Research Article

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# On the equivalence of three-dimensional differential systems

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**Abstract:** In this paper, firstly, we study the structural form of reflective integral for a given system. Then the sufficient conditions are obtained to ensure there exists the reflective integral with these structured form for such system. Secondly, we discuss the necessary conditions for the equivalence of such systems and a general three-dimensional differential system. And then, we apply the obtained results to the study of the behavior of their periodic solutions when such systems are periodic systems in $t$.

**Keywords:** three-dimensional polynomial system, reflective integral, equivalent systems, periodic solutions

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## 1 Introduction

Since Mironenko [1] created the theory of the reflective function, many experts and scholars have used this theory to study qualitative behaviors of the solutions of the differential systems and obtained many interesting results [1–13].

In the present section, we shall briefly introduce the related concepts which will be used throughout the rest of this article.

Consider differential system

$$x' = X(t, x), t \in R, x \in D \subset R^n, \tag{1.1}$$

which has a continuously differentiable right-hand side and a general solution $\varphi(t; t_0, x_o)$. For such a system, the reflective function is defined as $F(t, x) = \varphi(-t; t, x)$ [1].

A continuous differentiable vector function $F(t, x)$ on $R \times R^n$ is the reflective function of system (1.1) if and only if the following basic relation

$$F_t(t, x) + F_x(t, x)X(t, x) + X(-t, F(t, x)) = 0, F(0, x) = x \tag{1.2}$$

holds.

In particular, if $X(t + 2\omega, x) = X(t, x)$ ($\omega$ is a positive constant), and we can find the solution of the basic relation (1.2), then we can establish the Poincare mapping, so that the number of periodic solutions and the stability of periodic system (1.1) are solved. If the system

$$x' = Y(t, x), t \in R, x \in D \subset R^n \tag{1.3}$$

has the same reflective function $F(t, x)$ as system (1.1), then $X(0, x) = Y(0, x)$ and systems

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$$F_t(t,x) + F_x(t,x)X(t,x) + X(-t,F(t,x)) = 0,$$
$$F_t(t,x) + F_x(t,x)Y(t,x) + Y(-t,F(t,x)) = 0,$$
$$F(0,x) = x$$

are compatible. At this moment, we call system (1.3) is equivalent to system (1.1).

To check whether the above systems compatible, we can use the Frobenius theorem [2]. Doing this in practice, however, is a very hard task.

If we can neither solve the system (1.1) nor the problem (1.2), then it is good enough to construct any system (1.3), which is equivalent to system (1.1). To do this, sometimes we can use:

**Lemma 1.1.** [3] *Let the vector functions* $\Delta_k$ ($k = 1, 2, \dots, m$) *be solutions of the equation*

$$\Delta_t + \Delta_x X(t,x) = X_x \Delta \tag{1.4}$$

*and* $\alpha_k(t)$ ($k = 1, 2, \dots, m$) *are arbitrary continuous odd functions. Then all the perturbed systems of the form*

$$x' = X(t,x) + \sum_{k=1}^{n} \alpha_k(t)\Delta_k(t,x) \tag{1.5}$$

*are equivalent to each other and to system* (1.1) (*here m is any natural number or even* $m = \infty$).

Thus, if we find some solutions of equation (1.4), we can construct systems (1.5), which are equivalent to system (1.1). It can be seen that the solution of equation (1.4), that is, the reflective integral [3] of system (1.1), is particularly important to determine the equivalence of two differential systems. In addition, if these equivalent systems are $2\omega$-periodic with respect to $t$, then their Poincare mappings coincide, the initial conditions at $t = -\omega$ of their $2\omega$-periodic solutions and their stability characters are the same. Thus to study the equivalence of two differential systems is very important and interesting.

