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Research Article

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Meromorphic solutions of certain nonlinear difference equations

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Abstract: This paper focuses on finite-order meromorphic solutions of nonlinear difference equation $f^n(z) + q(z)e^{Q(z)}\Delta_c f(z) = p(z)$, where p, q, Q are polynomials, $n \geq 2$ is an integer, and $\Delta_c f$ is the forward difference of f. A relationship between the growth and zero distribution of these solutions is obtained. Using this relationship, we obtain the form of these solutions of the aforementioned equation. Some examples are given to illustrate our results.

Keywords: difference equations, meromorphic solutions, exponential polynomials

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1 Introduction and main results

In 2010, Yang and Laine [1] studied the existence of finite-order entire solutions of the differential-difference equation

$$f^n(z) + L(z, f) = h(z), \tag{1.1}$$

where $L(z, f)$ is a linear differential-difference polynomial in f with small meromorphic coefficients, $h(z)$ is a meromorphic function of finite order, and $n \geq 2$ is an integer. It is proved that if $n \geq 4$, then equation (1.1) has at most one admissible entire solution of finite order. In particular, the equation

$$f^2(z) + q(z)f(z + 1) = p(z),$$

where $p(z)$, $q(z)$ are polynomials, has no transcendental entire solutions of finite order. This paper results in an extensive focus on the integrability of nonlinear difference equations (see, e.g., [2–8]). For example, assuming that $h(z) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}$, the authors in [2–5] gave some conditions how to judge the existence of admissible finite-order entire solutions of (1.1), where $p_1, p_2, \alpha_1, \alpha_2$ are constants. If the condition admissible solution is omitted, that is, the coefficients of $L(z, f)$ are not small functions of f, Wen, Heittokangas, and Laine [7] proved that every finite-order meromorphic solution of the difference equation

$$f^n(z) + q(z)e^{Q(z)}f(z + c) = p(z) \tag{1.2}$$

is entire, and classified these entire solutions in terms of its growth and zero distribution, see Theorem A. For the convenience of our statement, we recall some notions and definitions.

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Let f be a meromorphic function in the complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the basic notations and the main results of the Nevanlinna theory, such as $m(r,f)$, $N(r,f)$, $T(r,f)$, the first and second main theorems, etc.; see, e.g., [9,10]. The notations $\sigma(f)$ and $\lambda(f)$ denote, respectively, the order and the exponent of convergence of zeros of f. An entire function of the form

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)}, \tag{1.3}$$

where $P_j(z)$, $Q_j(z)(j = 1,\ldots, k)$ are polynomials in z such that $\max\{\deg Q_j(z) : 1 \le j \le k\} = s \ge 1$ is called an exponential polynomial of degree s. Denote

$$\begin{aligned}\Gamma_1 &= \{e^{\alpha(z)} + d : d \in \mathbb{C},\ \alpha \ \text{polynomial},\ \alpha \not\equiv \text{const.}\},\\ \Gamma_0 &= \{e^{\alpha(z)} : \alpha \ \text{polynomial},\ \alpha \not\equiv \text{const.}\}.\end{aligned}$$

Theorem A. [7] *Let $n \ge 2$ be an integer, $c \in \mathbb{C}\backslash\{0\}$, and let $p(z)$, $q(z)(\not\equiv 0)$, $Q(z)$ be polynomials such that $Q(z)$ is not a constant. Then we identify finite-order entire solutions f of* (1.2) *as follows:*
(*a*) *every solution f satisfies $\sigma(f) = \deg Q$ and is of mean type;*
(*b*) *every solution f satisfies $\lambda(f) = \sigma(f)$ if and only if $p(z) \not\equiv 0$;*
(*c*) *a solution $f \in \Gamma_0$ if and only if $p(z) \equiv 0$. In particular, this is the case if $n \ge 3$;*
(*d*) *if a solution $f \in \Gamma_0$, and if g is any other finite-order entire solution of* (1.2), *then $f = \eta g$, where $\eta^{n-1} = 1$;*
(*e*) *if f is an exponential polynomial solution of the form* (1.3), *then $f \in \Gamma_1$. Moreover, if $f \in \Gamma_1\backslash\Gamma_0$, then $\sigma(f) = 1$.*

A variant of Theorem A was obtained by Liu [11]. Considering the differential-difference equation

$$f^n(z) + q(z)e^{Q(z)}f^{(k)}(z + c) = p(z), \tag{1.4}$$

Liu proved the following result.

