DE GRUYTER Open Mathematics 2020; 18: 1317–1331

Research Article

Vagif Guliyev, Hatice Armutcu* and Tahir Azeroglu

Characterizations for the potential operators on Carleson curves in local generalized Morrey spaces

https://doi.org/10.1515/math-2020-0102
received June 21, 2020; accepted September 25, 2020

Abstract: In this paper, we give a boundedness criterion for the potential operator $\mathcal{I}^{\alpha}$ in the local generalized Morrey space $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ and the generalized Morrey space $M_{p,\varphi}(\Gamma)$ defined on Carleson curves Γ, respectively. For the operator $\mathcal{I}^{\alpha}$, we establish necessary and sufficient conditions for the strong and weak Spanne-type boundedness on $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ and the strong and weak Adams-type boundedness on $M_{p,\varphi}(\Gamma)$.

Keywords: Carleson curve, local generalized Morrey space, potential operator, Adams-type inequalities

MSC 2020: 26A33, 42B25, 42B35, 47B38

1 Introduction

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le l \le \infty\}$ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ with arc-length measure $\nu(t) = s$, where $l = \nu\Gamma =$ lengths of Γ. We denote

$$\Gamma(t, r) := \Gamma \cap B(t, r), \quad t \in \Gamma, \quad r > 0,$$

where $B(t, r) = \{z \in \mathbb{C} : |z - t| < r\}$. We also denote for brevity $\nu\Gamma(t, r) = |\Gamma(t, r)|$.

A rectifiable Jordan curve Γ is called a Carleson curve if the condition

$$\nu\Gamma(t, r) \le c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r.

Let $f \in L_1^{\mathrm{loc}}(\Gamma)$. The maximal operator $\mathcal{M}$ and the potential operator $\mathcal{I}^{\alpha}$ on Γ are defined by

$$\mathcal{M}f(t) = \sup_{t>0}(\nu\Gamma(t, r))^{-1} \int\limits_{\Gamma(t,r)} |f(\tau)|\mathrm{d}\nu(\tau)$$

and

$$\mathcal{I}^{\alpha}f(t) = \int\limits_{\Gamma} \frac{f(\tau)\mathrm{d}\nu(\tau)}{|t - \tau|^{1-\alpha}}, \quad 0 < \alpha < 1,$$

respectively.

*** Corresponding author: Hatice Armutcu,** Gebze Technical University, Kocaeli, Turkey, e-mail: haticexarmutcu@gmail.com
Vagif Guliyev: Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan; Department of Mathematics, Dumlupinar University, Kutahya, Turkey; Institute of Mathematics and Mechanics, Baku, Azerbaijan, e-mail: vagif@guliyev.com
Tahir Azeroglu: Gebze Technical University, Kocaeli, Turkey, e-mail: aliyev@gtu.edu.tr

Maximal operator and potential operator in various spaces, in particular, defined on Carleson curves have been widely studied by many authors (see, for example, [1–14]).

The main purpose of this paper is to establish the boundedness of potential operator $\mathcal{I}^{\alpha}$, $0 < \alpha < 1$ in local generalized Morrey spaces $LM^{\{x_0\}}_{p,\varphi}(\Gamma)$ defined on Carleson curves Γ. We shall give characterizations for the strong and weak Spanne-type boundedness of the operator $\mathcal{I}^{\alpha}$ from $LM^{\{x_0\}}_{p,\varphi_1}(\Gamma)$ to $LM^{\{x_0\}}_{q,\varphi_2}(\Gamma)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha$ and from the space $LM^{\{x_0\}}_{1,\varphi_1}(\Gamma)$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2}(\Gamma)$, $1 < q < \infty$, $1 - 1/q = \alpha$. Also, we study Adams-type boundedness of the operator $\mathcal{I}^{\alpha}$ from generalized Morrey spaces $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$, $1 < p < q < \infty$, and from the space $M_{1,\varphi}(\Gamma)$ to the weak space $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$, $1 < q < \infty$. We shall give characterizations for the Adams-type boundedness of the operator $\mathcal{I}^{\alpha}$ in generalized Morrey spaces, including weak versions.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

Morrey spaces were introduced by C. B. Morrey [15] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Later, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients, and potential theory.

Let $L_p(\Gamma)$, $1 \le p < \infty$ be the space of measurable functions on Γ with finite norm

$$\|f\|_{L_p(\Gamma)} = \left(\int_{\Gamma} |f(t)|^p \, d\nu(t) \right)^{1/p}.$$

Definition 2.1. Let $1 \le p < \infty$, $0 \le \lambda \le 1$, $[r]_1 = \min\{1, r\}$. We denote by $L_{p,\lambda}(\Gamma)$ the Morrey space, and by $\tilde{L}_{p,\lambda}(\Gamma)$ the modified Morrey space, the set of locally integrable functions f on Γ with the finite norms

$$\|f\|_{L_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))}, \qquad \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))},$$

respectively.

Note that (see [16,17]) $L_{p,0}(\Gamma) = \tilde{L}_{p,0}(\Gamma) = L_p(\Gamma)$,

$$\tilde{L}_{p,\lambda}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_p(\Gamma) \text{ and } \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)} = \max\{\|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_p(\Gamma)}\}$$

and if $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(\Gamma) = \tilde{L}_{p,\lambda}(\Gamma) = \Theta$, where Θ is the set of all functions equivalent to 0 on Γ.

We denote by $WL_{p,\lambda}(\Gamma)$ the weak Morrey space, and by $W\tilde{L}_{p,\lambda}(\Gamma)$ the modified Morrey space, as the set of locally integrable functions f on Γ with finite norms

$$\|f\|_{WL_{p,\lambda}(\Gamma)} = \sup_{\beta>0} \beta \sup_{t\in\Gamma, r>0} \left(r^{-\lambda} \int_{\{\tau\in\Gamma(t,r): |f(\tau)|>\beta\}} d\nu(\tau) \right)^{1/p},$$

$$\|f\|_{W\tilde{L}_{p,\lambda}(\Gamma)} = \sup_{\beta>0} \beta \sup_{t\in\Gamma, r>0} \left([r]_1^{-\lambda} \int_{\{\tau\in\Gamma(t,r): |f(\tau)|>\beta\}} d\nu(\tau) \right)^{1/p}.$$

Samko [14] studied the boundedness of the maximal operator $\mathcal{M}$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces $L_{p,\lambda}(\Gamma)$:

Theorem A. *Let Γ be a Carleson curve, $1 < p < \infty$, $0 < \alpha < 1$ and $0 \le \lambda < 1$. Then $\mathcal{M}$ is bounded from $L_{p,\lambda}(\Gamma)$ to $L_{p,\lambda}(\Gamma)$.*

Kokilashvili and Meskhi [18] studied the boundedness of the operator $\mathcal{I}^\alpha$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces and proved the following:

Theorem B. *Let Γ be a Carleson curve, $1 < p < q < \infty$, $0 < \alpha < 1$, $0 < \lambda_1 < \frac{p}{q}$, $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Then the operator $\mathcal{I}^\alpha$ is bounded from the spaces $L_{p,\lambda_1}(\Gamma)$ to $L_{q,\lambda_2}(\Gamma)$.*

The following Adams boundedness (see [19]) of the operator $\mathcal{I}^\alpha$ in Morrey space defined on Carleson curves was proved in [20].