## 2 Main results

Now, we focus on the equivalence conditions for the general three-dimensional differential system

$$\begin{cases} x' = \sum\limits_{i+j+k=0}^{2} A_{ijk}(t)x^i y^j z^k \triangleq A(t,x,y,z), \\ y' = \sum\limits_{i+j+k=0}^{2} B_{ijk}(t)x^i y^j z^k \triangleq B(t,x,y,z), \\ z' = \sum\limits_{i+j+k=0}^{2} C_{ijk}(t)x^i y^j z^k \triangleq C(t,x,y,z) \end{cases} \tag{2.1}$$

and

$$\begin{cases} x' = p_1(t)x + p_2(t)x^2 \triangleq P(t,x), \\ y' = q_1(t)y + q_2(t)y^2 \triangleq Q(t,y), \\ z' = r_1(t)x + r_2(t)y + r_3(t)z + r_4(t)x^2 + r_5(t)xy + r_6(t)y^2 + r_7(t)xz + r_8(t)z^2 + r_9(t)yz \triangleq R(t,x,y,z) \end{cases} \tag{2.2}$$

as well as the characteristics of $A_{ijk}(t)$, $B_{ijk}(t)$ and $C_{ijk}(t)$, where $p_2(t) \neq 0$, $q_2(t) \neq 0$, and $p_i(t)$, $q_j(t)$, $r_k(t)$ ($i, j = 1, 2;\ k = 1, 2, \dots, 9$) are continuously differentiable functions.

Let us outline the investigation method. In accordance with Lemma 1.1, we seek systems equivalent to system (2.2) in the set of systems (1.5). Usually, however, we cannot find out all solutions of equation (1.4) in the case under consideration. Therefore, we seek only polynomial solutions $\Delta(t,x,y,z)$ of equation (1.4), namely, solutions of the form

$$\Delta(t,x,y,z)=\begin{pmatrix}\Delta_1(t,x,y,z)\\ \Delta_2(t,x,y,z)\\ \Delta_3(t,x,y,z)\end{pmatrix}=\begin{pmatrix}\sum\limits_{i+j+k=0}^{2}\mu_{ijk}(t)x^iy^jz^k\\ \sum\limits_{i+j+k=0}^{2}s_{ijk}(t)x^iy^jz^k\\ \sum\limits_{i+j+k=0}^{2}\gamma_{ijk}(t)x^iy^jz^k\end{pmatrix} \tag{2.3}$$

with differentiable coefficients $\mu_{ijk}(t)$, $s_{ijk}(t)$, and $\gamma_{ijk}(t)$, that is, we discuss when function $\Delta(t,x,y,z)$ of (2.3) is a solution of

$$\begin{pmatrix}\Delta_{1t}\\ \Delta_{2t}\\ \Delta_{3t}\end{pmatrix}+\begin{pmatrix}\Delta_{1x}&\Delta_{1y}&\Delta_{1z}\\ \Delta_{2x}&\Delta_{2y}&\Delta_{2z}\\ \Delta_{3x}&\Delta_{3y}&\Delta_{3z}\end{pmatrix}\begin{pmatrix}P\\ Q\\ R\end{pmatrix}=\begin{pmatrix}P_x&P_y&P_z\\ Q_x&Q_y&Q_z\\ R_x&R_y&R_z\end{pmatrix}\begin{pmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\end{pmatrix}. \tag{2.4}$$

In the following, we denote

$$\begin{aligned}&p=p_i(t),\ q=q_i(t),\ r=r_i(t);\\ &\mu_{ijk}=\mu_{ijk}(t),\ s_{ijk}=s_{ijk}(t),\ \gamma_{ijk}=\gamma_{ijk}(t);\\ &A_{ijk}=A_{ijk}(t),\ B_{ijk}=B_{ijk}(t),\ C_{ijk}=C_{ijk}(t);\\ &\Delta_k=\Delta_k(t,x,y,z)\ \ (k=1,2,3),\ldots,\ \text{etc.}\end{aligned}$$

**Theorem 2.1.** *Let* $p_2(t)\neq 0$, $q_2(t)\neq 0$, $r_8(t)\neq 0$, *and vector function* $\Delta(t,x,y,z)$ *of* (2.3) *is a reflective integral of system* (2.2), *then*

$$\Delta_1=\mu_{000}+\mu_{100}x+\mu_{200}x^2,$$

*where*

$$\mu_{000}=k_0 e^{\int p_1 dt}, \tag{2.5}$$

$$\mu_{100}=k_1+2k_0\int p_2 e^{\int p_1 dt}dt, \tag{2.6}$$

$$\mu_{200}=\left\{k_2+k_1\int p_2 e^{\int p_2 dt}dt+2k_0\left[\int\left(p_2 e^{\int p_1 dt}\cdot\int p_2 e^{\int p_1 dt}dt\right)dt\right]\right\}e^{-\int p_1 dt}, \tag{2.7}$$

