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Research Article

Wenxu Ge*, Weiping Li, and Feng Zhao

The integral part of a nonlinear form with a square, a cube and a biquadrate

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Abstract: In this paper, we show that if $\lambda_1, \lambda_2, \lambda_3$ are non-zero real numbers, and at least one of the numbers $\lambda_1, \lambda_2, \lambda_3$ is irrational, then the integer parts of $\lambda_1 n_1^2 + \lambda_2 n_2^3 + \lambda_3 n_3^4$ are prime infinitely often for integers n_1, n_2, n_3. This gives an improvement of an earlier result.

Keywords: Diophantine inequalities, primes, Davenport-Heilbronn method

MSC 2020: 11P32, 11P55, 11J25

1 Introduction

Gauss proved that every prime $p \equiv 1 (\bmod 4)$ can be represented as $n_1^2 + n_2^2$ using the imaginary quadratic field $\mathbb{Q}(i)$. In 1998, Friedlander and Iwaniec [1] proved that $n_1^2 + n_2^4$ are prime infinitely often for integers n_1, n_2. Furthermore, one can consider such problem with Diophantine inequalities. But at present there are essential difficulties to consider the problem of Friedlander and Iwaniec in Diophantine inequality, since the sequence is thin. In this paper, we consider a weaker result by adding a cube. We also conjecture that

Conjecture 1. Suppose that λ_1, λ_2 be non-zero real numbers, not all negative, and that at least one of the numbers λ_1, λ_2 is irrational. Then the integer parts of $\lambda_1 n_1^2 + \lambda_2 n_2^2$ are prime infinitely often for integers n_1, n_2.

Harman [2] proved that there is a number $\delta > 0$ such that, for θ a positive irrational and α an arbitrary real number, the equality

$$|\theta(n_1^2 + n_2^2) - p_1 p_2 + \alpha| < (n_1^2 + n_2^2)^{-\delta}$$

has infinitely many solutions with coprime integers n_1, n_2 and primes p_1, p_2. This implies that the integer parts of $|\theta(n_1^2 + n_2^2)|$ are just two prime factors infinitely often for integers n_1, n_2.

In 2015, Yang and Li [3] proved that if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are non-zero real numbers, at least one of the ratios λ_i/λ_j is irrational, then the integer parts of $\lambda_1 n_1^2 + \lambda_2 n_2^3 + \lambda_3 n_3^4 + \lambda_4 n_4^5$ are prime infinitely often for integers n_1, n_2, n_3, n_4. Later, Li, Ge and Wang [4] proved that the aforementioned result is still valid for primes p_1, p_2, p_3, p_4.

In this paper, we establish the following result.

*** Corresponding author: Wenxu Ge,** School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou, 450046, People's Republic of China, e-mail: gewenxu@ncwu.edu.cn
Weiping Li: School of Mathematics and Information Sciences, Henan University of Economics and Law, Zhengzhou, 450000, People's Republic of China
Feng Zhao: School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou, 450046, People's Republic of China

Theorem 1. *Let $\lambda_1, \lambda_2, \lambda_3$ be non-zero real numbers. Suppose that at least one of the real numbers $\lambda_1, \lambda_2, \lambda_3$ are irrational. Then the integer parts of $\lambda_1 n_1^2 + \lambda_2 n_2^3 + \lambda_3 n_3^4$ are prime infinitely often for integers n_1, n_2, n_3.*

Notation. Throughout the paper, the letter η denotes a sufficiently small, fixed positive number. The letter ε denotes an arbitrarily sufficiently small positive real number. The letter p, with or without subscript, denotes a prime number. Constants, both explicit and implicit, in Vinogradov symbols may depend on $\lambda_1, \lambda_2, \lambda_3$. We write $e(x) = \exp(2\pi i x)$.

