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Research Article

Congbian Ma* and Yanbo Ren

An equivalent quasinorm for the Lipschitz space of noncommutative martingales

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Abstract: In this paper, an equivalent quasinorm for the Lipschitz space of noncommutative martingales is presented. As an application, we obtain the duality theorem between the noncommutative martingale Hardy space $h_p^c(\mathcal{M})$ (resp. $h_p^r(\mathcal{M})$) and the Lipschitz space $\lambda_\beta^c(\mathcal{M})$ (resp. $\lambda_\beta^r(\mathcal{M})$) for $0 < p < 1$, $\beta = \frac{1}{p} - 1$. We also prove some equivalent quasinorms for $h_p^c(\mathcal{M})$ and $h_p^r(\mathcal{M})$ for $p = 1$ or $2 < p < \infty$.

Keywords: noncommutative space, martingale, Hardy space

MSC 2020: 46L53, 46L52, 60G42

1 Introduction

In the past two decades, due to the excellent work of Pisier and Xu on noncommutative martingale inequalities [1], the study of noncommutative martingale theory has attracted more and more attention. Especially in recent years, some meaningful research results on the noncommutative martingale theory have emerged continuously, and it has become a research hotspot in the field of noncommutative analysis.

The Lipschitz space was first introduced in the classical martingale theory by Herz and plays an important role in it. For instance, the Lipschitz space is the generalization of the *BMO* space and the dual space of the Hardy space h_p $(0 < p \leq 1)$. The noncommutative Lipschitz spaces $\lambda_0^c(\mathcal{M})$ and $\lambda_0^r(\mathcal{M})$ were first introduced in [2], and with their help, the atomic decomposition for the Hardy space $h_1(\mathcal{M})$ was proved. In this paper, we study the noncommutative Lipschitz spaces $\lambda_\beta^c(\mathcal{M})$ and $\lambda_\beta^r(\mathcal{M})$ for $\beta \geq 0$. We show that for $-\infty < q < 0$, $\beta = -\frac{1}{q}$, we have

$$\lambda_\beta^c(\mathcal{M}) = X_q^c(\mathcal{M}) \quad \text{and} \quad \lambda_\beta^r(\mathcal{M}) = X_q^r(\mathcal{M})$$

with equivalent norms. As its application, we have the duality equalities for $0 < p < 1$ and $\beta = \frac{1}{p} - 1$

$$(h_p^c(\mathcal{M}))^* = \lambda_\beta^c(\mathcal{M}) \quad \text{and} \quad (h_p^r(\mathcal{M}))^* = \lambda_\beta^r(\mathcal{M}).$$

This answers positively a question asked in [2]. The other main result of this paper concerns the equivalent quasinorms for $h_p^c(\mathcal{M})$ and $h_p^r(\mathcal{M})$ for $p = 1$ or $2 < p < \infty$. We prove the equalities

$$h_1^c(\mathcal{M}) = L_1^{2,c}(\mathcal{M}) \quad \text{and} \quad h_1^r(\mathcal{M}) = L_1^{2,r}(\mathcal{M}),$$

which give a new characterization of $h_1^c(\mathcal{M})$ and $h_1^r(\mathcal{M})$.

*** Corresponding author: Congbian Ma,** School of Mathematics and Information Science, Xinxiang University, Henan, People's Republic of China, e-mail: congbianm@whu.edu.cn
Yanbo Ren: School of Mathematics and Statistics, Henan University of Science and Technology, Henan, People's Republic of China, e-mail: ybren123@haust.edu.cn

This paper is organized as follows. Some definitions and notations are given in Section 2. An equivalent quasinorm for noncommutative martingale Lipschitz space is shown in Section 3. Equivalent quasinorms for $h_p^c(\mathcal{M})$ and $h_p^r(\mathcal{M})$ for $p = 1$ or $2 < p < \infty$ are considered in Section 4.

2 Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H and τ be a normal faithful finite trace on $\mathcal{M}$. We call $(\mathcal{M}, \tau)$ a noncommutative probability space. Let x be a positive operator on H. Then x admits a unique spectral decomposition: $x = \int_0^\infty \lambda \mathrm{d}e_\lambda(x)$. We will often use the spectral projection $e_{(\lambda,\infty)}(x)$ corresponding to the interval (λ, ∞). For $0 < p \leq \infty$, let $L_p(\mathcal{M})$ be the associated noncommutative L_p-space. Recall that the norm on $L_p(\mathcal{M})$ is defined by

$$\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}, \quad x \in L_p(\mathcal{M}),$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the usual modulus of x. Note that if $p = \infty$, $L_\infty(\mathcal{M})$ is just $\mathcal{M}$ with the usual operator norm. For more detailed discussions about noncommutative Banach function spaces, see [3–5].

Let us recall the general setup for noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq1}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$ and $\mathcal{E}_n$ (with $\mathcal{E}_0 = \mathcal{E}_1$) the conditional expectation with respect to $\mathcal{M}_n$. A sequence $x = (x_n)_{n\geq1}$ is said to be adapted if $x_n \in L_1(\mathcal{M}_n)$ for all $n \geq 1$, and predictable if $x_n \in L_1(\mathcal{M}_{n-1})$ for $n \geq 2$. A noncommutative martingale with respect to the filtration $(\mathcal{M}_n)_{n\geq1}$ is a sequence $x = (x_n)_{n\geq1}$ in $L_1(\mathcal{M})$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all } n \geq 1.$$

If additionally, $x = (x_n)_{n\geq1} \subset L_p(\mathcal{M})$ for some $0 \leq p < \infty$, we call x an $L_p(\mathcal{M})$-martingale. In this case, we set $\|x\|_p = \sup_n \|x_n\|_p$. If $\|x\|_p < \infty$, then x is called a bounded $L_p(\mathcal{M})$-martingale. Note that the space of all bounded L_p-martingales, equipped with $\|\cdot\|_p$, is isometric to $L_p(\mathcal{M})$ for $p > 1$. This permits us to not distinguish a martingale and its final value x_∞ (if the latter exists).