Theorem C. *Let Γ be a Carleson curve, $0 < \alpha < 1$, $0 \le \lambda < 1 - \alpha$ and $1 \le p < \frac{1-\lambda}{\alpha}$.*

(1) *If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $\mathcal{I}^\alpha$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.*

(2) *If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $\mathcal{I}^\alpha$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$.*

The following Adams boundedness of the operator $\mathcal{I}^\alpha$ in modified Morrey space $\tilde{L}_{p,\lambda}(\Gamma)$ defined on Carleson curves was proved in [16], see also [17].

Theorem D. *Let Γ be a Carleson curve, $0 < \alpha < 1$, $0 \le \lambda < 1 - \alpha$ and $1 \le p < \frac{1-\lambda}{\alpha}$.*

(1) *If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\alpha \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $\mathcal{I}^\alpha$ from $\tilde{L}_{p,\lambda}(\Gamma)$ to $\tilde{L}_{q,\lambda}(\Gamma)$.*

(2) *If $p = 1$, then the condition $\alpha \le 1 - \frac{1}{q} \le \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of $\mathcal{I}^\alpha$ from $\tilde{L}_{1,\lambda}(\Gamma)$ to $W\tilde{L}_{q,\lambda}(\Gamma)$.*

We use the following statement on the boundedness of the weighted Hardy operator:

$$H_w g(t) := \int_t^\infty g(s) w(s)\,\mathrm{d}s, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [21].

Theorem 2.1. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{2.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\,\mathrm{d}s}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (2.1).

3 Local generalized Morrey spaces

We find it convenient to define the local generalized Morrey spaces in the form as follows, see [21,22].

Definition 3.2. Let $1 \le p < \infty$ and $\varphi(t,r)$ be a positive measurable function on $\Gamma \times (0,\infty)$. Fixed $t_0 \in \Gamma$, we denote by $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ ($WLM^{\{t_0\}}_{p,\varphi}(\Gamma)$) the local generalized Morrey space (the weak local generalized Morrey space), the space of all functions $f \in L^{\text{loc}}_p(\Gamma)$ with finite quasinorm

$$\|f\|_{LM^{\{t_0\}}_{p,\varphi}(\Gamma)} = \sup_{r>0} \frac{1}{\varphi(t_0,r)} \frac{1}{(\nu\Gamma(t_0,r))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t_0,r))}$$

$$\left(\|f\|_{WLM^{\{t_0\}}_{p,\varphi}(\Gamma)} = \sup_{r>0} \frac{1}{\varphi(t_0,r)} \frac{1}{(\nu\Gamma(t_0,r))^{\frac{1}{p}}} \|f\|_{WL_p(\Gamma(t_0,r))} \right).$$

Definition 3.3. Let $1 \le p < \infty$ and $\varphi(t,r)$ be a positive measurable function on $\Gamma \times (0,\infty)$. The generalized Morrey space $M_{p,\varphi}(\Gamma)$ is defined as the set of all functions $f \in L^{\text{loc}}_p(\Gamma)$ by the finite norm

$$\|f\|_{M_{p,\varphi}} = \sup_{t\in\Gamma, r>0} \frac{1}{\varphi(t,r)} \frac{1}{(\nu\Gamma(t,r))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t,r))}.$$

Also, the weak generalized Morrey space $WM_{p,\varphi}(\Gamma)$ is defined as the set of all functions $f \in L^{\text{loc}}_p(\Gamma)$ by the finite norm

$$\|f\|_{WM_{p,\varphi}} = \sup_{t\in\Gamma, r>0} \frac{1}{\varphi(t,r)} \frac{1}{(\nu\Gamma(t,r))^{\frac{1}{p}}} \|f\|_{WL_p(\Gamma(t,r))}.$$

It is natural, first the set of all, to find conditions ensuring that the spaces $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ and $M_{p,\varphi}(\Gamma)$ are non-trivial, that is, consist not only of functions equivalent to 0 on Γ.

Lemma 3.1. [23] *Let $t_0 \in \Gamma$ and $\varphi(t,r)$ be a positive measurable function on $\Gamma \times (0,\infty)$. If*

$$\sup_{r<\tau<\infty} \frac{1}{\varphi(t_0,r)} \frac{1}{(\nu\Gamma(t_0,r))^{\frac{1}{p}}} = \infty \quad \text{for some } r > 0, \tag{3.2}$$

then $LM^{\{t_0\}}_{p,\varphi}(\Gamma) = \Theta$.

Remark 3.1. We denote by $\Omega_{p,\text{loc}}$ the set of all positive measurable functions φ on $\Gamma \times (0,\infty)$ such that for all $r > 0$,

$$\left\| \frac{1}{\varphi(t_0,\tau)} \frac{1}{(\nu\Gamma(t_0,\tau))^{\frac{1}{p}}} \right\|_{L_\infty(r,\infty)} < \infty.$$

In what follows, keeping in mind Lemma 1, for the non-triviality of the space $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ we always assume that $\varphi \in \Omega_{p,\text{loc}}$.

Lemma 3.2. [23] *Let $\varphi(t,r)$ be a positive measurable function on $\Gamma \times (0,\infty)$.*

(i) If

$$\sup_{r<\tau<\infty} \frac{1}{\varphi(t,\tau)} \frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}} = \infty \quad \text{for some } r > 0 \text{ and for all } t \in \Gamma, \tag{3.3}$$

then $M_{p,\varphi}(\Gamma) = \Theta$.

(ii) If

$$\sup_{0<\tau<r} \varphi(t,\tau)^{-1} = \infty \quad \text{for some } r > 0 \text{ and for all } t \in \Gamma, \tag{3.4}$$

then $M_{p,\varphi}(\Gamma) = \Theta$.

Remark 3.2. We denote by Ω_p the sets of all positive measurable functions φ on $\Gamma \times (0,\infty)$ such that for all $r > 0$,

$$\sup_{t\in\Gamma} \left\| \frac{1}{\varphi(t,\tau)} \frac{1}{(\nu\Gamma(t,\tau))^{\frac{1}{p}}} \right\|_{L_\infty(r,\infty)} < \infty \quad \text{and} \quad \sup_{t\in\Gamma} \|\varphi(t,\tau)^{-1}\|_{L_\infty(0,r)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2, we always assume that $\varphi \in \Omega_p$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \quad (\text{resp.} \quad \varphi(r) \ge C\varphi(s)) \ \text{ for } \ r \le s.$$

Let $1 \le p < \infty$. Denote by $\mathcal{G}_p$ the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto t^{\frac{1}{p}}\varphi(t) \in (0,\infty)$ is almost increasing.

Seemingly, the requirement $\varphi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, Nakai established that there exists a function ρ such that ρ itself is decreasing, that $\rho(t)t^{n/p} \le \rho(T)T^{n/p}$ for all $0 < t \le T < \infty$ and that $LM^{\{t_0\}}_{p,\varphi}(\Gamma) = LM^{\{t_0\}}_{p,\rho}(\Gamma)$, $M_{p,\varphi}(\Gamma) = M_{p,\rho}(\Gamma)$.

By elementary calculations we have the following, which shows particularly that the spaces $LM^{\{t_0\}}_{p,\varphi}$, $WLM^{\{t_0\}}_{p,\varphi}$, $M_{p,\varphi}(\Gamma)$ and $WM_{p,\varphi}(\Gamma)$ are not trivial, see, for example, [23–25].