2 Outline of the method

We follow the modification of the Hardy-Littlewood method first stated by Davenport and Heilbronn. We can assume that $\lambda_1, \lambda_2, \lambda_3$ are not all negative if necessary, n_2^3 is replaced by $(-n_2)^3$. Now let X be some (large) positive quantity, $L = \log X$. We define

$$K(\alpha) = \left(\frac{\sin\left(\frac{1}{2}\pi\alpha\right)}{\pi\alpha}\right)^2, \quad A(x) = \int_{\mathbb{R}} K(\alpha)e(\alpha x)\,\mathrm{d}\alpha. \tag{2.1}$$

Then, by [5], it is easy to show that

$$K(\alpha) \ll \min(1, |\alpha|^{-2}), \quad A(x) = \max\left(0, \frac{1}{2} - |x|\right). \tag{2.2}$$

Let

$$\mathcal{A} = \{1 \le n \le X^{1/3} : p|n \Rightarrow p \le X^{\delta}\},$$

where δ is a sufficiently small, fixed positive number. Obviously, we have

$$|\mathcal{A}| \gg X^{1/3-\delta}.$$

We write for $j = 1, 2$,

$$f(\alpha) = \sum_{\eta X < p \le X} (\log p)e(p\alpha); \quad g_3^*(\alpha) = \sum_{\substack{\eta X < n^3 \le X \\ n \in \mathcal{A}}} e(n^3\alpha); \tag{2.3}$$

$$g_2(\alpha) = \sum_{\eta X < n^2 \le X} e(n^2\alpha); \quad g_4(\alpha) = \sum_{\eta X < n^4 \le X} e(n^4\alpha); \tag{2.4}$$

$$I_j(\alpha) = \int_{(\eta X)^{1/j}}^{X^{1/j}} e(\alpha x^j)\,\mathrm{d}x,$$

where η is a sufficiently small, fixed positive number.

We define further

$$F(\alpha) := f(-\alpha)g_2(\lambda_1\alpha)g_3^*(\lambda_2\alpha)g_4(\lambda_3\alpha). \tag{2.5}$$

For any measurable subset $\mathfrak{X}$ of $\mathbb{R}$, we define

$$J(\mathfrak{X}) := \int_{\mathfrak{X}} F(\alpha)K(\alpha)e\left(-\frac{1}{2}\alpha\right)\mathrm{d}\alpha. \tag{2.6}$$

Then by (2.1), (2.2), (2.3) and (2.4), we have

$$\begin{aligned} J(\mathbb{R}) &= \sum_{\substack{\eta X<p,n_1^2,n_2^3,n_3^4\le X\\ n_2\in\mathcal{A}}} (\log p) A\left(\lambda_1 n_1^2+\lambda_2 n_2^3+\lambda_3 n_3^4-p-\frac{1}{2}\right)\\ &\le \frac{1}{2}L \sum_{\substack{\eta X<p,n_1^2,n_2^3,n_3^4\le X\\ n_2\in\mathcal{A}\\ \left|\lambda_1 n_1^2+\lambda_2 n_2^3+\lambda_3 n_3^4-p-\frac{1}{2}\right|<\frac{1}{2}}} 1\\ &\le \frac{1}{2}L \sum_{\substack{\eta X<p,n_1^2,n_2^3,n_3^4\le X\\ \left|\lambda_1 n_1^2+\lambda_2 n_2^3+\lambda_3 n_3^4-p-\frac{1}{2}\right|<\frac{1}{2}}} 1\\ &\le \frac{1}{2}L\mathcal{N}(X), \end{aligned} \tag{2.7}$$

where $\mathcal{N}(X)$ denote the number of solutions of the inequality

$$\left|\lambda_1 n_1^2+\lambda_2 n_2^3+\lambda_3 n_3^4-p-\frac{1}{2}\right| < \frac{1}{2} \tag{2.8}$$

in prime $\eta X < p \le X$ and positive integers $\eta X < n_1^2, n_2^3, n_3^4 \le X$.

To estimate the left integral in (2.7), we divide the real line into three parts: the major arc $\mathfrak{M}$, the minor arc $\mathfrak{m}$ and the trivial arc $\mathfrak{t}$, which are defined by

$$\mathfrak{M}=\{\alpha : |\alpha|\le\phi\},\quad \mathfrak{m}=\{\alpha : \phi<|\alpha|\le\xi\},\quad \mathfrak{t}=\{\alpha : |\alpha|>\xi\},$$

where $\phi = X^{-1/2-\varepsilon}$, $\xi = X^{2\delta}$.

