Research Article

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Tiny zero-sum sequences over some special groups

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Abstract: Let $S = g_1 \cdot \ldots \cdot g_n$ be a sequence with elements g_i from an additive finite abelian group G. S is called a tiny zero-sum sequence if S is non-empty, $g_1 + \ldots + g_n = 0$ and $k(S) \coloneqq \sum_{i=1}^{n} \frac{1}{\mathrm{ord}(g_i)} \le 1$. Let $t(G)$ be the smallest integer t such that every sequence of t elements (repetition allowed) from G contains a tiny zero-sum sequence. In this article, we mainly focus on the explicit value of $t(G)$ and compute this value for a new class of groups, namely ones of the form $G = C_3 \oplus C_{3p}$, where p is a prime number such that $p \ge 5$.

Keywords: zero-sum sequence, finite abelian group, cross number, tiny

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1 Introduction

Let G be an additively written finite abelian group with $\exp(G)$ its exponent and let $\mathcal{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathcal{F}(G)$ are called sequences over G. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = g_1 \cdot \ldots \cdot g_l = \Pi_{g \in G} g^{v_g(S)}, \quad \text{with } v_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

A sequence T is a subsequence of S if $v_g(T) \le v_g(S)$ for every $g \in G$, denoted by $T|S$. Let ST^{-1} denote the sequence obtained by deleting the terms of T from S. By $\sigma(S)$ we denote the sum of S, that is, $\sigma(S) = \sum_{i=1}^{l} g_i \in G$. If $\sigma(S) = 0$, then S is said to be a zero-sum sequence, sometimes, for simplicity we say S is zero-sum. The cross number $k(S)$ of a sequence S is defined by

$$k(S) = \sum_{i=1}^{l} \frac{1}{\mathrm{ord}(g_i)}.$$

We call a non-empty zero-sum sequence S a *tiny zero-sum sequence* if $k(S) \le 1$. The number $t(G)$ here is the least integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge t$ contains a tiny zero-sum subsequence. In present, the value of $t(G)$ is only known for seldom kinds of group G. We gather in one separate theorem without proof all values of the invariant that are either known or easy to be deduced from the existing literature.

Let $\eta(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge t$ contains a non-empty zero-sum subsequence $S'|S$ with $|S'| \le \exp(G)$. Such a subsequence is called a *short zero-sum subsequence*. Since $\mathrm{ord}(g_i) \le \exp(G)$, it is easy to find that a tiny zero-sum sequence must be a short zero-sum sequence. Therefore, $t(G) \ge \eta(G)$ for any finite abelian group G.

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Theorem 1. *Let p be a prime. Then, $t(G) = \eta(G)$ for the following groups.*
(1) $G \cong C_{2^\alpha}^r$ [1, Theorem 3.5],
(2) $G \cong C_m^3$ *with* $m = 3^\alpha$ *or* $m = 5^\beta$ [2, Theorem 1.7],
(3) $G \cong C_{p^\alpha} \oplus C_{p^\alpha}$ [1, Corollary 3.5], *and*
(4) $G \cong C_2 \oplus C_{2p}$ [3, Theorem 1.4].

Girard [1] proved that for any positive integer $r \geq 4$, there is a finite abelian group of rank r such that $t(G) > \eta(G)$. Concerning groups of rank 3, the author with coauthors found that $t(G) > \eta(G)$ if $G \cong C_2 \oplus C_2 \oplus C_{2p}$. For all finite abelian groups of rank 2, Girard [1] conjectured that $t(G) = \eta(G)$ and proved that $t(C_{p^\alpha}^2) = \eta(C_{p^\alpha}^2) = 3p^\alpha - 2$ for any prime p. This conjecture is widely open. The author with coauthors [3] proved that $t(C_2 \oplus C_{2p}) = \eta(C_2 \oplus C_{2p}) = 2p + 2$. This article mainly computes the value of $t(G)$ for $G \cong C_3 \oplus C_{3p}$ with $p \geq 5$.

For more information on $\eta(G)$ and $t(G)$ we refer to [1,4–13].

In Section 2, we give five lemmas for the proof of the following main theorem which is in Section 3.

Theorem 2. *Let p be a prime number with $p \geq 5$. Then, $t(G) = \eta(G) = 3p + 4$ for $G \cong C_3 \oplus C_{3p}$.*

2 Preliminaries

Throughout this article, p denotes a prime number. We list some definitions and results which will be used in the article as follows.

