Research Article

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Inequalities for the generalized trigonometric and hyperbolic functions

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Abstract: In this paper, the authors present some inequalities of the generalized trigonometric and hyperbolic functions which occur in the solutions of some linear differential equations and physics. By these results, some well-known classical inequalities for them are improved, such as Wilker inequality, Huygens inequality, Lazarević inequality and Cusa-Huygens inequality.

Keywords: generalized trigonometric and hyperbolic functions, Wilker inequality, Huygens inequality, Lazarević inequality, Cusa-Huygens inequality

MSC 2020: 26D05, 26D07

1 Introduction

The well-known Wilker inequality for trigonometric functions

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad \text{for } x \in \left(0, \frac{\pi}{2}\right) \tag{1}$$

was proposed by Wilker [1] and proved by Sumner et al. [2]. The hyperbolic counterpart of (1) was established in [3] as follows:

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2, \quad \text{for } x \in (0, \infty). \tag{2}$$

A related inequality that is of interest to us is the Huygens inequality [4,5]:

$$2\sin x + \tan x > 3x, \quad \text{for } x \in \left(0, \frac{\pi}{2}\right), \tag{3}$$

$$2\sinh x + \tanh x > 3x, \quad \text{for } x \in (0, \infty). \tag{4}$$

The Wilker inequalities (1), (2) and the Huygens inequalities (3), (4) have attracted much interest of many mathematicians. Many generalizations, improvements and refinements of the Wilker inequality and the Huygens inequality can be found in the literature [6,7] and references therein.

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In [6, Theorems 5 and 8], inequalities (1)–(4) were improved as

$$\left(\frac{\sin_p x}{x}\right)^2 + \frac{\tan_p x}{x} > 2, \quad \text{for } x \in \left(0, \frac{\pi_p}{2}\right) \text{ and } p \geq 2; \tag{5}$$

$$\left(\frac{\sinh_p x}{x}\right)^2 + \frac{\tanh_p x}{x} > 2, \quad \text{for } x \in (0, \infty) \text{ and } 1 < p \leq 2; \tag{6}$$

$$2\sin_p x + \tan_p x > 3x, \quad \text{for } x \in \left(0, \frac{\pi_p}{2}\right) \text{ and } p \geq 2; \tag{7}$$

$$2\sinh_p x + \tanh_p x > 3x, \quad \text{for } x \in (0, \infty) \text{ and } 1 < p \leq 2, \tag{8}$$

where $\pi_p = 2\operatorname{arcsin}_p 1 = 2\int_0^1 \frac{1}{(1-t^p)^{1/p}}\,\mathrm{d}t$. For $p = 2$, these inequalities coincide with (1)–(4).

In recent years, the following two-sided trigonometric inequality for hyperbolic functions

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{\cosh x + 2}{3}, \quad \text{for } x \in \left(0, \frac{\pi}{2}\right) \tag{9}$$

has attracted attention of several research studies. The left inequality of (9) is called the Lazarević inequality, which is obtained in [8]. The right inequality of (9) is the famous Cusa-Huygens inequality, which is obtained in [9], The counterpart of (9) for trigonometric functions

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3}, \quad \text{for } x \in \left(0, \frac{\pi}{2}\right) \tag{10}$$

is also well known. The left inequality (10) has been proven by Mitrinović [10], while the second one by Cusa and Huygens [4,11]. The aforementioned inequalities have also been obtained in [5].

The generalized trigonometric and hyperbolic functions depending on a parameter $p > 1$ were studied by Lindqvist in a highly cited paper [12]. Drábek and Manásevich [13] considered a certain (p, q)-eigenvalue problem with the Dirichlét boundary condition and found the complete solution to the problem. The solution of a special case is the function $\sin_{p,q}$, which is the first example of the so-called (p, q)-trigonometric function. Motivated by the (p, q)-eigenvalue problem, Takeuchi [14] has investigated the (p, q)-trigonometric functions depending on two parameters in which the case of $p = q$ coincides with the p-function of Lindqvist, and for $p = q = 2$ they coincide with familiar elementary functions.