Let $x = (x_n)_{n\geq1}$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n\geq1}$ with the usual convention that $\mathcal{E}_0 = \mathcal{E}_1$. Define $\mathrm{d}x_n = x_n - x_{n-1}$ for $n \geq 1$ with the usual convention that $x_0 = 0$. The sequence $\mathrm{d}x = (\mathrm{d}x_n)_{n\geq1}$ is called the martingale difference sequence of x. In the sequel, for any operator $x \in L_1(\mathcal{M})$ we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

Let $x = (x_n)_{n\geq1}$ be a finite martingale in $L_2(\mathcal{M})$. We set

$$s_{c,n}(x) = \left(\sum_{k=1}^{n} \mathcal{E}_{k-1}(|\mathrm{d}x_k|^2)\right)^{\frac{1}{2}}, \quad s_c(x) = \left(\sum_{n=1}^{\infty} \mathcal{E}_{n-1}(|\mathrm{d}x_n|^2)\right)^{\frac{1}{2}}$$

and

$$s_{r,n}(x) = \left(\sum_{k=1}^{n} \mathcal{E}_{k-1}(|\mathrm{d}x_k^*|^2)\right)^{\frac{1}{2}}, \quad s_r(x) = \left(\sum_{n=1}^{\infty} \mathcal{E}_{n-1}(|\mathrm{d}x_n^*|^2)\right)^{\frac{1}{2}}.$$

These will be called the column and row conditioned square functions, respectively. Let $0 < p < \infty$. Define $h_p^c(\mathcal{M})$ (resp. $h_p^r(\mathcal{M})$) as the completion of all finite L_∞-martingales under the (quasi) norm $\|x\|_{h_p^c(\mathcal{M})} = \|s_c(x)\|_p$ (resp. $\|x\|_{h_p^r(\mathcal{M})} = \|s_r(x)\|_p$). For $p = \infty$. We define $h_\infty^c(\mathcal{M})$ (resp. $h_\infty^r(\mathcal{M})$) as the Banach space of the L_∞-martingales x such that $\sum_{n=1}^{\infty} \mathcal{E}_{n-1}(|\mathrm{d}x_n|)^2$ (respectively $\sum_{n=1}^{\infty} \mathcal{E}_{n-1}(|\mathrm{d}x_n^*|)^2$) converge for the weak operator topology.

For more information of noncommutative martingales, see the seminal article of Pisier and Xu [3] and the sequels to it.

The main object of this paper is the noncommutative Lipschitz spaces λ_β^c and $\lambda_\beta^r(\mathcal{M})$.

Definition 2.1. Let $\beta \geq 0$. Set

$$\lambda_\beta^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\lambda_\beta^c(\mathcal{M})} < \infty\}$$

with

$$\|x\|_{\lambda_\beta^c(\mathcal{M})} = \max\left\{\sup_{e\in\mathcal{P}_1} \tau(e)^{-\beta}\|\mathcal{E}_1(x)\|_\infty, \sup_{n\geq 1}\sup_{e\in\mathcal{P}_n} \tau(e)^{-\frac{1}{2}-\beta}\tau(e\mathcal{E}_n(|x-x_n|^2))^{\frac{1}{2}}\right\},$$

where $\mathcal{P}_n$ denotes the lattice of projections of $\mathcal{M}_n$.

Similarly, we define

$$\lambda_\beta^r(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\lambda_\beta^r(\mathcal{M})} = \|x^*\|_{\lambda_\beta^c(\mathcal{M})} < \infty\}.$$

The classical martingale space L_1^2 which is defined in [6] has the following noncommutative analogue.

Definition 2.2. We define

$$L_1^{2,c}(\mathcal{M}) = \left\{x \in L^1(\mathcal{M}) : x = \sum_{k=1}^\infty y_k z_k, y_k \in L_2(\mathcal{M}_{m_k}), z_k \in L_2(\mathcal{M}), \mathcal{E}_{m_k}(z_k) = 0 \right.$$
$$\left. \text{and } \|x\|_{L_1^{2,c}(\mathcal{M})} = \inf\left\{\sum_{k=1}^\infty \|y_k\|_2\|z_k\|_2\right\} < \infty\right\},$$

where $(\mathcal{M}_{m_k})_{k\geq 1}$ and $(\mathcal{E}_{m_k})_{k\geq 1}$ are the subsequences of $(\mathcal{M}_k)_{k\geq 1}$ and $(\mathcal{E}_k)_{k\geq 1}$, the infimum runs over all decompositions of x as above. Similarly, define

$$L_1^{2,r}(\mathcal{M}) = \{x \in L_1(\mathcal{M}) : \|x\|_{L_1^{2,r}(\mathcal{M})} = \|x^*\|_{L_1^{2,c}(\mathcal{M})} < \infty\}.$$

It is clear that $L_1^{2,c}(\mathcal{M})$ and $L_1^{2,r}(\mathcal{M})$ are Banach spaces.

We will use the following definitions from [2,7].