$k_0$, $k_1$ *and* $k_2$ *are arbitrary constants.*

**Proof.** Using relation (2.4), we can get

$$\Delta_{1t}+\Delta_{1x}P+\Delta_{1y}Q+\Delta_{1z}R=P_x\Delta_1,$$

i.e.,

$$\begin{aligned}&\mu'_{000}+\mu'_{100}x+\mu'_{010}y+\mu'_{001}z+\mu'_{200}x^2+\mu'_{020}y^2+\mu'_{002}z^2+\mu'_{110}xy+\mu'_{101}xz+\mu'_{011}yz\\ &+(\mu_{100}+2\mu_{200}x+\mu_{110}y+\mu_{101}z)(p_1x+p_2x^2)+(\mu_{010}+\mu_{110}x+2\mu_{020}y+\mu_{011}z)(q_1y+q_2y^2)\\ &+(\mu_{001}+\mu_{101}x+\mu_{011}y+2\mu_{002}z)(r_1x+r_2y+r_3z+r_4x^2+r_5xy+r_6y^2+r_7xz+r_8z^2+r_9yz)\\ &\quad=(\mu_{000}+\mu_{100}x+\mu_{010}y+\mu_{001}z+\mu_{200}x^2+\mu_{020}y^2+\mu_{002}z^2+\mu_{110}xy+\mu_{101}xz+\mu_{011}yz)(p_1+2p_2x).\end{aligned}$$

Equating the coefficients of the same powers of $x$, $y$ and $z$, we have

$$\mu_{001}=\mu_{010}=\mu_{011}=\mu_{110}=\mu_{101}=\mu_{002}=\mu_{020}=0, \tag{2.8}$$

$$\mu'_{000}=p_1\mu_{000}, \tag{2.9}$$

$$\mu'_{100}=2p_2\mu_{000}, \tag{2.10}$$

$$\mu'_{200}+p_1\mu_{200}=p_2\mu_{100}. \tag{2.11}$$

Solving equations (2.9) and (2.10), we get (2.5) and (2.6). Substituting (2.6) into equation (2.11), we have (2.7).

Summarizing the above, the proof is finished. □

**Theorem 2.2.** *Let $p_2(t) \neq 0$, $q_2(t) \neq 0$, $r_8(t) \neq 0$, and vector function $\Delta(t,x,y,z)$ of* (2.3) *is a reflective integral of system* (2.2), *then*

$$\Delta_2 = s_{000} + s_{010}y + s_{020}y^2,$$

*where*

$$s_{000} = \omega_0 e^{\int q_1 dt}, \tag{2.12}$$

$$s_{010} = \left(\omega_1 + 2\omega_0 \int q_2 e^{2\int q_1 dt} dt\right) e^{-\int q_1 dt}, \tag{2.13}$$

$$s_{020} = \left\{\omega_2 - \omega_1 \int q_2 dt + 2\omega_0\left[\int\left(q_2 \cdot \int q_2 e^{2\int q_1 dt} dt\right) dt\right]\right\} e^{-\int q_1 dt}, \tag{2.14}$$

$\omega_0$, $\omega_1$ and $\omega_2$ are arbitrary constants.

**Proof.** Using relation (2.4), we can get

$$\Delta_{2t} + \Delta_{2x}P + \Delta_{2y}Q + \Delta_{2z}R = Q_y\Delta_2,$$

i.e.,

$$\begin{aligned}
&s'_{000} + s'_{100}x + s'_{010}y + s'_{001}z + s'_{200}x^2 + s'_{020}y^2 + s'_{002}z^2 + s'_{110}xy + s'_{101}xz + s'_{011}yz\\
&+(s_{100} + 2s_{200}x + s_{110}y + s_{101}z)(p_1x + p_2x^2) + (s_{010} + s_{110}x + 2s_{020}y + s_{011}z)(q_1y + q_2y^2)\\
&+(s_{001} + s_{101}x + s_{011}y + 2s_{002}z)(r_1x + r_2y + r_3z + r_4x^2 + r_5xy + r_6y^2 + r_7xz + r_8z^2 + r_9yz)\\
&\quad = (s_{000} + s_{100}x + s_{010}y + s_{001}z + s_{200}x^2 + s_{020}y^2 + s_{002}z^2 + s_{110}xy + s_{101}xz + s_{011}yz)(q_1 + 2q_2y).
\end{aligned}$$