In [15], the relations of generalized trigonometric and hyperbolic functions of two parameters with their inverse functions were studied. In [16], some inequalities for (p, q)-trigonometric were obtained and a few conjectures for them were posed. Recently, a conjecture posed in [16] was verified in [17]. In [18], the power mean inequality for generalized trigonometric and hyperbolic functions with two parameters was presented.

Motivated by these results on the trigonometric functions, we make a contribution to the subject by showing some Wilker inequalities, Huygens inequalities, Lazarević inequalities and Cusa-Huygens inequalities for the (p, q)-trigonometric and hyperbolic functions.

2 Definitions and formulas

For the formulation of our main results, we give the following definitions of (p, q)-trigonometric and hyperbolic functions, such as the generalized (p, q)-cosine function, the generalized (p, q)-tangent function and their inverses, and also the corresponding hyperbolic functions.

For $1 < p,\ q < \infty$, the increasing function $\operatorname{arcsin}_{p,q} x : [0, 1] \to [0, \pi_{p,q}/2]$ is defined by

$$\operatorname{arcsin}_{p,q} x = \int_0^x \frac{1}{(1-t^q)^{1/p}}\,\mathrm{d}t \tag{11}$$

and

$$\frac{\pi_{p,q}}{2} = \arcsin_{p,q}1 = \int_0^1 \frac{1}{(1-t^q)^{1/p}}\,\mathrm{d}t. \tag{12}$$

The inverse of $\arcsin_{p,q}$ on $[0, \pi_{p,q}/2]$ is called the generalized (p, q)-sine function, denoted by $\sin_{p,q} : [0, \pi_{p,q}/2] \to [0, 1]$.

The generalized (p, q)-cosine function $\cos_{p,q}x : [0, \pi_{p,q}/2] \to [0, 1]$ is defined as

$$\cos_{p,q}x = \frac{\mathrm{d}}{\mathrm{d}x}\sin_{p,q}x. \tag{13}$$

If $x \in [0, \pi_{p,q}/2]$, then by (2.6) in [19]

$$\sin_{p,q}^q x + \cos_{p,q}^p x = 1. \tag{14}$$

The generalized (p, q)-tangent function $\tan_{p,q} : (0, \pi_{p,q}/2) \to (0, \infty)$ is defined as

$$\tan_{p,q}x = \frac{\sin_{p,q}x}{\cos_{p,q}x}. \tag{15}$$

Similarly, for $x \in (0, \infty)$, the inverse of the generalized (p, q)-hyperbolic sine function [15] is defined by

$$\operatorname{arcsinh}_{p,q}x = \int_0^x \frac{1}{(1+t^q)^{1/p}}\,\mathrm{d}t$$

and also other corresponding (p, q)-hyperbolic functions, such as (p, q)-hyperbolic cosine and tangent functions, are defined by

$$\cosh_{p,q}x = \frac{\mathrm{d}}{\mathrm{d}x}\sinh_{p,q}x, \quad \tanh_{p,q}x = \frac{\sinh_{p,q}x}{\cosh_{p,q}x}$$

for $x \in [0, \infty)$, respectively.

The definitions show that

$$|\cosh_{p,q}x|^p - |\sinh_{p,q}x|^q = 1, \quad x \in (0, \infty). \tag{16}$$

It is clear that all these generalized functions coincide with the classical ones when $p = q = 2$.