Definition 2.3. [2] Let $0 < p \leq 2$. Denote the index class W_1 which consists of sequences $(\omega_n)_{n\geq 1}$ such that $(\omega_n^{2/p-1})_{n\geq 1}$ is nondecreasing with each $\omega_n \in L_1^+(\mathcal{M}_n)$ invertible with bounded inverse and $\|\omega_n\|_1 \leq 1$. Define

$$X_p^c(\mathcal{M}) = \left\{x \in L_2(\mathcal{M}) : \|x\|_{X_p^c(\mathcal{M})} = \inf_{W_1}\left(\tau\left(\sum_{n=1}^\infty |\omega_n|^{1-2/p}|\mathrm{d}x_{n+1}|^2\right)\right)^{\frac{1}{2}} < \infty\right\}.$$

For $2 < p \leq \infty$ or $-\infty \leq p < 0$. Denote W_2 which consists of sequences $(\omega_n)_{n\geq 1}$ such that $(\omega_n^{1-2/p})_{n\geq 1}$ is nondecreasing with each $\omega_n \in L_1^+(\mathcal{M}_n)$ invertible with bounded inverse and $\|\omega_n\|_1 \leq 1$. Define

$$X_p^c(\mathcal{M}) = \left\{x \in L_2(\mathcal{M}) : \|x\|_{X_p^c(\mathcal{M})} = \sup_{W_2}\left(\tau\left(\sum_{n=1}^\infty |\omega_n|^{1-2/p}|\mathrm{d}x_{n+1}|^2\right)\right)^{\frac{1}{2}} < \infty\right\}.$$

Similarly, define

$$X_p^r(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{X_p^r(\mathcal{M})} = \|x^*\|_{X_p^c(\mathcal{M})} < \infty\}.$$

Remark that for $2 < p \leq \infty$, $X_p^c(\mathcal{M})$ can be rewritten in the following form. Given $(\omega_n)_{n\geq 1} \in W_2$, we set

$$g_n = \left(\omega_n^{\frac{2}{\alpha}} - \omega_{n-1}^{\frac{2}{\alpha}}\right)^{\frac{1}{2}}, \quad \forall n \geq 1,$$

where $\alpha = \frac{1}{2} - \frac{1}{p}$. It is clear that

$$(g_n) \in G_\alpha = \left\{u = (u_n)_{n\geq 0} : u_n \in L_\alpha(\mathcal{M}_n), \tau\left(\left(\sum_{n=1}^\infty |u_n|^2\right)^{\frac{\alpha}{2}}\right) \leq 1\right\}. \tag{3.1}$$

Then

$$\|x\|_{X_p^c(\mathcal{M})} = \sup_{(u_n)\in G_\alpha} \left(\tau\left(\sum_{n=1}^{\infty} |u_n|^2 \mathcal{E}_n(|x - x_n|^2)\right)\right)^{\frac{1}{2}}.$$

We recall the definition of the space $L_p(\mathcal{M};\ell_\infty)$, with $1 \le p \le \infty$. A sequence $(x_n)_{n\ge1}$ in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ if $(x_n)_{n\ge1}$ admits a factorization $x_n = ay_nb$, with $a, b \in L_{2p}(\mathcal{M})$ and $(y_n)_{n\ge1} \in L_\infty(\mathcal{M})$. The norm of $(x_n)_{n\ge1}$ is then defined as

$$\|(x_n)_{n\ge1}\|_{L_p(\mathcal{M};\ell_\infty)} = \inf_{x_n=ay_nb} \{\|a\|_{2p} \sup_{n\ge1} \|y_n\|_\infty \|b\|_{2p}\}.$$

We usually write $\|(x_n)_{n\ge1}\|_{L_p(\mathcal{M};\ell_\infty)} = \|\sup_n{}^{+} x_n\|_p$.

Definition 2.4. [7] Let $2 < p < \infty$. Define the space

$$L_p^c mo(\mathcal{M}) = \left\{a \in L_2(\mathcal{M}) : \|\sup{}^{+}\mathcal{E}_n(|a - \mathcal{E}_n(a)|^2)\|_{\frac{p}{2}} < \infty\right\}$$

equipped with the norm

$$\|a\|_{L_p^c mo(\mathcal{M})} = \max\left\{\|a\|_p, \left(\|\sup{}^{+}\mathcal{E}_n(|a - \mathcal{E}_n(a)|^2)\|_{\frac{p}{2}}\right)^{\frac{1}{2}}\right\}.$$

Then $(L_p^c mo(\mathcal{M}), \|\cdot\|_{L_p^c mo(\mathcal{M})})$ is a Banach space. Similarly, we set

$$L_p^r mo(\mathcal{M}) = \{a : a^* \in L_p^c mo(\mathcal{M})\}$$

equipped with the norm

$$\|a\|_{L_p^r mo(\mathcal{M})} = \|a^*\|_{L_p^c mo(\mathcal{M})}.$$

3 An equivalent quasinorm for the Lipschitz space of noncommutative martingales

In this section, we prove the noncommutative equivalent quasinorms for Lipschitz spaces $\lambda_\beta^c(\mathcal{M})$ and $\lambda_\beta^r(\mathcal{M})$ $(\beta \ge 0)$. As its application, we obtain the duality of the Hardy space $h_p^c(\mathcal{M})$ (resp. $h_p^r(\mathcal{M})$) and the Lipschitz space $\lambda_\beta^c(\mathcal{M})$ (resp. $\lambda_\beta^r(\mathcal{M})$) for $0 < p < 1$.

Theorem 3.1. *For* $-\infty < q < 0$ *and* $\beta = -\frac{1}{q}$, *we have* $\lambda_\beta^c(\mathcal{M}) = X_q^c(\mathcal{M})$ *with equivalent norms. More precisely, for any* $x \in L_2(\mathcal{M})$, *we have*

$$\left(1 - \frac{2}{q}\right)^{-\frac{1}{2}} \|x\|_{X_q^c(\mathcal{M})} \le \|x\|_{\lambda_\beta^c(\mathcal{M})} \le \|x\|_{X_q^c(\mathcal{M})}.$$

Similarly, $\lambda_\beta^r(\mathcal{M}) = X_q^r(\mathcal{M})$ *with the same equivalence constants.*

The following Lemma is the key ingredient of our proof.

Lemma 3.2. *For* $x \in \lambda_\beta^c(\mathcal{M})$ $(\beta \ge 0)$, *we have*

$$\mathcal{E}_n(|x - x_n|^2) \le C^2\left(\sum_{e\in\mathcal{B}_n} \tau(e)^{2\beta} e\right) n \ge 1, \tag{3.1}$$

where $C = \|x\|_{\lambda_\beta^c(\mathcal{M})}$ *and* $\mathcal{B}_n$ *is the set consisting of all minimal projections with respect to* $\mathcal{M}_n$.

