congruence we use that

$$(k\circ\alpha_{1,0})(\tau)^p=k\left(\frac{\tau}{p}\right)^p=\left(\sum_{m\geq1}c_mq^{\frac{m}{p}}\right)^p\equiv\sum_{m\geq1}c_m^pq^m\equiv\sum_{m\geq1}c_mq^m\pmod{1-\zeta_p}.$$

On the other hand,

$$r=-\sum_{s\in S_{1,\infty}\cap S_{n,0}}\mathrm{ord}_sk(\tau)=\begin{cases}0 & \text{if } p\equiv\pm1\pmod{10},\\ p & \text{if } p\equiv\pm3\pmod{10}.\end{cases}$$

Hence,

$$\begin{aligned}F_p(X,k(\tau))&=k(\tau)^r\prod_{a=1,p}\prod_{0\leq b\leq p/a}(X-(k\circ\alpha_{a,b})(\tau))\\&\equiv k(\tau)^r(X^p-k(\tau))(X-(k\circ\alpha_{p,0})(\tau))\\&\equiv k(\tau)^r(X^p-k(\tau))(X-(k\circ\sigma_p)(p\tau))\\&\equiv\begin{cases}(X^p-k(\tau))(X-k(p\tau)) & \text{if } p\equiv\pm1\pmod{10},\\ k(\tau)^p(X^p-k(\tau))\left(X+\frac{1}{k(p\tau)}\right) & \text{if } p\equiv\pm3\pmod{10}\end{cases}\\&\equiv\begin{cases}(X^p-k(\tau))(X-k(\tau)^p) & \text{if } p\equiv\pm1\pmod{10},\\ (X^p-k(\tau))(k(\tau)^pX+1) & \text{if } p\equiv\pm3\pmod{10},\end{cases}\end{aligned}$$

where $a\equiv b$ means that $a\equiv b\pmod{1-\zeta_p}$, because $k(p\tau)=\sum c_mq^{pm}\equiv\sum c_m^pq^{pm}\equiv(\sum c_mq^m)^p=k(\tau)^p\pmod{p}$. We have that $1-\zeta_p$ divides $F_p(X,Y)-H_p(X,Y)$ in $\mathbb{Z}[\zeta_p]$ and $F_p(X,Y)\in\mathbb{Z}[X,Y]$, where

$$H_p(X,Y)=\begin{cases}(X^p-Y)(X-Y^p) & \text{if } p\equiv\pm1\pmod{10},\\ (X^p-Y)(XY^p+1) & \text{if } p\equiv\pm3\pmod{10}.\end{cases}$$

Our result follows since p is the smallest integer which is divisible by $1-\zeta_p$ in $\mathbb{Z}[\zeta_p]$. □

We study the property of $F_p(X,Y)$ of odd prime level $p\neq5$ in Proposition 3.5. It is helpful for finding the modular equations in practice; it reduces the computation cost.

Proposition 3.5. *Let p be an odd prime with $p\neq5$ and $F_p(X,Y)=\sum_{0\leq i,j\leq p+1}C_{i,j}X^iY^j$ be a modular equation of level p. Then $F_p(X,Y)$ satisfies the following:*
(1) *If $p\equiv\pm1\pmod{10}$, then*
  (a) $C_{p+1,0}\neq0$ *and* $C_{0,p+1}\neq0$,
  (b) $C_{p+1,j}=C_{j,p+1}=0$ *for* $j=1,2,\dots,p+1$,
  (c) $C_{0,j}=C_{j,0}=0$ *for* $j=0,1,\dots,p$.
(2) *If $p\equiv\pm3\pmod{10}$, then*
  (a) $C_{p+1,p}\neq0$, $C_{0,1}\neq0$, $C_{1,p+1}\neq0$ *and* $C_{p,0}\neq0$,
  (b) $C_{0,j}=C_{j,p+1}=0$ *for* $j=0,2,\dots,p+1$,
  (c) $C_{p+1,j}=C_{j,0}=0$ *for* $j=0,1,\dots,p-1,p+1$.