Theorem B. [11] *Let $n \ge 2$, $k \ge 1$ be an integer, $c \in \mathbb{C}\backslash\{0\}$, and let $p(z)$, $q(z)(\not\equiv 0)$, $Q(z)$ be polynomials such that $Q(z)$ is not a constant. Then every finite-order transcendental entire solution f of* (1.4) *should satisfy results* (*a*), (*b*), *and* (*d*) *of Theorem A. Moreover,*
(*i*) *a solution $f \in \{B(z)e^{\alpha(z)} : B, \alpha$ polynomials, $\alpha \not\equiv$ const.$\}$ if and only if $p(z) \equiv 0$. In particular, this is case if $n \ge 3$;*
(*ii*) *if f is an exponential polynomial solution of the form* (1.3), *then $f \in \{B(z)e^{\alpha(z)} + h(z) : B, \alpha, h$ polynomials, $\alpha \not\equiv$ const.$\}$.*

The purpose of this paper is to classify finite-order meromorphic solutions of the difference equation

$$f^n(z) + q(z)e^{Q(z)}\Delta_c f(z) = p(z), \tag{1.5}$$

where n, p, q, Q, c satisfy the hypotheses of Theorem A, $\Delta_c f(z) = f(z + c) - f(z)$. Our results show that meromorphic solutions of (1.5) have different structure from equations (1.2) and (1.4). Note that every finite-order meromorphic solution of (1.5) is entire by comparing the poles of both sides of (1.5). Now we state our results as follows.

Theorem 1.1. *Let $n \ge 2$ be an integer, $c \in \mathbb{C}\backslash\{0\}$, and let $p(z)$, $q(z)(\not\equiv 0)$, $Q(z)$ be polynomials such that $Q(z)$ is not a constant. Then every finite-order nonconstant entire solution f of* (1.5) *has the following properties.*
(*i*) $\sigma(f) = \deg(Q(z))$ *and f is of mean type*;
(*ii*) $\lambda(f) = \sigma(f)$ *if and only if $p(z) \not\equiv 0$*;
(*iii*) $\lambda(f) = \sigma(f) - 1$ *if and only if $p(z) \equiv 0$. In particular, this is the case if $n \ge 3$.*

Theorem 1.2 is a direct result of the proof of result (iii) of Theorem 1.1, see Section 2. It gives the expression of finite-order entire solutions of (1.5) in the case $n \ge 3$ or $p(z) \equiv 0$.

Theorem 1.2. *Let n, $p(z)$, $q(z)$, $Q(z)$, c satisfy the hypotheses of Theorem 1.1. If $n \geq 3$ or $p(z) \equiv 0$, then every finite-order nonconstant entire solution f of* (1.5) *is of the form*

$$f(z) = A(z)e^{\omega z^s},$$

where $s = \deg Q$, ω is a nonzero constant, and $A(z)(\not\equiv 0)$ is an entire function satisfying $\sigma(A) = \lambda(A) = \deg Q - 1$. In particular, if $\deg Q = 1$, then $A(z)$ reduces to a polynomial.

Remark 1.1. Examples 1.1–1.3 show that the difference equation (1.5) does have such solution defined as Theorem 1.2.

Example 1.1. The equation $f^2(z) + e^{z^2}\Delta_c f(z) = 0$ has a solution $f(z) = (1 - e^{2cz})e^{z^2}$, where c is a nonzero constant such that $e^{c^2} = 1$. Here, $\lambda(f) = \lambda(1 - e^{2cz}) = 1 = \sigma(f) - 1$.

Example 1.2. The equation $f^2(z) + (z+1)^2 e^{\beta z}\Delta_c f(z) = 0$ has a solution $f(z) = -c(z+1)e^{\beta z}$, where c, β are nonzero constants such that $e^{\beta c} = 1$. Here, $\lambda(f) = 0 = \sigma(f) - 1$.

Example 1.3. The equation $f^2(z) + 2e^z\Delta_c f(z) = 0$ has a solution $f(z) = e^z$, where $c = -\log 2$. Here, $\lambda(f) = 0 = \sigma(f) - 1$.

Now we consider the expression of finite-order entire solution of (1.5) in the case $n = 2$ and $p(z) \not\equiv 0$. From Theorem 1.1, it follows that every finite-order entire solution f satisfies $\lambda(f) = \sigma(f) = \deg Q$. So, it is plausible to consider the exponential polynomial solutions of (1.5).

Theorem 1.3. *Let $p(z)(\not\equiv 0)$, $q(z)(\not\equiv 0)$, $Q(z)$ be polynomials such that $Q(z)$ is not a constant. If f is an exponential polynomial solution of (1.5) of the form (1.3), then f is of the form*

$$f(z) = H_1(z)e^{\omega_1 z} + H_0(z), \tag{1.6}$$

where H_1, H_0 are nonconstant polynomials, ω_1 is a nonzero constant satisfying $e^{\omega_1 c} = 1$.

Remark 1.2. Example 1.4 shows that entire solutions of the form (1.6) do exist.

Example 1.4. The equation $f^2(z) + (z+1)^2 e^{2\pi iz}(f(z+1) - f(z)) = (z^2 + 3z + 2)^2$ has a solution $f(z) = z^2 + 3z + 2 - (z+1)e^{2\pi iz}$.

The remainder of this paper is organized as follows. In Section 2, we use some known results to give the proof of Theorem 1.1. The proof of Theorem 1.3 is given in Section 3, and its key reasoning relies on Steinmetz's results on the Nevanlinna characteristic of exponential polynomials.

2 Proof of Theorem 1.1

The following lemmas will be used to prove Theorem 1.1, in which, Lemma 2.1 is the difference version of the logarithmic derivative lemma due to Chiang and Feng; Lemma 2.4 is called the Hadamard factorization theorem.