Equating the coefficients of the same powers of $x$, $y$ and $z$, we have

$$s_{001} = s_{100} = s_{011} = s_{110} = s_{101} = s_{002} = s_{020} = 0,$$

$$s'_{000} = q_1 s_{000}, \tag{2.15}$$

$$s'_{010} + q_1 s_{010} = 2q_2 s_{000}, \tag{2.16}$$

$$s'_{020} + q_1 s_{020} + q_2 s_{010} = 0. \tag{2.17}$$

Solving equation (2.15), we get (2.12). Substituting (2.12) into equation (2.16), we get (2.13). Substituting (2.13) into equation (2.17), we have (2.14).

Summarizing the above, the proof is finished. □

**Theorem 2.3.** *Suppose that vector function $\Delta(t,x,y,z)$ of* (2.3) *is a reflective integral of system* (2.2) *and* $p_2(t) \neq 0$, $q_2(t) \neq 0$, $r_8(t) \neq 0$, $r_7(t) \neq 0$, $r_6(t) \neq 0$, $p_2 - r_7 \neq 0$, $q_2 - r_9 \neq 0$, $2p_2r_7 - r_7^2 + 4r_4r_8 \neq 0$, $2q_2r_9 - r_9^2 + 4r_6r_8 \neq 0$, $r_7^2 - p_2r_7 - 4r_4r_8 \triangleq \lambda_1 \neq 0$, $r_9^2 - q_2r_9 - 4r_6r_8 \triangleq \lambda_2 \neq 0$, $(r_7 - p_2)\left(\frac{r_7}{r_8}\right)' - 2r_8\left(\frac{r_4}{r_8}\right)' \triangleq \beta_1$, $(r_9 - q_2)\left(\frac{r_9}{r_8}\right)' - 2r_8\left(\frac{r_6}{r_8}\right)' \triangleq \beta_2$, $r_7\left(\frac{r_4}{r_8}\right)' - 2r_4\left(\frac{r_7}{r_8}\right)' \triangleq \delta_1$, $r_9\left(\frac{r_6}{r_8}\right)' - 2r_6\left(\frac{r_9}{r_8}\right)' \triangleq \delta_2$, *then*

$$\Delta(t,x,y,z) = \begin{pmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\end{pmatrix} = \begin{pmatrix}\sum\limits_{i+j+k=0}^{2}\mu_{ijk}(t)x^iy^jz^k\\ \sum\limits_{i+j+k=0}^{2}s_{ijk}(t)x^iy^jz^k\\ \sum\limits_{i+j+k=0}^{2}\gamma_{ijk}(t)x^iy^jz^k\end{pmatrix} = \begin{pmatrix}\mu_{000} + \mu_{100}x + \mu_{200}x^2\\ s_{000} + s_{010}y + s_{020}y^2\\ \gamma_{000} + \gamma_{100}x + \gamma_{010}y + \gamma_{001}z + \frac{\gamma_{002}}{r_8}(r_4x^2 + r_5xy + r_6y^2 + r_7xz + r_8z^2 + r_9yz)\end{pmatrix},$$

*where*

$$\mu_{100} = \frac{\gamma'_{002}}{r_8} + \left(\frac{\beta_1}{\lambda_1} + \frac{p_1}{r_8}\right)\gamma_{002}, \tag{2.18}$$

$$s_{010} = \frac{\gamma'_{002}}{r_8} + \left(\frac{\beta_2}{\lambda_2} + \frac{q_1}{r_8}\right)\gamma_{002}, \tag{2.19}$$

$$\mu_{200} = \frac{p_2}{r_8}\gamma_{002}, \tag{2.20}$$

$$s_{020} = \frac{q_2}{r_8}\gamma_{002}, \tag{2.21}$$

$$\gamma'_{000} = r_1\mu_{000} + r_2 s_{000} + r_3\gamma_{000}, \tag{2.22}$$

$$\gamma'_{001} = r_7\mu_{000} + r_9 s_{000} + 2r_8\gamma_{000}, \tag{2.23}$$

$$\gamma_{001} = \frac{1}{r_8}(\gamma'_{002} + r_3\gamma_{002}), \tag{2.24}$$

$$\gamma_{100} = \left(\frac{\delta_1}{\lambda_1} + \frac{r_1}{r_8}\right)\gamma_{002}, \tag{2.25}$$