Proof. Let $\Gamma=\Gamma_1(10)\cap\Gamma_0(10p)$. Since all statements below also hold for the case $p=3$, we deal with the only case of the prime $p\geq7$.

Assume that the cusp a/c of Γ with $\mathrm{ord}_xk(\tau)$ or $\mathrm{ord}_x\,k(p\tau)$ is nonzero. Then c is a multiple of 10. Let $f_1(\tau)=k(\tau)$ and $f_2(\tau)=k(p\tau)$. Define the sets $S_{j,\infty}$ and $S_{j,0}$ for $j=1,2$ as Lemma 2.2. Then we know that

$$S_{1,\infty} = \left\{\frac{3}{10}, \frac{3}{10p}\right\}, \quad S_{1,0} = \left\{\frac{1}{10}, \frac{1}{10p}\right\}$$

and

$$S_{2,\infty} = \begin{cases} \left\{\frac{3}{10}, \frac{3}{10p}\right\} & \text{if } p \equiv \pm 1 \pmod{10}, \\ \left\{\frac{1}{10}, \frac{3}{10p}\right\} & \text{if } p \equiv \pm 3 \pmod{10}, \end{cases} \qquad S_{2,0} = \begin{cases} \left\{\frac{1}{10}, \frac{1}{10p}\right\} & \text{if } p \equiv \pm 1 \pmod{10}, \\ \left\{\frac{3}{10}, \frac{1}{10p}\right\} & \text{if } p \equiv \pm 3 \pmod{10}. \end{cases}$$

Moreover,

$$\mathrm{ord}_{3/10p} f_1(\tau) = -1, \quad \mathrm{ord}_{3/10} f_1(\tau) = -p, \quad \mathrm{ord}_{1/10p} f_1(\tau) = 1, \quad \mathrm{ord}_{1/10} f_1(\tau) = p.$$

We calculate $\mathrm{ord}_x f_2(\tau)$ $(x = 1/10, 3/10, 1/10p, 3/10p)$ as follows:

(1) $\mathrm{ord}_{1/10p} f_2(\tau) = \mathrm{ord}_\infty k(p\tau) = p.$

(2) $\mathrm{ord}_{3/10p} f_2(\tau) = \mathrm{ord}_q\left(k \circ (p\ \ 00\ \ 1)\begin{pmatrix} 3 & b \\ 10p & c \end{pmatrix}(\tau)\right) = \mathrm{ord}_q\left(k \circ \begin{pmatrix} 3 & bp \\ 10 & c \end{pmatrix}(p\tau)\right) = \mathrm{ord}_q \frac{1}{k(p\tau)} = -p,$

where b and c are integers such that $3c - 10bp = 1$.

(3) $\mathrm{ord}_{1/10} f_2(\tau) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{10}, \\ -1 & \text{if } p \equiv \pm 3 \pmod{10}, \end{cases}$

because the width of 1/10 is p and

$$\begin{aligned} f_2 \circ \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}(\tau) &= k \circ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}(\tau) \\ &= k \circ \begin{pmatrix} p & 0 \\ 10 & 1 \end{pmatrix}(\tau) = k \circ \begin{pmatrix} p & -t \\ 10 & y \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix}(\tau) \\ &= \begin{cases} k \circ \begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix}(\tau) = \zeta_p^t q^{1/p} + \cdots, & \text{if } p \equiv \pm 1 \pmod{10}, \\ 1/k \circ \begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix}(\tau) = \zeta_p^{-t} q^{-1/p} + \cdots & \text{if } p \equiv \pm 3 \pmod{10}, \end{cases} \end{aligned}$$

where $t, y \in \mathbb{Z}$ such that $10t + yp = 1$ and $0 \le t \le p - 1$.