Proof. By the definition of $\lambda_\beta^c(\mathcal{M})(\beta \geq 0)$, we have for any $e \in \mathcal{B}_n$

$$\tau(e\mathcal{E}_n(|x - x_n|^2))^{\frac{1}{2}} \leq C\tau(e)^{\frac{1}{2}+\beta}. \tag{3.2}$$

Let P be a projection with respect to $\mathcal{M}_n$ and there does not exist any minimal projections e' such that $e' \leq P$. Let $y = \mathcal{E}_n(|x - x_n|^2)P$. Then for any $k \geq 1$

$$ye_{\left(\frac{1}{k},\infty\right)}(y) \geq \frac{1}{k}e_{\left(\frac{1}{k},\infty\right)}(y).$$

Thus by (3.2), we get that

$$\frac{1}{\sqrt{k}}\tau\left(e_{\left(\frac{1}{k},\infty\right)}(y)\right)^{\frac{1}{2}} \leq \tau\left(ye_{\left(\frac{1}{k},\infty\right)}(y)\right)^{\frac{1}{2}} = \tau\left(\mathcal{E}_n(|x - x_n|^2)Pe_{\left(\frac{1}{k},\infty\right)}(y)\right)^{\frac{1}{2}} \leq C\tau\left(e_{\left(\frac{1}{k},\infty\right)}(y)P\right)^{\frac{1}{2}+\beta}.$$

When $\tau(e_{\left(\frac{1}{k},\infty\right)}(y)) \neq 0$, we have that

$$\frac{1}{\sqrt{k}} \leq \tau\left(e_{\left(\frac{1}{k},\infty\right)}(y)\right)^{\beta}. \tag{3.3}$$

Note that $e_{\left(\frac{1}{k},\infty\right)}(y) \leq P$ and P can be divided into infinite small pieces. Thus, $\tau(e_{\left(\frac{1}{k},\infty\right)}(y)) = 0$, which contradicts (3.3). Therefore, we obtain $\tau(e_{\left(\frac{1}{k},\infty\right)}(y)) = 0$ for every $k \geq 1$ which implies $\mathcal{E}_n(|x - x_n|^2)P = 0$.

Now we prove (3.1) holds. Let $e \in \mathcal{B}_n$ and $a = C\tau(e)^\beta$. Let $\varepsilon > 0$ and $e_0 = e_{((a+\varepsilon)^2,\infty)}(\mathcal{E}_n(|x - x_n|^2))$. Then we have that

$$ee_0\mathcal{E}_n(|x - x_n|^2) \geq (a + \varepsilon)^2 e_0 e.$$

Thus using (3.2), we get that

$$(a + \varepsilon)\tau(e_0 e)^{\frac{1}{2}} \leq \tau(e_0 eE_n(|x - x_n|^2))^{\frac{1}{2}} \leq a\tau(e_0 e)^{\frac{1}{2}}.$$

It is easy to see that $e_0 e = 0$, which implies that $e \leq e_0$ does not hold. Using the preceding result, we have that

$$\mathcal{E}_n(|x - x_n|^2)e_0 = 0.$$

Let $e_1 = e_{(0,a^2)}(\mathcal{E}_n(|x - x_n|^2))$. Then we have

$$\mathcal{E}_n(|x - x_n|^2) = \mathcal{E}_n(|x - x_n|^2)e_1 \leq a^2 e_1.$$

It follows that

$$\mathcal{E}_n(|x - x_n|^2)e \leq C^2\tau(e)^{2\beta}e. \tag{3.4}$$

Note that $\mathcal{E}_n(|x - x_n|^2)(1 - \sum_{e\in\mathcal{B}_n} e) = 0$. Thus by (3.4), we have

$$\mathcal{E}_n(|x - x_n|^2) = \mathcal{E}_n(|x - x_n|^2)\sum_{e\in\mathcal{B}_n} e \leq C^2\left(\sum_{e\in\mathcal{B}_n}\tau(e)^{2\beta}e\right).$$

The proof is complete. □

We will also need the following well-known lemma from [8].

Lemma 3.3. *Let f be a function in $\mathbb{C}^1(\mathbb{R}^+)$ and $x, y \in \mathcal{M}^+$. Then*

$$\tau(f(x + y) - f(x)) = \tau\left(\int_0^1 f'(x + ty)y\,\mathrm{d}t\right).$$

Proof of Theorem 3.1. Let $x \in X_q^c(\mathcal{M})$ and $\frac{1}{\alpha} = \frac{1}{2} - \frac{1}{q}$. Fix an integer $m \geq 0$ and let e be a projection with respect to $\mathcal{M}_m$. Define

$$u_n = \begin{cases} e\tau(e)^{-\frac{1}{\alpha}}, & m = n; \\ 0, & m \neq n. \end{cases}$$

Noting that $(u_n)_{n\geq 0} \in G_\alpha$, we have

$$\left(\tau(e)^{-\frac{2}{\alpha}}\tau(e\mathcal{E}_n|x - x_n|^2)\right)^{\frac{1}{2}} = \left(\tau\left(\sum_{n=1}^{\infty} |u_n|^2\mathcal{E}_n(|x - x_n|^2)\right)\right)^{\frac{1}{2}} \leq \|x\|_{X_q^c(\mathcal{M})}.$$

Thus, we have that $x \in \lambda_\beta^c(\mathcal{M})$ and $\|x\|_{\lambda_\beta^c(\mathcal{M})} \leq \|x\|_{X_q^c(\mathcal{M})}$.