$$\gamma_{010} = \left(\frac{\delta_2}{\lambda_2} + \frac{r_2}{r_8}\right)\gamma_{002}, \tag{2.26}$$

$$2r_4\mu_{000} + r_5 s_{000} + r_7\gamma_{000} = \left[\left(\frac{\delta_1}{\lambda_1} + \frac{r_1}{r_8}\right)\gamma_{002}\right]' + \frac{(p_1 - r_3)\delta_1 - r_1\beta_1}{\lambda_1}\gamma_{002}, \tag{2.27}$$

$$r_5\mu_{000} + 2r_6 s_{000} + r_9\gamma_{000} = \left[\left(\frac{\delta_2}{\lambda_2} + \frac{r_2}{r_8}\right)\gamma_{002}\right]' + \frac{(q_1 - r_3)\delta_2 - r_2\beta_2}{\lambda_2}\gamma_{002}. \tag{2.28}$$

**Proof.** Using relation (2.4), we can get

$$\Delta_{3t} + \Delta_{3x}P + \Delta_{3y}Q + \Delta_{3z}R = R_x\Delta_1 + R_y\Delta_2 + R_z\Delta_3,$$

i.e.

$$\begin{aligned}
&\gamma'_{000} + \gamma'_{100}x + \gamma'_{010}y + \gamma'_{001}z + \gamma'_{200}x^2 + \gamma'_{020}y^2 + \gamma'_{002}z^2 + \gamma'_{110}xy + \gamma'_{101}xz + \gamma'_{011}yz\\
&+ (\gamma_{100} + 2\gamma_{200}x + \gamma_{110}y + \gamma_{101}z)(p_1x + p_2x^2) + (\gamma_{010} + \gamma_{110}x + 2\gamma_{020}y + \gamma_{011}z)(q_1y + q_2y^2)\\
&+ (\gamma_{001} + \gamma_{101}x + \gamma_{011}y + 2\gamma_{002}z)(r_1x + r_2y + r_3z + r_4x^2 + r_5xy + r_6y^2 + r_7xz + r_8z^2 + r_9yz)\\
&= (r_1 + 2r_4x + r_5y + r_7z)(\mu_{000} + \mu_{100}x + \mu_{200}x^2) + (r_2 + r_5x + 2r_6y + r_9z)(s_{000} + s_{010}y + s_{020}y^2)\\
&\quad + (r_3 + r_7x + r_9y + 2r_8z)(\gamma_{000} + \gamma_{100}x + \gamma_{010}y + \gamma_{001}z + \gamma_{200}x^2 + \gamma_{020}y^2 + \gamma_{002}z^2 + \gamma_{110}xy + \gamma_{101}xz\\
&\quad + \gamma_{011}yz).
\end{aligned} \tag{2.29}$$

Equating the coefficients of $x^0$, $z$ and $z^2$ in (2.29), we have (2.22), (2.23), and (2.24). Equating the coefficients of $xz^2$ and $yz^2$ in (2.29), we get

$$\gamma_{101} = \frac{r_7}{r_8}\gamma_{002}, \tag{2.30}$$

$$\gamma_{011} = \frac{r_9}{r_8}\gamma_{002}. \tag{2.31}$$

Equating the coefficients of remaining items in (2.29), we obtain

$$\gamma'_{100} + p_1\gamma_{100} + r_1\gamma_{001} = r_1\mu_{100} + 2r_4\mu_{000} + r_5 s_{000} + r_3\gamma_{100} + r_7\gamma_{000}, \tag{2.32}$$

$$\gamma'_{010} + q_1\gamma_{010} + r_2\gamma_{001} = r_5\mu_{000} + 2r_6 s_{000} + r_2 s_{010} + r_3\gamma_{010} + r_9\gamma_{000}, \tag{2.33}$$

$$y'_{200} + (p_2 - r_7)y_{100} + (2p_1 - r_3)y_{200} + r_4y_{001} + r_1y_{101} = r_1\mu_{200} + 2r_4\mu_{100}, \tag{2.34}$$