3 Preliminaries and proofs

In this section, we give three Lemmas needed in the proofs of our main results. First, let us recall the following well-known formulas [15,19]: for $p, q \in (1, \infty)$,

$$x < \sinh_{p,q}x, \quad \text{for } x \in (0, \infty), \tag{17}$$

$$x < \tan_{p,q}x, \quad \text{for } x \in (0, \pi_{p,q}/2), \tag{18}$$

$$x > \tanh_{p,q}x, \quad \text{for } x \in (0, \infty), \tag{19}$$

$$\frac{2}{\pi_{p,q}} \le \frac{\sin_{p,q}x}{x} \le 1, \quad \text{for } x \in (0, \pi_{p,q}/2] \tag{20}$$

and the Jacobsthal inequality [20]

$$na^{n-1}b \le (n-1)a^n + b^n, \ (a, b > 0). \tag{21}$$

The following Lemmas will be frequently applied later.

Lemma 1. [15, Lemma 1] [19, Proposition 3.1] *For all* $p, q \in (1, \infty)$, $x \in (0, \pi_{p,q}/2)$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos_{p,q}x = -\frac{q}{p}(\cos_{p,q}x)^{2-p}(\sin_{p,q}x)^{q-1}\,, \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan_{p,q}x = 1 + \frac{q}{p}(\sin_{p,q}x)^{q}(\cos_{p,q}x)^{-p}\,; \tag{23}$$

and for all $x \in (0, \infty)$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh_{p,q}x = \frac{q}{p}(\cosh_{p,q}x)^{2-p}(\sinh_{p,q}x)^{q-1}\,, \tag{24}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh_{p,q}x = 1 - \frac{q}{p}(\sinh_{p,q}x)^{q}(\cosh_{p,q}x)^{-p}\,. \tag{25}$$

Lemma 2.

(1) *For* $p, q \in (1, \infty)$, *the function*

$$f_1(x) = \frac{\log(\sin_{p,q}x/x)}{\log\cos_{p,q}x}$$

is strictly decreasing from $(0, \pi_{p,q}/2)$ *to* $(0, 1/(1+q))$. *In particular, for all* $p, q \in (1, \infty)$ *and* $x \in (0, \pi_{p,q}/2)$,

$$\cos^{\alpha}_{p,q}x < \frac{\sin_{p,q}x}{x} < 1 \tag{26}$$

with the best constant $\alpha = 1/(1+q)$.

(2) *For* $p, q \in (1, \infty)$, *the function*

$$f_2(x) = \frac{\log(\sinh_{p,q}x/x)}{\log\cosh_{p,q}x}$$

is strictly increasing from $(0, \infty)$ *to* $(1/(1+q), 1)$. *In particular, for all* $p, q \in (1, \infty)$ *and* $x \in (0, \infty)$,

$$\cosh^{\alpha}_{p,q}x < \frac{\sinh_{p,q}x}{x} < \cosh^{\beta}_{p,q}x \tag{27}$$

with the best constant $\alpha = 1/(1+q)$ *and* $\beta = 1$.

Proof.

(1) Let $f_{11}(x) = \log(\sin_{p,q}x/x)$ and $f_{12}(x) = \log\cos_{p,q}x$. Clearly, $f_{11}(0^+) = f_{12}(0) = 0$, by (13) and (22), we have

$$\frac{f'_{11}(x)}{f'_{12}(x)} = \frac{p}{q}\,\frac{\tan_{p,q}x - x}{x\cos^{-p}_{p,q}x\cos^{q}_{p,q}x} = \frac{p}{q}\,\frac{f_{13}(x)}{f_{14}(x)}$$

with $f_{13}(x) = \tan_{p,q}x - x$, $f_{14}(x) = x\cos^{-p}_{p,q}x\sin^{q}_{p,q}x$ and $f_{13}(0) = f_{14}(0) = 0$. By (14), (22) and (23), we have

$$\frac{f'_{13}(x)}{f'_{14}(x)} = \frac{q}{p}\,\frac{1}{1 + qg_1(x)}$$

with

$$g_1(x) = \frac{x}{\sin_{p,q}x\cos^{p-1}_{p,q}x},$$

which is strictly increasing. Using the monotone form of l'Hôpital rule [21, Theorem 1.25], we see that $f_1(x)$ is strictly decreasing. We can easily obtain the limiting values, too. The assertion on f_1 is clear.