(4) $\mathrm{ord}_{3/10} f_2(\tau) = \begin{cases} -1 & \text{if } p \equiv \pm 1 \pmod{10}, \\ 1 & \text{if } p \equiv \pm 3 \pmod{10}, \end{cases}$

because the width of 3/10 is p and

$$\begin{aligned} f_2 \circ \begin{pmatrix} 3 & 2 \\ 10 & 7 \end{pmatrix}(\tau) &= k \circ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 10 & 7 \end{pmatrix}(\tau) = k \circ \begin{pmatrix} 3p & 2p \\ 10 & 7 \end{pmatrix}(\tau) = k \circ \begin{pmatrix} 3p & 2 - 3t \\ 10 & y \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix}(\tau) \\ &= \begin{cases} 1/k \circ \begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix}(\tau) = \zeta_p^{-t} q^{-1/p} + \cdots & \text{if } p \equiv \pm 1 \pmod{10}, \\ k \circ \begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix}(\tau) = \zeta_p^t q^{1/p} + \cdots, & \text{if } p \equiv \pm 3 \pmod{10}, \end{cases} \end{aligned}$$

where $t, y \in \mathbb{Z}$ such that $10t + yp = 7$ and $0 \le t \le p - 1$.

Hence, the total degree of poles of $f_i(\tau)$ is $p + 1$ for $i = 1, 2$, and the modular equation $F_p(X, Y)$ of $k(\tau)$ of level p is written as:

$$F_p(X, Y) = \sum_{0 \le i,j \le p+1} C_{i,j} X^i Y^j.$$

Now we have the information on $S_{j,0}$ and $S_{j,\infty}$ for $j = 1, 2$ as follows:

(1) $S_{1,\infty} \cup S_{1,0} = S_{2,\infty} \cup S_{2,0}$.

(2) $S_{1,\infty} \cap S_{2,0} = \begin{cases} \phi & \text{if } p \equiv \pm 1 \pmod{10}, \\ \left\{\frac{3}{10}\right\} & \text{if } p \equiv \pm 3 \pmod{10}. \end{cases}$

(3) $S_{1,0} \cap S_{2,0} = \begin{cases} \left\{\frac{1}{10}, \frac{1}{10p}\right\} & \text{if } p \equiv \pm 1 \pmod{10}, \\ \left\{\frac{1}{10p}\right\} & \text{if } p \equiv \pm 3 \pmod{10}. \end{cases}$

Thus, we have

$$C_{p+1,0} \neq 0, \quad C_{0,p+1} \neq 0,$$
$$C_{p+1,1} = C_{p+1,2} = \cdots = C_{p+1,p+1} = 0,$$

and

$$C_{0,0} = C_{0,1} = \cdots = C_{0,p} = 0$$

if $p \equiv \pm 1 \pmod{10}$. Moreover,

$$C_{p+1,p} \neq 0, \quad C_{0,1} \neq 0,$$
$$C_{p+1,0} = \cdots = C_{p+1,p-1} = C_{p+1,p+1} = 0,$$

and

$$C_{0,0} = C_{0,2} = \cdots = C_{0,p+1} = 0$$

if $p \equiv \pm 3 \pmod{10}$.

By letting $f_1(\tau) = k(p\tau)$ and $f_2(\tau) = k(\tau)$ and repeating this similar calculation for $S_{j,\infty}$ and $S_{j,0}$, the result follows. □

We find the modular equations of levels 2 and 5 explicitly in Proposition 3.6 using the method presented in Theorem 1.1 (1).

