Research Article

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On triangular matroids induced by n_3-configurations

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Abstract: A triangular matroid is a rank-3 matroid whose ground set consists of the points of an n_3-configuration and whose bases are the point triples corresponding to non-triangles within the configuration. Raney previously enumerated the n_3-configurations which induce triangular matroids for $7 \leq n \leq 15$. In this work, the enumeration is extended to configurations having up to 18 points. Several examples of such configurations and their symmetry groups are presented, as well as geometric representations of the triangular matroids induced by these configurations.

Keywords: classification, configuration, matroid, triangle

MSC 2020: 05B30, 05B35

1 Introduction

A (finite) *incidence structure* $\mathcal{X}$ is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is a set of points, $\mathcal{L}$ is a set of lines (or blocks), and $\mathcal{I}$ is an incidence relation on $\mathcal{P}$ and $\mathcal{L}$, i.e., $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. A pair $(p, L) \in \mathcal{I}$ is called a *flag*. In that case, we say that the point p and the line L are incident. We may also say that p lies on L or that L contains p.

A (*combinatorial*) *configuration* C of type (v_r, n_k) is an incidence structure of v distinct points and n distinct lines. We further require that each point is incident with exactly r lines, each line is incident with exactly k points, and any pair of distinct points is contained in at most one line. Consequently, any pair of distinct lines has at most one common point.

If C can be (realized) embedded in the real projective plane in the sense of points and straight lines, then we say that C is *geometric*. Clearly, every geometric configuration is combinatorial, but the converse is not valid, see [1].

A configuration (v_r, n_k) with $v = n$ (and by a simple counting argument also $r = k$) is called a *symmetric configuration*. In this case, we simply write n_k-configuration.

A *triangle* in a configuration is a triple of non-collinear points $\{p_1, p_2, p_3\}$ in which every pair of distinct points is contained in a line. For the sake of simplicity, we write $p_1p_2p_3$ to denote the triangle $\{p_1, p_2, p_3\}$. A configuration with no triangles is called a *triangle-free* configuration. In Figure 1, we see that each non-collinear triple of points in the Fano configuration (left) forms a triangle. On the other hand, the Cremona-Richmond configuration (right) has no triangles.

Two configurations $C_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $C_2 = (\mathcal{P}_2, \mathcal{L}_2)$ are said to be *isomorphic* if there is a bijection $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ which maps $\mathcal{L}_1$ onto $\mathcal{L}_2$. Here, a line $L = \{p_1,\ldots, p_k\} \in \mathcal{L}_1$ is mapped onto $L^\alpha = \{p_1^\alpha,\ldots, p_k^\alpha\}$. Thus, isomorphisms are incidence preserving maps. That is, $p^\alpha \in L^\alpha$ if and only if $p \in L$. In this case,

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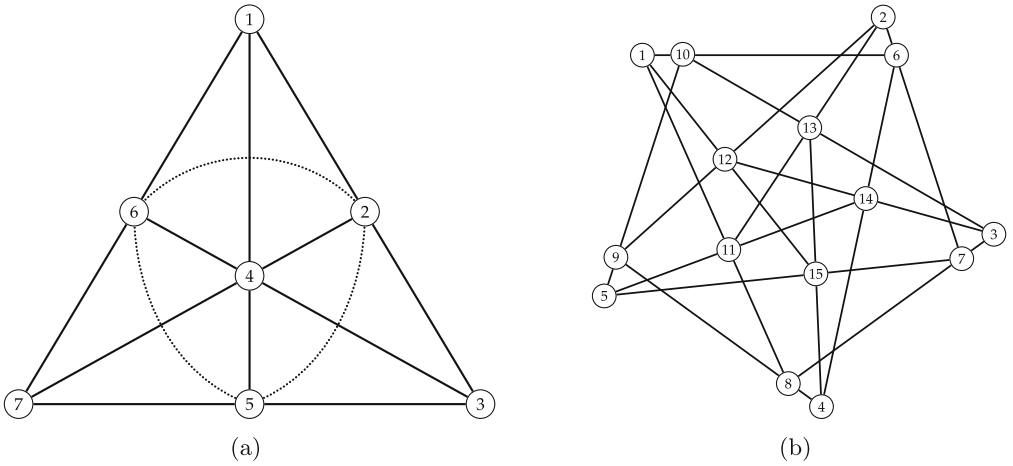


Figure 1: (a) The Fano 7_3-configuration and (b) the Cremona-Richmond 15_3-configuration.

α is called an isomorphism mapping. If furthermore $C = C_1 = C_2$, α is called an automorphism. The automorphism group $\mathrm{Aut}(C)$ is formed by all automorphisms.

A subgroup $G \leq \mathrm{Aut}(C)$ may be seen as a group acting on points, lines, and flags. A *G-orbit* on points, lines, or flags is called a point-, line-, flag-orbit, respectively. The configuration C is called point-, line-, or flag-transitive, if $\mathrm{Aut}(C)$ is transitive on $\mathcal{P}$, $\mathcal{L}$, or the set of flags, respectively. Clearly, flag-transitivity implies both point- and line-transitivity. If a configuration C has h_1 orbits of points and h_2 orbits of lines, we say that C has (h_1, h_2)-orbits. If further $h = h_1 = h_2$, then we simply say that C has *h-orbits*. If $\mathrm{Aut}(C)$ contains a cyclic subgroup acting transitively on the points, then C is called *cyclic*.

Furthermore, a configuration C is called k-cyclic if there exists an automorphism α of order k such that all orbits on points and lines under α are of the same size. In this setting, a cyclic n_3-configuration is an n-cyclic configuration. See [2] for more details.

To each (v_r, n_k)-configuration $C = (\mathcal{P}, \mathcal{L})$, we assign another configuration known as the *dual* configuration. It is the (n_k, v_r)-configuration $C^\star = (\mathcal{L}, \mathcal{P})$, with the roles of points and lines reversed, but with the same incidences. It is clear that C and $C^\star$ have the same *Levi* graphs, except that the colors of classes (the classes of points and lines) are reversed. If furthermore C is isomorphic to its dual $C^\star$, we say that C is *self-dual* and the corresponding isomorphism is called a *duality*. Moreover, a *polarity* is a duality of order 2. A configuration admitting a polarity is called *self-polar*.

We now turn our attention to the relation between n_3-configurations and the class of *rank-3 matroids*. In particular, we consider the class of rank-3 matroids called *triangular matroids*, defined recently by Raney [3].

In the literature, a matroid is a structure that is related directly to the notion of *linear independence* in *vector spaces*. A matroid can be defined in many equivalent ways. Here, we define a matroid in terms of its *bases*.