Lemma 2.1. [12] *Let $f(z)$ be a finite-order meromorphic function in the complex plane, and c be a fixed nonzero constant. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\sigma(f)-1+\varepsilon}) + O(\log r).$$

Lemma 2.2. [13] *Let $f(z)$ be a transcendental meromorphic function of finite order. Then*

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f(z)),$$

where $S(r, f(z))$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside of a set of finite logarithmic measure.

Lemma 2.3. [14] *Let $f_j(z), g_j(z)$ $(j = 1, \ldots, n)$ $(n \geq 2)$ be entire functions satisfying the following conditions.*
(*i*) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(*ii*) *The order of f_j is less than that of $e^{g_t(z)-g_k(z)}$ for $1 \leq j \leq n$, $1 \leq t < k \leq n$.*
Then $f_j(z) \equiv 0$, $(j = 1, \ldots, n)$.

Lemma 2.4. [14] *Let $f(z)$ be an entire function of finite order. Then $f(z) = u(z)e^{v(z)}$, where $u(z)$ is the canonical product of $f(z)$ formed with the zeros of f, and $v(z)$ is a polynomial of degree at most $\sigma(f)$.*

Proof of Theorem 1.1. Let f be a finite-order nonconstant entire solution of (1.5). If $\Delta_c f \equiv 0$, then we get f is transcendental, which contradicts that $f^n = p$ is a polynomial by (1.5). So we have $\Delta_c f \not\equiv 0$.

Proof of (i). From Lemma 2.1 and (1.5), it follows that for each $\varepsilon > 0$,

$$\begin{aligned} nm(r, f) &= m(r, p(z) - q(z)e^{Q(z)}\Delta_c f) \\ &\leq m(r, e^{Q(z)}) + m\left(r, \frac{\Delta_c f(z)}{f(z)}\right) + m(r, f(z)) + O(\log r) \\ &\leq m(r, e^{Q(z)}) + m(r, f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r), \end{aligned} \tag{2.1}$$

$$\begin{aligned} m(r, e^{Q(z)}) &= m\left(r, \frac{p(z) - f^n(z)}{q(z)\Delta_c f(z)}\right) \\ &\leq m\left(r, \frac{1}{\Delta_c f(z)}\right) + m(r, f^n(z)) + O(\log r) \\ &\leq T(r, \Delta_c f(z)) + nm(r, f(z)) + O(\log r) \\ &\leq (n+1)m(r, f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r). \end{aligned} \tag{2.2}$$

Hence, by (2.1) and (2.2), we get $\sigma(f) = \deg(Q(z))$, and f is of mean type, that is, its type $\tau(f) = \overline{\lim\limits_{r\to\infty}} \frac{T(r,f)}{r^{\sigma(f)}} \in (0, +\infty)$.

Proof of (ii) "⇒." Suppose that $P(z) \equiv 0$, then (1.5) can be represented as

$$f^{n-1}(z) + q(z)e^{Q(z)}\left(\frac{f(z+c)}{f(z)} - 1\right) = 0. \tag{2.3}$$

By (2.3), we get

$$N\left(r, \frac{f(z+c)}{f(z)}\right) = O(\log r).$$

Then, combining Lemma 2.1, we get for each $\varepsilon > 0$,

$$T\left(r, \frac{f(z+c)}{f(z)}\right) = O(r^{\sigma(f)-1+\varepsilon}) + O(\log r). \tag{2.4}$$

Hence, from (2.3) and (2.4), it follows that for each $\varepsilon > 0$,

$$(n-1)N\left(r, \frac{1}{f(z)}\right) = N\left(r, \frac{1}{q(z)e^{Q(z)}\frac{\Delta_c f(z)}{f(z)}}\right) \leq T\left(r, \frac{\Delta_c f(z)}{f(z)}\right) + O(\log r) = O(r^{\sigma(f)-1+\varepsilon}) + O(\log r). \tag{2.5}$$

By (2.5), we get $\lambda(f) < \sigma(f)$, which contradicts the condition $\lambda(f) = \sigma(f)$. So $P(z) \not\equiv 0$.

Proof of (ii) "⇐." Since $P(z) \not\equiv 0$, from Lemma 2.1, (1.5), and the second main theorem related to three small functions [9, Theorem 2.5], we get for each $\varepsilon > 0$,

$$\begin{aligned} nT(r,f) = T(r,f^n) &\le \bar{N}\left(r,\frac{1}{f^n}\right) + \bar{N}(r,f^n) + \bar{N}\left(r,\frac{1}{f^n - P}\right) + S(r,f) \\ &\le \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{\Delta_c f}\right) + S(r,f) \\ &\le \bar{N}\left(r,\frac{1}{f}\right) + T(r,\Delta_c f) + S(r,f) \\ &\le \bar{N}\left(r,\frac{1}{f}\right) + m(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f). \end{aligned} \tag{2.6}$$

(2.6) implies that $\sigma(f) \le \lambda(f)$. Then from the fact $\sigma(f) \ge \lambda(f)$, we get $\sigma(f) = \lambda(f)$.