Lemma 3.3. [23] *Let $\varphi \in \mathcal{G}_p$, $1 \le p < \infty$, $\Gamma_0 = \Gamma(t_0, r_0)$ and χ_{Γ_0} be the characteristic function of the ball Γ_0, then $\chi_{\Gamma_0} \in LM^{\{t_0\}}_{p,\varphi}(\Gamma) \cap M_{p,\varphi}(\Gamma)$. Moreover, there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\Gamma_0}\|_{WLM^{\{t_0\}}_{p,\varphi}} \le \|\chi_{\Gamma_0}\|_{LM^{\{t_0\}}_{p,\varphi}} \le \frac{C}{\varphi(r_0)}$$

and

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\Gamma_0}\|_{WM_{p,\varphi}} \le \|\chi_{\Gamma_0}\|_{M_{p,\varphi}} \le \frac{C}{\varphi(r_0)}.$$

4 Maximal operator in the spaces $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ and $M_{p,\varphi}(\Gamma)$

We denote by $L_{\infty,v}(0,\infty)$ the set of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\bar{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\bar{S}_u g)(t) := \|ug\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [26].

Theorem 4.2. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a non-negative continuous function on $(0, \infty)$.*

Then the operator $\bar{S}_u$ is bounded from $L_{\infty,v_1}(0, \infty)$ to $L_{\infty,v_2}(0, \infty)$ on the cone $\mathbb{A}$ if and only if

$$\|v_2 \bar{S}_u(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)})\|_{L_\infty(0,\infty)} < \infty. \tag{4.5}$$

The following Guliyev-type local estimate for the maximal operator $\mathcal{M}$ is true, see for example, [27,28].

Lemma 4.4. *Let Γ be a Carleson curve, $1 \le p < \infty$ and $t_0 \in \Gamma$. Then for $p > 1$ and any $r > 0$ the inequality*

$$\|\mathcal{M}f\|_{L_p(\Gamma(t_0,r))} \lesssim \|f\|_{L_p(\Gamma(t_0,2r))} + r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-1} \|f\|_{L_1(\Gamma(t_0,\tau))} \tag{4.6}$$

holds for all $f \in L_p^{\mathrm{loc}}(\Gamma)$.

Moreover, for $p = 1$ the inequality

$$\|\mathcal{M}f\|_{WL_1(\Gamma(t_0,r))} \lesssim \|f\|_{L_1(\Gamma(t_0,2r))} + r \sup_{\tau>2r} \tau^{-1} \|f\|_{L_1(\Gamma(t_0,\tau))} \tag{4.7}$$

holds for all $f \in L_1^{\mathrm{loc}}(\Gamma)$.

Proof. Let $1 < p < \infty$. For arbitrary ball $\Gamma(t_0, r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{\Gamma(t_0,2r)}$ and $f_2 = f\chi_{\complement(\Gamma(t_0,2r))}$.

$$\|\mathcal{M}f\|_{L_p(\Gamma(t_0,r))} \le \|\mathcal{M}f_1\|_{L_p(\Gamma(t_0,r))} + \|\mathcal{M}f_2\|_{L_p(\Gamma(t_0,r))}.$$

By the continuity of the operator $\mathcal{M} : L_p(\Gamma) \to L_p(\Gamma)$ from Theorem A we have

$$\|\mathcal{M}f_1\|_{L_p(\Gamma(t_0,r))} \lesssim \|f\|_{L_p(\Gamma(t_0,2r))}.$$

Let y be an arbitrary point from $\Gamma(t_0, \tau)$. If $\Gamma(y, \tau) \cap {}^\complement(\Gamma(t_0, 2r)) \neq \emptyset$, then $\tau > r$. Indeed, if $z \in \Gamma(y, \tau) \cap {}^\complement(\Gamma(t_0, 2r))$, then $\tau > |y - z| \ge |t - z| - |t - y| > 2r - r = r$.

On the other hand, $\Gamma(y, \tau) \cap {}^\complement(\Gamma(t_0, 2r)) \subset \Gamma(t_0, 2\tau)$. Indeed, $z \in \Gamma(y, \tau) \cap {}^\complement(\Gamma(t_0, 2r))$, then we get $|t - z| \le |y - z| + |t - y| < \tau + r < 2\tau$.

Hence,

$$\mathcal{M}f_2(y) \le 2 \sup_{\tau>r} \frac{1}{\nu\Gamma(t_0, 2\tau)} \int_{\Gamma(t_0,2\tau)} |f(z)|d\nu(z) = 2 \sup_{\tau>2r} \frac{1}{\nu\Gamma(t_0, \tau)} \int_{\Gamma(t_0,\tau)} |f(z)|d\nu(z) \le 2 \sup_{\tau>2r} \tau^{-1} \int_{\Gamma(t_0,\tau)} |f(z)|d\nu(z).$$

Therefore, for all $y \in \Gamma(t_0, \tau)$ we have

$$\mathcal{M}f_2(y) \le 2 \sup_{\tau>2r} \tau^{-1} \int_{\Gamma(t_0,\tau)} |f(z)|d\nu(z). \tag{4.8}$$

Thus,

$$\|\mathcal{M}f\|_{L_p(\Gamma(t_0,r))} \lesssim \|f\|_{L_p(\Gamma(t_0,2r))} + r^{\frac{1}{p}} \left(\sup_{\tau>2r} \tau^{-1} \int_{\Gamma(t_0,\tau)} |f(z)|d\nu(z) \right).$$

Let $p = 1$. It is obvious that for any ball $\Gamma(t_0, r)$

$$\|\mathcal{M}f\|_{WL_1(\Gamma(t_0,r))} \le \|\mathcal{M}f_1\|_{WL_1(\Gamma(t_0,r))} + \|\mathcal{M}f_2\|_{WL_1(\Gamma(t_0,r))}.$$

By the continuity of the operator $\mathcal{M} : L_1(\Gamma) \to WL_1(\Gamma)$ from Theorem A we have

$$\|\mathcal{M}f_1\|_{WL_1(\Gamma)} \lesssim \|f\|_{L_1(\Gamma(t_0,2r))}.$$

Then by (4.8) we get inequality (4.7). □

Lemma 4.5. *Let* Γ *be a Carleson curve,* $1 \le p < \infty$ *and* $t_0 \in \Gamma$. *Then for* $p > 1$ *and any* $r > 0$ *in* Γ, *the inequality*

$$\|\mathcal{M}f\|_{L_p(\Gamma(t_0,r))} \lesssim r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t_0,\tau))} \tag{4.9}$$

holds for all $f \in L_p^{\mathrm{loc}}(\Gamma)$.

Moreover, for $p = 1$ *the inequality*

$$\|\mathcal{M}f\|_{WL_1(\Gamma(t_0,r))} \lesssim r \sup_{\tau>2r} \tau^{-1} \|f\|_{L_1(\Gamma(t_0,\tau))} \tag{4.10}$$

holds for all $f \in L_1^{\mathrm{loc}}(\Gamma)$.