A (finite) matroid M is an ordered pair $(\mathcal{E}, \mathcal{B})$ of a finite set of elements $\mathcal{E}$, called the *ground* set of M, and a nonempty collection $\mathcal{B}$ of subsets of $\mathcal{E}$, called *bases* (also called maximal independent sets), satisfying the following so-called "*basis exchange property*:" If B_1 and B_2 are two distinct bases in $\mathcal{B}$, then for every $a \in B_1 - B_2$, there exists $b \in B_2 - B_1$ such that $(B_1 - \{a\}) \cup \{b\}$ is a basis in $\mathcal{B}$.

Applying the basis exchange property repeatedly, one can show that any two bases in $\mathcal{B}$ share the same cardinality. A matroid with bases of cardinality r is said to have rank r.

Matroid theory has been extensively related and applied to different areas of mathematics including *geometry*, *topology*, *group theory*, and *coding theory*.

If $C = (\mathcal{P}, \mathcal{L})$ is any n_3-configuration, then C defines a rank-3 *linear* (or *vector*) matroid $M(C) = (\mathcal{E}, \mathcal{B})$. Here $\mathcal{E} = \mathcal{P} = \{p_1, \ldots, p_n\}$ and $\mathcal{B}$ consists of all of the non-collinear point triples $p_i p_j p_k$ from $\mathcal{P}$. A simple counting argument shows that $\mathcal{B}$ contains $\binom{n}{3} - n$ triples. In this case, we say that the linear matroid $M(C)$ is *induced* by C. It is worth mentioning that $M(C)$ is of rank 3 since each basis in $\mathcal{B}$ is formed by a triple of non-collinear points in $\mathcal{E}$.

Table 1: n_3-configurations inducing triangular matroids and the number of non-isomorphic triangular matroids induced by n_3-configurations

N	$\#_c(n)$	$\#_{\text{tri}}(n)$	$\#_{\text{mat}}(n)$
7	1	0	0
8	1	0	0
9	3	0	0
10	10	1	1
11	31	0	0
12	229	1	1
13	2,036	1	1
14	21,399	4	4
15	245,342	220	*173*
16	**3,004,881**	**6,053**	**2,634**
17	**38,904,499**	**166,286**	**19,930**
18	530,452,205	**4,126,028**	**101,910**

A fundamental example of a matroid is a *uniform* matroid. Let $\mathcal{E}$ be a ground set of n elements and $\mathcal{B}$ be the set of all possible k-subsets of $\mathcal{E}$. Then $(\mathcal{E}, \mathcal{B})$ defines a matroid called the uniform matroid $\mathcal{U}_{k,n}$.

In [3], Raney presented a special class of matroids, called triangular matroids, defined as follows:

Definition 1. Let $C = (\mathcal{P}, \mathcal{L})$ be an n_3-configuration. A triangular matroid $M_{\text{tri}}(C)$, if it exists, is a matroid $(\mathcal{E}, \mathcal{B})$ such that: (I) the ground set $\mathcal{E}$ is the set of points $\mathcal{P}$ of C and (II) the collection of bases $\mathcal{B}$ is the set of all non-triangular point triples in C.

In this setting, the work in [3] intended to answer the question: if C induces $M_{\text{tri}}(C)$, then what are the conditions on the triangles of C that must be satisfied?

We note that if C is any triangle-free n_3-configuration, then C induces $M_{\text{tri}}(C)$. In fact, $M_{\text{tri}}(C)$ is isomorphic to $\mathcal{U}_{3,n}$. This is because each triple of points in C forms a basis for $M_{\text{tri}}(C)$. Hence, the uniform matroid $\mathcal{U}_{3,n}$ is constructed. See [3] for more details.

It is possible that two or more non-isomorphic n_3-configurations produce isomorphic copies of some induced triangular matroids. This is discussed in some detail in Section 2.

The search in [3] was established on n_3-configurations for $n \leq 15$. In this work, we extend that list to up to $n \leq 18$. Furthermore, we present some structural properties for the constructed configurations. We also correct one miscalculated value which appeared in [3] (written in italics in Table 1).

Table 1 presents the main results of the search for triangular matroids induced by n_3-configurations. The new results are highlighted in bold. Here, $\#_c(n)$ denotes the number of all non-isomorphic (combinatorial) n_3-configurations, $\#_{\text{tri}}(n)$ denotes the number of n_3-configurations which induce triangular matroids, and $\#_{\text{mat}}(n)$ denotes the number of non-isomorphic triangular matroids induced from these n_3-configurations. Note that the entry for $\#_c$ (18), namely, 530,452,205, was faded to indicate that a special search was done in that case. That special search was used also for the cases $n = 7$ through $n = 17$ for further validation of the results. This search is explained in Section 3.

2 Properties

We now discuss some of the main properties of n_3-configurations which induce triangular matroids. In order to keep the present work self-contained, we present some of the theoretical results related to triangular matroids. For proofs and further discussion, the reader can see [3].

A *complete quadrangle* in a configuration is a set of four points a, b, c, and d, no three collinear, for which all possible lines connecting each pair of distinct points exist. A *near-complete quadrangle* in a config-

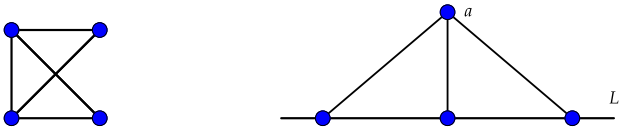


Figure 2: A near-quadrangle and a near-pencil in a configuration. (a) Near quadrangle, (b) near-pencil.

uration is a complete quadrangle missing exactly one line connecting one pair of points. A *near-pencil* in a configuration consists of the points of a line L and a point a not on L for which a is incident to all points on L. Figure 2 shows a near-complete quadrangle and a near-pencil.

Theorem 1. *An n_3-configuration C induces a triangular matroid $M_{\text{tri}}(C)$ if and only if C does not contain either a near-complete quadrangle or a near-pencil.*

If C satisfies the conditions of Theorem 1, then it induces $M_{\text{tri}}(C) = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is the set of points in C, and $\mathcal{B}$ is the set of all non-triangular triples of points in C.

Theorem 2. *Let C be an n_3-configuration with n triangles. If every point in C is incident to exactly three triangles and no pair of points is incident to more than one triangle, then C is isomorphic to C_{tri}, where C_{tri} is the configuration formed by the n triangles in C.*

As an example of Theorem 2, we present one of the four 14_3-configurations inducing triangular matroids. It is the cyclic 14_3-configuration (Figure 3) whose automorphism group is C_{14} of order 14:

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}	l_{14}
1	1	1	2	2	3	3	4	5	5	6	7	10	11
2	4	6	4	9	6	9	13	7	9	8	8	11	12
3	5	7	8	10	11	12	14	12	13	10	14	13	14

This configuration possesses 14 triangles, triangles $t_1, t_2, \ldots, t_{14}$. Every point is contained in exactly three triangles, and no pair of points is incident to more than one triangle:

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}
1	1	1	2	2	3	4	4	5	6	6	7	9	11
2	3	5	3	8	11	5	8	9	7	10	12	10	13
4	6	7	9	10	12	13	14	12	8	11	14	13	14

These triangles form another configuration on 14 points which is isomorphic to the cyclic one. It is an isomorphic copy of the configuration in Figure 3.