Proposition 3.6. *Let $U := k(\tau)$, $V := k(2\tau)$, and $W := k(5\tau)$. Then*
(*1*) (*modular equation of level 2*)

$$U^2 - (1 - 2U - U^2)V + V^2 = 0.$$

(*2*) (*modular equation of level 5*)

$$\begin{aligned} U^5 &- (1 - 5U + 15U^3 - 2U^5)W - (2 - 5U - 15U^2 - 10U^3 + 5U^4 + U^5)W^2 \\ &+ (1 - 5U - 10U^2 + 15U^3 - 5U^4 - 2U^5)W^3 + (2 - 15U^2 + 5U^4 + U^5)W^4 - W^5. \end{aligned}$$

Proof.
(1) We consider $\Gamma_1(10) \cap \Gamma_0(20)$. If $a/c \in \mathbb{Q}$ satisfies $a \equiv \pm 3 \pmod{10}$ and $c \equiv 0 \pmod{20}$, then a/c is equivalent to either $3/10$ or $3/20$ under $\Gamma_1(10) \cap \Gamma_0(20)$. Similarly, if $a/c \in \mathbb{Q}$ satisfies $a \equiv \pm 1 \pmod{10}$ and $c \equiv 0 \pmod{20}$, then a/c is equivalent to either $1/10$ or $1/20$ under $\Gamma_1(10) \cap \Gamma_0(20)$. Moreover, the widths of $1/20$, $3/20$, $1/10$, and $3/10$ are all 1. Hence, we have

$$\mathrm{ord}_{1/10} k(\tau) = \mathrm{ord}_{1/20} k(\tau) = 1, \quad \mathrm{ord}_{3/10} k(\tau) = -1, \quad \mathrm{ord}_{3/20} k(\tau) = -1$$

and

$$\mathrm{ord}_{1/20} k(2\tau) = 2, \quad \mathrm{ord}_{3/20} k(2\tau) = -2.$$

By Lemma 2.2, the modular equation $F_2(X, Y)$ is given by

$$F_2(X, Y) = \sum_{0 \le i,j \le 2} C_{i,j} X^i Y^j.$$

Since there is no point x such that $\mathrm{ord}_x\, k(\tau) < 0$ and $\mathrm{ord}_x\, k(2\tau) > 0$, by Lemma 2.2 (1), $C_{2,0}$ is nonzero, so we may assume that $C_{2,0}$ is 1. By substituting $X := k(\tau)$ and $Y := k(2\tau)$ as q-expansions, we get that

$$C_{0,1} = -1, \quad C_{1,1} = 2, \quad C_{2,1} = 1, \quad C_{0,2} = 1.$$

(2) Under the group $\Gamma_1(1) \cap \Gamma_0(50)$, we may consider the points

$$x = \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{1}{20}, \frac{3}{20}, \frac{7}{20}, \frac{9}{20}, \frac{1}{50}, \frac{3}{50}$$

in $\mathbb{Q}$ such that $\mathrm{ord}_x\, k(\tau)$ is nonzero. Note that all widths of these points are 1. Hence,

$$\mathrm{ord}_{3/10} k(\tau) = \mathrm{ord}_{7/10} k(\tau) = \mathrm{ord}_{3/20} k(\tau) = \mathrm{ord}_{7/20} k(\tau) = \mathrm{ord}_{3/50} k(\tau) = -1,$$

$$\mathrm{ord}_{1/10} k(\tau) = \mathrm{ord}_{9/10} k(\tau) = \mathrm{ord}_{1/20} k(\tau) = \mathrm{ord}_{9/20} k(\tau) = \mathrm{ord}_{1/50} k(\tau) = 1,$$

and

$$\mathrm{ord}_{3/50} k(5\tau) = -5, \quad \mathrm{ord}_{1/50} k(5\tau) = 5.$$

Thus, we get the modular equation $F_5(X, Y) = \sum_{0 \le i,j \le 5} C_{i,j} X^i Y^j$ and $C_{5,0}$ is nonzero because there is no point x such that $\mathrm{ord}_x k(\tau) < 0$ and $\mathrm{ord}_x k(5\tau) > 0$ by Lemma 2.2 (1). When we put $C_{5,0} = 1$ and substitute $X := k(\tau)$ and $k(5\tau)$ in $F_5(X, Y)$, we conclude that

$$\begin{aligned} F_5(X, Y) = {} & X^5 - (1 - 5X + 15X^3 - 2X^5)Y - (2 - 5X - 15X^2 - 10X^3 + 5X^4 + X^5)Y^2 \\ & + (1 - 5X - 10X^2 + 15X^3 - 5X^4 - 2X^5)Y^3 + (2 - 15X^2 + 5X^4 + X^5)Y^4 - Y^5. \end{aligned}$$