Now, let $x \in \lambda_\beta^c(\mathcal{M})$ and $C = \|x\|_{\lambda_\beta^c(\mathcal{M})}$. Let $\mathcal{B}_n$ be the set consisting of all minimal projections with respect to $\mathcal{M}_n$. Then by Lemma 3.3, we have that

$$\mathcal{E}_n(|x - x_n|^2) \leq C^2\left(\sum_{e\in\mathcal{B}_n} \tau(e)^{2\beta}e\right). \tag{3.5}$$

Let $u = (u_n) \in G_\alpha$. Denote $r_n' = \left(\sum_{k\leq n} |u_k|^2\right)^{\frac{\alpha}{2}}$ and $r_n = r_n' e_{(0,\infty)}(r_n')$. Then r_n is invertible and $\tau(r_n) \leq 1$. Let $r_n = \int_0^\infty \lambda \mathrm{d}e_\lambda$ be the spectral decomposition of r_n. Let $\mathrm{d}\mu = \mathrm{d}\tau(e_\lambda)$. Then we have $\int_0^\infty \lambda \mathrm{d}\mu \leq 1$. Observe that

$$\sum_{\lambda_0\in(0,\infty)} \mu(\lambda_0)\chi_{\{\lambda_0\}}(\lambda)\cdot \lambda \leq 1,$$

where $\chi_{\{\lambda_0\}}(\lambda)$ is the characteristic function at point λ_0. It follows that

$$\sum_{\lambda_0\in(0,\infty)} \mu(\lambda_0)^{2\beta}\chi_{\{\lambda_0\}}(\lambda) \leq \lambda^{-2\beta}.$$

By the continuous function calculus, we have

$$\sum_{e\in\mathcal{B}_n} \tau(e)^{2\beta}e \leq r_n^{-2\beta}. \tag{3.6}$$

Using (3.5) and (3.6),

$$\left(\tau\left(\sum_{n=1}^{\infty} |u_n|^2\mathcal{E}_n(|x - x_n|^2)\right)\right)^{\frac{1}{2}} \leq C\left(\tau\left(\sum_{n=1}^{\infty} |u_n|^2 \sum_{e\in\mathcal{B}_n} \tau(e)^{2\beta}e\right)\right)^{\frac{1}{2}} \leq C\left(\tau\left(\sum_{n=1}^{\infty} (r_n^{\frac{2}{\alpha}} - r_{n-1}^{\frac{2}{\alpha}})r_n^{1-\frac{2}{\alpha}}\right)\right)^{\frac{1}{2}}. \tag{3.7}$$

Applying Lemma 3.3 with $f(t) = t^{\frac{\alpha}{2}}$, $x + y = r_n^{\frac{2}{\alpha}}$ and $x = r_{n-1}^{\frac{2}{\alpha}}$, we obtain

$$\tau\left(\left(r_n^{\frac{2}{\alpha}} - r_{n-1}^{\frac{2}{\alpha}}\right)r_n^{1-\frac{2}{\alpha}}\right) \leq \tau\left(\int_0^1 \left(r_{n-1}^{\frac{2}{\alpha}} + t\left(r_n^{\frac{2}{\alpha}} - r_{n-1}^{\frac{2}{\alpha}}\right)\right)^{\frac{\alpha}{2}-1}\left(r_n^{\frac{2}{\alpha}} - r_{n-1}^{\frac{2}{\alpha}}\right)\mathrm{d}t\right) = \frac{2}{\alpha}\tau(r_n - r_{n-1}),$$

where we have used the fact that the operator function $a \mapsto a^{\frac{\alpha}{2}-1}$ is nonincreasing for $-1 < \frac{\alpha}{2} - 1 \leq 0$. Taking the sum over n leads to $\left(\tau\left(\sum_{n=1}^{\infty} |u_n|^2\mathcal{E}_n(|x - x_n|^2)\right)\right)^{\frac{1}{2}} \leq \left(\frac{2}{\alpha}\right)^{\frac{1}{2}}\|x\|_{\lambda_\beta^c(\mathcal{M})}$. Taking supremum over all $(u_n) \in G_\alpha$, we get

$$\|x\|_{X_p^c(\mathcal{M})} \leq \left(\frac{2}{\alpha}\right)^{\frac{1}{2}}\|x\|_{\lambda_\beta(\mathcal{M})} = \left(1 - \frac{2}{p}\right)^{\frac{1}{2}}\|x\|_{\lambda_\beta(\mathcal{M})}.$$

The proof is complete. □

Using the dual result in Theorem 3.2 in [2], we will describe the dual space of $h_p(\mathcal{M})(0 < p < 1)$ as the Lipschitz space.

Corollary 3.4. *Let* $0 < p < 1$ *and* $\beta = \frac{1}{p} - 1$. *Then we have*

$$(h_p^c(\mathcal{M}))^* = \lambda_\beta^c(\mathcal{M})$$

and

$$(h_p^r(\mathcal{M}))^* = \lambda_\beta^r(\mathcal{M})$$

with equivalent norms.

4 Equivalent quasinorms for $h_1^c(\mathcal{M})$ and $h_p^c(\mathcal{M})$ $(2 < p < \infty)$

In this section, we first describe an equivalent quasinorm for $h_1^c(\mathcal{M})$. As in the classical case, the spaces h_1 and L_1^2 are equivalent (see [6]). We will transfer this to the noncommutative martingales.

Theorem 4.1. *We have that*

$$h_1^c(\mathcal{M}) = L_1^{2,c}(\mathcal{M}) \quad \textit{and} \quad h_1^r(\mathcal{M}) = L_1^{2,r}(\mathcal{M})$$

with equivalent norms.

For the proof we need the following lemmas.