Let G be an abelian group and let $S = g_1 \cdot \ldots \cdot g_l = \Pi_{g \in G} g^{v_g(S)}$, where $l \in \mathbb{N}$ and $g_1, \ldots, g_l \in G$. We call $v_g(S)$ the multiplicity of g in S, $|S| = l = \sum_{g \in G} v_g(S)$ the length of S, $\mathrm{supp}(S) = \{g \in G | v_g(S) > 0\}$ the support of S, and denote by $h(S) = \max_{g \in G} v_g(S)$ the maximum multiplicity of an element in S. We usually call $h(S)$ the height of S.

We call S zero-sum free if any non-empty subsequence of S is not zero-sum, and minimal zero-sum if it is zero-sum and each proper subsequence is zero-sum free.

Let $D(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence. $D(G)$ is called the Davenport constant, which is a very important invariant especially in the zero-sum theory.

Lemma 3. [14, Corollary 4.4] *Let $m, n \in \mathbb{N}$ with $m|n$. Then, $\eta(C_m \oplus C_n) = 2m + n - 2$ and $D(C_m \oplus C_n) = m + n - 1$.*

Lemma 4. [10, Theorem 5.4.5] *Let $S \in \mathcal{F}(C_p)$ be a sequence of length $p - 1$. If S is zero-sum free, then $S = g^{p-1}$ for some $g \in C_p \backslash \{0\}$.*

Lemma 5. *Let $S \in \mathcal{F}(C_p)$ be a sequence of length $2p - 1$. If S contains no two disjoint non-empty zero-sum subsequences, then $S = g^{2p-1}$ for some $g \in C_p \backslash \{0\}$.*

Proof. Let T be an arbitrary subsequence of S such that $|T| = p - 1$. Then, $|ST^{-1}| = p = D(C_p)$. Therefore, ST^{-1} contains a non-empty zero-sum subsequence. It follows from the hypothesis of the lemma that T is zero-sum free. Hence, $T = g^{p-1}$ for some $g \in C_p \backslash \{0\}$ by Lemma 4. Now the result follows from the arbitrariness of the choice of T. □

Lemma 6. *Let $G \cong C_3^2$ and suppose that S is a minimal zero-sum sequence over G.*
(1) *If $|S| = 4$, then $S = g_1^2 g_2 g_3$ for pairwise distinct elements $g_1, g_2, g_3 \in G \backslash \{0\}$. In particular, $h(S) = 2$.*
(2) *If $|S| = 5$, then $S = g_1^2 g_2^2 g_3$ for pairwise distinct elements $g_1, g_2, g_3 \in G \backslash \{0\}$.*

Proof. Note that for every short zero-sum sequence W over G, we have $h(W) = 1$ or $W = h^3$ for some $h \in G$. Let S be a minimal zero-sum sequence of length ≥ 4 over G. Then, $h(S) \leq 2$.

Assume to the contrary that $|\mathrm{supp}(S)| \geq 4$. Suppose $g_1, g_2, g_3, g_4 \in \mathrm{supp}(S)$ are pairwise distinct elements. It follows by $\eta(G) = 7$ that $g_1^2 g_2^2 g_3^2 g_4^2$ have a short zero-sum subsequence T with $h(T) = 1$, whence T is a short zero-sum subsequence of S, a contradiction. Thus, $|\mathrm{supp}(S)| \leq 3$.

(1) Suppose $|S| = 4$. It follows by $h(S) \leq 2$ and $|\mathrm{supp}(S)| \leq 3$ that S must be of the form $S = g_1^2 g_2 g_3$ or $S = g_1^2 g_2^2$, where $g_1, g_2, g_3 \in G \setminus \{0\}$ are pairwise distinct elements. If $S = g_1^2 g_2^2$, then $g_1 g_2$ is a zero-sum subsequence of S, a contradiction. Thus, the assertion follows.

(2) Suppose $|S| = 5$. The assertion follows by $h(S) \leq 2$ and $|\mathrm{supp}(S)| \leq 3$. □

The following lemma is a special case of Theorem 10.7 in [15], but here we give a direct proof.