□

4 Class fields and evaluations: Proofs of Theorems 1.2, 1.3, and 1.4

For an imaginary quadratic field K with discriminant d_K and positive integer N, let $K_{(N)}$ be the ray class field modulo N over K. Let $\tau \in K \cap \mathfrak{H}$ be a root of a primitive equation $ax^2 + bx + c = 0$ satisfying $b^2 - 4ac = d_K$, where a, b, and c are integers. In this section, we show that $k(\tau)$ generates $K_{(10)}$ over K, and we work on evaluation of $k(\tau)$.

Lemma 4.1. *Let K be an imaginary quadratic field with discriminant d_K and $\tau \in K \cap \mathfrak{H}$ be a root of a primitive equation $ax^2 + bx + c = 0$ such that $b^2 - 4ac = d_K$ and $a, b, c \in \mathbb{Z}$. Let Γ' be a congruence subgroup such that $\Gamma(N) \subset \Gamma' \subset \Gamma_1(N)$. Suppose that $(N, a) = 1$. Then the field generated over K by all the values $h(\tau)$, where $h \in \mathcal{K}(\Gamma')_{\mathbb{Q}}$ is defined and finite at τ, is the ray class field modulo N over K.*

Proof. See [16, Corollary 5.2]. □

Proof of Theorem 1.2. By Proposition 1.1, we have $\mathcal{K}(\Gamma_1(10))_{\mathbb{Q}} = \mathbb{Q}(k(\tau))$. Hence, $k(\tau)$ satisfies the condition of Lemma 4.1 when $N = 10$, and so we can conclude that for an imaginary quadratic field K with discriminant d_K, the field $K(k(\tau))$ is the ray class field modulo 10 over K when $\tau \in K \cap \mathfrak{H}$ satisfies $a\tau^2 + b\tau + c = 0$, $b^2 - 4ac = d_K$, and $(a, 10) = 1$. □

Now we evaluate $k(\tau)$ for an imaginary quadratic quantity τ by proving Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Let $j^{(5)}$ and $j^{(10)}$ be the Hauptmoduln which are given in Table 1. We claim that the following two identities are true:

$$j^{(5)}(\tau) = \frac{j^{(10)}(\tau)^3 - 7j^{(10)}(\tau)^2 + 8j^{(10)}(\tau) + 16}{j^{(10)}(\tau)^2} \tag{4.1}$$

and

$$j(\tau) = \frac{(j^{(5)}(\tau)^2 + 250j^{(5)}(\tau) + 3{,}125)^3}{j^{(5)}(\tau)^5}. \tag{4.2}$$

First we show (4.1). Since $[\Gamma_0(5) : \Gamma_0(10)] = [\Gamma_0(1) : \Gamma_0(10)]/[\Gamma_0(1) : \Gamma_0(5)] = 18/6 = 3$, a Hauptmodul $j^{(5)}(\tau)$ of level 5 is written as the fraction $P_1(j^{10}(\tau))/P_2(j^{(10)}(\tau))$, where $P_1(X)$ and $P_2(X)$ are polynomials and $\max\{\deg_X P_1(X), \deg_X P_2(X)\} = 3$. Note that both $j^{(10)}(\tau)$ and $j^{(5)}(\tau)$ have a simple pole at ∞. Hence $\deg_X P_1(X) > \deg_X P_2(X)$ and $\deg_X P_1(X) = 3$. This implies that there is a polynomial $A(X, Y) = P_2(X)Y - P_1(X)$ such that $A(j^{(10)}(\tau), j^{(5)}(\tau)) = 0$, $P_2(X) = a_2X^2 + a_1X + a_0$, and $P_1(X) = a_3'X^3 + a_2'X^2 + a_1'X + a_0'$.