Note that there exists no one-to-one correspondence between the triangular matroids themselves and the n_3-configurations which induce them. For instance, there are several triangle-free n_3-configurations (for $n \geq 18$) inducing the same exact uniform matroid on n points, see [4,5].

But this same idea is also applicable for smaller n_3-configurations. For that purpose, we present two 15_3-configurations ($15A$ and $15B$) in Figure 4.

Each of these configurations consists of six triangles inducing two isomorphic triangular matroids. Figure 5 shows the geometric representations of the two isomorphic triangular matroids induced by configurations $15A$ and $15B$. In the geometric representation, each collinear triple (that is, each non-basis element) corresponds to a triangle in the configuration it was induced by.

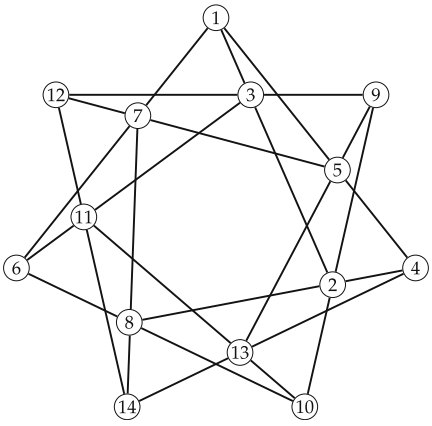


Figure 3: The cyclic 14_3-configuration whose automorphism group order is 14.

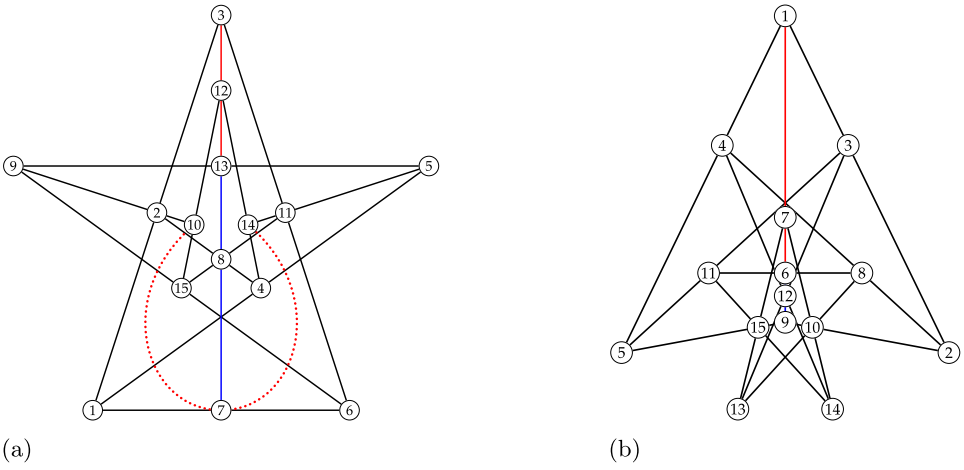


Figure 4: Two configurations of automorphism groups $C_2 \times C_2$ (of order 4) and C_2 (of order 2). (a) 15A, (b) 15B.

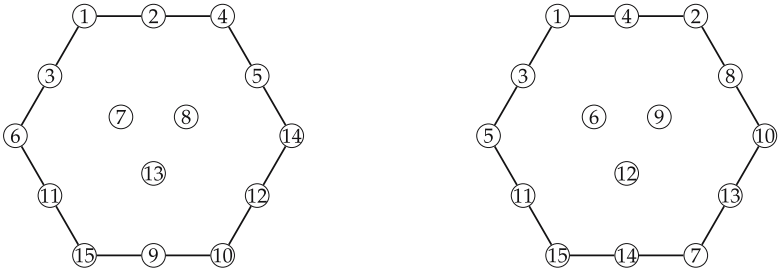


Figure 5: Two isomorphic triangular matroids induced by 15*A* (left) and 15*B* (right).

3 The search

Assume that a list of all n_3-configurations C (for some $n \geq 7$) is available at hand. One can test the configurations in the list for the tests stated in Theorem 1. This creates a (smaller or identical) list of all non-isomorphic n_3-configurations inducing all triangular matroids $M_{\text{tri}}(C)$. We call this step "*INDUCE*." This list still can have isomorphic copies of triangular matroids (induced by several non-isomorphic configurations). Hence, we do one last test to clean up the isomorphic copies of these triangular matroids. We call this step "*CLEAN UP*." These two steps were implemented independently by the two authors for $7 \leq n \leq 15$, and by the first author for $n = 16$ and $n = 17$. The results were in full agreement after all.

Another approach was to directly generate all n_3-configurations inducing triangular matroids without having (storing) the whole list of non-isomorphic n_3-configurations. In what follows, we describe the main steps of this (special) approach. This approach was initially applied for the case $n = 18$ as indicated in Table 1. The same approach was used again for the cases $n = 7$ through $n = 17$ for further validation.

An n_3-configuration C can be represented by a $\{0, 1\}$-incidence matrix A with n rows and n columns corresponding to points and lines, respectively. The (i, j)-entry of A is 1 if the point indexed i is incident with the line indexed j. It is 0 otherwise.

As A is representing an n_3-configuration, it has exactly three ones in each row and exactly three ones in each column. Moreover, the dot product of two distinct rows (or two distinct columns) is at most one. Two incidence matrices (and hence their corresponding configurations) are said to be isomorphic if one can be obtained from the other by permuting the rows and the columns.

We now describe the search algorithm to classify the n_3 configurations inducing non-isomorphic triangular matroids. This algorithm can be considered as an example of the *orderly generation* method, see [6]. It has two main parts: the *generation* and the *isomorphism* test.

The search first starts with the generation procedure which carries out a row-by-row (or a point-by-point) backtrack search to consider all possible incidence matrices of n_3-configurations having neither a near-quadrangle nor a near-pencil. It starts initially with the empty matrix (all-zero entries), and it starts to build a row at each step.

Once a row is constructed by the generation procedure, the algorithm performs another test to check whether the created incidence matrix agrees with the *canonical* one. Here, canonical might have a different meaning depending on what our canonical matrix is defined to be. One example (which we chose) is to choose the lexicographically least form of the incidence matrix. If the incidence matrix is canonical, we proceed to the next row and continue the search. Otherwise, we reject it and backtrack.

The algorithm ends up with a list of non-isomorphic n_3-configurations which induce triangular matroids. This list is similar (up to isomorphism) to the list created by the INDUCE step. At this point, we take that list again to the CLEAN UP step to produce all of the non-isomorphic triangular matroids induced by these non-isomorphic n_3-configurations. This final CLEAN UP step is crucial since it is possible to produce isomorphic copies of induced triangular matroids from non-isomorphic copies of n_3-configurations as discussed in Section 2.