(2) Write $f_{21}(x) = \log(\sinh_{p,q}x/x)$ and $f_{22}(x) = \log\cosh_{p,q}x$, then $f_{21}(0^+) = f_{22}(0) = 0$. By (24), we have

$$\frac{f_{21}'(x)}{f_{22}'(x)} = \frac{p}{q}\frac{x - \tanh_{p,q}x}{x\cosh_{p,q}^{-p}x\sinh_{p,q}^{q}x} = \frac{p}{q}\frac{f_{23}(x)}{f_{24}(x)}$$

with $f_{23}(x) = x - \tanh_{p,q}x, f_{24}(x) = x\cosh_{p,q}^{-p}x\sinh_{p,q}^{q}x$ and $f_{23}(0) = f_{24}(0) = 0$. By differentiation and by (16),

$$\frac{f_{23}'(x)}{f_{24}'(x)} = \frac{q}{p}\frac{1}{1 + qg_2(x)}$$

with

$$g_2(x) = \frac{x}{\sinh_{p,q}x\cosh_{p,q}^{p-1}x},$$

which is strictly decreasing by [15, Lemma 5]. Hence, the function f_2 is strictly increasing by [21, Theorem 1.25]. So the other results follow. □

Remark. The left inequalities of (26) and (27) are Lazarević inequalities.

Lemma 3. *For $p \geq q > 2$, the function*

$$g(x) = \frac{\sin_{p,q}^{q-2}x}{\cos_{p,q}^{p-2}x} - \frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-2}x}$$

is strictly increasing in $(0, \pi_{p,q}/2)$.

Proof. By differentiation, we have

$$g'(x) = \frac{\sin_{p,q}^{q-3}x}{p\cos_{p,q}^{p-3}x}\left[p(q-2) + q(p-2)\frac{\sin_{p,q}^{q}x}{\cos_{p,q}^{p}x}\right] - \frac{\sinh_{p,q}^{q-3}x}{p\cosh_{p,q}^{p-3}x}\left[p(q-2) - q(p-2)\frac{\sinh_{p,q}^{q}x}{\cosh_{p,q}^{p}x}\right]. \quad (28)$$

Case (i). For $p \geq q \geq 3$.
In this case, by (28), we obtain

$$\begin{aligned} g'(x) &\geq (q-2)\frac{\sin_{p,q}^{q-3}x}{\cos_{p,q}^{p-3}x} - (q-2)\frac{\sinh_{p,q}^{q-3}x}{\cosh_{p,q}^{p-3}x} = (q-2)\left[\frac{\sin_{p,q}^{q-3}x}{\cos_{p,q}^{p-3}x} - \frac{\sinh_{p,q}^{q-3}x}{\cosh_{p,q}^{p-3}x}\right] \\ &\geq (q-2)\left[\frac{\sin_{p,q}^{q-3}x}{\cos_{p,q}^{q-3}x} - \frac{\sinh_{p,q}^{q-3}x}{\cosh_{p,q}^{q-3}x}\right] = (q-2)\left[\tan_{p,q}^{q-3}x - \tanh_{p,q}^{q-3}x\right] > 0, \end{aligned}$$

which is true by (18) and (19).