$$y'_{020} + (q_2 - r_9)y_{100} + (2q_1 - r_2)y_{020} + r_6y_{001} + r_2y_{011} = r_2s_{020} + 2r_6s_{010}, \tag{2.35}$$

$$y'_{110} + (p_1 + q_1 - r_3)y_{110} + r_1y_{011} + r_5(y_{001} - \mu_{100} - s_{010}) + r_2y_{101} = r_7y_{010} + r_9y_{100}, \tag{2.36}$$

$$y'_{101} + p_1y_{101} + 2r_1y_{002} = r_7\mu_{100} + 2r_8y_{100}, \tag{2.37}$$

$$y'_{011} + q_1y_{011} + 2r_2y_{002} = r_9s_{010} + 2r_8y_{010}, \tag{2.38}$$

$$(q_2 - r_9)y_{110} + r_6y_{101} + r_5y_{011} = r_5s_{020} + r_7y_{020}, \tag{2.39}$$

$$(2p_2 - r_7)y_{200} + r_4y_{101} = 2r_4\mu_{200}, \tag{2.40}$$

$$p_2y_{101} + 2r_4y_{002} - 2r_8y_{200} = r_7\mu_{200}, \tag{2.41}$$

$$(2q_2 - r_9)y_{020} + r_6y_{011} = 2r_6s_{020}, \tag{2.42}$$

$$2r_8y_{020} - q_2y_{011} - 2r_6y_{002} = -r_9s_{020}, \tag{2.43}$$

$$(p_2 - r_7)y_{110} = r_5\mu_{200} + r_9y_{200} - r_5y_{101} - r_4y_{011}. \tag{2.44}$$

Eliminating $s_{020}$ from (2.42) and (2.43), and using identity (2.31), we get

$$(2q_2r_9 - r_9^2 + 4r_6r_8)r_8y_{020} = [4r_6^2r_8 + (2q_2r_6 - r_6r_9)r_9]y_{002}.$$

Since $(2q_2r_9 - r_9^2 + 4r_6r_8)r_8 \neq 0$, then

$$y_{020} = \frac{r_6}{r_8}y_{002}. \tag{2.45}$$

Substituting (2.31) and (2.45) into relation (2.42), and using $r_6 \neq 0$, we get

$$s_{020} = \frac{q_2}{r_8}y_{002}. \tag{2.46}$$

Eliminating $\mu_{200}$ from (2.40) and (2.41), and using identity (2.30), we get

$$(2p_2r_7 - r_7^2 + 4r_4r_8)r_8y_{200} = r_4(4r_4r_8 + 2p_2r_7 - r_7^2)y_{002}.$$

Since $(2p_2r_7 - r_7^2 + 4r_4r_8)r_8 \neq 0$, then

$$y_{200} = \frac{r_4}{r_8}y_{002}. \tag{2.47}$$

Since $r_7 \neq 0$, from (2.30), (2.47), and (2.41), we can get

$$\mu_{200} = \frac{p_2}{r_8}y_{002}. \tag{2.48}$$

Since $p_2 - r_7 \neq 0$, substituting (2.30), (2.31), (2.47), and (2.48) into relation (2.44), we have

$$y_{110} = \frac{r_5}{r_8}y_{002}. \tag{2.49}$$

(2.25) can be obtained by using identities (2.24), (2.34), (2.48), (2.30), (2.37), and $\lambda_1 \neq 0$. Similarly, using identities (2.35), (2.38), (2.31), (2.45), (2.24), and $\lambda_2 \neq 0$, we get (2.26). Substituting (2.25) and (2.30) into (2.37), we get (2.18). Similarly, substituting (2.26) and (2.31) into (2.38), we obtain (2.19). Substituting (2.18), (2.24), and (2.25) into (2.32), we get (2.27). (2.28) can be obtained by using identities (2.18), (2.24), (2.25), and (2.33). The calculation shows that identities (2.36) and (2.39) are identical.