We remark that we compute the lexicographically least representative of the isomorphism class of a matrix using our own algorithm. The complexity of this algorithm is exponential in the size of the input. No fast algorithm to solve this problem is known.

Moreover, the lexicographically least representative can be replaced by the *canonical representative* which can be computed using the idea of *canonical augmentation* due to McKay [7]. In almost all cases these representatives are different. We also tried this method using *nauty* [8] to compute the canonical representative. We found that orderly generation using the lexicographically least representative worked better for us. This may not be seen as a critique of "canonical augmentation." We did not try very hard to make it work, so a comparison is unfair. Again in either methods, no fast (i.e. polynomial) algorithm to solve this problem is known.

4 Results

The main results of the search described in Section 3 are presented in Table 1.

The search was done on a single Mac Laptop (with a Processor 2.2 GHz). The CPU needed for the search on $n = 15, 16, 17$, and 18 was 2 min, 26 min, 9 h, and 9.25 days, respectively.

We note that all the presented figures in this paper were produced manually. The main purpose of these drawings is to emphasize some aspects such as symmetry of the groups and geometric realizations, if possible. Further study is advised here to study in more detail whether the objects can be realized geometrically. But this direction is not the purpose of the presented work.

We present some of the n_3-configurations inducing triangular matroids that were constructed using the algorithm of Section 3 in what follows. But first, we present Table 2 which shows the triangle distribution of n_3-configurations inducing triangular matroids. Here, #tri(n) and #mat(n) are defined in the same way as in Table 1. The two columns denoted by $\Delta_{\text{tri}}(n)$ and $\Delta_{\text{mat}}(n)$ present the number (and multiplicity) of the triangles of the related n_3-configurations. That is, an entry a^x means that there are x n_3 configurations inducing triangular matroids each of which have a triangles. For instance, the first entry written as 20^1 means that the 10_3-configuration has 20 triangles and this type occurs once.

We might have (in the same table row) a^x and a^y in the third and fourth columns, respectively, with $y \leq x$. This means that we have several $(y - x + 1)$ non-isomorphic n_3-configurations, each of whose induced triangular matroids is shared by at least one other n_3-configuration. For instance, in the row of $n = 15$ we have 6^4 (in $\Delta_{\text{tri}}(n)$) and 6^3 (in $\Delta_{\text{mat}}(n)$). This means that there exist two non-isomorphic 15_3-configurations, each having six triangles, that produce isomorphic copies of the same induced triangular matroid. This fact has already been discussed in Section 2 (Figure 4).

Table 3 gives the distribution of non-trivial automorphism groups of n_3-configurations which only produce non-isomorphic induced triangular matroids. The '*ago*' column presents the automorphism group orders while the '*Aut*' column presents the automorphism group type. An entry a^x in the *ago* column means that we have x geometries with automorphism group order a. In the same corresponding position of the *Aut* column, the group type is shown. The group types listed in the last column of Table 3 can be described using the Gap-command "StructureDescription," see [9] for further details.

We use C_n and D_n to denote the cyclic group of order n and the dihedral group of order $2n$, respectively. For groups H and K, let $H : K$ be a split extension of H by K (with normal subgroup H).

Table 2: Triangle distribution of n_3-configurations inducing triangular matroids

n	#tri(n)	$\Delta_{\text{tri}}(n)$	$\Delta_{\text{mat}}(n)$	#mat(n)
10	1	20^1	20^1	1
12	1	12^1	12^1	1
13	1	13^1	13^1	1
14	4	$6^1; 10^2; 14^1$	$6^1; 10^2; 14^1$	4
15	220	$0^1; 4^2; 6^4; 7^9; 8^{36}; 9^{38}; 10^{64}; 11^{17}; 12^{20}; 13^{18}; 14^{10}; 15^1$	$0^1; 4^2; 6^3; 7^9; 8^{29}; 9^{34}; 10^{58}; 11^{14}; 12^{11}; 13^7; 14^4; 15^1$	173
16	6,053	$3^1; 4^4; 5^{26}; 6^{106}; 7^{401}; 8^{1053}; 9^{1602}; 10^{1501}; 11^{796}; 12^{324}; 13^{139}; 14^{62}; 15^{20}; 16^{18}$	$3^1; 4^4; 5^{12}; 6^{31}; 7^{97}; 8^{277}; 9^{587}; 10^{834}; 11^{531}; 12^{171}; 13^{47}; 14^{24}; 15^7; 16^{11}$	2,634
17	166,286	$0^1; 2^6; 3^{18}; 4^{172}; 5^{901}; 6^{4595}; 7^{14986}; 8^{32868}; 9^{44726}; 10^{38145}; 11^{19183}; 12^{6514}; 13^{2303}; 14^{1140}; 15^{523}; 16^{185}; 17^{18}; 18^2$	$0^1; 2^2; 3^2; 4^8; 5^{22}; 6^{77}; 7^{240}; 8^{876}; 9^{2685}; 10^{6036}; 11^{6296}; 12^{2777}; 13^{619}; 14^{176}; 15^{72}; 16^{34}; 17^6; 18^1$	19,930
18	4,126,028	$0^4; 1^9; 2^{109}; 3^{856}; 4^{6293}; 5^{35717}; 6^{148976}; 7^{430096}; 8^{836226}; 9^{1070498}; 10^{881174}; 11^{463553}; 12^{171309}; 13^{55145}; 14^{17702}; 15^{5685}; 16^{2017}; 17^{534}; 18^{87}; 19^{27}; 20^9; 21^1; 22^1$	$0^1; 1^1; 2^2; 3^5; 4^{14}; 5^{30}; 6^{93}; 7^{306}; 8^{1239}; 9^{5013}; 10^{16845}; 11^{33667}; 12^{29435}; 13^{11366}; 14^{2879}; 15^{703}; 16^{216}; 17^{67}; 18^{18}; 19^5; 20^3; 21^1; 22^1$	101,910

Table 3: Automorphism group types (excluding trivial groups)

n	*ago*	Aut
10	120	S_5
12	72	$(C3 \times A4) : C2$
13	39	$C13 : C3$
14	4, 8, 14	C_4, D_4, C_{14}
15	2^{25}, 3^9, 4^6, 4, 5, 6^3, 6^5, $8,8^2$, 10, 24, 30, 48^2, 720	C_2, C_3, $C_2 \times C_2$, C_4, C_5, C_6, S_3, D_4, $C_2 \times C_2 \times C_2$, D_5, $C_2 \times A_4$, $C_3 \times D_5$, $C_2 \times S_4$, S_6
16	2^{114}, 3^{19}, 4^8, 4, 6, 6^4, 8^2, 8, 12, 16, 32, 32, 96	C_2, C_3, $C_2 \times C_2$, C_4, C_6, S_3, D_4, $C_4 \times C_2$, D_6, C_{16}, $(C_8 \times C_2) : C_2$, SD_{16}, $((C_4 \times C_4) : C_3) : C_2$
17	2^{288}, 3^4, 4^{19}, 4, 6, 12^2, 17	C_2, C_3, $C_2 \times C_2$, C_4, S_3, D_6, C_{17}
18	2^{568}, 3^{49}, 4^{13}, 4^9, 6^7, 6^4, 8^2, 12^2, 18^2, 24, 48^2, 72, 144^2	C_2, C_3, $C_2 \times C_2$, C_4, S_3, C_6, $C_2 \times C_2 \times C_2$, D_6, C_{18}, $C_2 \times C_2 \times S_3$, $C_2 \times S_4$, $(S_3 \times S_3) : C_2$, $S_3 \times S_4$

We now present some properties of some of the n_3-configurations inducing non-isomorphic triangular matroids. In what follows, we write n_3^x-configuration to denote an n_3-configuration whose automorphism group order is x.