3 Preliminary lemmas

Lemma 1. *We have*

$$\int_0^1 |f(\alpha)|^2 \mathrm{d}\alpha \ll X^{1+\varepsilon},\qquad \int_{-\infty}^{+\infty} |f(\alpha)|^2 K(\alpha)\mathrm{d}\alpha \ll X^{1+\varepsilon}; \tag{3.1}$$

$$\int_0^1 |g_2(\alpha)^4| K(\alpha)\mathrm{d}\alpha \ll X^{1+\varepsilon},\qquad \int_{-\infty}^{+\infty} |g_2(\alpha)^4| K(\alpha)\mathrm{d}\alpha \ll X^{1+\varepsilon}; \tag{3.2}$$

$$\int_0^1 |g_2(\alpha)^2 g_4(\alpha)^4| \mathrm{d}\alpha \ll X^{1+\varepsilon},\qquad \int_{-\infty}^{+\infty} |g_2(\alpha)^2 g_4(\alpha)^4| K(\alpha)\mathrm{d}\alpha \ll X^{1+\varepsilon}. \tag{3.3}$$

Proof. Estimates (3.1) and (3.2) follow from Hua's Lemma (see Lemma 2.5 of [6]). Estimate (3.3) follows from Lemma 4.2 of [7]. □

Lemma 2. [8, Theorem 1.4] *Suppose that* $\delta > 0$ *and* $k \ge 7.5906$. *Then one has*

$$\int_0^1 |g_3^*(\alpha)|^k \mathrm{d}\alpha \ll X^{k/3-1},\qquad \int_{-\infty}^{+\infty} |g_3^*(\alpha)|^k K(\alpha)\mathrm{d}\alpha \ll X^{k/3-1}.$$

Lemma 3. ([6], see also [9, Corollary 3.2]) *Suppose that $X \geq Z \geq X^{4/5+\varepsilon}$ and that $|f(\alpha)| > Z$. Then there are coprime integers a, q satisfying*

$$1 \leq q \ll X^{2+\varepsilon}Z^{-2}, \quad |q\alpha - a| \ll X^{1+\varepsilon}Z^{-2}.$$

Lemma 4. [10] *Suppose that $X^{1/2} \geq Z \geq X^{1/4+\varepsilon}$ and that $|g_2(\alpha)| > Z$. Then there are coprime integers a, q satisfying*

$$1 \leq q \ll X^{1+\varepsilon}Z^{-2}, \quad |q\alpha - a| \ll X^{\varepsilon}Z^{-2}.$$

Lemma 5. [11, Lemma 2.2] *Suppose that α is a real number and suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$, $q \leq X^{2/3}$ and $|q\alpha - a| < X^{-2/3}$. Then*

$$g_3^*(\alpha) \ll X^{3/10+\varepsilon} + \frac{X^{1/3+\varepsilon}}{q^{1/6}(1 + X|\alpha - a/q|)^{1/3}}.$$

Corollary 1. *Suppose that $X^{1/3} \geq Z \geq X^{3/10+\varepsilon}$ and that $|g_3^*(\alpha)| > Z$. Then there are coprime integers a, q satisfying*

$$1 \leq q \ll X^{2+\varepsilon}Z^{-6}, \quad |q\alpha - a| \ll X^{1+\varepsilon}Z^{-6}.$$

Proof. By Lemma 5, we have

$$Z < |g_3^*(\alpha)| \ll X^{3/10+\varepsilon/6} + \frac{X^{1/3+\varepsilon/6}}{q^{1/6}(1 + X|\alpha - a/q|)^{1/3}}.$$

Thus, we have

$$q^{1/6} \ll X^{1/3+\varepsilon/6}Z^{-1}, \quad q^{1/6}X^{1/3}|\alpha - a/q|^{1/3} \ll X^{1/3+\varepsilon/6}Z^{-1}.$$

Then

$$q \ll X^{2+\varepsilon}Z^{-6}, \quad |q\alpha - a| \ll q^{1/2}X^{\varepsilon/2}Z^{-3} \ll X^{1+\varepsilon}Z^{-6}.$$ □

4 The major arc

Lemma 6. *For $\varepsilon > 0$ and $j = 1, 2$, we have*

$$\int_{\mathfrak{M}} |f(\alpha) - I_1(\alpha)|^2 \mathrm{d}\alpha \ll XL^{-2}, \quad \int_{\mathfrak{M}} |I_j(\alpha)|^2 \mathrm{d}\alpha \ll X^{2/j-1}, \tag{4.1}$$

$$\int_{\mathfrak{M}} |g_2(\lambda_1\alpha) - I_2(\lambda_1\alpha)|^2 \mathrm{d}\alpha \ll X^{-\varepsilon}, \quad \int_{\mathfrak{M}} |g_2(\lambda_1\alpha)|^2 \mathrm{d}\alpha \ll 1. \tag{4.2}$$