Case (ii). For $2 < q \leq p < 3$.
In this case, by (14), (16), (17) and (28), we have

$$\begin{aligned} g'(x) &= \frac{\sin_{p,q}^{q-3}x}{p}\frac{q(p-2) + 2(q-p)\cos_{p,q}^{p}x}{\cos_{p,q}^{2p-3}x} - \frac{\sinh_{p,q}^{q-3}x}{p}\frac{q(p-2) + 2(q-p)\cosh_{p,q}^{p}x}{\cosh_{p,q}^{2p-3}x} \\ &\geq \frac{\sinh_{p,q}^{q-3}x}{p}\left[\frac{q(p-2) + 2(q-p)\cos_{p,q}^{p}x}{\cos_{p,q}^{2p-3}x} - \frac{q(p-2) + 2(q-p)\cosh_{p,q}^{p}x}{\cosh_{p,q}^{2p-3}x}\right] \\ &= \frac{\sinh_{p,q}^{q-3}x}{p}\left[q(p-2)\left(\frac{1}{\cos_{p,q}^{2p-3}x} - \frac{1}{\cosh_{p,q}^{2p-3}x}\right) + 2(q-p)(\cos_{p,q}^{3-p}x - \cosh_{p,q}^{3-p}x)\right] > 0, \end{aligned}$$

which is true since $\cos_{p,q}x < 1 < \cosh_{p,q}x$. This completes the proof. □

Lemma 4. *For $p \geq q > 1$, the function $h(x) = \cos_{p,q}x \cdot \cosh_{p,q}x$ is strictly decreasing from $(0, \pi_{p,q}/2)$ to $(0, 1)$. In particular, for all $p \geq q > 1$ and $x \in (0, \pi_{p,q}/2)$,*

$$\cos_{p,q}x < \frac{1}{\cosh_{p,q}x}.$$

Proof. By differentiation, we have

$$\begin{aligned} h'(x) &= \frac{q}{p}\cos_{p,q}x \cdot \cosh_{p,q}x\left(\frac{\sinh_{p,q}^{q-1}x}{\cosh_{p,q}^{p-1}x} - \frac{\sin_{p,q}^{q-1}x}{\cos_{p,q}^{p-1}x}\right) \\ &\leq \frac{q}{p}\cos_{p,q}x \cdot \cosh_{p,q}x\left(\frac{\sinh_{p,q}^{q-1}x}{\cosh_{p,q}^{q-1}x} - \frac{\sin_{p,q}^{q-1}x}{\cos_{p,q}^{q-1}x}\right) \\ &= \frac{q}{p}\cos_{p,q}x \cdot \cosh_{p,q}x(\tanh_{p,q}^{q-1}x - \tan_{p,q}^{q-1}x) < 0. \end{aligned}$$

It is true by (18) and (19), which implies that h is strictly decreasing. Hence, the other conclusion for h is clear. □

4 Main results

Theorem 1.

(1) *If $x \in (0, \pi_{p,q}/2)$, $p, q > 1$ and $n \in \mathbb{N}^+$, $(n-1)\alpha - q\beta \geq 0$, $\beta \leq 0$, then*

$$(n-1)\left(\frac{x}{\sin_{p,q}x}\right)^{\alpha} + \left(\frac{x}{\tan_{p,q}x}\right)^{\beta} > n. \tag{29}$$

(2) *If $x \in (0, \infty)$, $p, q > 1$ and $n \in \mathbb{N}^+$, $(n-1)\alpha - q\beta \leq 0$, $\beta \leq 0$, then*

$$(n-1)\left(\frac{x}{\sinh_{p,q}x}\right)^{\alpha} + \left(\frac{x}{\tanh_{p,q}x}\right)^{\beta} > n. \tag{30}$$

Proof.

(1) Taking $a = \left(\frac{x}{\sin_{p,q}x}\right)^{\frac{\alpha}{n}}$, $b = \left(\frac{x}{\tan_{p,q}x}\right)^{\frac{\beta}{n}}$ in Jacobsthal inequality (21), by (20) and (26) in Lemma 2,