4.1 14_3-configurations inducing triangular matroids

For the 14_3-configurations, there is a unique 14_3-configuration which induces a triangular matroid with a transitive group. Its automorphism group is C_{14} of order 14. Hence, it has one orbit on points (and one orbit on lines as well). It is the 14_3^{14}-configuration presented in Figure 3. It is cyclic as it has a transitive automorphism group C_{14}.

Figure 6 shows one of the four 14_3-configurations (namely, 14_3^8 with six triangles) which induces the triangular matroid described by its geometric representation. This configuration has 4-orbits with automorphism group D_4 of order 8. Points of the same color belong to the same point-orbit.

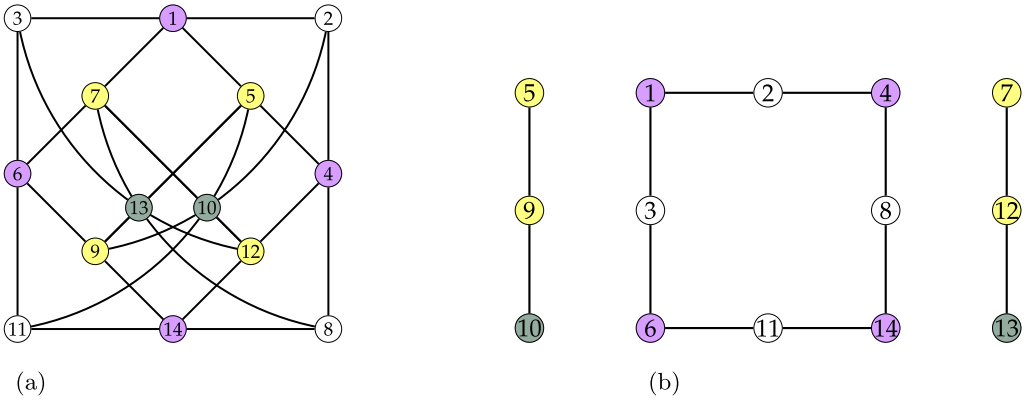


Figure 6: The 14_3^8-configuration which induces a triangular matroid. (a) 14_3^8-configuration, (b) induced a triangular matroid.

4.2 15_3-configurations inducing triangular matroids

There are 220 15_3-configurations inducing triangular matroids. Among these we found several examples of configurations inducing isomorphic triangular matroids. Thus, we cleaned up this list to get exactly 173 15_3-configurations which induce non-isomorphic triangular matroids. There are only two configurations among these with transitive groups. Namely, these are the ones with automorphism groups S_6 and $C_3 \times D_5$ of orders 720 and 30, respectively.

An example of a 15_3-configuration which induces a triangular matroid is already presented in Figure 1. It is the Cremona-Richmond 15_3^{720}-configuration. Its automorphism group (S_6) is transitive on its flags (hence on points and on lines). This configuration is the smallest triangle-free configuration. Hence, its induced triangular matroid is the uniform matroid on 15 points.

Figure 7 presents the unique 15_3^5-configuration. It induces a triangular matroid whose geometric representation is also presented. As it can be seen, it has ten triangles. It has five points incident with three triangles, five points incident with two triangles, and five points incident with only one triangle. This configuration has 3-orbits as it has three point-orbits (of the same size 5) under the action of C_5. The three point-orbits are shown in the figure as different colors. Its automorphism group can be generated by (1 6 14 12 9)(2 5 11 3 7)(4 8 13 15 10).

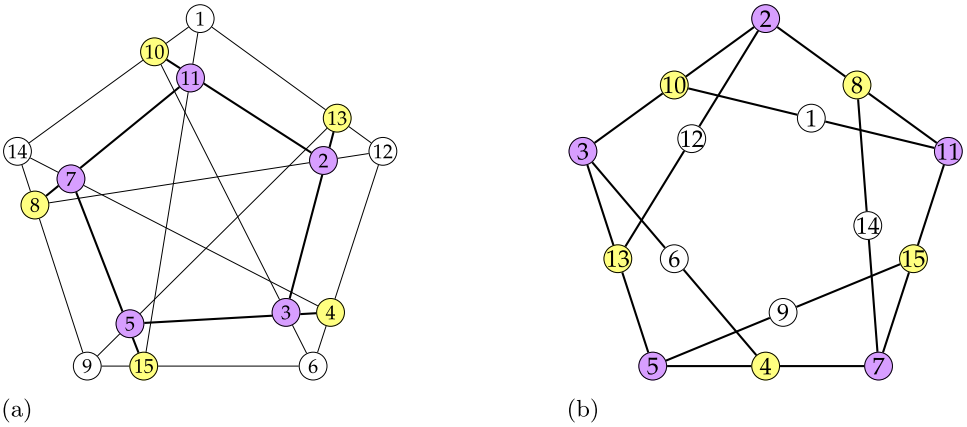
We next present the unique 15_3^{10}-configuration which induces a triangular matroid. These two structures are shown in Figure 8. The automorphism group here is D_5 of ago 10. Some points in the diagram of the configuration were duplicated to show a better presentation of the structure of the group which can be generated by

$$(1\ 2)(3\ 14)(4\ 7)(6\ 15)(8\ 9)(10\ 12)(11\ 13) \text{ and } (1\ 3)(2\ 11)(4\ 9)(5\ 6)(7\ 13)(8\ 14)(12\ 15).$$

It induces two point-orbits of lengths 10 and 5. These two orbits are shown in two different colors in the diagram. Each point of the configuration is incident to two triangles where the number of triangles is 10. The geometric representation of the induced triangular matroid is also presented. As it can be seen, it has ten triangles. It is in fact an $(15_2, 10_3)$-configuration.

There is a unique 15_3^{12}-configuration which induces a triangular matroid. It has 14 triangles. Figure 9 shows this configuration along with its induced triangular matroid. It has $(4, 3)$-orbits under its automorphism group D_6 of order 12. Again, we give different colors to points in different orbits.