Proof. The first estimate of (4.1) follows from Lemma 2 of [12] and that

$$\left| I_1(\alpha) - \sum_{\eta X < n \leq X} e(\alpha n) \right| \ll 1 + |\alpha| X.$$

The second estimate of (4.1) follows from that

$$|I_j(\alpha)| \ll X^{1/j} \min(1, |\alpha|^{-1}X^{-1}). \tag{4.3}$$

By Theorem 4.1 of [6], we have $|g_2(\alpha) - I_2(\alpha)| \ll (1 + |\alpha|X)^{1/2}$. Thus,

$$\int_{\mathfrak{M}} |g_2(\alpha) - I_2(\alpha)|^2 \mathrm{d}\alpha \ll X\phi^2 \ll X^{-\varepsilon}$$

and

$$\int_{\mathfrak{M}} |g_2(\alpha)|^2 \mathrm{d}\alpha \ll \int_{\mathfrak{M}} |g_2(\alpha) - I_2(\alpha)|^2 \mathrm{d}\alpha + \int_{\mathfrak{M}} |I_2(\alpha)|^2 \mathrm{d}\alpha \ll 1. \qquad \square$$

Lemma 7. *Let*

$$G(\alpha) = I_1(-\alpha) I_2(\lambda_1\alpha) g_3^*(\lambda_2\alpha) g_4(\lambda_3\alpha).$$

Then we have

$$\int_{\mathfrak{M}} |F(\alpha) - G(\alpha)| K(\alpha) \mathrm{d}\alpha = o(X^{3/4}|\mathcal{A}|). \tag{4.4}$$

Proof. By (2.2), (2.3), (2.4) and Lemma 6, we have

$$\begin{aligned}
\int_{\mathfrak{M}} |F(\alpha) - G(\alpha)| K(\alpha) \mathrm{d}\alpha &\ll X^{1/4}|\mathcal{A}| \int_{\mathfrak{M}} |f(-\alpha) g_2(\lambda_1\alpha) - I_1(-\alpha) I_2(\lambda_1\alpha)| \mathrm{d}\alpha \\
&\ll X^{1/4}|\mathcal{A}| \int_{\mathfrak{M}} |f(-\alpha) g_2(\lambda_1\alpha) - I_1(-\alpha) g_2(\lambda_1\alpha)| \mathrm{d}\alpha \\
&\quad + X^{1/4}|\mathcal{A}| \int_{\mathfrak{M}} |I_1(\alpha) g_2(\lambda_1\alpha) - I_1(-\alpha) I_2(\lambda_1\alpha)| \mathrm{d}\alpha \\
&\ll X^{1/4}|\mathcal{A}| \left(\int_{\mathfrak{M}} |f(\alpha) - I_1(\alpha)|^2 \mathrm{d}\alpha \right)^{1/2} \left(\int_{\mathfrak{M}} |g_2(\lambda_1\alpha)|^2 \mathrm{d}\alpha \right)^{1/2} \\
&\quad + X^{1/4}|\mathcal{A}| \left(\int_{\mathfrak{M}} |g_2(\lambda_1\alpha) - I_2(\lambda_1\alpha)|^2 \mathrm{d}\alpha \right)^{1/2} \left(\int_{\mathfrak{M}} |I_1(\alpha)|^2 \mathrm{d}\alpha \right)^{1/2} \\
&\ll X^{3/4}|\mathcal{A}| L^{-1}. \qquad \square
\end{aligned}$$

Lemma 8. *We have*

$$J(\mathfrak{M}) \gg X^{3/4}|\mathcal{A}|. \tag{4.5}$$

Proof. We only need to prove that

$$\int_{\mathfrak{M}} G(\alpha) K(\alpha) e\left(-\frac{1}{2}\alpha\right) \mathrm{d}\alpha \gg X^{3/4}|\mathcal{A}|. \tag{4.6}$$