Proof of (iii). First, we prove that if $n \ge 3$, then $p(z) \equiv 0$. In fact, if $p(z) \not\equiv 0$, then by (2.6), we get $T(r,f) = O(r^{\sigma(f)-1+\varepsilon}) + S(r,f)$, which yields $\sigma(f) \le \sigma(f) - 1$. This is a contradiction.

Second, from result (ii) of Theorem 1.1, it is easy to obtain that $\lambda(f) < \sigma(f)$ if and only if $P(z) \equiv 0$. So, we only prove $\lambda(f) = \sigma(f) - 1$ provided $P(z) \equiv 0$.

Since $P(z) \equiv 0$, we get $\lambda(f) < \sigma(f)$. From this equality and Lemma 2.4, it yields

$$f(z) = A(z)e^{\alpha(z)}, \tag{2.7}$$

where $\alpha(z)$ is a polynomial, $A(z)(\not\equiv 0)$ is the canonical product of f formed with the zeros of f, such that

$$\sigma(A) = \lambda(A) = \lambda(f) < \sigma(f), \deg \alpha = \sigma(f) = \deg Q := s. \tag{2.8}$$

Substituting (2.7) into (1.5), we get

$$A^n(z)e^{n\alpha_0(z)}e^{na_s z^s} + q(z)e^{\alpha_0(z)+Q_0(z)}(A(z+c)e^{\alpha(z+c)-\alpha(z)} - A(z))e^{(a_s+b_s)z^s} = 0, \tag{2.9}$$

where $\alpha(z) = a_s z^s + \alpha_0(z)$, $Q(z) = b_s z^s + Q_0(z)$. Here, a_s, b_s are nonzero constants, α_0, Q_0 are polynomials of degree $\le s-1$. By (2.8), (2.9), and Lemma 2.3, we get $na_s = a_s + b_s$, and

$$A^n(z) + q(z)e^{Q_0(z)+(1-n)\alpha_0(z)}(A(z+c)e^{\alpha(z+c)-\alpha(z)} - A(z)) = 0. \tag{2.10}$$

Next, we divide three cases to complete the proof.

Case 1. $s - 1 > 0$ and $\sigma(A) < s - 1$. If $\deg(Q_0(z) + (1-n)\alpha_0(z)) < s - 1$, then by $\deg(\alpha(z+c) - \alpha(z)) = s - 1$, Lemmas 2.2, 2.3, and (2.10), we get $q(z)A(z+c) \equiv 0$, which contradicts the fact that $q(z) \not\equiv 0$ and $A(z) \not\equiv 0$. Similarly, if $\deg(Q_0(z) + (1-n)\alpha_0(z)) = s - 1$, then by Lemmas 2.2, 2.3, and (2.10), it yields $q(z)A(z) \equiv 0$, and we again have a contradiction.

Case 2. $0 < s - 1 \le \sigma(A)$. It follows from (2.10) that

$$N\left(r,\frac{A(z+c)}{A(z)}\right) \le N\left(r,\frac{1}{q(z)}\right) = O(\log r).$$

Then by Lemma 2.1, we get for each $\varepsilon > 0$,

$$T\left(r,\frac{A(z+c)}{A(z)}\right) = O(r^{\sigma(A)-1+\varepsilon}) + O(\log r). \tag{2.11}$$

Combining (2.10) and (2.11), we get for each $\varepsilon > 0$,

$$\begin{aligned} (n-1)N\left(r,\frac{1}{A(z)}\right) &\le N\left(r,\frac{1}{\frac{A(z+c)}{A(z)}e^{\alpha(z+c)-\alpha(z)} - 1}\right) + O(\log r) \\ &\le N\left(r,\frac{1}{e^{\alpha(z+c)-\alpha(z)} - \frac{A(z)}{A(z+c)}}\right) + O(r^{\sigma(A)-1+\varepsilon}) + O(\log r) \\ &\le T(r,e^{\alpha(z+c)-\alpha(z)}) + O(r^{\sigma(A)-1+\varepsilon}) + O(\log r). \end{aligned} \tag{2.12}$$

Hence, from $\deg(\alpha(z+c)-\alpha(z)) = s-1$, (2.8), and (2.12), it follows $\lambda(A) \le s-1$. Since $\lambda(A) = \sigma(A) \ge s-1$, we get $\lambda(f) = \lambda(A) = s-1 = \sigma(f)-1$.

Case 3. $s-1=0$. By (2.8), we get $\sigma(A) = \lambda(A) < 1$. If $A(z)$ is transcendental, then $A(z)$ has infinitely many zeros. But (2.12) implies $N\left(r, \frac{1}{A(z)}\right) = O(\log r)$. This is a contradiction. So $A(z)$ is a polynomial and $\lambda(f) = \lambda(A) = 0 = \sigma(f)-1$. □

3 Proof of Theorem 1.3

Following Steinmetz [15], the exponential polynomial (1.3) can be normalized in the form

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^s} + \cdots + H_m(z)e^{\omega_m z^s}, \tag{3.1}$$

where $m \le k$, $\omega_1,\ldots,\omega_m$ are pairwise different leading coefficients of the polynomials $Q_j(z)(1 \le j \le k)$ of degree s, $H_j(z)(0 \le j \le m)$ are either exponential polynomials of degree $\le s-1$ or ordinary polynomials in z. So, we have $H_j(z) \not\equiv 0 (1 \le j \le m)$.