Summarizing the above, Theorem 2.1 and Theorem 2.2, the proof is finished. □

From Lemma 1.1, we have thereby constructed the set of three-dimensional differential systems

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + \sum_{k=1}^{m} \alpha_k(t)\Delta_k(t, x, y, z) \tag{2.50}$$

equivalent to system (2.2) in the sense of the reflective integral. Here $\alpha_k(t)$ are arbitrary scalar continuous odd functions, and $\Delta_k(t,x,y,z)$ are vector functions described in Theorem 2.3.

Therefore, if some system can be represented in the form (2.50), then it is equivalent to the original system (2.2).

**Theorem 2.4.** *Suppose that the general three-dimensional differential system* (2.1) *can be represented in the form* (2.50), *i.e.*

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + \sum_{k=1}^{m} \alpha_k(t)\Delta_k(t,x,y,z), \tag{2.51}$$

*where $\alpha_k(t)$ are arbitrary continuous scalar odd functions, continuously differentiable vector functions $\Delta_k(t,x,y,z)$ $(k = 1,2,\ldots,m)$ are reflective integrals of system* (2.2). *Then system* (2.1) *is equivalent to the system* (2.2), *and*

$$\begin{aligned} &A_{000}(0) = B_{000}(0) = C_{000}(0) = 0, \\ &A_{010} = A_{001} = A_{020} = A_{002} = A_{110} = A_{101} = A_{011} = 0, \end{aligned} \tag{2.52}$$

$$\begin{aligned} &A_{100} = p_1 + \sum_{k=1}^{m} \alpha_k \mu_{100k}, \quad A_{200} = p_2 + \sum_{k=1}^{m} \alpha_k \mu_{200k}, \\ &B_{100} = B_{001} = B_{200} = B_{002} = B_{110} = B_{101} = B_{011} = 0, \\ &B_{010} = q_1 + \sum_{k=1}^{m} \alpha_k s_{010k}, \quad B_{020} = q_2 + \sum_{k=1}^{m} \alpha_k s_{020k}, \\ &C_{100} = r_1 + \sum_{k=1}^{m} \alpha_k \gamma_{100k}, \quad C_{010} = r_2 + \sum_{k=1}^{m} \alpha_k \gamma_{010k}, \\ &C_{001} = r_3 + \sum_{k=1}^{m} \alpha_k \gamma_{001k}, \quad C_{110} = \frac{r_5}{r_8} C_{002}, \quad C_{200} = \frac{r_4}{r_8} C_{002}, \\ &C_{020} = \frac{r_6}{r_8} C_{002}, \quad C_{101} = \frac{r_7}{r_8} C_{002}, \quad C_{011} = \frac{r_9}{r_8} C_{002}. \end{aligned} \tag{2.53}$$

**Proof.** Using relation (2.51), we can get

$$A = P + \sum_{k=1}^{m} \alpha_k \Delta_{1k},$$

i.e.

$$\begin{aligned} &A_{000} + A_{100}x + A_{010}y + A_{001}z + A_{200}x^2 + A_{020}y^2 + A_{002}z^2 + A_{110}xy + A_{101}xz + A_{011}yz \\ &= (p_1x + p_2x^2) + \textstyle\sum_{k=1}^{m} \alpha_k(\mu_{000k} + \mu_{100k}x + \mu_{200k}x^2). \end{aligned}$$

Equating the coefficients of the same powers of $x, y$ and $z$, we have (2.52) and (2.53). Since $\alpha_k(t) + \alpha_k(-t) = 0$ $(k = 1,2,\ldots,m)$, then $A_{000}(0) = 0$.

Similarly, the rest of conclusions in this theory can be obtained from $B = Q + \sum_{k=1}^{m} \alpha_k \Delta_{2k}$ and $C = R + \sum_{k=1}^{m} \alpha_k \Delta_{3k}$.

Summarizing the above, the proof is finished. □

**Remark 2.5.** We have thereby shown that a general three-dimensional differential system (2.1) that can be represented in the form (2.50) necessarily has the form

$$\begin{cases} x' = A_{000} + A_{100}x + A_{200}x^2, \\ y' = B_{000} + B_{010}y + B_{020}y^2, \\ z' = C_{000} + C_{100}x + C_{010}y + C_{001}z + \frac{C_{002}}{r_8}(r_4x^2 + r_5xy + r_6y^2 + r_7xz + r_8z^2 + r_9yz), \end{cases}$$

where $A_{000}(0) = B_{000}(0) = C_{000}(0) = 0$.