Note that in the geometric representation of the induced triangular matroid, four collinear points in the geometric representation means that each of the four possible point triples taken from these four points defines a triangle in the configuration. For instance, any point triples taken from the four collinear points (complete quadrangle) $\{3, 4, 12, 15\}$ defines a triangle in the configuration.



(a) (b)

Figure 7: The 15_3^5-configuration with ago 5 inducing a triangular matroid. (a) 15_3^5-configuration, (b) induced triangular matroid.

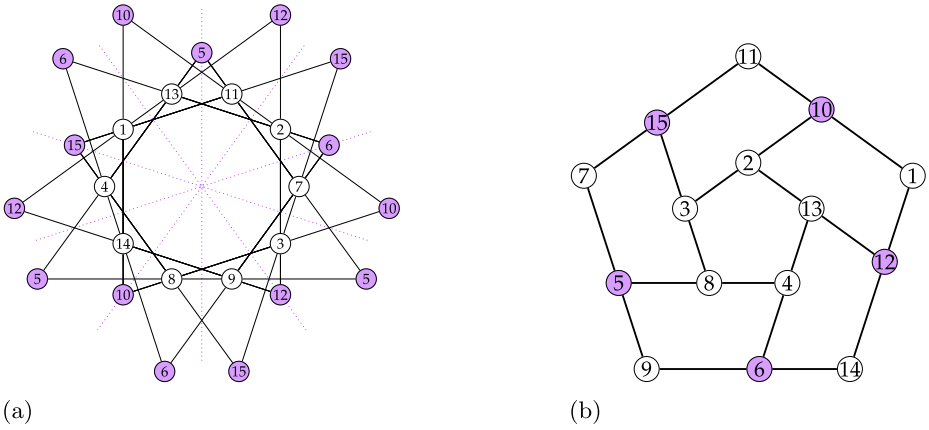


Figure 8: The 15_3^{10}-configuration along with its induced triangular matroid. (a) 15_3^{10}-configuration, (b) induced triangular matroid.

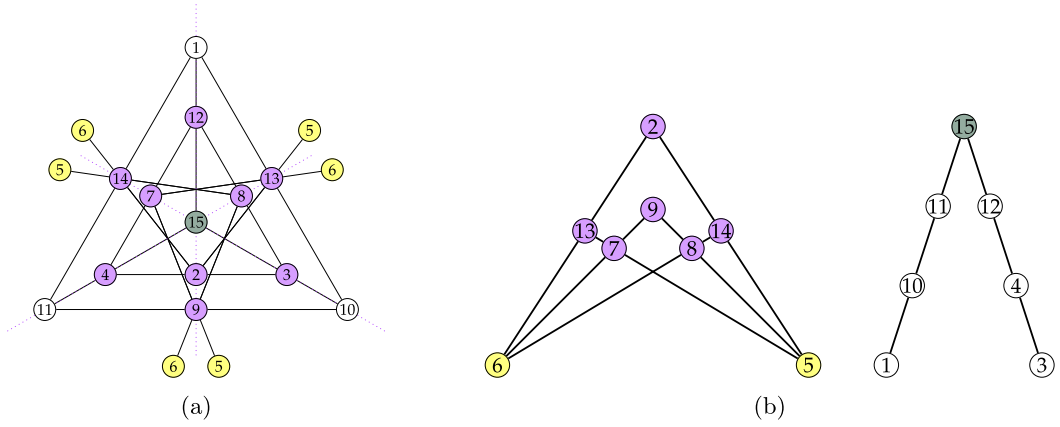


Figure 9: The 15_3^{12}-configuration along with its induced triangular matroid. (a) 15_3^{12}-configuration, (b) induced triangular matroid.

The unique 15_3^{30}-configuration is drawn in Figure 10. The automorphism group $C_3 \times D_5$ is transitive on this configuration. It has one orbit on points and one orbit on lines.

This configuration has 15 triangles. Each point is incident to three triangles, with no pair of points incident to a triangle more than once. Its triangles are blocks of another 15_3 configuration isomorphic to this 15_3^{30}-configuration, see Theorem 2.

The 15 triangles of this configuration are as follows:

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}
1	1	1	2	2	2	3	3	3	4	5	5	6	6	7
10	12	13	4	5	11	4	6	10	12	7	8	7	9	8
11	14	15	9	12	15	8	13	14	13	11	14	10	15	9

4.3 16_3-configurations inducing triangular matroids

There are 2,634 16_3-configurations inducing non-isomorphic copies of triangular matroids. Among these there are four configurations with transitive automorphism groups. Their automorphism group orders are 16, 32, 32, and 96.

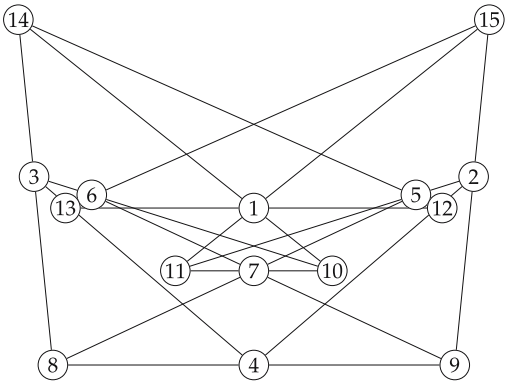


Figure 10: The unique 15_3^{30}-configuration.

One of the four configurations with a transitive automorphism group is the cyclic 16_3^{16}-configuration. This configuration is presented in Figure 11. Its automorphism group is C_{16} and it is flag-transitive. It has 16 triangles inducing an isomorphic 16_3-configuration.

There are two 16_3^{32}-configurations ($16A$ and $16B$) inducing non-isomorphic triangular matroids. The automorphism groups for $16A$ and $16B$ are $(C_8 \times C_2) : C_2$ and SD_{16} (the semidihedral group of order 32), respectively. Both configurations have 16 triangles in the setting of Theorem 2. These two groups are transitive (but not cyclic). Figure 12 presents these two configurations.

The 16_3-configuration inducing a triangular matroid with the largest automorphism group order is presented in Figure 13. Its automorphism group $((C_4 \times C_4) : C_3) : C_2$ of order 96 is flag-transitive. It has 16 triangles.

4.4 17_3-configurations inducing triangular matroids

In this case, we found 166,286 17_3-configurations inducing triangular matroids. Applying the CLEAN UP procedure results in exactly 19,930 configurations inducing non-isomorphic triangular matroids. Among these, we found two 17_3^{12}-configurations with the automorphism group D_6 of order 12. One is the unique triangular-free (self-polar) configuration on 17 points. The other one is a self-polar configuration with 14 triangles. Both have 4-orbits and the length of their orbits on points and lines is $6, 6, 3$, and 2. Figure 14 shows their incidence matrices.