Lemma 4.2. *Let* $1 \le p \le 2$ *and* p' *be the conjugate index of* p. *For* $x, y \in L^2(\mathcal{M})$, *we have*

$$|\tau(xy^*)| \le \left(\frac{2}{p}\right)^{\frac{1}{2}} \|x\|_{h_p^c(\mathcal{M})} \|y\|_{X_{p'}^c(\mathcal{M})}.$$

Proof. Let $(r_n) \in W_1$. Then by the Cauchy-Schwarz inequality and Lemma 3.1, we have that

$$\begin{aligned}\tau(xy^*) &= \sum_{k=1}^{\infty} \tau\left(\left(dx_k r_{k-1}^{\frac{1}{2}-\frac{1}{p}}\right)\left(dy_k r_{k-1}^{\frac{1}{2}-\frac{1}{p'}}\right)^*\right) \le \left(\sum_{k=1}^{\infty} \tau\left(r_{k-1}^{1-\frac{2}{p}} |dx_k|^2\right)\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \tau\left(r_{k-1}^{1-\frac{2}{p'}} |dy_k|^2\right)\right)^{\frac{1}{2}} \\ &\le \|x\|_{X_p^c(\mathcal{M})} \|y\|_{X_{p'}^c(\mathcal{M})} \le \left(\frac{2}{p}\right)^{\frac{1}{2}} \|x\|_{h_p^c(\mathcal{M})} \|y\|_{X_{p'}^c(\mathcal{M})}.\end{aligned}$$

Note that the set G_α defined in (3.1) can be reduced to the following one:

$$\widetilde{G_\alpha} = \left\{ \text{finite sequences } u = (u_n)_{n\ge1} : u_n \in L_\infty(\mathcal{M}_n), \ \tau\left(\left(\sum_{n\ge1} |u_n|^2\right)^{\frac{\alpha}{2}}\right) \le 1 \right\}.$$

Indeed, for any $(u_n)_{n\ge0} \in G_\alpha$ set

$$u_n^{(N)} = \begin{cases} |u_n| e_{[0,N]}(|u_n|), & n \le N; \\ 0, & n > N. \end{cases}$$

Then we have that

$$\lim_{N\to\infty} \left(\tau\left(\sum_{n\ge1} |u_n^{(N)}|^2 \mathcal{E}_n(|x - x_n|^2)\right)\right)^{\frac{1}{2}} = \left(\tau\left(\sum_{n\ge1} |u_n|^2 \mathcal{E}_n(|x - x_n|^2)\right)\right)^{\frac{1}{2}}.$$

Thus, the set G_α can be reduced to $\widetilde{G_\alpha}$. □

Lemma 4.3. *Let $y \in L^2(\mathcal{M})$ and $2 \le p \le \infty$. Then we have*

$$\|y\|_{X_p^c(\mathcal{M})} = \sup_{x \in F_2^p} |\tau(xy^*)|,$$

where

$$F_2^p = \left\{ x = \sum_{n\ge1} a_n b_n : (a_n) \in \widetilde{G_\alpha}, \mathcal{E}_n(b_n) = 0, \left(\tau\left(\sum_{n\ge1} |b_n|^2\right)\right)^{\frac{1}{2}} \le 1 \right\}.$$

Proof. For any finite sequences $(a_n) \in \widetilde{G_\alpha}$, set

$$A_{a,y} = \left(\tau\left(\sum_{n\ge1} |a_n|^2 |y^n|^2\right)\right)^{\frac{1}{2}},$$

where $y^n = y - y_n$. Then we have $A_{a,y} < \infty$ and

$$A_{a,y} = \frac{A_{a,y}^2}{A_{a,y}} = \tau\left(\sum_{n\ge1} a_n y^{n*} \frac{y^n a_n^*}{A_{a,y}}\right) = \tau\left(\sum_{n\ge1} a_n y^{n*} z_n\right), \tag{4.1}$$

where $z_n = \frac{y^n a_n^*}{A_{a,y}}$. Note that $\mathcal{E}_n(z_n) = \frac{1}{A_{a,y}} \mathcal{E}_n(y^n) a_n^* = 0$ and

$$\tau\left(\sum_{n\ge1} |z_n|^2\right) = \frac{1}{A_{a,y}^2} \tau\left(\sum_{n\ge1} |y^n a_n^*|^2\right) = \frac{1}{A_{a,y}^2} \tau\left(\sum_{n\ge1} |a_n|^2 |y^n|^2\right) = 1.$$

Let $x = \sum_{n\ge1} a_n z_n$. It is clear that $x \in F_2^p$ and

$$\left\|\sum_{n\ge1} a_n z_n\right\|_2 \le \sum_{n\ge1} \|a_n\| \|z_n\|_2 \le \sup_{n\ge1} \|a_n\| \sum_{n\ge1} \|z_n\|_2 < \infty.$$

Thus,

$$\left|\tau\left(\sum_{n\ge1} a_n z_n y^{n*}\right)\right| \le \left\|\sum_{n\ge1} a_n z_n\right\|_2 \left\|\sum_{n\ge1} y^* - y_n^*\right\|_2 < \infty.$$

Therefore, by the definition of $X_p^c(\mathcal{M})$ and (4.1), we have that

$$\begin{aligned}\|y\|_{X_p^c(\mathcal{M})} &= \sup_{(a_n)\in\widetilde{G_\alpha}} A_{a,y} = \sup_{x\in F_2^p} \left|\tau\left(\sum_{n\ge1} a_n z_n y^{n*}\right)\right| = \sup_{x\in F_2^p} \left|\sum_{n\ge1} \tau(\mathcal{E}_n(a_n z_n (y^* - y_n^*)))\right| \\ &= \sup_{x\in F_2^p} \left|\sum_{n\ge1} \tau(a_n z_n y^*) - \sum_{n\ge1} \tau(a_n \mathcal{E}_n(z_n) y_n^*)\right| = \sup_{x\in F_2^p} |\tau(xy^*)|.\end{aligned}$$

□

Proof of Theorem 4.1. First let $x = zy$ and there exists $n \ge 1$ such that $y \in L_2(\mathcal{M}_n)$, $z \in L_2(\mathcal{M})$, $\mathcal{E}_n(z) = 0$. Then we have that

$$\mathcal{E}_m(zy) = \mathcal{E}_m \mathcal{E}_n(zy) = 0; \quad m \le n. \tag{4.2}$$