Since

$$j^{(10)}(\tau) = q^{-1} + 1 + q + 2q^2 + 2q^3 - 2q^4 - q^5 - 4q^7 - 2q^8 + O(q^9)$$

and

$$j^{(5)}(\tau) = q^{-1} - 6 + 9q + 10q^2 - 30q^3 + 6q^4 - 25q^5 + 96q^6 + 60q^7 - 250q^8 + O(q^9),$$

by substituting the q-expansions of $X := j^{(5)}(\tau)$ and $Y := j^{(10)}(\tau)$ to $A(X, Y)$, we get the relation between $j^{(10)}(\tau)$ and $j^{(5)}(\tau)$.

Now we show (4.2). Note that $[\mathbb{C}(j^{(5)}(\tau)) : \mathbb{C}(j(\tau))] = [\mathcal{K}(\Gamma_0(1)) : \mathcal{K}(\Gamma_0(5))] = 6$; so, we can take the polynomials of degree less than or equal to 6 similar to $P_1(X)$ and $P_2(X)$ in the case of (4.1). Since

$$j(\tau) = q^{-1} + 744 + 196{,}884q + 21{,}493{,}760q^2 + 864{,}299{,}970q^3 + 20{,}245{,}856{,}256q^4 + 333{,}202{,}640{,}600q^5 + O(q^6),$$

(4.2) follows.

Combining (1.5), (4.1), and (4.2), we obtain our assertion. □

Remark 4.2. Identity (4.1) can be obtained by using the third formula in [11, Theorem 10.5]

$$\frac{\eta(5\tau)^6}{\eta(\tau)^6} = \left(\frac{k}{1 + k - k^2}\right)\left(\frac{1 - k^2}{1 - 4k - k^2}\right)^2$$

and (1.5) because $j^{(5)}(\tau) = \eta(\tau)^6/\eta(5\tau)^6$. The function field $\mathcal{K}(\Gamma_0(10))$ is generated by $j^{(10)}(\tau)$ and $\mathcal{K}(\Gamma_0(5)) \subset \mathcal{K}(\Gamma_0(10))$. Thus, $j^{(5)}(2\tau)$ is contained in $\mathcal{K}(\Gamma_0(10))$, and so it is also written in terms of $j^{(10)}(\tau)$:

$$j^{(5)}(2\tau) = (j^{(10)}(\tau) + 1)^2\left(\frac{j^{(10)}(\tau) - 4}{j^{(10)}(\tau)}\right).$$

Furthermore, $j^{(5)}(2\tau)$ belongs to the bigger field $\mathcal{K}(\Gamma_1(10))$ which is generated by $k = k(\tau)$, and thus it can be written as the rational function in k as follows (this appears in [11, Theorem 10.5]):

$$j^{(5)}(2\tau) = \left(\frac{1 + k - k^2}{k}\right)^2\left(\frac{1 - 4k - k^2}{1 - k^2}\right).$$

Remark 4.3. [11, Theorem 5.26] contains (4.2). It also contains the relation between $j(5\tau)$ and $j^{(5)}(\tau)$:

$$j(5\tau) = \frac{(5 + 10j^{(5)}(\tau) + j^{(5)}(\tau)^2)^3}{j^{(5)}(\tau)}$$

because $j(5\tau)$ is a modular function on $\Gamma_0(5)$.

Example 4.4. By Theorem 1.3, we can compute $k\left(\frac{1+\sqrt{-3}}{2}\right)$ and $k(i)$.