Lemma 7. *Let $G \cong C_3^2$ be a finite abelian group and let S be a sequence of length 6 over G. If S does not contain a short zero-sum subsequence, then $S = g_1^2 g_2^2 g_3^2$ for pairwise distinct elements $g_1, g_2, g_3 \in G \setminus \{0\}$.*

Proof. Since S has no short zero-sum subsequence, we obtain $h(S) \leq 2$ and $0 \notin \mathrm{supp}(S)$. Assume to the contrary that there exists $g \in \mathrm{supp}(S)$ such that $v_g(S) = 1$. It follows by $\eta(G) = 7$ that $S \cdot g$ has a short zero-sum subsequence T, whence $v_g(T) = v_g(S \cdot g) = 2$. Let $T = g^2 h$, where $h \in \mathrm{supp}(S)$. Then, $h = g$ and hence $v_g(T) = 3$, a contradiction. Therefore, $v_g(S) = 2$ for every $g \in \mathrm{supp}(S)$ and the assertion follows. □

3 Proof of Theorem 2

Before we begin to prove the theorem, we list some results about $D(\cdot)$ and $\eta(\cdot)$ which will be used frequently in the following proof. Let G be a finite abelian group with $G \cong C_3 \oplus C_{3p} \cong C_3^2 \oplus C_p$, where $p \geq 5$ is a prime. By Lemma 3, we see that

$$\eta(C_3^2) = 7,\ \eta(G) = 3p + 4, \text{ and } \eta(C_p) = p.$$

Let H_1 and H_2 be subgroups of G such that $H_1 \cong C_3^2$ and $H_2 \cong C_p$. Let φ_1, φ_2 be the projections from G to H_1 and H_2, respectively. If $g \in G$, then $\varphi_1(g) = 0$ implies that $g \in H_2$ and $\varphi_2(g) = 0$ implies that $g \in H_1$. Furthermore, $g = 0$ if and only if $\varphi_1(g) = 0$ and $\varphi_2(g) = 0$.

It is referred to the introduction that $t(G) \geq \eta(G)$ and so it suffices to prove that

$$t(G) \leq \eta(G) = 3p + 4.$$

Let S be a sequence of length $3p + 4$ over G. We want to show that S contains a tiny zero-sum subsequence. Denote the subsequence of S consisting of elements of order i by S_i. Then, S shall be written as the following form $S = S_3 \cdot S_p \cdot S_{3p}$, whence

$$|S_3| + |S_p| + |S_{3p}| = 3p + 4. \tag{1}$$

We proceed by the following three claims.

Claim A. *If $|S_p| = 0$ and there exists a subsequence W_0 of S_{3p} such that $|W_0| = p - 1$ and W_0^3 divides S_{3p}, then S has a tiny zero-sum subsequence.*

Proof of Claim A. Since $|S_p| = 0$, we have $|S_3| + |S_{3p}| = 3p + 4$. If $|S_3| \geq 7$, then the assertion follows by $t(H_1) = \eta(H_1) = \eta(C_3^2) = 7$. Therefore, we may assume $|S_3| \leq 6$ and $|S_{3p}| \geq 3p - 2$. Suppose $S_{3p} \cdot (W_0^3)^{-1} = g_1 \cdot \ldots \cdot g_t$, where $t \in \mathbb{N}$ and $g_1, \ldots, g_t \in G$. Let $W_i = W_0 \cdot g_i$ for all $i \in [1, t]$. Then, for each $i \in [1, t]$, we have $|W_i| = p$, and hence there exists a non-empty subsequence T_i of W_i such that $\varphi_2(T_i)$ has sum zero. Therefore, the sequence $U = \sigma(T_1) \cdot \ldots \cdot \sigma(T_t) \cdot S_3$ is a sequence over H_1 of length $t + |S_3| = 7$, whence U has a short zero-sum subsequence. Therefore, there exist a subset $I \subset [1, t]$ and a subsequence S' of S_3 such that $S' \cdot \Pi_{i \in I} T_i$ is zero-sum and $1 \le |I| + |S'| \le 3$. It follows that $S' \cdot \Pi_{i\in I} T_i$ is a non-empty zero-sum subsequence of S and $k(S' \cdot \Pi_{i\in I} T_i) = \frac{|S'|}{3} + \frac{\sum_{i\in I}|T_i|}{3p} \le \frac{|S'|+|I|}{3} \le 1$. □

Claim B. *If $|S_3| = 3$, $|S_{3p}| \ge 3p$, and there exists a subsequence W_0 of S_{3p} such that $|W_0| = p-2$ and W_0^3 divides S_{3p}, then S has a tiny zero-sum subsequence.*

Proof of Claim B. Suppose$S_{3p} \cdot (W_0^3)^{-1} = g_1 \cdot \ldots \cdot g_t$, where $t \in [6, 7]$ and $g_1, \ldots, g_t \in G$. Let $W_i = W_0 g_i$ for all $i \in [1, t]$. We distinguish four cases.