Thus, we have that

$$\mathcal{E}_m(zy) = \mathcal{E}_m(z)y; \quad m \ge 1. \tag{4.3}$$

Using (4.2), (4.3), and the fact $y \in L_2(\mathcal{M}_n)$, we find

$$s_c(x)^2 = \sum_{k\ge1} \mathcal{E}_{k-1}|\mathrm{d}z_k y|^2 = \sum_{k>n} \mathcal{E}_{k-1}|\mathrm{d}z_k y|^2 = y^*\left(\sum_{k>n} \mathcal{E}_{k-1}|\mathrm{d}z_k|^2\right)y.$$

Thus, we deduce that

$$\|x\|_{h_1^c(\mathcal{M})} = \|s_c(x)\|_1 = \|s_c(z)y\|_1 \le \|s_c(z)\|_2\|y\|_2 = \|z\|_2\|y\|_2.$$

Therefore, we get that $\|x\|_{h_1^c(\mathcal{M})} \le \|x\|_{L_1^{2,c}(\mathcal{M})}$. Now we consider the general case. Let

$$x = \sum_{k=1}^{\infty} z^k y^k,$$

where for every $k \ge 1$, $y^k \in L_2(\mathcal{M}_{m_k})$, $z^k \in L_2(\mathcal{M})$, and $\mathcal{E}_{m_k}(z^k) = 0$. Then by (4.3), we have that

$$\|x\|_{h_1^c(\mathcal{M})} = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} z^k y^k \right\|_{h_1^c(\mathcal{M})} \le \lim_{n\to\infty} \sum_{k=1}^{n} \|z^k y^k\|_{h_1^c(\mathcal{M})} \le \sum_{k=1}^{\infty} \|z^k\|_2 \|y^k\|_2.$$

It follows that $\|x\|_{h_1^c(\mathcal{M})} \le \|x\|_{L_1^{2,c}(\mathcal{M})}$.

We turn to the converse inequality. Let $x \in L_1^{2,c}(\mathcal{M})$. Then by Lemma 4.2 and Lemma 4.3,

$$\begin{aligned} \|x\|_{L_1^{2,c}(\mathcal{M})} &= \sup_{y\in L_1^{2,c}(\mathcal{M})^*, \|y\|_{L_1^{2,c}(\mathcal{M})^*}\le 1} |\tau(xy^*)| \\ &\le \sup_{y\in L_1^{2,c}(\mathcal{M})^*, \|y\|_{L_1^{2,c}(\mathcal{M})^*}\le 1} \|x\|_{h_1^c(\mathcal{M})} \|y\|_{X_\infty^c(\mathcal{M})} \\ &\le \|x\|_{h_1^c(\mathcal{M})} \sup_{y\in L_1^{2,c}(\mathcal{M})^*, \|y\|_{L_1^{2,c}(\mathcal{M})^*}\le 1} \sup_{a\in F_2^p} |\tau(ay^*)|. \end{aligned} \tag{4.4}$$

We will show

$$\sup_{a\in F_2^p} |\tau(ay^*)| \le \sup_{a\in L_1^{2,c}(\mathcal{M}), \|a\|_{L_1^{2,c}(\mathcal{M})}\le 1} |\tau(ay^*)|. \tag{4.5}$$

Indeed, let $a \in F_2^p$. Then a can be decomposed as

$$a = \sum_{n\ge 1} r_n b_n, \tag{4.6}$$

where $(r_n) \in \widetilde{G_\alpha}$, $\mathcal{E}_n(b_n) = 0$ and $(\tau(\sum_{n\ge1} |b_n|^2))^{\frac12} \le 1$. Thus, by the Cauchy-Schwarz inequality we have that

$$\sum_{n\ge1} \|r_n\|_2 \|b_n\|_2 \le \left(\sum_{n\ge1} \|r_n\|_2^2\right)^{\frac12} \left(\sum_{n\ge1} \|b_n\|_2^2\right)^{\frac12} \le 1.$$

Taking the infimum as in (4.6), we obtain $a \in L_1^{2,c}(\mathcal{M})$ and $\|a\|_{L_1^{2,c}(\mathcal{M})} \le 1$, which imply that (4.5) holds. Combining (4.4) and (4.5), we have that $\|x\|_{L_1^{2,c}(\mathcal{M})} \le \|x\|_{h_1^c(\mathcal{M})}$. Thus, we have that $h_1^c(\mathcal{M}) = L_1^{2,c}(\mathcal{M})$.

Similarly, we have that $h_1^r(\mathcal{M}) = L_1^{2,r}(\mathcal{M})$. The proof of the theorem is complete. □

Let $2 < p < \infty$. Our second result of this section concerns the equivalent quasinorms for $h_p^c(\mathcal{M})$ and $h_p^r(\mathcal{M})$.

Theorem 4.4. *Let $2 < p < \infty$. Then we have*

$$h_p^c(\mathcal{M}) = X_p^c(\mathcal{M}) = L_p^c mo(\mathcal{M})$$

with equivalent norms. More precisely,

$$\|x\|_{X_p^c(\mathcal{M})} \le \|x\|_{L_p^c mo(\mathcal{M})} \le \|x\|_{h_p^c(\mathcal{M})} \le \left(\frac{p}{2}\right)^{\frac12} \|x\|_{X_p^c(\mathcal{M})}.$$

Similarly,

$$h_p^r(\mathcal{M}) = X_p^r(\mathcal{M}) = L_p^r mo(\mathcal{M})$$

with equivalent norms.