Then Lemma 8 immediately follows from Lemma 7. By (4.3), we have

$$\begin{aligned}
\int_{\mathfrak{M}} G(\alpha) K(\alpha) e\left(-\frac{1}{2}\alpha\right) \mathrm{d}\alpha &= \int_{-\infty}^{+\infty} G(\alpha) K(\alpha) e\left(-\frac{1}{2}\alpha\right) \mathrm{d}\alpha + O\left(\int_{|\alpha|>\phi} |G(\alpha)| K(\alpha) \mathrm{d}\alpha \right) \\
&= \int_{-\infty}^{+\infty} G(\alpha) K(\alpha) e\left(-\frac{1}{2}\alpha\right) \mathrm{d}\alpha + O\left(X^{1/4}|\mathcal{A}| \int_{|\alpha|>\phi} |I_1(\alpha) I_2(\lambda_1\alpha)| \mathrm{d}\alpha \right)
\end{aligned} \tag{4.7}$$

$$= \int_{-\infty}^{+\infty} G(\alpha)K(\alpha)e\left(-\frac{1}{2}\alpha\right)\mathrm{d}\alpha + O\left(X^{1/4}|\mathcal{A}|X^{-1/2}\int_{|\alpha|>\phi}|\alpha|^{-2}\mathrm{d}\alpha\right)$$
$$= \int_{-\infty}^{+\infty} G(\alpha)K(\alpha)e\left(-\frac{1}{2}\alpha\right)\mathrm{d}\alpha + O(X^{1/4+\varepsilon}|\mathcal{A}|).$$

Then claim (4.6) follows by showing that

$$\int_{-\infty}^{+\infty} G(\alpha)K(\alpha)e\left(-\frac{1}{2}\alpha\right)\mathrm{d}\alpha \gg X^{3/4}|\mathcal{A}|,$$

which follows as Lemma 10 of [5]. □

5 The trivial arc

By Lemmas 1, 2 and (2.2), we have

$$\begin{aligned}
J(\mathrm{t}) &\ll \int_{|\alpha|>\xi} |f(\alpha)g_2(\lambda_1\alpha)g_3^*(\lambda_2\alpha)g_4(\lambda_3\alpha)|K(\alpha)\mathrm{d}\alpha \\
&\ll \left(\int_{|\alpha|>\xi}|f(\alpha)|^2\alpha^{-2}\mathrm{d}\alpha\right)^{1/2}\left(\int_{|\alpha|>\xi}|g_2(\lambda_1\alpha)^2g_4(\lambda_3\alpha)^4|\alpha^{-2}\mathrm{d}\alpha\right)^{1/4} \\
&\quad\times\left(\int_{|\alpha|>\xi}|g_2(\lambda_1\alpha)|^4\alpha^{-2}\mathrm{d}\alpha\right)^{1/8}\left(\int_{|\alpha|>\xi}|g_3^*(\lambda_2\alpha)|^8\alpha^{-2}\mathrm{d}\alpha\right)^{1/8} \\
&\ll \left(\sum_{k\geq[\xi]}\frac{1}{k^2}\int_0^1|f(\alpha)|^2\mathrm{d}\alpha\right)^{1/2}\left(\sum_{k\geq[\xi]}\frac{1}{k^2}\int_0^1|g_2(\alpha)^2g_4(\alpha)^4|\mathrm{d}\alpha\right)^{1/4} \\
&\quad\times\left(\sum_{k\geq[\xi]}\frac{1}{k^2}\int_0^1|g_2(\alpha)|^4\mathrm{d}\alpha\right)^{1/8}\left(\sum_{k\geq[\xi]}\frac{1}{k^2}\int_0^1|g_3^*(\alpha)|^8\mathrm{d}\alpha\right)^{1/8} \\
&\ll \xi^{-1}\left(\int_0^1|f(\alpha)|^2\mathrm{d}\alpha\right)^{1/2}\left(\int_0^1|g_2(\alpha)^2g_4(\alpha)^4|\mathrm{d}\alpha\right)^{1/4} \\
&\quad\times\left(\int_0^1|g_2(\alpha)|^4\mathrm{d}\alpha\right)^{1/8}\left(\int_0^1|g_3^*(\alpha)|^8\mathrm{d}\alpha\right)^{1/8} \\
&\ll \xi^{-1}X^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{5}{24}+\varepsilon} \ll X^{\frac{13}{12}-2\delta+\varepsilon}.
\end{aligned}$$

Thus, we have

$$J(\mathrm{t}) = o(X^{3/4}|\mathcal{A}|), \tag{5.1}$$

since $|\mathcal{A}| \gg X^{1/3-\delta}$.