$$\begin{aligned} (n-1)\left(\frac{x}{\sin_{p,q}x}\right)^{\alpha} + \left(\frac{x}{\tan_{p,q}x}\right)^{\beta} &\geq n\left(\frac{x}{\sin_{p,q}x}\right)^{\frac{(n-1)\alpha}{n}}\left(\frac{x}{\tan_{p,q}x}\right)^{\frac{\beta}{n}} \\ &= n\left(\frac{x}{\sin_{p,q}x}\right)^{\frac{(n-1)\alpha+\beta}{n}}\left(\frac{x}{\sin_{p,q}x}\right)^{-\frac{\beta}{n}}\left(\frac{x}{\tan_{p,q}x}\right)^{\frac{\beta}{n}} \\ &= n\left(\frac{x}{\sin_{p,q}x}\right)^{\frac{(n-1)\alpha+\beta}{n}}\left(\cos_{p,q}x\right)^{\frac{\beta}{n}} \\ &> n\left(\frac{x}{\sin_{p,q}x}\right)^{\frac{(n-1)\alpha+\beta}{n}}\left(\frac{\sin_{p,q}x}{x}\right)^{\frac{(q+1)\beta}{n}} \\ &= n\left(\frac{x}{\sin_{p,q}x}\right)^{\frac{(n-1)\alpha-q\beta}{n}} > n. \end{aligned}$$

(2) Taking $a = \left(\frac{x}{\sinh_{p,q}x}\right)^{\frac{\alpha}{n}}$, $b = \left(\frac{x}{\tanh_{p,q}x}\right)^{\frac{\beta}{n}}$ in Jacobsthal inequality (21), the proof of the inequality (30) is similar to that of the proof of inequality (29). □

Remark.

(1) Put $p = q \geq 2$, $n = 2$, $\alpha = -2$, $\beta = -1$ or put $p = q \geq 2$, $n = 3$, $\alpha = -1$, $\beta = -1$ in (29), inequality (29) becomes inequality (5) or (7).

(2) Put $1 < p = q \leq 2$, $n = 2$, $\alpha = -2$, $\beta = -1$ or put $1 < p = q \leq 2$, $n = 3$, $\alpha = -1$, $\beta = -1$ in (30), inequality (30) becomes inequality (6) or (8).

In particular,

(3) Taking $p = q = 2$, $n = 2$, $\alpha = -2$, $\beta = -1$ or put $p = q = 2$, $n = 3$, $\alpha = -1$, $\beta = -1$ in (29) and (30), the inequalities turn into the Wilker inequality (1), (2) or the Huygens inequality (3), (4).

Theorem 2.

(1) For $x \in (0, \pi_{p,q}/2)$, $p, q \in (1, 2]$, we have

$$\frac{\sin_{p,q}x}{x} < \frac{\cos_{p,q}x + q}{1 + q} \leq \frac{\cos_{p,q}x + 2}{3}; \tag{31}$$

(2) For $x \in (0, \pi_{p,q}/2)$, $p \geq q \geq 2$, we have

$$\frac{\sin_{p,q}x}{x} < \frac{x}{\sinh_{p,q}x}; \tag{32}$$

(3) For $x \in (0, \pi_{p,q}/2)$, $p \geq q > 1$, we have

$$\frac{\sin_{p,q}x}{x} > \frac{x}{\tan_{p,q}x}. \tag{33}$$

Proof.

(1) Let $F_1(x) = x(\cos_{p,q} + q) - (1 + q)\sin_{p,q}x$. Then by differentiation, we have

$$F_1'(x) = q - \frac{q}{p}x \sin_{p,q}^{q-1}x \cos_{p,q}^{2-p}x - q\cos_{p,q}x$$

and

$$F_1''(x) = \frac{q}{p}\frac{\cos_{p,q}^{3-p}x}{\sin_{p,q}^{2-q}x}\left[(q-1)(\tan_{p,q}x - x) + \frac{q(2-p)}{p}x\frac{\sin_{p,q}^{q}x}{\cos_{p,q}^{p}x}\right] > 0,$$

which is true by (18) for $p, q \in (1, 2]$. Hence, $F_1(x)$ is increasing with $F_1(0) = 0$, or equivalently,

$$x(\cos_{p,q}x + q) > (1 + q)\sin_{p,q}x.$$

The right inequality in (31) is clear. It is easy to verify that the left inequality in (31) is true.