Set $W = \{\overline{\omega_1},\ldots,\overline{\omega_m}\}$, $W_0 = W \cup \{0\}$. The convex hull of the set W, denoted by $co(W)$, is the intersection of all convex sets containing W. If W contains finitely many elements, then $co(W)$ is either a compact polygon with nonempty interior or a line segment. We denote by $C(co(W))$ the perimeter of $co(W)$. When $co(W)$ is a line segment, $C(co(W))$ equals to twice the length of this line segment.

Theorem C. [15] *Let f be given by* (3.1). *Then*

$$T(r,f) = C(co(W_0))\frac{r^s}{2\pi} + o(r^s).$$

Theorem D. [15] *Let f be given by* (3.1). *If $H_0(z) \not\equiv 0$, then*

$$m\left(r, \frac{1}{f}\right) = o(r^s);$$

while if $H_0(z) \equiv 0$, then

$$N\left(r, \frac{1}{f}\right) = C(co(W))\frac{r^s}{2\pi} + o(r^s).$$

In order to prove Theorem 1.3, we need Lemmas 3.1 and 3.2. As one may observe, the proof of Lemma 3.1 is somewhat similar to that of [7, Lemma 4.5].

Lemma 3.1. *Let f be given by* (3.1), *and be a solution of* (1.5), *where $n=2$. If the points $0, \omega_1,\ldots,\omega_m$ are collinear, then $m=1$.*

Proof. Suppose contrary to the assertion that $m \ge 2$. Without loss of generality, we assume that $\omega_i = k_i\omega$ $(i=1,\ldots,m)$, where ω is a nonzero complex number, k_i are distinct nonzero real numbers such that $k_i > k_j$ for $i > j$. Set $Q(z) = b_s z^s + Q_0(z)$, where $Q_0(z)$ is a polynomial of degree $\le s-1$. Substituting the expression (3.1) of f into (1.5), we get

$$\sum_{i,j=0}^{m} H_i(z)H_j(z)e^{(k_i+k_j)\omega z^s} = P(z) - q(z)e^{Q_0(z)}\left\{l_0(z)e^{b_s z^s} + \sum_{t=1}^{m} l_t(z)e^{(b_s+k_t\omega)z^s}\right\}, \tag{3.2}$$

where $k_0 = 0$, $l_0(z) = H_0(z+c) - H_0(z)$, $l_t(z) = H_t(z+c)e^{\omega_t(z+c)^s - \omega_t z^s} - H_t(z)$ $(t=1,\ldots,m)$.

Now we divide two cases $k_m > 0$ and $k_m < 0$ to derive a contradiction. Considering the similarity of proofs in these two cases, we only discuss the case $k_m > 0$ in detail.

Let $A = \{k_i + k_j : i, j = 0, 1, \ldots, m\}$, we have

$$\max A = 2k_m, \quad \min(A - \{0\}) = \begin{cases} k_1, & k_1 > 0, \\ 2k_1, & k_1 < 0. \end{cases} \tag{3.3}$$

From the proof of Theorem 1.1, we get $\Delta_c f \not\equiv 0$, which implies that at least one of $l_m, \ldots, l_0$ is not vanishing.

Case 1. $l_m(z) = \cdots = l_1(z) \equiv 0$. Then we have $l_0(z) \not\equiv 0$, which implies $H_0(z) \not\equiv 0$. If $2k_m\omega \neq b_s$, then by (3.2), (3.3), and Lemma 2.3, we get $H_m^2(z) \equiv 0$, which contradicts $H_m(z) \not\equiv 0$. If $2k_m\omega = b_s$, then by (3.2), (3.3), and Lemma 2.3, we get $H_1(z) \equiv 0$, which also contradicts $H_1(z) \not\equiv 0$.

Case 2. At least one of $l_m, \ldots, l_1$ is not vanishing. Set $\Lambda = \{t \in \{0, 1, \ldots, m\} : l_t(z) \not\equiv 0\}$, then Λ is a nonempty set. Denote $t_0 = \max \Lambda$.

Subcase 2.1. $l_0(z) \not\equiv 0$. If there exists some $j_0 \in \Lambda \backslash \{t_0\}$ such that $2k_m\omega = b_s + k_{j_0}\omega$, then by (3.2), (3.3), Lemma 2.3, and the fact

$$2k_m - k_{j_0} + k_{t_0} > \max\{2k_m - k_{j_0} + k_i : i = 0, \ldots, t_0 - 1\},$$

we get $q(z)e^{Q_0(z)}l_{t_0}(z) \equiv 0$, which contradicts $q(z) \not\equiv 0$. If $2k_m\omega = b_s + k_{t_0}\omega$, then by (3.2), (3.3), Lemma 2.3, and the fact

$$\min\{2k_m - k_{t_0} + k_i : i = 0, \ldots, t_0\} > \min(A - \{0\}),$$

we get $H_1(z) \equiv 0$, which contradicts $H_1(z) \not\equiv 0$.