6 The proof of Theorem 1 for the cases λ_1 or λ_3 irrational

The cases λ_1 irrational and λ_3 irrational are similar, and we only list the proof of the case λ_1 irrational here. First, we divide the minor arc $\mathfrak{m}$ into three parts. Let $0 < \Delta < 1/10$ and $\mathfrak{m}_3 = \mathfrak{m}\backslash(\mathfrak{m}_1 \cup \mathfrak{m}_2)$, where

$$\mathfrak{m}_1 = \{\alpha \in \mathfrak{m} : |f(\alpha)| \ll X^{1-\Delta+\varepsilon}\}, \quad \mathfrak{m}_2 = \{\alpha \in \mathfrak{m} : |g_2(\alpha)| \ll X^{\frac{1}{2}(1-\Delta)+\varepsilon}\}. \tag{6.1}$$

For convenience, we take $s = 7.5906$. Then by Lemmas 1 and 2, we have

$$\begin{aligned} J(\mathfrak{m}_1) &\ll \max_{\alpha\in\mathfrak{m}_1}\left\{|f(\alpha)|^{\frac{2}{s}-\frac{1}{4}}\right\}\left(\int_{-\infty}^{+\infty}|f(\alpha)|^2K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{5}{8}-\frac{1}{s}}\left(\int_{-\infty}^{+\infty}|g_3^*(\alpha)|^sK(\alpha)\,\mathrm{d}\alpha\right)^{1/s} \\ &\quad\times\left(\int_{-\infty}^{+\infty}|g_2(\alpha)^2g_4(\alpha)^4|K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{-\infty}^{+\infty}|g_2(\alpha)^4|K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{8}} \\ &\ll X^{3/4}X^{1/3-\Delta\left(\frac{2}{s}-\frac{1}{4}\right)+\varepsilon} \end{aligned}$$

and

$$\begin{aligned} J(\mathfrak{m}_2) &\ll \max_{\alpha\in\mathfrak{m}_2}\left\{|g_2(\alpha)|^{\frac{4}{s}-\frac{1}{2}}\right\}\left(\int_{-\infty}^{+\infty}|f(\alpha)|^2K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}|g_3^*(\alpha)|^sK(\alpha)\,\mathrm{d}\alpha\right)^{1/s} \\ &\quad\times\left(\int_{-\infty}^{+\infty}|g_2(\alpha)^2g_4(\alpha)^4|K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{-\infty}^{+\infty}|g_2(\alpha)^4|K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}-\frac{1}{s}} \\ &\ll X^{3/4}X^{1/3-\Delta\left(\frac{2}{s}-\frac{1}{4}\right)+\varepsilon}. \end{aligned}$$

Since δ is a sufficiently small positive number, we take $0 < \delta < \Delta\left(\frac{2}{s} - \frac{1}{4}\right)$. Thus, we have

$$J(\mathfrak{m}_1 \cup \mathfrak{m}_2) \ll X^{3/4}X^{1/3-\Delta\left(\frac{2}{s}-\frac{1}{4}\right)+\varepsilon} \ll X^{3/4}X^{1/3-\delta} = o(X^{3/4}|\mathcal{A}|). \tag{6.2}$$

It remains to discuss the set $\mathfrak{m}_3$. We use the method of Harman [13]. We divide $\mathfrak{m}_3$ into disjoint sets such that for $\alpha \in \mathcal{B}(Z_1, Z_2, y)$, we have

$$Z_1 \le |f(\alpha)| < 2Z_1, \quad Z_2 \le |g_2(\lambda_1\alpha)| < 2Z_2, \quad y \le |\alpha| < 2y,$$

where $Z_1 = X^{1-\Delta+\varepsilon}2^{t_1}$, $Z_2 = X^{1/2-\Delta/2+\varepsilon}2^{t_2}$, $y = \phi 2^u$ for some positive integers t_1, t_2, u. Thus, by Lemmas 3 and 4, there exist two pairs of coprime integers (a_1, q_1), (a_2, q_2) with $a_1a_2 \neq 0$ and

$$\begin{aligned} 1 \le q_1 &\ll X^{2+\varepsilon}Z_1^{-2}, \quad |q_1\alpha - a_1| \ll X^{1+\varepsilon}Z_1^{-2}; \\ 1 \le q_2 &\ll X^{1+\varepsilon}Z_2^{-2}, \quad |q_2\lambda_1\alpha - a_2| \ll X^{\varepsilon}Z_2^{-2}. \end{aligned}$$