Suppose there exist distinct $i, j \in [1, t]$ such that neither $\varphi_2(W_i)$ nor $\varphi_2(W_j)$ is zero-sum free, say $i = 1$ and $j = 2$. Then, there exist non-empty subsequences T_1 of W_1 and T_2 of W_2 such that $\varphi_2(T_1)$ and $\varphi_2(T_2)$ are zero-sum. Moreover, as $\eta(H_2) = \eta(C_p) = p$ there are non-empty subsequences T_3 of $W_0 g_3 g_4$ and T_4 of $W_0 g_5 g_6$ such that $\varphi_2(T_3)$ and $\varphi_2(T_4)$ are zero-sum. Now consider the sequence $U = \sigma(T_1) \cdot \ldots \cdot \sigma(T_4) \cdot S_3$, which is a sequence of length 7 over H_1, whence U has a short zero-sum subsequence. Therefore, there exist a subset $I \subset [1, 4]$ and a subsequence S' of S_3 such that $S' \cdot \prod_{i\in I} T_i$ is zero-sum and $1 \le |I| + |S'| \le 3$. It follows that $S' \cdot \prod_{i\in I} T_i$ is a non-empty zero-sum subsequence of S and $k(S' \cdot \prod_{i\in I} T_i) = \frac{|S'|}{3} + \frac{\sum_{i\in I}|T_i|}{3p} \le \frac{|S'|+|I|}{3} \le 1$.

Suppose $|S_{3p}| = 3p + 1$ and there is precisely one $i \in [1, 7]$, say $i = 1$, such that $\varphi_2(W_i)$ is not zero-sum free. Let V_1 be a non-empty subsequence of W_1 with $\varphi_2(V_1)$ is zero-sum and let $V_2 = W_0 g_2 g_3$, $V_3 = W_0 g_4 g_5$, and $V_4 = W_0 g_6 g_7$. Since $\varphi_2(W_i)$ is zero-sum free for every $i \in [2, 7]$, it follows by Lemma 4 that $|\mathrm{supp}(\varphi_2(W_0 \cdot g_2 \cdot \ldots \cdot g_7))| = 1$, whence $\varphi_2(V_k)$ is zero-sum for every $k \in [1, 4]$. Now consider the sequences $U = \sigma(V_1) \cdot \ldots \cdot \sigma(V_4) \cdot S_3$, which is a sequence of length 7 over H_1, whence U has a short zero-sum subsequence. Therefore, there exist a subset $I \subset [1, 4]$ and a subsequence S' of S_3 such that $S' \cdot \prod_{i\in I} V_i$ is zero-sum and $1 \le |I| + |S'| \le 3$. It follows that $S' \cdot \prod_{i\in I} V_i$ is a non-empty zero-sum subsequence of S and $k(S' \cdot \prod_{i\in I} V_i) = \frac{|S'|}{3} + \frac{\sum_{i\in I}|V_i|}{3p} \le \frac{|S'|+|I|}{3} \le 1$.

Suppose $|S_{3p}| = 3p + 1$ and $\varphi_2(W_i)$ is zero-sum free for all $i \in [1, 7]$. Then, by Lemma 4 we have $|\mathrm{supp}(\varphi_2(W_0 \cdot g_1 \cdot \ldots \cdot g_7))| = 1$. We claim that $|\mathrm{supp}(g_1 \cdot \ldots \cdot g_7)| = 1$. Assume to the contrary that there are two distinct $k_1, k_2 \in [1, 7]$, say $k_1 = 1$ and $k_2 = 2$, such that $g_1 \ne g_2$. Let $V_1 = W_0 g_1 g_3$, $V_2 = W_0 g_4 g_5$, $V_3 = W_0 g_6 g_7$, and $V_4 = W_0 g_2 g_3$. Then, $\varphi_2(V_k)$ is zero-sum for every $k \in [1, 4]$. Now consider the sequences $U = \sigma(V_1) \cdot \sigma(V_2) \cdot \sigma(V_3) \cdot S_3$ and $U' = \sigma(V_4) \cdot \sigma(V_2) \cdot \sigma(V_3) \cdot S_3$. If U has a short zero-sum subsequence, then there exist a subset $I \subset [1, 3]$ and a subsequence S' of S_3 such that $S' \cdot \prod_{i\in I} V_i$ is zero-sum and $1 \le |I| + |S'| \le 3$. It follows that $S' \cdot \prod_{i\in I} V_i$ is a non-empty zero-sum subsequence of S and $k(S' \cdot \prod_{i\in I} T_i) = \frac{|S'|}{3} + \frac{\sum_{i\in I}|V_i|}{3p} \le \frac{|S'|+|I|}{3} \le 1$, then we are done. If U' has a short zero-sum subsequence, then we also get a tiny zero-sum subsequence of S. Hence, we can assume that both U and U' have no short zero-sum subsequence, whence by Lemma 7 and the structures of U and U' we obtain $\sigma(V_1) = \sigma(V_4)$ and hence $g_1 = g_2$, a contradiction. Thus, $|\mathrm{supp}(g_1 \cdot \ldots \cdot g_7)| = 1$ and so we have $(W_0 g_1)^3 | S_{3p}$. Hence, by Claim A we have done.