(2) Let $F_2(x) = x^2 - \sin_{p,q}x\sinh_{p,q}x$, we have

$$F_2'(x) = 2x - \cos_{p,q}x\sinh_{p,q}x - \sin_{p,q}x\cosh_{p,q}x$$

and

$$F_2''(x) = 2 + \frac{q}{p}\sin_{p,q}x\sinh_{p,q}x\left(\frac{\sin_{p,q}^{q-2}x}{\cos_{p,q}^{p-2}x} - \frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-2}x}\right) - 2\cos_{p,q}x\cosh_{p,q}x,$$

then F_2'' is increasing in x with $F_2''(0) = 0$ by Lemmas 3 and 4. Hence, F_2' is increasing in x with $F_2'(0) = 0$, which implies the monotonicity of F_2 with $F_2(0) = 0$. Therefore, we obtain inequality (32).

(3) Let $F_3(x) = \sin_{p,q}x \tan_{p,q}x - x^2$, then by (14), differentiation gives

$$F_3'(x) = 2\sin_{p,q}x - 2x + \frac{q}{p}\frac{\sin_{p,q}^{q+1}x}{\cos_{p,q}^{p}x},$$

$$F_3''(x) = \frac{q}{p}\frac{\sin_{p,q}^{q}x}{\cos_{p,q}^{2p-1}x}(\cos_{p,q}^{p}x + q) + 2\cos_{p,q}x - 2$$

and

$$\begin{aligned} F_3'''(x) &= \frac{q}{p}\frac{\sin_{p,q}^{q-1}x}{\cos_{p,q}^{2p-2}x}\left[\frac{pq^2+pq-q}{p} - \frac{2p-q}{p}\cos_{p,q}^{p}x + \frac{q^2(2p-1)}{p}\frac{\sin_{p,q}^{q}x}{\cos_{p,q}^{p}x}\right] \\ &= \frac{q}{p}\frac{\sin_{p,q}^{q-1}x}{\cos_{p,q}^{2(p-1)}x}\left[(q^2+q-2) + \left(2-\frac{q}{p}\right)\sin_{p,q}^{q} + \frac{q^2(2p-1)}{p}\frac{\sin_{p,q}^{q}x}{\cos_{p,q}^{p}x}\right] > 0. \end{aligned}$$

Hence $F_3''(x) > F_3''(0) = 0$, then F_3' is strictly increasing with $F_3'(0) = 0$, and $F_3(x) > F_3(0) = 0$, inequality (33) follows. □

Theorem 3.

(1) *For* $x > 0$, $p, q \in (1, 2]$, *we have*

$$\frac{\sinh_{p,q}x}{x} < \frac{\cosh_{p,q}x + q}{1+q}; \tag{34}$$

(2) *For* $x > 0$, $q \geq p \geq 2$, *we have*

$$\frac{\sinh_{p,q}x}{x} < \frac{\cosh_{p,q}x + 2}{3}; \tag{35}$$

(3) *For* $x > 0$, $q \geq p > 1$, *we have*

$$\frac{\sinh_{p,q}x}{x} > \frac{(p+1)\cosh_{p,q}x}{q\cosh_{p,q}x + 1}, \tag{36}$$

or equivalently,

$$\frac{\tanh_{p,q}x}{x} > \frac{p+1}{q\cosh_{p,q}x + 1}. \tag{37}$$

Proof.