Subcase 2.2. $l_0(z) \equiv 0$ and $H_0(z) \not\equiv 0$. Using the similar proof to subcase 2.1, we get a contradiction.

Subcase 2.3. $l_0(z) \equiv 0$ and $H_0(z) \equiv 0$. Then by (3.2), (3.3), and Lemma 2.3, we get there exists some $j_0 \in \Lambda \backslash \{0\}$ such that $2k_m\omega = b_s + k_{j_0}\omega$. Otherwise, we have $H_m^2(z) \equiv 0$, which contradicts $H_m(z) \not\equiv 0$. While if $2k_m\omega = b_s + k_{j_0}\omega$, then by (3.2), (3.3) and the fact

$$2k_1 = \min\{2k_1, \ldots, 2k_m, k_i + k_j : 1 \leq i < j \leq m\} < \min\{2k_m - k_{j_0} + k_i : i = 1, \ldots, t_0\},$$

we get $H_1^2(z) \equiv 0$, which contradicts $H_1(z) \not\equiv 0$. □

Lemma 3.2. *Let f be given by* (3.1). *If f is a solution of* (1.5), *where* $n = 2$, *then* $m = 1$.

Proof. Suppose contrary to the assertion that $m \geq 2$. Substituting the expression (3.1) of f into (1.5), we get

$$F(z) := f^2(z) - p(z) = M(z) + \sum_{\substack{i,j=0 \\ \omega_i + \omega_j \neq 0}}^{m} H_i(z)H_j(z)e^{(\omega_i+\omega_j)z^s} = -q(z)e^{Q(z)}\Delta_c f(z) = -q(z)e^{Q(z)} \sum_{j=0}^{m} l_j(z)e^{\omega_j z^s}, \tag{3.4}$$

where $M(z)$ is either an exponential polynomial of degree $\leq s - 1$ or ordinary polynomial in z, $l_j(z) = H_j(z + c)e^{\omega_j(z+c)^s - \omega_j z^s} - H_j(z)$ $(j = 0, 1, \ldots, m)$, $\omega_0 = 0$. From the proof of Theorem 1.1, we get $\Delta_c f \not\equiv 0$, which implies that at least one of $l_m, \ldots, l_0$ is not vanishing.

Set

$$X_1 = \{\overline{\omega_1}, \ldots, \overline{\omega_m}, \overline{\omega_i} + \overline{\omega_j} : \overline{\omega_i} + \overline{\omega_j} \neq 0, i, j = 1, \ldots, m\},$$

$$X_2 = \{\overline{\omega_i} + \overline{\omega_j} : \overline{\omega_i} + \overline{\omega_j} \neq 0, i, j = 1, \ldots, m\},$$

$$Y_1 = \{\overline{\omega_1}, \ldots, \overline{\omega_m}, 2\overline{\omega_1}, \ldots, 2\overline{\omega_m}\}, \quad Y_2 = \{2\overline{\omega_1}, \ldots, 2\overline{\omega_m}\}.$$

It is obvious by convexity that

$$C(co(X_2)) \leq C(co(X_1)), C(co(Y_2)) \leq C(co(Y_1)), C(co(X_j)) = C(co(Y_j)), (j = 1, 2). \tag{3.5}$$

If $l_m(z) = \cdots = l_1(z) \equiv 0$, then we have $l_0(z) \not\equiv 0$, which implies $H_0(z) \not\equiv 0$. From the fact $F(z) = -q(z)l_0(z)e^{Q(z)}$ and Theorem C, we get

$$N\left(r, \frac{1}{F}\right) = o(r^s).$$

On the other hand, by (3.4), Theorems C and D, we get

$$N\left(r, \frac{1}{F}\right) = \begin{cases} C(co(X_1))\frac{r^s}{2\pi} + o(r^s), & M(z) \equiv 0, \\ 2C(co(W_0))\frac{r^s}{2\pi} + o(r^s), & M(z) \not\equiv 0. \end{cases}$$

This yields a contradiction. So, there exists some $t_0 \in \{1,\dots,m\}$ such that $l_{t_0}(z) \not\equiv 0$. Set $\Lambda_1 = \{t \in \{1,\dots,m\}: l_t(z) \not\equiv 0\}$. Denote by T the set consisted of nonzero elements $\bar{\omega}_t (t \in \Lambda_1)$ and $T_0 = T \cup \{0\}$. It is obvious that

$$C(co(T)) \le C(co(W)), \; C(co(T_0)) \le C(co(W_0)).$$

By (3.4), Theorems C and D, we get

$$N\left(r, \frac{1}{F(z)}\right) = N\left(r, \frac{1}{\Delta_c f(z)}\right) + O(\log r) = \begin{cases} C(co(T))\frac{r^s}{2\pi} + o(r^s), & l_0(z) \equiv 0, \\ C(co(T_0))\frac{r^s}{2\pi} + o(r^s), & l_0(z) \not\equiv 0. \end{cases} \tag{3.6}$$