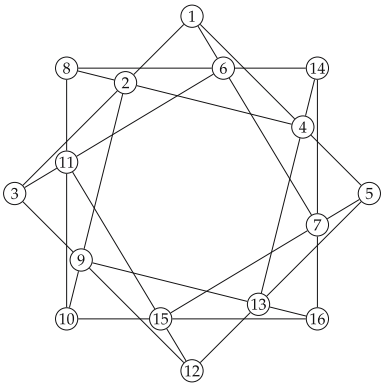


Figure 11: The unique cyclic 16_3^{16}-configuration inducing a triangular matroid.

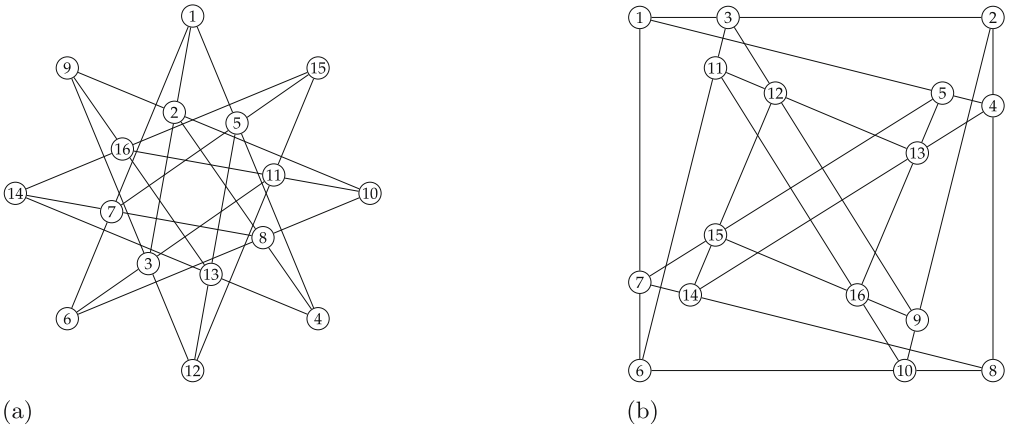


Figure 12: The two 16_3^{32}-configurations. (a) 16A, (b) 16B.

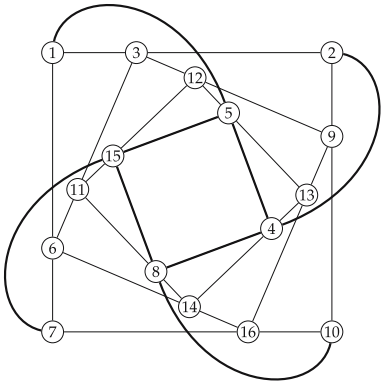


Figure 13: The unique 16_3^{96}-configuration inducing a triangular matroid.

	l_1	l_3	l_5	l_{13}	l_{16}	l_{17}	l_2	l_4	l_8	l_{10}	l_{11}	l_{12}	l_6	l_{15}	l_7	l_9	l_{14}
1	X	X					X										
2	X		X					X									
6		X		X								X					
11			X		X						X						
16					X	X			X								
17				X		X				X							
3	X												X		X		
7		X												X			X
10			X											X		X	
12				X									X			X	
13					X								X				X
15						X								X	X		
4							X		X							X	
9								X		X							X
14											X	X			X		
5							X			X	X						
8								X	X			X					

(a)

	l_1	l_2	l_4	l_{15}	l_{16}	l_{17}	l_3	l_5	l_8	l_{11}	l_{12}	l_{13}	l_6	l_{14}	l_7	l_9	l_{10}
1	X	X					X										
2	X		X					X									
4		X	X						X								
15				X	X					X							
16				X		X					X						
17					X	X						X					
3	X												X		X		
5		X											X			X	
8			X										X				X
11				X										X	X		
12					X									X		X	
13						X								X			X
6							X	X	X								
14										X	X	X					
7							X			X							X
9								X			X					X	
10									X			X			X		

(b)

Figure 14: The incidence matrices of (a) the unique triangle-free 17_3-configuration and (b) a self-polar 17_3-configuration with 14 triangles.

4.5 18_3-configurations inducing triangular matroids

In this case, we see a considerable increase in the number of n_3-configurations inducing triangular matroids. We found (up to isomorphism) 4,126,028 18_3-configurations which induce triangular matroids, and 101,910 non-isomorphic triangular matroids in total. There are 23 triangle distributions in this case.

We first present a 6-cyclic (in the terminology of [2]) 18_3^6-configuration which induces a triangular matroid. Its automorphism group is C_6 of order 6. It has three orbits on points and three orbits on lines where each orbit has length 6. Figure 15 shows this configuration which happens to be a self-polar configuration. The point-orbits are presented as different colors. We also present the geometric representation of the induced triangular matroid.

We found three flag-transitive configurations among the 4,126,028 18_3-configurations. Their automorphism groups are C_{18}, C_{18}, and $(S_3 \times S_3) : C_2$ with orders 18, 18, and 72, respectively. We also found one point-transitive and another line-transitive configuration. Both of these have $S_3 \times S_4$ as their automorphism group with order 144.

Figure 16 shows a realization for the 18_3^{72}-configuration which is flag transitive. It has six triangles where each point is incident to exactly one triangle.

The other two flag-transitive configurations have automorphism group C_{18}. They are cyclic configurations having 18 triangles each, but their respective triangular matroids are non-isomorphic. We present these two configurations in Figure 17.

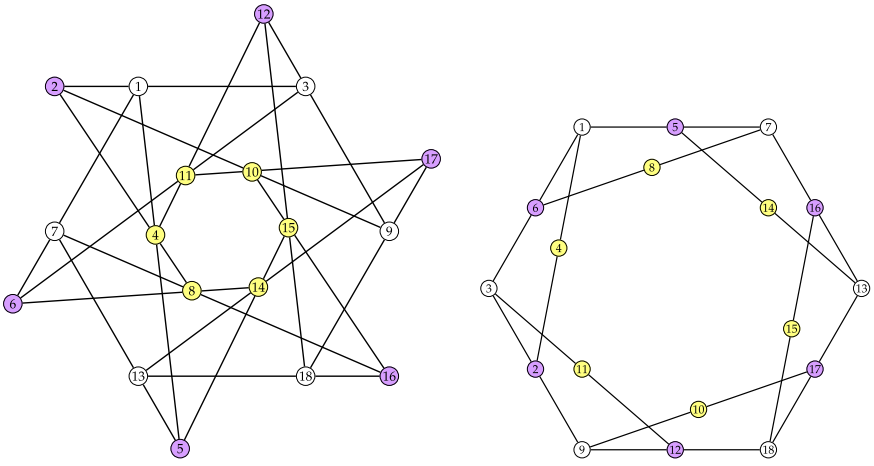


Figure 15: Two cyclic 18_3^{18}-configurations each of them with 18 triangles inducing non-isomorphic triangular matroids.