**Theorem 2.6.** *If system* (2.1) *is equivalent to system* (2.2), $p_i$, $q_i$ *and* $r_i$ *are arbitrary constants and* $p_1 \neq 0$, *then system* (2.1) *has no periodic solutions.*

**Proof.** When $p_1 \neq 0$, the general solution of

$$x' = p_1(t)x + p_2(t)x^2$$

is

$$p_1 t = \ln|x| - \ln|p_1 + p_2 x| + \varpi_3,$$

where $\varpi_3$ is arbitrary constant. Obviously, it is not a periodic solution. Therefore, system (2.2) has no periodic solutions. Considering equivalence between systems (2.1) and (2.2), system (2.1) has no periodic solutions too. □

**Example 1.** Consider the three-dimensional differential system

$$\begin{cases} x' = 1 - x\sin 2t + z\sin 2t + x^2\cos 2t + 2xz\sin^2 t, \\ y' = 1 - y\sin 2t + z\sin 2t + y^2\cos 2t + 2yz\sin^2 t, \\ z' = 1 + \sin 3t\cos t + x\sin 3t\sin t + xz\cos 2t + 2z^2\sin^2 t. \end{cases} \tag{2.54}$$

Set $t = 0$, we have the corresponding stationary system

$$\begin{cases} x' = 1 + x^2, \\ y' = 1 + y^2, \\ z' = 1 + xz. \end{cases} \tag{2.55}$$

Let a continuously differentiable vector function

$$F(t, x, y, z) = (F_1(t, x, y, z), F_2(t, x, y, z), F_3(t, x, y, z))^T$$

be reflective function of system (2.55). Since $F_1(t, x, y, z) = F(t, x) = \varphi(-t; t, x)$, where $x = \varphi(t; t_0, x_0)$ is the general solution of equation $x' = 1 + x^2$ in Cauchy form, we have

$$F_1(t, x, y, z) = \frac{x\cos 2t - \sin 2t}{\cos 2t + x\sin 2t}.$$

Similarly, we obtained

$$F_2(t, x, y, z) = \frac{y\cos 2t - \sin 2t}{\cos 2t + y\sin 2t}.$$

Let $F = (F_1, F_2, F_3)^T$ be the reflective function of system (2.54), by relation (1.2), we have

$$\begin{aligned} &\begin{pmatrix} F_{1t} \\ F_{2t} \\ F_{3t} \end{pmatrix} \begin{pmatrix} F_{1x} & 0 & 0 \\ 0 & F_{2y} & 0 \\ F_{3x} & F_{3y} & F_{3z} \end{pmatrix} \begin{pmatrix} 1 - x\sin 2t + z\sin 2t + x^2\cos 2t + 2xz\sin^2 t \\ 1 - y\sin 2t + z\sin 2t + y^2\cos 2t + 2yz\sin^2 t \\ 1 + \sin 3t\cos t + x\sin 3t\sin t + xz\cos 2t + 2z^2\sin^2 t \end{pmatrix} \\ &+ \begin{pmatrix} 1 + F_1\sin 2t - F_3\sin 2t + F_1^2\cos 2t + 2F_1F_3\sin^2 t \\ 1 + F_2\sin 2t - F_3\sin 2t + F_2^2\cos 2t + 2F_2F_3\sin^2 t \\ 1 - \sin 3t\cos t - F_1\sin 3t\sin t + F_1F_3\cos 2t + 2F_3^2\sin^2 t \end{pmatrix} = 0. \end{aligned} \tag{2.56}$$

From the first equation in system (2.56), we obtain

$$F_3(t, x, y, z) = \frac{x(\cos 2t - 1) + z - \sin 2t}{\cos 2t + x\sin 2t}.$$

Substitute $F_1$ and $F_3$ into the third equation in system (2.56), we obtain an identity. Since

$$F_{3t} + F_{3x}(1 + x^2) + F_{3z}(1 + xz) + 1 + F_1F_3 = 0,$$

then function $F = (F_1, F_2, F_3)^T$ is also the reflective function of system (2.55), i.e. systems (2.54) and (2.55) are equivalent. Obviously, system (2.54) has no periodic solutions. Considering equivalence between systems (2.54) and (2.55), system (2.54) has no periodic solutions too.

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