Proof. Let $1 < p < \infty$. Denote

$$\mathcal{M}_1 := r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-1} \int\limits_{\Gamma(t_0,r)} |f(z)|\,d\nu(z), \quad \mathcal{M}_2 := \|f\|_{L_p(\Gamma(t_0,2r))}.$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1 \lesssim r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{\frac{1}{p}} \Bigg(\int\limits_{\Gamma(t_0,\tau)} |f(z)|^p\,d\nu(z) \Bigg)^{\frac{1}{p}}.$$

On the other hand,

$$r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{\frac{1}{p}} \Bigg(\int\limits_{\Gamma(t_0,\tau)} |f(z)|^p\,d\nu(z) \Bigg)^{\frac{1}{p}} \gtrsim r^{\frac{1}{p}} \Big(\sup_{\tau>2r} \tau^{\frac{1}{p}} \Big) \|f\|_{L_p(\Gamma(t_0,2r))} \approx \mathcal{M}_2.$$

Since by Lemma 4.4

$$\|\mathcal{M}f\|_{L_p(\Gamma(t_0,r))} \le \mathcal{M}_1 + \mathcal{M}_2,$$

we arrive at (4.9).

Let $p = 1$. The inequality (4.10) directly follows from (4.7). □

The following theorem is valid.

Theorem 4.3. *Let* Γ *be a Carleson curve,* $1 \le p < \infty$, $t_0 \in \Gamma$ *and* (φ_1, φ_2) *satisfies the condition*

$$\sup_{r<\tau<\infty} \tau^{-\frac{1}{p}} \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t_0, s) s^{\frac{1}{p}} \le C\varphi_2(t_0, r), \tag{4.11}$$

where C *does not depend on* r. *Then for* $p > 1$ *the operator* $\mathcal{M}$ *is bounded from* $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma)$ *to* $LM_{p,\varphi_2}^{\{t_0\}}(\Gamma)$ *and for* $p = 1$ *the operator* $\mathcal{M}$ *is bounded from* $LM_{1,\varphi_1}^{\{t_0\}}(\Gamma)$ *to* $WLM_{1,\varphi_2}^{\{t_0\}}(\Gamma)$.

Proof. By Theorem 4.2 and Lemma 4.5, we get

$$\|\mathcal{M}f\|_{LM_{p,\varphi_2}^{\{t_0\}}(\Gamma)} \lesssim \sup_{r>0} \varphi_2(t_0, r)^{-1} \sup_{\tau>r} \tau^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t_0,\tau))} \lesssim \sup_{r>0} \varphi_1(t, r)^{-1} r^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t_0,r))} = \|f\|_{LM_{p,\varphi_1}^{\{t_0\}}(\Gamma)}$$

if $p \in (1, \infty)$ and

$$\|\mathcal{M}f\|_{WLM_{p,\varphi_2}^{\{t_0\}}(\Gamma)} \lesssim \sup_{r>0} \varphi_2(t_0, r)^{-1} \sup_{\tau>r} \tau^{-1} \|f\|_{L_1(\Gamma(t_0,r))} \lesssim \sup_{r>0} \varphi_1(t, r)^{-1} r^{-1} \|f\|_{L_1(\Gamma(t_0,r))} = \|f\|_{LM_{1,\varphi_1}^{\{t_0\}}(\Gamma)}$$

if $p = 1$. □

From Theorem 4.3, we get the following.

Corollary 4.1. *Let* Γ *be a Carleson curve,* $1 \le p < \infty$ *and* $\varphi_1, \varphi_2 \in \Omega_p$ *satisfies the condition*

$$\sup_{r<\tau<\infty} \tau^{-\frac{1}{p}} \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t,s) s^{\frac{1}{p}} \le C\varphi_2(t,r), \tag{4.12}$$

where C does not depend on t and r. Then for $p > 1$ *the operator* $\mathcal{M}$ *is bounded from* $M_{p,\varphi_1}(\Gamma)$ *to* $M_{p,\varphi_2}(\Gamma)$ *and for* $p = 1$ *the operator* $\mathcal{M}$ *is bounded from* $M_{1,\varphi_1}(\Gamma)$ *to* $WM_{1,\varphi_2}(\Gamma)$.

Corollary 4.2. *Let* Γ *be a Carleson curve,* $1 \le p < \infty$ *and* $\varphi \in \mathcal{G}_p$. *Then for* $p > 1$ *the operator* $\mathcal{M}$ *is bounded on* $M_{p,\varphi}(\Gamma)$ *and for* $p = 1$ *the operator* $\mathcal{M}$ *is bounded from* $M_{1,\varphi}(\Gamma)$ *to* $WM_{1,\varphi}(\Gamma)$.

5 Fractional integral operator in the spaces $LM^{\{t_0\}}_{p,\varphi}(\Gamma)$ and $M_{p,\varphi}(\Gamma)$

5.1 Spanne-type results

The following local estimate is true, see for example, [28].

Theorem 5.4. *Let* Γ *be a Carleson curve,* $1 \le p < \infty$, $t_0 \in \Gamma$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$ *and* $f \in L^{\mathrm{loc}}_p(\Gamma)$. *Then for* $p > 1$

$$\|\mathcal{I}^\alpha f\|_{L_q(\Gamma(t_0,r))} \le C r^{\frac{1}{q}} \int_{2r}^{\infty} \tau^{-\frac{1}{q}-1} \|f\|_{L_p(\Gamma(t_0,\tau))} \, d\tau \tag{5.13}$$

and for $p = 1$

$$\|\mathcal{I}^\alpha f\|_{WL_q(\Gamma(t_0,r))} \le C r^{\frac{1}{q}} \int_{2r}^{\infty} \tau^{-\frac{1}{q}-1} \|f\|_{L_1(\Gamma(t_0,\tau))} \, d\tau, \tag{5.14}$$

where C does not depend on f, $t_0 \in \Gamma$ *and* $r > 0$.

Proof. For a given ball $\Gamma(t_0, r)$, we split the function f as $f = f_1 + f_2$, where $f_1 = f\chi_{\Gamma(t_0,2r)}$, $f_2 = f\chi_{\complement(\Gamma(t_0,2r))}$, and then

$$\mathcal{I}^\alpha f(t) = \mathcal{I}^\alpha f_1(t) + \mathcal{I}^\alpha f_2(t).$$

Let $1 < p < \infty$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. Since $f_1 \in L_p(\Gamma)$, by the boundedness of the operator $\mathcal{I}^\alpha$ from $L_p(\Gamma)$ to $L_q(\Gamma)$ (see Theorem B) it follows that

$$\|\mathcal{I}^\alpha f_1\|_{L_q(\Gamma)} \le C\|f_1\|_{L_p(\Gamma)} = C\|f\|_{L_p(\Gamma(t_0,2r))} \le C r^{\frac{1}{q}} \int_{2r}^{\infty} \tau^{-\frac{1}{q}-1} \|f\|_{L_p(\Gamma(t_0,\tau))} \, d\tau, \tag{5.15}$$

where the constant C is independent of f.