Suppose $|S_{3p}| = 3p$ and there are at most one $i \in [1, 6]$ such that $\varphi_2(W_i)$ is not zero-sum free. We may assume that $\varphi_2(W_j)$ is zero-sum free for all $j \in [2, 6]$. Then, by Lemma 4 we have $|\mathrm{supp}(\varphi_2(W_0 \cdot g_2 \cdot \ldots \cdot g_6))| = 1$. If $\varphi_1(g_1 \cdot \ldots \cdot g_6)$ has a short zero-sum subsequence, then $\varphi_1(S_{3p})$ has pairwise disjoint $p-1$ short zero-sum subsequences. Together with $|S_p| = 1$ and $\eta(H_2) = p$, we can find a tiny zero-sum subsequence of S. Otherwise, by Lemma 7 we have $\varphi_1(g_1 \cdot \ldots \cdot g_6) = h_1^2 h_2^2 h_3^2$, where $h_1, h_2, h_3 \in H_1$ are pairwise distinct. After renumbering if necessary, we may assume $\varphi_1(g_1 g_2) = h_1^2$, $\varphi_1(g_3 g_4) = h_2^2$, and $\varphi_1(g_5 g_6) = h_3^2$. Let $V_1 = W_0 g_1 g_2$, $V_2 = W_0 g_3 g_4$, $V_3 = W_0 g_5 g_6$, $V_4 = W_0 g_3 g_5$, and $V_5 = W_0 g_4 g_6$. Then, $\varphi_2(V_2)$, $\varphi_2(V_3)$, $\varphi_2(V_4)$, and $\varphi_2(V_5)$ are all zero-sum sequences. Furthermore, there exists a non-empty subsequence T_1 of V_1 such that $\varphi_2(T_1)$ is zero-sum. Consider the sequences $U = \sigma(T_1) \cdot \sigma(V_2) \cdot \sigma(V_3) \cdot S_3$ and $U' = \sigma(T_1) \cdot \sigma(V_4) \cdot \sigma(V_5) \cdot S_3$, which are both sequences of length 6 over H_1. If U or U' has a short zero-sum subsequence, then the corresponding subsequence of S is a tiny zero-sum sequence. Otherwise, by Lemma 7 we know that both U and U' have the same structures. Together with $\sigma(V_4) = \sigma(V_5)$, we obtain $\sigma(V_2) = \sigma(V_4)$, and then $g_4 = g_5$. However, $\varphi_1(g_4) = h_2$, $\varphi_2(g_5) = h_3$, and $h_2 \ne h_3$, which is a contradiction. □

Assume to the contrary that S has no tiny zero-sum subsequence. It follows from $\eta(H_1) = 7$ and $\eta(H_2) = p$ that $|S_3| \le 6$ and $|S_p| \le p-1$. So, we see that

$$|S_{3p}| = |S| - |S_p| - |S_3| \ge 3p + 4 - (p-1) - 6 = 2p - 1.$$

If $|\varphi_1(S_{3p})| \ge \eta(H_1)$, then we can choose a subsequence denoted by U of S_{3p} such that $\sigma(\varphi_1(U)) = 0 \in C_3^2$ and $|U| \le 3$. Now let m be the maximum number of disjoint sequences denoted by $U_1, \dots, U_m$ of S_{3p} like U. Obviously,