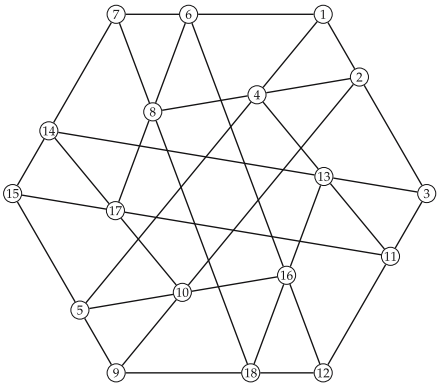


Figure 16: The unique 18_3^{72}-configuration inducing a triangular matroid.

Figure 18 shows two incidence matrices associated with the two 18_3^{144}-configurations; both configurations have the automorphism group $S_3 \times S_4$. The configurations 18L and 18P are line-transitive ((2,1)-orbits) and point transitive ((1,2)-orbits), respectively. Configuration 18L has two point-orbits of length 12 and 6. On the other hand, configuration 18P has two line-orbits of length 12 and 6. In fact, these two configurations are dual. By dual configuration, we mean that the roles of points and lines in the dual configuration are

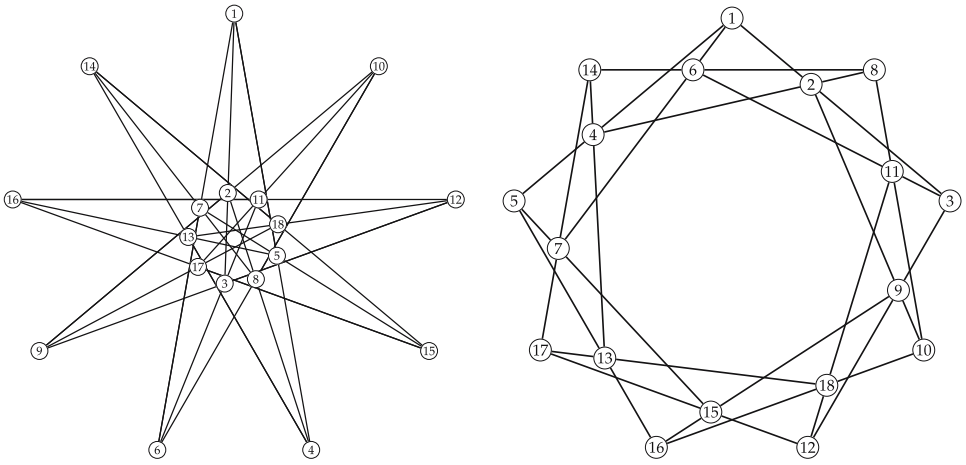
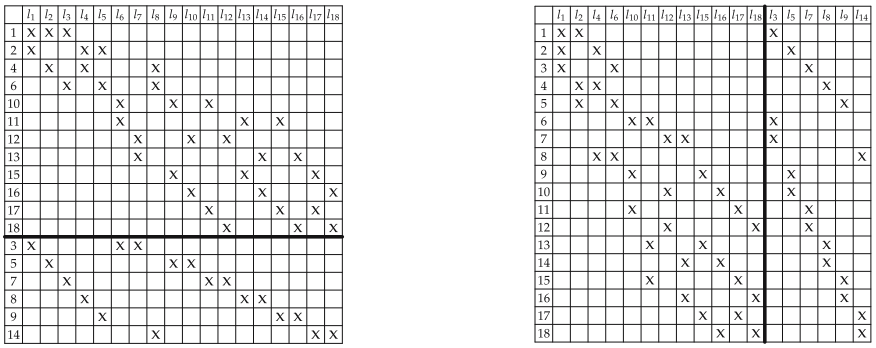
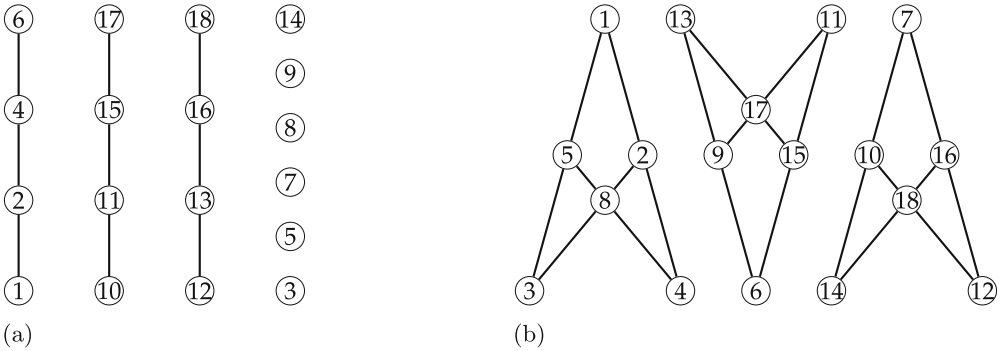


Figure 17: Two cyclic 18_3^{18}-configurations with 18 triangles inducing non-isomorphic triangular matroids.



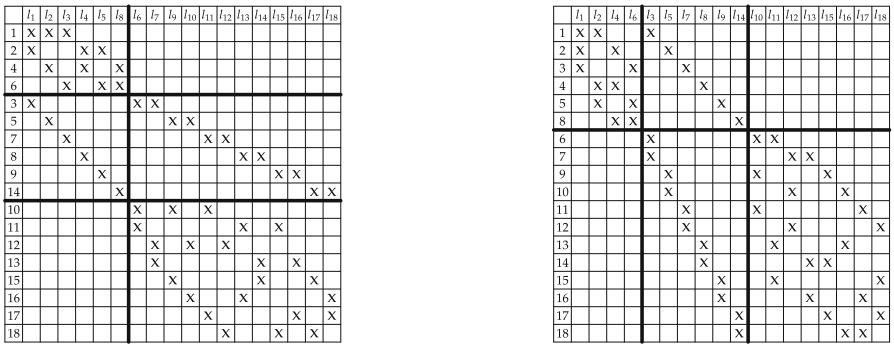
(a) (b)

Figure 18: Two dual 18_3^{144}-configurations inducing non-isomorphic triangular matroids. (a) 18L, (b) 18P.



(a) (b)

Figure 19: Two geometric representations of triangular matroids induced by (a) 18L and (b) 18P.



(a) (b)

Figure 20: A pair of dual 18_3^{48}-configurations. (a) An incidence matrix associated with an 18_3^{48}-configuration inducing a triangular matroid. It is (3, 2)-orbits and it consists one of the six 3-regular graphs on 8 points. (b) The dual of (a).

reversed. The configuration $18L$ consists of three $(4_3, 6_2)$-subconfigurations (or complete quadrangles), while configuration $18P$ consists of three $(6_2, 4_3)$-subconfigurations (or complete quadrilaterals).

The geometric representations of the induced triangular matroids by the dual pair $18L$ and $18P$ are presented in Figure 19.

Another example of a dual pair of 18_3-configurations inducing non-isomorphic triangular matroids is presented in Figure 20. This time we provide the incidence matrices associated with this dual pair of 18_3^{48}-configurations.

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