Observe that the conditions $z \in \Gamma(t_0, r)$, $y \in^{\complement} (\Gamma(t_0, 2r))$ imply

$$\frac{1}{2}|z-y| \le |t-y| \le \frac{3}{2}|t-z|.$$

Then for all $z \in \Gamma(t_0, r)$ we get

$$|\mathcal{I}^\alpha f_2(z)| \le \left(\frac{3}{2}\right)^{1-\alpha} \int_{\complement(\Gamma(t_0,2r))} |t-y|^{\alpha-1} |f(y)| \, d\nu(y).$$

By Fubini's theorem, we have

$$\int_{\complement(\Gamma(t_0,2r))} |t-y|^{\alpha-1}|f(y)|\,d\nu(y) \approx \int_{\complement(\Gamma(t_0,2r))} |f(y)|\,d\nu(y) \int_{|t-y|}^{\infty} \tau^{\alpha-2}\,d\tau \approx \int_{2r}^{\infty} \int_{2r\le|t-y|<\tau} |f(y)|\,d\nu(y)\tau^{\alpha-2}\,d\tau$$
$$\lesssim \int_{2r}^{\infty} \int_{\Gamma(t_0,\tau)} |f(y)|\,d\nu(y)\tau^{\alpha-2}\,d\tau.$$

Applying Hölder's inequality, we get

$$\int_{\complement(\Gamma(t_0,2r))} |t-y|^{\alpha-1}|f(y)|\,d\nu(y) \lesssim \int_{2r}^{\infty} \|f\|_{L_p(\Gamma(t_0,\tau))}\tau^{-\frac{1}{q}-1}\,d\tau$$

and for all $z \in \Gamma(t_0, r)$

$$|\mathcal{I}^\alpha f_2(z)| \lesssim \int_{2r}^{\infty} \|f\|_{L_p(\Gamma(t_0,\tau))}\tau^{-\frac{1}{q}-1}\,d\tau. \tag{5.16}$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\|\mathcal{I}^\alpha f_2\|_{L_q(\Gamma(t_0,r))} \lesssim r^{\frac{1}{q}} \int_{2r}^{\infty} \tau^{-\frac{1}{q}-1}\|f\|_{L_p(\Gamma(t_0,\tau))}\,d\tau \tag{5.17}$$

is valid. Thus, from (5.15) and (5.17) we get inequality (5.13).

Finally, in the case $p = 1$ by the weak $(1, q)$-boundedness of the operator $\mathcal{I}^\alpha$ (see Theorem B) it follows that

$$\|\mathcal{I}^\alpha f_1\|_{WL_q(\Gamma(t_0,r))} \le C\|f_1\|_{L_1(\Gamma)} \le Cr^{\frac{1}{q}} \int_{2r}^{\infty} \tau^{-\frac{1}{q}-1}\|f\|_{L_1(\Gamma(t_0,\tau))}\,d\tau, \tag{5.18}$$

where C does not depend on t_0 and r. Then from (5.17) and (5.18) we get inequality (5.14). □

Theorem 5.5. *Let Γ be a Carleson curve, $1 \le p < \infty$, $t_0 \in \Gamma$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, $\varphi_1 \in \Omega_{p,\mathrm{loc}}$, $\varphi_2 \in \Omega_{q,\mathrm{loc}}$ and the pair (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t_0,s)s^{\frac{1}{p}}}{\tau^{\frac{1}{q}}}\frac{d\tau}{\tau} \le C\varphi_2(t_0, r), \tag{5.19}$$

where C does not depend on t_0 and r. Then for $p > 1$ the operator $\mathcal{I}^\alpha$ is bounded from $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ to $LM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$ and for $p = 1$ the operator $\mathcal{I}^\alpha$ is bounded from $LM^{\{t_0\}}_{1,\varphi_1}(\Gamma)$ to $WLM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$.

Proof. By Theorems 2.1 and 5.4 with $v_2(r) = \varphi_2(t_0, r)^{-1}$, $v_1(r) = \varphi_1(t_0, r)^{-1}r^{-\frac{1}{p}}$ and $w(r) = r^{-\frac{1}{q}}$ we have for $p > 1$

$$\|\mathcal{I}^\alpha f\|_{LM^{\{t_0\}}_{q,\varphi_2}(\Gamma)} \lesssim \sup_{r>0} \varphi_2(t_0, r)^{-1} \int_r^{\infty} s^{-\frac{1}{q}-1}\|f\|_{L_p(\Gamma(t_0,s))}\,ds \lesssim \sup_{r>0} \varphi_1(t_0, r)^{-1}r^{-\frac{1}{p}}\|f\|_{L_p(\Gamma(t_0,r))} = \|f\|_{LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)}$$

and for $p = 1$

$$\|\mathcal{I}^\alpha f\|_{WLM^{\{t_0\}}_{q,\varphi_2}(\Gamma)} \lesssim \sup_{r>0} \varphi_2(t_0, r)^{-1} \int_r^{\infty} s^{-\frac{1}{q}-1}\|f\|_{L_1(\Gamma(t_0,s))}\,ds \lesssim \sup_{r>0} \varphi_1(t_0, r)^{-1}r^{-Q}\|f\|_{L_1(\Gamma(t_0,r))} = \|f\|_{LM^{\{t_0\}}_{1,\varphi_1}(\Gamma)}.$$ □

From Theorem 4.3 we get the following.

Corollary 5.3. *Let* Γ *be a Carleson curve,* $1 \le p < \infty$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ *and the pair* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{\tau<s<\infty} \varphi_1(t,s) s^{\frac{1}{p}}}{\tau^{\frac{1}{q}}} \frac{d\tau}{\tau} \le C\varphi_2(t,r), \tag{5.20}$$

where C does not depend on t and r. Then for $p > 1$ *the operator* $\mathcal{I}^\alpha$ *is bounded from* $M_{p,\varphi_1}(\Gamma)$ *to* $M_{q,\varphi_2}(\Gamma)$ *and for* $p = 1$ *the operator* $\mathcal{I}^\alpha$ *is bounded from* $M_{1,\varphi_1}(\Gamma)$ *to* $WM_{q,\varphi_2}(\Gamma)$.

For proving our main results, we need the following estimate.

Lemma 5.6. *Let* Γ *be a Carleson curve and* $\Gamma_0 := \Gamma(t_0, r_0)$, *then* $r_0^\alpha \lesssim \mathcal{I}^\alpha \chi_{\Gamma_0}(t)$ *for every* $t \in \Gamma_0$.

Proof. If $t, y \in \Gamma_0$, then $|t - y| \le |t - t_0| + |t_0 - y| < 2r_0$. Since $0 < \alpha < 1$, we get $r_0^{\alpha-1} \le 2^{1-\alpha}|t - y|^{\alpha-Q}$. Therefore,

$$\mathcal{I}^\alpha \chi_{\Gamma_0}(t) = \int_\Gamma \chi_{\Gamma_0}(y)|t - y|^{\alpha-1} d\nu(y) = \int_{\Gamma_0} |t - y|^{\alpha-1} d\nu(y) \ge c_0 2^{1-\alpha} r_0^\alpha. \qquad \square$$

The following theorem is one of our main results.

Theorem 5.6. *Let* Γ *be a Carleson curve,* $0 < \alpha < 1$, $t_0 \in \Gamma$ *and* $p, q \in [1, \infty)$.