$$m \ge \frac{|S_{3p}| - (\eta(H_1) - 1)}{3}. \tag{2}$$

On the other hand,

$$m + |S_p| \le \eta(H_2) - 1, \tag{3}$$

otherwise $|\sigma(U_1) \cdot \ldots \cdot \sigma(U_m) \cdot S_p| = m + |S_p| \ge \eta(H_2) = p$. Hence, there exists a subsequence denoted by W of $\sigma(U_1) \cdot \ldots \cdot \sigma(U_m) \cdot S_p$ such that $|W| \le p$ and $\sigma(\varphi_2(W)) = 0$. Suppose $W = \prod_{i \in I} \sigma(U_i) \cdot \prod_{j \in J} g_j$ with $I \subset [1, m]$, $J \subset [1, |S_p|]$ and $|I| + |J| = |W| \le p$, then

$$\sigma(\varphi_1(W)) = \sum_{i \in I} \varphi_1(\sigma(U_i)) + \sum_{j \in J} \varphi_1(g_j) = \sum_{i \in I} \sigma(\varphi_1(U_i)) + \sum_{j \in J} \varphi_1(g_j) = 0 + 0 = 0 \in C_3^2.$$

And

$$k(W) \le \frac{|I|}{p} + \frac{|J|}{p} = \frac{|W|}{p} \le 1.$$

Therefore, the sequence W is a tiny zero-sum subsequence of S, which contradicts the assumption of S.

Now we remark that $\varphi_1(S_{3p}(U_1 \cdot \ldots \cdot U_m)^{-1})$ has no short zero-sum subsequence and $\sigma(U_1) \cdot \ldots \cdot \sigma(U_m) \cdot S_p$ is zero-sum free. Furthermore, we have the following claim.

Claim C. *Suppose* $m + |S_p| = \eta(H_2) - 1 = p - 1$. *Let* $h \in \operatorname{supp}(\varphi_1(S_{3p}))$ *with* $v_h(\varphi_1(S_{3p})) > 3$ *and let* $g_1, g_2 \in \operatorname{supp}(S_{3p})$. *If* $\varphi_1(g_1) = \varphi_1(g_2) = h$, *then* $g_1 = g_2$.

Proof of Claim C. Assume to the contrary that $g_1 \ne g_2$. Note that $S_{3p} = U_1 \cdot \ldots \cdot U_m \cdot U_0$, where $U_0 = S_{3p}(U_1 \cdot \ldots \cdot U_m)^{-1}$ with $h(U_0) \le 2$. Since the decomposition only depends on $\varphi_1(S_{3p})$, without loss of generality we may assume $g_1 | U_1$ and $g_2 \nmid U_1$.

If $g_2 | U_0$, then let $U_1' = U_1 g_2 g_1^{-1}$. Thus, both sequences $\sigma(U_1) \cdot \ldots \cdot \sigma(U_m) \cdot S_p$ and $\sigma(U_1') \cdot \ldots \cdot \sigma(U_m) \cdot S_p$ are zero-sum free and of length $p - 1$. It follows by Lemma 4 and $p - 1 \ge 4$ that $\sigma(U_1) = \sigma(U_1')$ and hence $g_1 = g_2$, a contradiction.

If $g_2 \nmid U_0$, say $g_2 | U_2$, then let $U_1' = U_1 g_2 g_1^{-1}$ and $U_2' = U_2 g_1 g_2^{-1}$. Thus, both sequences $\sigma(U_1) \cdot \sigma(U_2) \cdot \ldots \cdot \sigma(U_m) \cdot S_p$ and $\sigma(U_1') \cdot \sigma(U_2') \cdot \ldots \cdot \sigma(U_m) \cdot S_p$ are zero-sum free and of length $p - 1$. It follows by Lemma 4 and $p - 1 \ge 4$ that $\sigma(U_1) = \sigma(U_1')$ and hence $g_1 = g_2$, a contradiction. □

By inequalities (2) and (3), we see that

$$\frac{|S_{3p}| - 6}{3} + |S_p| \le p - 1. \tag{4}$$

Combining inequalities (1) and (4), we obtain that

$$2|S_p| + 1 \le |S_3|, \tag{5}$$

and so it follows from $|S_3| \le 6$ and $|S_p| \in \mathbb{N}$ that $|S_p| \le 2$. Therefore,

$$|S_{3p}| = |S| - |S_p| - |S_3| \ge 3p + 4 - 2 - 6 = 3p - 4.$$

It follows by (2) and $m \in \mathbb{N}$ that

$$m \ge p - 3. \tag{6}$$

Similarly, if $|\varphi_2(S_{3p})| \geq \eta(H_2)$, then we can find a subsequence denoted by V of S_{3p} such that $\sigma(\varphi_2(V)) = 0 \in C_p$ and $|V| \leq p$. Now let n be the maximum number of disjoint subsequences denoted by $V_1, \ldots, V_n$ of S_{3p} like V. Obviously,

$$n \geq \frac{|S_{3p}| - (p-1)}{p} \tag{7}$$

and

$$n + |S_3| \leq \eta(C_3^2) - 1 = 6, \tag{8}$$

which could be deduced by the similar method in (3).