Let $\mathcal{B} = \mathcal{B}(Z_1, Z_2, y, Q_1, Q_2)$ be the subset of $\mathcal{B}(Z_1, Z_2, y)$ for which $q_j \sim Q_j$. Then, by a familiar argument (see p. 147 of [6] for example),

$$\begin{aligned} |a_2q_1 - \lambda_1a_1q_2| &= \left|\frac{a_2(q_1\alpha - a_1) + a_1(a_2 - q_2\lambda_1\alpha)}{\alpha}\right| \\ &\ll Q_2X^{1+\varepsilon}Z_1^{-2} + Q_1X^{\varepsilon}Z_2^{-2} \\ &\ll X^{2+2\varepsilon}Z_1^{-2}Z_2^{-2} \\ &\ll X^{-1+3\Delta-2\varepsilon}. \end{aligned}$$

Also,

$$|a_1q_2| \ll yQ_1Q_2.$$

Since λ_1 is irrational, we let a/q be a convergent to λ_1 and put $X = q^{\frac{1}{1-3\Delta}}$. Thus, we have

$$\|\lambda_1 a_1 q_2\| \le \frac{1}{4q}, \quad q_2 \sim Q_2, \quad a_1 \approx yQ_1, \tag{6.3}$$

since X is sufficiently large. Then by the pigeon-hole principle and the Legendre law of best approximation for continued fractions, the aforementioned inequality (6.3) has $\ll yQ_1Q_2q^{-1}$ solutions of $|a_1q_2|$. Clearly, each value of $|a_1q_2|$ corresponds to $\ll X^{\varepsilon}$ values of a_1, a_2, q_1, q_2 by the well-known bound on the divisor function. Hence, we conclude that

$$\begin{aligned}\mu(\mathcal{B}) &\ll X^{\varepsilon}\frac{yQ_1Q_2}{q}\min(X^{1+\varepsilon}Z_1^{-2}Q_1^{-1}, X^{\varepsilon}Z_2^{-2}Q_2^{-1})\\ &\ll X^{\varepsilon}\frac{yQ_1Q_2}{q}\frac{X^{1/2+\varepsilon}}{Z_1Z_2Q_1^{1/2}Q_2^{1/2}} \ll \frac{X^{1/2+2\varepsilon}yQ_1^{1/2}Q_2^{1/2}}{qZ_1Z_2} \ll \frac{X^{2+2\varepsilon}y}{qZ_1^2Z_2^2},\end{aligned} \tag{6.4}$$

where $\mu(\mathcal{B})$ is the Lebesgue measure of $\mathcal{B}$. Thus, we have

$$\begin{aligned}\int_{\mathcal{B}} |f(\alpha)g_2(\alpha)g_3^*(\alpha)g_4(\alpha)|K(\alpha)\,\mathrm{d}\alpha &\ll Z_1Z_2|\mathcal{A}|X^{1/4}\min(1,|y|^{-2})\frac{X^{2+2\varepsilon}y}{qZ_1^2Z_2^2}\\ &\ll |\mathcal{A}|X^{1/4}\frac{X^{2+2\varepsilon}y}{qZ_1Z_2} \ll |\mathcal{A}|X^{9\Delta/2-1/4+\varepsilon}.\end{aligned}$$

Summing over all possible values of Z_1, Z_2, y, Q_1, Q_2, we conclude that

$$J(\mathfrak{m}_3) \ll |\mathcal{A}|X^{9\Delta/2-1/4+6\varepsilon} = o(X^{3/4}|\mathcal{A}|). \tag{6.5}$$

From (2.7), (4.5), (5.1), (6.2) and (6.5), we have

$$\mathcal{N}(X) \gg X^{3/4}|\mathcal{A}|L^{-1}.$$

Since λ_1 is irrational, there are infinitely many pairs of integers q, a. So $X = q^{\frac{1}{1-3\Delta}} \to \infty$, as $q \to \infty$, and this implies that the integer parts of $\lambda_1 n_1^2 + \lambda_2 n_2^3 + \lambda_3 n_3^4$ are prime infinitely often for integers n_1, n_2, n_3. This completes the proof of Theorem 1 for the case λ_1 irrational.