1. *If* $1 \le p < \frac{1}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \alpha$, *then condition* (5.20) *is sufficient for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ *to* $WLM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$. *Moreover, if* $1 < p < \frac{1}{\alpha}$, *condition* (5.20) *is sufficient for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ *to* $LM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$.
2. *If the function* $\varphi_1 \in \mathcal{G}_p$, *then the condition*
$$r^\alpha \varphi_1(r) \le C\varphi_2(r), \tag{5.21}$$
for all $r > 0$, *where* $C > 0$ *does not depend on r, is necessary for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ *to* $WLM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$ *and* $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ *to* $LM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$.
3. *Let* $1 \le p < \frac{1}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \alpha$. *If* $\varphi_1 \in \mathcal{G}_p$ *satisfies the regularity condition*
$$\int_r^\infty s^{\alpha-1}\varphi_1(s) ds \le Cr^\alpha \varphi_1(r), \tag{5.22}$$
for all $r > 0$, *where* $C > 0$ *does not depend on r, then condition* (5.21) *is necessary and sufficient for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ *to* $WLM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$. *Moreover, if* $1 < p < \frac{Q}{\alpha}$, *then condition* (5.21) *is necessary and sufficient for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma)$ *to* $LM^{\{t_0\}}_{q,\varphi_2}(\Gamma)$.

Proof. The first part of the theorem is proved in Theorem 5.3.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(t_0, r_0)$ and $t \in \Gamma_0$. By Lemma 5.6, we have $r_0^\alpha \le C\mathcal{I}^\alpha \chi_{\Gamma_0}(r)$. Therefore, by Lemmas 3.3 and 5.6

$$r_0^\alpha \lesssim (\nu(\Gamma_0))^{-\frac{1}{p}} \|\mathcal{I}^\alpha \chi_{\Gamma_0}\|_{L_q(\Gamma_0)} \lesssim \varphi_2(r_0) \|\mathcal{I}^\alpha \chi_{\Gamma_0}\|_{M_{q,\varphi_2}} \lesssim \varphi_2(r_0) \|\chi_{\Gamma_0}\|_{M_{p,\varphi_1}} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)}$$

or

$$r_0^\alpha \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)} \text{ for all } r_0 > 0 \Leftrightarrow r_0^\alpha \varphi_1(r_0) \lesssim \varphi_2(r_0) \text{ for all } r_0 > 0.$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. $\square$

Remark 5.3. If we take $\varphi_1(r) = r^{\frac{\lambda-1}{p}}$ and $\varphi_2(r) = r^{\frac{\mu-1}{q}}$ at Theorem 5.6, then conditions (5.22) and (5.21) are equivalent to $0 < \lambda < 1 - \alpha p$ and $\frac{\lambda}{p} = \frac{\mu}{q}$, respectively. Therefore, we get Theorem C from Theorem 5.6.

5.2 Adams-type results

The following pointwise estimate plays a key role where we prove our main results.

Theorem 5.7. *Let Γ be a Carleson curve, $1 \le p < \infty$, $0 < \alpha < 1$ and $f \in L_p^{\mathrm{loc}}(\Gamma)$. Then*

$$|\mathcal{I}^\alpha f(t)| \le Cr^\alpha \mathcal{M}f(t) + C\int_r^\infty s^{\alpha-\frac{1}{p}-1}\|f\|_{L_p(\Gamma(t,s))}\,\mathrm{d}s, \tag{5.23}$$

where C does not depend on f, $t \in \Gamma$ and $r > 0$.

Proof. Write $f = f_1 + f_2$, where $f_1 = f\chi_{\Gamma(t,2r)}$ and $f_2 = f\chi_{\complement(\Gamma(t,2r))}$. Then

$$\mathcal{I}^\alpha f(t) = \mathcal{I}^\alpha f_1(t) + \mathcal{I}^\alpha f_2(t).$$

For $\mathcal{I}^\alpha f_1(t)$, following Hedberg's trick (see for instance [29, p. 354]), for all $z \in \Gamma$ we obtain $|\mathcal{I}^\alpha f_1(z)| \le C_1 r^\alpha \mathcal{M}f(z)$. For $\mathcal{I}^\alpha f_2(z)$ with $z \in D$ from (5.16) we have

$$|\mathcal{I}^\alpha f_2(z)| \le \int_{\complement(\Gamma(t,2r))} |t-y|^{\alpha-1}|f(y)|\,\mathrm{d}y \le C\int_{2r}^\infty s^{\alpha-\frac{1}{p}-1}\|f\|_{L_p(\Gamma(t,s))}\,\mathrm{d}s, \tag{5.24}$$

which proves (5.23). □

The following is a result of Adams type for the fractional integral on Carleson curves (see [28]).

Theorem 5.8. (Adams-type result) *Let Γ be a Carleson curve, $1 \le p < q < \infty$, $0 < \alpha < \frac{1}{p}$ and let $\varphi \in \Omega_p$ satisfy condition*

$$\sup_{r<\tau<\infty} \tau^{-1} \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi(t,s)s \le C\varphi(t,r), \tag{5.25}$$

and

$$\int_r^\infty \tau^{\alpha-1}\varphi(t,\tau)^{\frac{1}{p}}\,\mathrm{d}\tau \le Cr^{-\frac{\alpha p}{q-p}}, \tag{5.26}$$

where C does not depend on $t \in \Gamma$ and $r > 0$. Then for $p > 1$ the operator $\mathcal{I}^\alpha$ is bounded from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$ and for $p = 1$ the operator $\mathcal{I}^\alpha$ is bounded from $M_{1,\varphi}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

Proof. Let $1 \le p < \infty$ and $f \in M_{p,\varphi}(\Gamma)$. By Theorem 5.7, inequality (5.23) is valid. Then from condition (5.26) and inequality (5.23) we get

$$\begin{aligned} |\mathcal{I}^\alpha f(t)| &\lesssim r^\alpha \mathcal{M}f(t) + \int_r^\infty s^{\alpha-\frac{1}{p}-1}\|f\|_{L_p(\Gamma(t,s))}\,\mathrm{d}s \\ &\le r^\alpha \mathcal{M}f(t) + \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \int_r^\infty s^{\alpha-1}\varphi(t,s)^{\frac{1}{p}}\mathrm{d}s \\ &\le r^\alpha \mathcal{M}f(t) + r^{-\frac{\alpha p}{q-p}}\|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}. \end{aligned} \tag{5.27}$$

Hence, choosing $r = \left(\frac{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}{\mathcal{M}f(t)}\right)$ for every $t \in \Gamma$, we have

$$|\mathcal{I}^{\alpha}f(t)| \lesssim (\mathcal{M}f(t))^{\frac{p}{q}} \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}.$$

Hence, the statement of the theorem follows in view of the boundedness of the maximal operator $\mathcal{M}$ in $M_{p,\varphi}(\Gamma)$ provided by Theorem 3, by virtue of condition (5.25).

$$\|\mathcal{I}^{\alpha}f\|_{M_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \sup_{t\in\Gamma, r>0} \varphi(t,r)^{-\frac{p}{q}} r^{-\frac{1}{q}} \|\mathcal{M}f\|^{\frac{p}{q}}_{L_p(\Gamma(t,r))} \lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \|\mathcal{M}f\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}$$

if $1 < p < q < \infty$ and

$$\|\mathcal{I}^{\alpha}f\|_{WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \lesssim \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}(\Gamma)} \sup_{t\in\Gamma, r>0} \varphi(t,r)^{-\frac{1}{q}} r^{-\frac{1}{q}} \|\mathcal{M}f\|^{\frac{1}{q}}_{WL_1(\Gamma(t,r))} \lesssim \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}(\Gamma)} \|\mathcal{M}f\|^{\frac{1}{q}}_{M_{1,\varphi}(\Gamma)} \lesssim \|f\|_{M_{1,\varphi}(\Gamma)}$$

if $p = 1 < q < \infty$. □

The following theorem is another of our main results.