Now we remark that $\varphi_2(S_{3p}(V_1 \cdot \ldots \cdot V_n)^{-1})$ is zero-sum free and $\sigma(V_1) \cdot \ldots \cdot \sigma(V_n) \cdot S_3$ has no short zero-sum subsequence.

By inequalities (7) and (8), we get that

$$|S_{3p}| \leq (6 - |S_3|)p + p - 1. \tag{9}$$

Combining inequalities (1) and (9), we obtain that

$$|S_3| \leq \frac{4(p-1) - 1 + |S_p|}{p-1}.$$

Since $|S_p| \leq 2, p > 2$ and $|S_3| \in \mathbb{N}$, it follows that $|S_3| \leq 4$. Together with (5) and (9), we obtain that $2|S_p| + 1 \leq |S_3| \leq 4$ and $|S_{3p}| \leq (6 - |S_3|)p + p - 1$. Then, we can distinguish four cases depending on the previous inequalities and $|S_3| + |S_p| + |S_{3p}| = 3p + 4$.

Case 1. $|S_3| = 4$, $|S_p| = 1$, and $|S_{3p}| = 3p - 1$.

By inequalities (7) and (8), we obtain that $n = 2$, i.e., there are exactly two pairwise disjoint subsequences V_1, V_2 of S_{3p} such that $\sigma(\varphi_2(V_i)) = 0 \in H_2$ and $|V_i| \leq p$ for $i = 1, 2$. If $|V_1| + |V_2| \leq 2p - 1$, then $|S_{3p}(V_1V_2)^{-1}| \geq p$ and so there exists one more subsequence disjoint with V_1 and V_2, a contradiction. Therefore, $|V_1| = |V_2| = p$, $|S_{3p}(V_1V_2)^{-1}| = p - 1$, and $\varphi_2(S_{3p}(V_1V_2)^{-1})$ are zero-sum free over H_2. By Lemma 4, we may assume that $\varphi_2(S_{3p}(V_1V_2)^{-1}) = b^{p-1}$, $\varphi_2(V_1) = b_1^p$, and $\varphi_2(V_2) = b_2^p$, where $b, b_1, b_2 \in H_2 \backslash \{0\}$. Note that both $b^{p-1}b_1^p$ and $b^{p-1}b_2^p$ contain no two disjoint zero-sum subsequences over H_2. It follows from Lemma 5 that $b = b_1 = b_2$. Hence, we may assume that $\varphi_2(S_{3p}) = b^{3p-1}$.

Next, we assert that $S_{3p} = g^{3p-1}$ for some $g \in G$, which implies that $m + |S_p| \geq \left\lfloor \frac{3p-1}{3} \right\rfloor + 1 \geq p$, a contradiction to inequality (3). Assume to the contrary that there exist $g_1, g_2 \in \text{supp}(S_{3p})$ such that $g_1 \neq g_2$. Let T_1 be a subsequence of $S_{3p}(g_1g_2)^{-1}$ with length $p - 1$ and let T_2 be a subsequence of $S_{3p}(T_1g_1g_2)^{-1}$ with length p. Then, $W_1 = \sigma(T_1g_1) \cdot \sigma(T_2) \cdot S_3$ and $W_2 = \sigma(T_1g_2) \cdot \sigma(T_2) \cdot S_3$ are both sequences over H_1 of length 6. It follows by our assumptions S has no tiny zero-sum subsequence that W_1 and W_2 have no short zero-sum subsequence. Hence, by Lemma 7 $\sigma(T_1g_1) = \sigma(T_1g_2)$, i.e., $g_1 = g_2$, a contradiction.

Case 2. $|S_3| = 3$, $|S_p| = 1$, and $|S_{3p}| = 3p$.