7 The proof of Theorem 1 for the case λ_2 irrational

Let $\mathfrak{m}_3' = \mathfrak{m}\backslash(\mathfrak{m}_1 \cup \mathfrak{m}_2')$, where $\mathfrak{m}_1$ is defined in (6.1) and

$$\mathfrak{m}_2' = \left\{\alpha \in \mathfrak{m} : |g_3^*(\alpha)| \ll X^{\frac{1}{3}(1-\Delta)+\varepsilon}\right\}.$$

Then by Lemmas 1 and 2, we have

$$\begin{aligned}J(\mathfrak{m}_2') &\ll \max_{\alpha\in\mathfrak{m}_2}\left\{g_3^*(\alpha)^{1-\frac{s}{8}}\right\}\left(\int_{-\infty}^{+\infty}|f(\alpha)|^2K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}|g_3^*(\alpha)|^sK(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{8}}\\ &\quad\times\left(\int_{-\infty}^{+\infty}|g_2(\alpha)^2g_4(\alpha)^4|K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{-\infty}^{+\infty}|g_2(\alpha)^4|K(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{8}}\\ &\ll X^{3/4}X^{1/3-\Delta\left(\frac{1}{3}-\frac{s}{24}\right)+\varepsilon} = o(X^{3/4}|\mathcal{A}|),\end{aligned}$$

since $0 < \delta < \Delta\left(\frac{2}{s}-\frac{1}{4}\right) < \Delta\left(\frac{1}{3}-\frac{s}{24}\right)$.

It remains to discuss the set $\mathfrak{m}_3'$. We divide $\mathfrak{m}_3'$ into disjoint sets such that for $\alpha \in \mathcal{B}'(Z_1, Z_2, y)$, we have

$$Z_1 \le |f(\alpha)| < 2Z_1, \quad Z_2 \le |g_3^*(\lambda_1\alpha)| < 2Z_2, \quad y \le |\alpha| < 2y,$$

where $Z_1 = X^{1-\Delta+\varepsilon}2^{t_1}$, $Z_2 = X^{\frac{1}{3}(1-\Delta)+\varepsilon}2^{t_2}$, $y = \phi 2^u$ for some positive integers t_1, t_2, u. Thus, by Lemma 3 and Corollary 1, there exist two pairs of coprime integers (a_1, q_1), (a_2, q_2) with $a_1 a_2 \neq 0$ and

$$1 \le q_1 \ll X^{2+\varepsilon} Z_1^{-2}, \quad |q_1\alpha - a_1| \ll X^{1+\varepsilon} Z_1^{-2},$$
$$1 \le q_2 \ll X^{2+\varepsilon} Z_2^{-6}, \quad |q_2\lambda_2\alpha - a_2| \ll X^{1+\varepsilon} Z_2^{-6}.$$

Let $\mathcal{B}' = \mathcal{B}'(Z_1, Z_2, y, Q_1, Q_2)$ be the subset of $\mathcal{B}'(Z_1, Z_2, y)$ for which $q_j \sim Q_j$. Then

$$|a_2 q_1 - \lambda_2 a_1 q_2| \ll X^{3+2\varepsilon} Z_1^{-2} Z_2^{-6} \ll X^{-1+4\Delta-6\varepsilon}.$$

Also,

$$|a_1 q_2| \ll y Q_1 Q_2.$$

Let a'/q' be a convergent to the irrational number λ_3 and put $X = q'^{\frac{1}{1-4\Delta}}$. Thus, we have

$$\mu(\mathcal{B}') \ll X^{\varepsilon} \frac{y Q_1 Q_2}{q'} \min(X^{1+\varepsilon} Z_1^{-2} Q_1^{-1}, X^{1+\varepsilon} Z_2^{-6} Q_2^{-1}) \ll \frac{X^{3+3\varepsilon} y}{q' Z_1^2 Z_2^6}. \tag{7.1}$$

Then

$$\int\limits_{\mathcal{B}'} |f(\alpha) g_2(\alpha) g_3^*(\alpha) g_4(\alpha)| K(\alpha) \mathrm{d}\alpha \ll Z_1 Z_2 X^{1/2+1/4} \min(1, |y|^{-2}) \frac{X^{3+3\varepsilon} y}{q' Z_1^2 Z_2^6} \ll X^{1/12+20\Delta/3+\varepsilon}.$$

Summing over all possible values of Z_1, Z_2, y, Q_1, Q_2, we conclude that

$$J(\mathfrak{m}_3') \ll X^{1/12+20\Delta/3+6\varepsilon}.$$

Then Theorem 1 for the case λ_2 irrational follows by repeating the works in Section 6.

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