Theorem 5.9. *Let Γ be a Carleson curve, $0 < \alpha < 1$, $1 \le p < q < \infty$ and $\varphi \in \Omega_p$.*

1. *If $\varphi(t,r)$ satisfies condition (5.25), then condition (5.26) is sufficient for the boundedness of the operator $\mathcal{I}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$. Moreover, if $1 < p < q < \infty$, then condition (5.26) is sufficient for the boundedness of the operator $\mathcal{I}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.*
2. *If $\varphi \in \mathcal{G}_p$, then the condition*
$$r^{\alpha}\varphi(r)^{\frac{1}{p}} \le Cr^{-\frac{\alpha p}{q-p}}, \tag{5.28}$$
for all $r > 0$, where $C > 0$ does not depend on r, is necessary for the boundedness of the operator $\mathcal{I}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$ and from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.
3. *If $\varphi \in \mathcal{G}_p$ satisfies the regularity condition*
$$\int_r^{\infty} s^{\alpha-1}\varphi(s)^{\frac{1}{p}} ds \le Cr^{\alpha}\varphi(r)^{\frac{1}{p}}, \tag{5.29}$$
for all $r > 0$, where $C > 0$ does not depend on r, then condition (5.28) is necessary and sufficient for the boundedness of the operator $\mathcal{I}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$. Moreover, if $1 < p < q < \infty$, then condition (5.28) is necessary and sufficient for the boundedness of the operator $\mathcal{I}^{\alpha}$ from $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ to $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

Proof. The first part of the theorem is a corollary of Theorem 5.8.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(t_0, r_0)$ and $t \in \Gamma_0$. By Lemma 5.6, we have $r_0^{\alpha} \lesssim \mathcal{I}^{\alpha}\chi_{\Gamma_0}(t)$. Therefore, by Lemmas 3.3 and 5.6 we have

$$r_0^{\alpha} \lesssim (\nu(\Gamma_0))^{-\frac{1}{q}} \|\mathcal{I}^{\alpha}\chi_{\Gamma_0}\|_{L_q(\Gamma_0)} \lesssim \varphi(r_0)^{\frac{1}{q}} \|\mathcal{I}^{\alpha}\chi_{\Gamma_0}\|_{M_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \lesssim \varphi(r_0)^{\frac{1}{q}} \|\chi_{\Gamma_0}\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \lesssim \varphi(r_0)^{\frac{1}{q}-\frac{1}{p}}$$

or

$$r_0^{\alpha}\varphi(r_0)^{\frac{1}{p}-\frac{1}{q}} \lesssim 1 \text{ for all } r_0 > 0 \Leftrightarrow r_0^{\alpha}\varphi(r_0)^{\frac{1}{p}} \lesssim r_0^{-\frac{\alpha p}{q-p}}.$$

Since this is true for every $t \in \Gamma$ and $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

The following is a result of Adams type for the fractional integral on Carleson curves.

Theorem 10. (Adams-type result). *Let* Γ *be a Carleson curve,* $0 < \alpha < 1$, $1 \le p < q < \infty$ *and* $\varphi \in \Omega_p$ *satisfy condition* (5.25) *and*

$$r^\alpha \varphi(t,r) + \int_r^\infty s^{\alpha-1}\varphi(t,s)\,ds \le C\varphi(t,r)^{\frac{p}{q}}, \tag{5.30}$$

where C *does not depend on* $t \in \Gamma$ *and* $r > 0$. *Then for* $p > 1$ *the operator* $\mathcal{I}^\alpha$ *is bounded from* $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ *to* $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$ *and for* $p = 1$ *the operator* $\mathcal{I}^\alpha$ *is bounded from* $M_{1,\varphi}(\Gamma)$ *to* $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

Proof. Let $1 \le p < \infty$ and $f \in M_{p,\varphi}(\Gamma)$. By Theorem 5.7, inequality (5.23) is valid. Then from condition (5.26) and inequality (5.23), we get

$$|\mathcal{I}^\alpha f(t)| \lesssim r^\alpha \mathcal{M}f(t) + \int_r^\infty s^{\alpha-\frac{1}{p}-1}\|f\|_{L_p(\Gamma(t,s))}\,ds \le r^\alpha \mathcal{M}f(t) + \|f\|_{M_{p,\varphi}(\Gamma)}\int_r^\infty s^{\alpha-1}\varphi(t,s)\,ds. \tag{5.31}$$

Thus, by (5.30) and (5.31) we obtain

$$\begin{aligned} |\mathcal{I}^\alpha f(t)| &\lesssim \min\left\{\varphi(t,r)^{\frac{p}{q}-1}\mathcal{M}f(t), \varphi(t,r)^\beta \|f\|_{M_{p,\varphi}(\Gamma)}\right\} \lesssim \sup_{r>0}\min\left\{r^{\frac{p}{q}-1}\mathcal{M}f(t), r^{\frac{p}{q}}\|f\|_{M_{p,\varphi}(\Gamma)}\right\} \\ &= (\mathcal{M}f(t))^{\frac{p}{q}}\|f\|^{1-\frac{p}{q}}_{M_{p,\varphi}(\Gamma)}, \end{aligned} \tag{5.32}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Theorem 4.3 and (5.32), we get

$$\|\mathcal{I}^\alpha f\|_{M_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)}\|\mathcal{M}f\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)},$$

if $1 < p < q < \infty$ and

$$\|\mathcal{I}^\alpha f\|_{WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \lesssim \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi}(\Gamma)}\|\mathcal{M}f\|^{\frac{1}{q}}_{M_{1,\varphi}(\Gamma)} \lesssim \|f\|_{M_{1,\varphi}(\Gamma)},$$

if $p = 1 < q < \infty$. □

The following theorem is another of our main results.

Theorem 5.11. *Let* Γ *be a Carleson curve,* $0 < \alpha < 1$, $1 \le p < q < \infty$ *and* $\varphi \in \Omega_p$.

1. *If* $\varphi(t,r)$ *satisfies condition* (5.25), *then condition* (5.30) *is sufficient for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ *to* $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$. *Moreover, if* $1 < p < q < \infty$, *then condition* (5.30) *is sufficient for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ *to* $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.
2. *If* $\varphi \in \mathcal{G}_p$, *then the condition*

$$r^\alpha \varphi(r)^{\frac{1}{p}} \le C\varphi(r)^{\frac{1}{q}}, \tag{5.33}$$

   for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is necessary for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ *to* $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$ *and from* $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ *to* $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.
3. *If* $\varphi \in \mathcal{G}_p$ *satisfies the regularity condition* (5.29), *then condition* (5.33) *is necessary and sufficient for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ *to* $WM_{q,\varphi^{\frac{1}{q}}}(\Gamma)$. *Moreover, if* $1 < p < q < \infty$, *then condition* (5.33) *is necessary and sufficient for the boundedness of the operator* $\mathcal{I}^\alpha$ *from* $M_{p,\varphi^{\frac{1}{p}}}(\Gamma)$ *to* $M_{q,\varphi^{\frac{1}{q}}}(\Gamma)$.