Case 1. Suppose that $M(z) \not\equiv 0$, then by (3.4) and Theorem D, we get

$$m\left(r, \frac{1}{F(z)}\right) = o(r^s). \tag{3.7}$$

So, by Theorem C, (3.4), (3.6), and (3.7), we get

$$\begin{aligned} 2C(co(W_0))\frac{r^s}{2\pi} + o(r^s) &= 2T(r, f(z)) = T(r, F(z)) + O(\log r) = T\left(r, \frac{1}{F(z)}\right) + O(\log r) \\ &= \begin{cases} C(co(T))\frac{r^s}{2\pi} + o(r^s), & l_0(z) \equiv 0, \\ C(co(T_0))\frac{r^s}{2\pi} + o(r^s), & l_0(z) \not\equiv 0, \end{cases} \end{aligned}$$

which contradicts $C(co(W_0)) \ge C(co(T_0)) \ge C(co(T))$.

Case 2. Suppose that $M(z) \equiv 0$, then by (3.4) and Theorem D, we get

$$N\left(r, \frac{1}{F(z)}\right) = N\left(r, \frac{1}{f^2(z) - p(z)}\right) = \begin{cases} C(co(X_1))\frac{r^s}{2\pi} + o(r^s), & H_0(z) \not\equiv 0, \\ C(co(X_2))\frac{r^s}{2\pi} + o(r^s), & H_0(z) \equiv 0. \end{cases} \tag{3.8}$$

Subcase 2.1. If $H_0(z) \equiv 0$, then we must have $l_0(z) \equiv 0$. So, by (3.6) and (3.8), we get $C(co(T)) = C(co(X_2))$. From this equality and (3.5), it follows that

$$\frac{1}{2}C(co(Y_2)) = C(co(W)) \ge C(co(T)) = C(co(X_2)) = C(co(Y_2)),$$

which is a contradiction.

Subcase 2.2. If $H_0(z) \not\equiv 0$ and $l_0(z) \equiv 0$, then by (3.6) and (3.8), we get $C(co(T)) = C(co(X_1))$. From this equality and (3.5), we get

$$\frac{1}{2}C(co(Y_2)) = C(co(W)) \ge C(co(T)) = C(co(X_1)) = C(co(Y_1)) \ge C(co(Y_2)),$$

which is a contradiction.

Subcase 2.3. If $H_0(z)l_0(z) \not\equiv 0$, then by (3.6) and (3.8), we get

$$C(co(T_0)) = C(co(X_1)). \tag{3.9}$$

By Lemma 3.1 and $m \geq 2$, we know that $co(W_0)$ cannot reduce to a line segment. Hence, $co(W_0)$ is a polygon with nonempty interior.

If 0 is not a boundary point of $co(W_0)$, then we have $co(W_0) = co(W)$. From this equality, (3.5) and (3.9), it follows that

$$C(co(Y_2)) \leq C(co(Y_1)) = C(co(X_1)) \leq C(co(W_0)) = C(co(W)) = \frac{1}{2}C(co(Y_2)),$$

which is a contradiction. So, 0 is a boundary point of $co(W_0)$. Since $co(W_0)$ is a polygon with nonempty interior, we denote the nonzero corner points of $co(W_0)$ by $u_1,\ldots,u_t (t \leq m)$, such that $0 \leq \arg u_1 < \arg u_2 < \cdots < \arg u_t < 2\pi$. Hence,

$$C(co(W_0)) = |u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + |u_t|. \tag{3.10}$$

Set $Y_3 = \{u_1, 2u_1,\ldots, 2u_t, u_t\}$. Since $u_1,\ldots, u_t$ are corner points of $co(W_0)$ and $\arg u_j < \arg u_{j+1} (j = 1,\ldots, t-1)$, we know that the points $2u_1,\ldots, 2u_t$ must be corner points of $co(Y_3)$, and $co(Y_3)$ may have other corner points. So by (3.10), we get

$$C(co(Y_3)) > |2u_1 - u_1| + |2u_2 - 2u_1| + \cdots + |2u_t - 2u_{t-1}| + |2u_t - u_t| > C(co(W_0)). \tag{3.11}$$

Then, by $Y_3 \subset Y_1$, (3.5), and (3.11), we get

$$C(co(X_1)) = C(co(Y_1)) \geq C(co(Y_3)) > C(co(W_0)) \geq C(co(T_0)),$$

which contradicts (3.9). □

Proof of Theorem 1.3. The condition $p(z) \not\equiv 0$ and result (iii) of Theorem 1.1 imply $n = 2$. Then by Lemma 3.2, we get $m = 1$, that is,