(1) Let $G_1(x) = x(\cosh_{p,q}x + q) - (1+q)\sinh_{p,q}x$, then

$$G_1'(x) = q + \frac{q}{p}x\sinh_{p,q}^{q-1}x\cosh_{p,q}^{2-p}x - q\cosh_{p,q}x$$

and

$$G_1''(x) = \frac{q}{p}\frac{\cosh_{p,q}^{3-p}x}{\sinh_{p,q}^{2-q}x}\left[(q-1)(x - \tanh_{p,q}x) + \frac{q(2-p)}{p}x\frac{\sinh_{p,q}^{q}x}{\cosh_{p,q}^{p}x}\right] > 0,$$

which is true by (19) for $p, q \in (1, 2]$. Hence, G_1' is increasing in x with $G_1'(0) = 0$. So $G_1(x) > G_1(0) = 0$. Therefore, inequality (34) is obtained.

(2) Let $G_2(x) = x(\cosh_{p,q}x + 2) - 3\sinh_{p,q}x$, then

$$G_2'(x) = 2 + \frac{q}{p}x\sinh_{p,q}^{q-1}x\cosh_{p,q}^{2-p}x - 2\cosh_{p,q}x,$$

and by (16), (24), differentiation gives

$$\begin{aligned} G_2''(x) &= \frac{q}{p}\frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-3}x}\left[(q-1)x - \tanh_{p,q}x + \frac{q(2-p)}{p}x\frac{\sinh_{p,q}^{q}x}{\cosh_{p,q}^{p}x}\right] \\ &= \frac{q}{p}\frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-3}x}\left[(q-1)x\left(1-\frac{\sinh_{p,q}^{q}x}{\cosh_{p,q}^{p}x}\right) - \tanh_{p,q}x + \frac{2q-p}{p}x\frac{\sinh_{p,q}^{q}x}{\cosh_{p,q}^{p}x}\right] \\ &= \frac{q}{p}\frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-3}x}\left[(q-1)\frac{x}{\cosh_{p,q}^{p}x} - \tanh_{p,q}x + \frac{2q-p}{p}x\frac{\sinh_{p,q}^{q}x}{\cosh_{p,q}^{p}x}\right] \\ &\geq \frac{q}{p}\frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-3}x}\left[\frac{2q-p}{p}\frac{x}{\cosh_{p,q}^{p}x} - \tanh_{p,q}x + \frac{2q-p}{p}x\frac{\sinh_{p,q}^{q}x}{\cosh_{p,q}^{p}x}\right] \\ &= \frac{q}{p}\frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-3}x}\left[\frac{2q-p}{p}\frac{x}{\cosh_{p,q}^{p}x}(1+\sinh_{p,q}^{q}x) - \tanh_{p,q}x\right] \\ &= \frac{q}{p}\frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-3}x}\left[\frac{2q-p}{p}x - \tanh_{p,q}x\right] \\ &\geq \frac{q}{p}\frac{\sinh_{p,q}^{q-2}x}{\cosh_{p,q}^{p-3}x}[x - \tanh_{p,q}x] \geq 0, \end{aligned}$$

which is true by (19). So $G_2'(x) > G_2'(0) = 0$. Hence, G_2 is increasing in x with $G_2(x) > G_2(0) = 0$. Then inequality (35) is obtained.

(3) Set $G_3(x) = (p+1)x - \tanh_{p,q}x(q\cosh_{p,q}x + 1)$. Then

$$G_3'(x) = p - q\cosh_{p,q}x + \frac{q}{p}\frac{\sinh_{p,q}^{q}x}{\cosh_{p,q}^{p}x},$$

and by (16) and (24), differentiation gives

$$G_3''(x) = \frac{q^2}{p}\frac{\sinh_{p,q}^{q-1}x}{\cosh_{p,q}^{p-1}x}\left[\frac{1}{\cosh_{p,q}^{p}x} - \cosh_{p,q}x\right] < 0.$$

Hence, G_3' is strictly decreasing with $G_3'(0) = p - q \leq 0$, and $G_3(x) < G_3(0) = 0$. Inequalities (36) and (37) are proved. □

Remark. The left inequalities of (31), (34) and (35) are Cusa-Huygens inequalities.

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