Proof. Step 1: Let $x \in X_p^c(\mathcal{M})$. Fix a positive integer N, we will show that $s_{c,N}(x) \in L_p(\mathcal{M})$. Let $1 \le n \le N$. Since the dual space of $L_{\frac{p}{2}}(\mathcal{M}_n)$ is $L_{\frac{p}{p-2}}(\mathcal{M}_n)$,

$$\|\mathcal{E}_n(|dx_{n+1}|)^2\|_{\frac{p}{2}} = \sup_{\substack{y_n \in L^+_{\frac{p}{p-2}}(\mathcal{M}_n) \\ \|y_n\|_{L_{\frac{p}{p-2}}(\mathcal{M})} \le 1}} \tau(y_n \mathcal{E}_n(|dx_{n+1}|^2)).$$

Let $w_n = y_n^{\frac{p}{p-2}}$. Then $w_n \ge 0$ and $\tau(w_n) \le 1$. Thus, we have that

$$\|\mathcal{E}_n(|dx_{n+1}|)^2\|_{\frac{p}{2}} = \sup_{w_n \ge 0, \tau(w_n) \le 1} \tau\left(w_n^{1-\frac{2}{p}} \mathcal{E}_n(|dx_{n+1}|)^2\right) \le \|x\|^2_{X_p^c(\mathcal{M})}.$$

Therefore, we obtain that

$$\left\| \left(\sum_{n=1}^N \mathcal{E}_{n-1}(|dx_n|)^2 \right)^{\frac{1}{2}} \right\|_p = \left\| \sum_{n=1}^N \mathcal{E}_{n-1}(|dx_n|)^2 \right\|_{\frac{p}{2}}^{\frac{1}{2}} \le \sum_{n=1}^N \|\mathcal{E}_{n-1}(|dx_n|)^2\|_{\frac{p}{2}}^{\frac{1}{2}} \le N\|x\|_{X_p^c(\mathcal{M})} < \infty.$$

Assume $\|s_{c,N}(x)\|_p = 1$. Then the sequence $\{s_{c,N}^p(x), s_{c,N}^p(x), \ldots, s_{c,N}^p(x), \ldots\} \in W_2$. Set $s_{c,0}(x) = 0$. Thus, we have that

$$\|x\|^2_{X_p^c(\mathcal{M})} \ge \tau\left(\sum_{n=1}^N (s_{c,N}^p(x))^{1-\frac{2}{p}} (s_{c,n}^2(x) - s_{c,n-1}^2(x)) \right) = \tau\left(s_{c,N}^{p-2}(x) \sum_{n=1}^N (s_{c,n}^2(x) - s_{c,n-1}^2(x)) \right) = \tau(s_{c,N}^p(x)).$$

It follows that

$$\|x\|_{h_p^c(\mathcal{M})} \le \|x\|_{X_p^c(\mathcal{M})}.$$

Now let $x \in h_p^c(\mathcal{M})$ and $(w_n) \in W_2$. Set $s_{c,0}(x) = 0$. Then for any n

$$\left(\tau\left(\sum_{k=1}^n w_k^{1-\frac{2}{p}} (s_{c,k+1}^2(x) - s_{c,k}^2(x)) \right) \right)^{\frac{1}{2}} = \left(\tau\left(w_n^{1-\frac{2}{p}} (s_{c,n+1}^2(x)) \right) \right)^{\frac{1}{2}} \le (\tau(w_n))^{1-\frac{2}{p}} (\tau(s_{c,n+1}^p))^{\frac{1}{p}} \le \|s_c(x)\|_p.$$

Thus, we get that

$$\left(\tau\left(\sum_{k=1}^\infty w_k^{1-\frac{2}{p}} (s_{c,k+1}^2(x) - s_{c,k}^2(x)) \right) \right)^{\frac{1}{2}} \le \|s_c(x)\|_p.$$

Therefore, the inequality $\|x\|_{X_p^c(\mathcal{M})} \le \|x\|_{h_p^c(\mathcal{M})}$ holds.

Step 2: Let $x \in h_p^c(\mathcal{M})$. Then for any $n \ge 0$

$$\mathcal{E}_n(|x - \mathcal{E}_n(x)|^2) = \mathcal{E}_n\left(\sum_{k=n+1}^\infty \mathcal{E}_{k-1}(|dx_k|^2) \right) \le \mathcal{E}_n(s_c^2(x)).$$

Thus, we have that

$$\|x\|_{L_p^c mo(\mathcal{M})} \le \|x\|_{h_p^c(\mathcal{M})}.$$

Now let $x \in L_p^c mo(\mathcal{M}))$ and $\frac{1}{p'} + \frac{1}{p} = 1$. Note that

$$\|\sup{}^+(\mathcal{E}_n(|x - \mathcal{E}_n(x)|)^2)\|_{\frac{p}{2}} = \sup\left\{ \sum_{n \ge 1} \tau(\mathcal{E}_n(|x - \mathcal{E}_n(x)|)^2 y_n) : y_n \in L_{\frac{p'}{2}} \text{ and } \left\| \sum_{n \ge 1} y_n \right\|_{\frac{p'}{2}} \le 1 \right\}.$$

Thus, we have that

$$\|x\|_{X_p^c(\mathcal{M})} = \sup_{(u_n)\in G_\alpha}\left(\tau\left(\sum_{n=1}^{\infty}|u_n|^2\mathcal{E}_n(|x - x_n|^2\right)\right)^{\frac{1}{2}} \leq \|\sup{}^+(\mathcal{E}_n(|x - \mathcal{E}_n(x)|)^2)\|_{\frac{p}{2}} \leq \|x\|_{L_p^c mo(\mathcal{M})}.$$

The proof of the theorem is complete. □

The following is an immediate consequence of Theorem 4.4 and Theorem 3.3 in [2] (or Theorem 3.1 in [7]).

Corollary 4.5. *Let* $1 < p < 2$ *and let* q *be the index conjugate to* p*. Then*

$$(h_p^c(\mathcal{M}))^* = h_q^c(\mathcal{M})$$

and

$$(h_p^r(\mathcal{M}))^* = h_q^r(\mathcal{M})$$

with equivalent norms.

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