Proof. The first part of the theorem is a corollary of Theorem 5.10.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(t_0, r_0)$ and $t \in \Gamma_0$. By Lemma 5.6 we have $r_0^\alpha \le C\mathcal{I}^\alpha\chi_{\Gamma_0}(t)$. Therefore, by Lemmas 3.3 and 5.6 we have

$$r_0^\alpha \lesssim (\nu(\Gamma_0))^{-\frac{1}{q}}\|\mathcal{I}^\alpha\chi_{\Gamma_0}\|_{L_q(\Gamma_0)} \lesssim \varphi(r_0)^{\frac{1}{q}}\|\mathcal{I}^\alpha\chi_{\Gamma_0}\|_{M_{q,\varphi^{\frac{1}{q}}}(\Gamma)} \lesssim \varphi(r_0)^{\frac{1}{q}}\|\chi_{\Gamma_0}\|_{M_{p,\varphi^{\frac{1}{p}}}(\Gamma)} \lesssim \varphi(r_0)^{\frac{1}{q}-\frac{1}{p}}$$

or

$$r_0^\alpha \varphi(r_0)^{\frac{1}{p}-\frac{1}{q}} \lesssim 1 \text{ for all } r_0 > 0 \Leftrightarrow r_0^\alpha \varphi(r_0)^{\frac{1}{p}} \lesssim \varphi(r_0)^{\frac{1}{q}}.$$

Since this is true for every $t \in \Gamma$ and $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

Remark 5.4. If we take $\varphi(r) = r^{\lambda-1}$ in Theorem 5.9, then condition (5.29) is equivalent to $0 < \lambda < 1 - \alpha p$ and condition (5.28) is equivalent to $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$. Therefore, from Theorem 5.9 we get Theorem C.

Remark 5.5. If we take $\varphi(r) = [r]_1^{\lambda-1}$ in Theorem 5.9, then condition (5.29) is equivalent to $0 < \lambda < 1 - \alpha$ and condition (5.28) is equivalent to $\alpha \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{1-\lambda}$. Therefore, from Theorem 5.9 we get Theorem D.

Acknowledgments: The authors thank the referee(s) for carefully reading the paper and useful comments. The research of V. Guliyev was supported by cooperation Program 2532 TUBITAK RFBR (RUSSIAN FOUNDATION FOR BASIC RESEARCH) under grant no. 119N455 and by the Grant of 1st Azerbaijan-Russia Joint Grant Competition (Agreement Number no. EIF-BGM-4-RFTF-1/2017-21/01/1-M-08). The research of H. Armutcu was supported by program TUBİTAK-BİDEB 2211-A (general domestic doctoral scholarship) under the application no. 1649B031800400.

References

[1] R. P. Agarwal, S. Gala, and M. A. Ragusa, *A regularity criterion in weak spaces to Boussinesq equations*, Mathematics **8** (2020), no. 6, 920.
[2] A. Böttcher and Yu. I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Basel, Boston, Berlin: Birkhäuser Verlag, 1997.
[3] A. Böttcher and Yu. I. Karlovich, *Toeplitz operators with PC symbols on general Carleson Jordan curves with arbitrary Muckenhoupt weights*, Trans. Amer. Math. Soc. **351** (1999), 3143–3196.
[4] A. Eridani, V. Kokilashvili, and A. Meskhi, *Morrey spaces and fractional integral operators*, Expo. Math. **27** (2009), no. 3, 227–239.
[5] A. Eroglu and J. V. Azizov, *A note on the fractional integral operators in generalized Morrey spaces on the Heisenberg group*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **37** (2017), no. 1, 86–91.
[6] V. S. Guliyev, R. V. Guliyev, and M. N. Omarova, *Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces*, Appl. Comput. Math. **17** (2018), no. 1, 56–71.
[7] A. Yu. Karlovich, *Maximal operators on variable Lebesgue spaces with weights related to oscillations of Carleson curves*, Math. Nachr. **283** (2010), no. 1, 85–93.
[8] V. Kokilashvili, *Fractional integrals on curves*, (Russian) Trudy Tbliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR **95** (1990), 56–70.
[9] V. Kokilashvili and A. Meskhi, *Fractional integrals on measure spaces*, Frac. Calc. Appl. Anal. **4** (2001), no. 1, 1–24.
[10] V. Kokilashvili and S. Samko, *Boundedness of maximal operators and potential operators on Carleson curves in Lebesgue spaces with variable exponent*, Acta Math. Sin. (Engl. Ser.) **24** (2008), no. 11, 1775–1800.
[11] M. A. Ragusa, *Operators in Morrey type spaces and applications*, Eurasian Math. J. **3** (2012), no. 3, 94–109.
[12] M. A. Ragusa and A. Scapellato, *Mixed Morrey spaces and their applications to partial differential equations*, Nonlinear Anal. **151** (2017), 51–65.
[13] M. A. Ragusa and A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), no. 1, 710–728.
[14] N. Samko, *Weighted Hardy and singular operators in Morrey spaces*, J. Math. Anal. Appl. **350** (2009), no. 1, 56–72.
[15] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
[16] I. B. Dadashova, C. Aykol, Z. Cakir, and A. Serbetci, *Potential operators in modified Morrey spaces defined on Carleson curves*, Trans. A. Razmadze Math. Inst. **172** (2018), no. 1, 15–29.
[17] V. S. Guliyev, J. J. Hasanov, and Y. Zeren, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. **5** (2011), no. 4, 491–506.
[18] V. Kokilashvili and A. Meskhi, *Boundedness of maximal and singular operators in Morrey spaces with variable exponent*, Arm. J. Math. (Electronic) **1** (2008), no. 1, 18–28.

[19] D. R. Adams, *A note on Riesz potentials*, Duke Math. **42** (1975), 765–778.

[20] A. Eroglu and I. B. Dadashova, *Potential operators on Carleson curves in Morrey spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **66** (2018), no. 2, 188–194.

[21] V. S. Guliyev, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) **193** (2013), no. 2, 211–227.

[22] V. S. Guliyev, M. N. Omarova, M. A. Ragusa, and A. Scapellato, *Commutators and generalized local Morrey spaces*, J. Math. Anal. Appl. **457** (2018), no. 2, 1388–1402.

[23] H. Armutcu, A. Eroglu, and F. Isayev, *Characterizations for the fractional maximal operators on Carleson curves in local generalized Morrey spaces*, Tbil. Math. J. **13** (2020), no. 1, 23–38.

[24] A. Eridani, M. I. Utoyo, and H. Gunawan, *A characterization for fractional integrals on generalized Morrey spaces*, Anal. Theory Appl. **28** (2012), no. 3, 263–268.

[25] A. Eroglu, V. S. Guliyev, and C. V. Azizov, *Characterizations for the fractional integral operators in generalized Morrey spaces on Carnot groups*, Math. Notes **102** (2017), no. 5, 127–139.

[26] V. Burenkov, A. Gogatishvili, V. S. Guliyev, and R. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. **55** (2010), no. 8–10, 739–758.

[27] A. Akbulut, V. S. Guliyev, and R. Mustafayev, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. **137** (2012), no. 1, 27–43.

[28] V. S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009 (**2009**), 503948.

[29] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.