(1) $k\left(\frac{1+\sqrt{-3}}{2}\right) = \frac{1}{3\alpha_0}(5\alpha_0^2 + 1{,}320 + 590\sqrt{5} + 25\sqrt{5}\alpha_0 + 59\alpha_0$
$- \sqrt{10{,}491{,}900 + 455{,}760\alpha_0 + 203{,}820\sqrt{5}\alpha_0 + 8{,}850\sqrt{5}\alpha_0^2 + 4{,}692{,}120\sqrt{5} + 19{,}815\alpha_0^2}),$
where $\alpha_0 := \sqrt[3]{2{,}728\sqrt{5} + 6{,}100}$ by using $j\left(\frac{1+\sqrt{-3}}{2}\right) = 0$, $j^{(5)}\left(\frac{1+\sqrt{-3}}{2}\right) = -125 - 50\sqrt{5}$, and $j^{(10)}\left(\frac{1+\sqrt{-3}}{2}\right) = -\frac{10\alpha_0}{3} - \frac{2{,}640 + 1{,}180\sqrt{5}}{3\alpha_0} - \frac{50\sqrt{5}}{3} - \frac{118}{3}$.

(2) $k(i) = -67 - 30\sqrt{5} - 5\beta_0 + \sqrt{17{,}990 + 8{,}045\sqrt{5} + 670\beta_0 + 300\sqrt{5}\beta_0}$,
where $\beta_0 := \sqrt{360 + 161\sqrt{5}}$ by using $j(i) = 1728$, $j^{(5)}(i) = 250 + 125\sqrt{5}$, and $j^{(10)}(i) = 134 + 60\sqrt{5} + 10\sqrt{360 + 161\sqrt{5}}$.

(3) $k\left(\frac{1+i}{2}\right) = \frac{1}{2}\left(11 + 5\sqrt{5} - \sqrt{250 - 110\sqrt{5}}\right)$
by using $j\left(\frac{1+i}{2}\right) = 1728$, $j^{(5)}\left(\frac{1+i}{2}\right) = 250 - 125\sqrt{5}$, and $j^{(10)}\left(\frac{1+i}{2}\right) = -11 - 5\sqrt{5}$.

Proof of Theorem 1.4. We choose positive integers a and b such that $r = a/b$ and $(a, b) = 1$. When p is a prime factor of r, let $\tau_0 = p\tau$ (resp. τ/p) if $p|a$ (resp. $p|b$). By finding the modular equation $F_p(X, Y)$ in Appendix A, let the polynomial $P(T) := F_p(k(\tau), T)$ (resp. $F_p(T, k(\tau))$) if $p|a$ (resp. $p|b$). Then the solutions $t_1, \ldots, t_m$ of the equation $P(T) = 0$ which can be written in terms of radicals are candidates for the value $k(\tau_0)$. Among them, we choose t_j to have the smallest absolute value

$$\left| t_j - q_0 \prod_{n=1}^{M} \frac{(1 - q_0^{10n-9})(1 - q_0^{10n-8})(1 - q_0^{10n-2})(1 - q_0^{10n-1})}{(1 - q_0^{10n-7})(1 - q_0^{10n-6})(1 - q_0^{10n-4})(1 - q_0^{10n-3})} \right|$$

for $q_0 := e^{2\pi i\tau_0}$; then t_k is the value $k(\tau_0)$. One can repeat this procedure until getting $k(r\tau)$. □

Example 4.5. By Theorem 1.4, we have the value

$$k(\sqrt{-3}) = \frac{1}{2} - \alpha - \frac{\alpha^2}{2} - \frac{\sqrt{1 - 4\alpha - 2\alpha^2 + 4\alpha^3 + \alpha^4}}{2},$$

where $\alpha := k\left(\frac{1+\sqrt{-3}}{2}\right)$ is the value found in Example 4.4

$$\alpha = -\frac{1}{3\alpha_0}(5\alpha_0^2 - 1{,}320 - 590\sqrt{5} - 25\sqrt{5}\alpha_0 - 59\alpha_0 + \sqrt{10{,}491{,}900 + 455{,}760\alpha_0 + 203{,}820\sqrt{5}\alpha_0 + 8{,}850\sqrt{5}\alpha_0^2 + 4{,}692{,}120\sqrt{5} + 19{,}815\alpha_0^2}),$$

and

$$\alpha_0 := \sqrt[3]{2{,}728\sqrt{5} + 6{,}100}.$$

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