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^s}, \tag{3.12}$$

where $H_0(z), H_1(z) (\not\equiv 0)$ are either exponential polynomials of degree $\leq s-1$ or ordinary polynomials in z. Substituting (3.12) into (1.5), we get

$$H_1^2(z)e^{2\omega_1 z^s} + 2H_0(z)H_1(z)e^{\omega_1 z^s} + q(z)e^{Q_0(z)}l_0(z)e^{b_s z^s} + q(z)e^{Q_0(z)}l_1(z)e^{(\omega_1+b_s)z^s} = p(z) - H_0^2(z), \tag{3.13}$$

where $Q(z) = b_s z^s + Q_0(z)$, $Q_0(z)$ is a polynomial of degree $\leq s-1$, $l_0(z) = H_0(z+c) - H_0(z)$, $l_1(z) = H_1(z+c)e^{\omega_1(z+c)^s - \omega_1 z^s} - H_1(z)$. From the proof of Theorem 1.1, we get $\Delta_c f \not\equiv 0$, which implies that at least one of l_1, l_0 is not vanishing.

Claim I. We claim that $l_1(z) \not\equiv 0$. Otherwise, we have $l_0(z) \not\equiv 0$, which implies $H_0(z) \not\equiv 0$. If $b_s = 2\omega_1$, then by (3.13) and Lemma 2.3, we get $H_0(z)H_1(z) \equiv 0$, which yields $H_1(z) \equiv 0$. This contradicts $H_1(z) \not\equiv 0$. So $b_s \neq 2\omega_1$. Then by (3.13) and Lemma 2.3, we get $H_1^2(z) \equiv 0$, which contradicts $H_1(z) \not\equiv 0$.

Claim II. We claim that $H_0(z) \not\equiv 0$. Otherwise, we have $l_0(z) \equiv 0$. If $b_s = \omega_1$, then by (3.13) and Lemma 2.3, we get $p(z) \equiv 0$, which contradicts the hypotheses of Theorem 1.3. So $b_s \neq \omega_1$. Then by (3.13) and Lemma 2.3, we get $H_1^2(z) \equiv 0$, which contradicts $H_1(z) \not\equiv 0$.

Now we divide two cases to complete the proof of Theorem 1.3.

Case 1. $l_0(z) \equiv 0$. Since $\omega_1 \notin \{2\omega_1, \omega_1 + b_s\}$, by (3.13) and Lemma 2.3, we get $H_0(z)H_1(z) \equiv 0$, which contradicts Claims I or II.

Case 2. $l_0(z) \not\equiv 0$. If $\omega_1 + b_s = 0$, then by $2\omega_1 \notin \{\omega_1, -\omega_1\}$, (3.13), and Lemma 2.3, we get $H_1^2(z) \equiv 0$, which contradicts $H_1(z) \not\equiv 0$. So, we have $\omega_1 + b_s \neq 0$.

Subcase 2.1. Suppose that $b_s \neq \omega_1$, then we get $\omega_1 \notin \{2\omega_1, b_s, \omega_1 + b_s\}$. Hence, by (3.13) and Lemma 2.3, we get $H_0(z)H_1(z) \equiv 0$. This is a contradiction.

Subcase 2.2. Suppose that $b_s = \omega_1$, then by (3.13) and Lemma 2.3, we get

$$\begin{cases} H_1^2(z) + q(z)e^{Q_0(z)}\{H_1(z+c)e^{\omega_1(z+c)^s-\omega_1 z^s} - H_1(z)\} = 0, \\ 2H_0(z)H_1(z) + q(z)e^{Q_0(z)}\{H_0(z+c) - H_0(z)\} = 0, \\ p(z) - H_0^2(z) = 0. \end{cases} \tag{3.14}$$

From the second and third equations of (3.14), we obtain that $H_0(z)$ is a nonconstant polynomial. Solving the second equation of (3.14) yields that

$$H_1(z) = G(z)e^{Q_0(z)}, \tag{3.15}$$

where $G(z) = \frac{H_0(z) - H_0(z+c)}{2H_0(z)}q(z)$ is a nonzero polynomial. Substituting (3.15) into the first equation of (3.14), we get

$$G^2(z) + q(z)\{G(z+c)e^{\omega_1(z+c)^s-\omega_1 z^s+Q_0(z+c)-Q_0(z)} - G(z)\} = 0. \tag{3.16}$$

If $s \geq 2$, then by (3.16), Lemma 2.3, and the fact that

$$\deg(\omega_1(z+c)^s - \omega_1 z^s) = s - 1, \deg(Q_0(z+c) - Q_0(z)) = \deg\{Q_0(z)\} - 1 \leq s - 2,$$

we get $q(z)G(z+c) \equiv 0$, which contradicts $q(z) \not\equiv 0$. So $s = 1$ and $H_1(z)$ is a polynomial. Suppose that H_1 is a constant, it follows from Claim I that $e^{\omega_1 c} \neq 1$. Then by the first and second equations of (3.14), we obtain that q is a constant and $\deg H_0 = \deg H_0 - 1$. But the latter equality cannot hold. So H_1 is a nonconstant polynomial. Finally, comparing the degree of polynomials of the first and second equations of (3.14), we get $\deg H_1 = \deg q - 1$ and $e^{\omega_1 c} = 1$. Theorem 1.3 is thus proved. □

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