Because $|S_p| = 1$, together with inequalities (2) and (3) we obtain that $m = p - 2$ and $S_{3p} = U_1 \cdot \ldots \cdot U_{p-2} \cdot U_0$, where $U_0 = S_{3p}(U_1 \cdot \ldots \cdot U_{p-2})^{-1}$. Since $\varphi_1(U_0)$ does not contain a short zero-sum subsequence over H_1 and $\eta(H_1) = 7$, it follows that

$$6 \geq |U_0| = |S_{3p}| - \sum_{i=1}^{p-2} |U_i| \geq 3p - 3(p-2) = 6,$$

i.e., $|U_0| = 6$, whence, $|U_1| = \ldots = |U_{p-2}| = 3$. By Lemma 7, we may assume that $\varphi_1(U_0) = h_1^2h_2^2h_3^2$ for three pairwise distinct elements $h_1, h_2, h_3 \in H_1 \backslash \{0\}$. We claim that $\varphi_1(U_i) \in \{h_1^3, h_2^3, h_3^3\}$ for every $i \in [1, p-2]$. Assume to the contrary that there exists $i \in [1, m]$, say $i = 1$, such that $\varphi_1(U_1) \notin \{h_1^3, h_2^3, h_3^3\}$, whence there exists $h_4 \in H_1 \backslash \{0, h_1, h_2, h_3\}$ such that $h_4 | \varphi_1(U_1)$. If there exists $h_5 \in H_1 \backslash \{0, h_1, h_2, h_3\}$ such that $h_4h_5 | \varphi_1(U_1)$.

Then, $\eta(H_1) = 7$ implies that both $h_1^2h_2^2h_3^2h_4$ and $h_1^2h_2^2h_3^2h_5$ have short zero-sum subsequences, whence $h_1^2h_2^2h_3^2h_4h_5$ has two disjoint short zero-sum subsequences, a contradiction to the maximality of m. Thus, by symmetry we may assume $\varphi_1(U_1) = h_4h_1h_2$. It follows that $\varphi_1(U_1)h_1^2h_2^2h_3^2$ has two disjoint short zero-sum subsequences, a contradiction to the maximality of m.

Now we can assume that

$$\varphi_1(S_{3p}) = h_1^{3x_1+2}h_2^{3x_2+2}h_3^{3x_3+2}, \quad \text{where } x_1, x_2, x_3 \in N_0 \text{ with } x_1 + x_2 + x_3 = p - 2.$$

By Claim C, for every $i \in [1, 3]$ with $x_i \geq 1$, there exists $g_i \in G$ such that $\varphi_1(g_i) = h_i$ and $g_i^{3x_i+2} \mid S_{3p}$. If there exists $j \in [1, 3]$ such that $x_j = 0$, we choose an arbitrary $g_j \in G$. Let $W = g_1^{x_1}g_2^{x_2}g_3^{x_3}$. Then, $W^3|S_{3p}$, a contradiction to Claim B.

Case 3. $|S_3| = 3$, $|S_p| = 0$, and $|S_{3p}| = 3p + 1$.

Assume to the contrary that $\varphi_1(S_{3p})$ has a zero-sum subsequence $\varphi_1(U_1)$ of length 2. Since $|S_{3p}U_1^{-1}| = 3p - 1$ and $\eta(H_1) = 7$, there exist disjoint short zero-sum subsequences $\varphi_1(U_2), \ldots, \varphi_1(U_{p-1})$ of $\varphi_1(S_{3p}U_1^{-1})$. Let $U_0' = S_{3p}(U_1 \cdot \ldots \cdot U_{p-1})^{-1}$. Then, $|U_0'| \geq 3p + 1 - 2 - 3(p - 2) = 5$. It follows by $D(H_1) = 5$ that $\varphi_1(U_0')$ has a non-empty zero-sum subsequence $\varphi_1(U_0)$. Since $W_0 = \sigma(U_0)\sigma(U_1) \ldots \sigma(U_{p-1})$ is a sequence of length p over H_2, there exists a subset $I \subset [0, p - 1]$ such that $\sum_{i \in I} \sigma(U_i) = 0$, i.e., $\prod_{i \in I} U_i$ is zero-sum. If $|\prod_{i \in I} U_i| \leq 3p$, then $\prod_{i \in I} U_i$ is a tiny zero-sum subsequence, a contradiction. If $|\prod_{i \in I} U_i| = 3p + 1$, then S_{3p} is zero-sum and hence $n \geq 4$, a contradiction to inequality (8).

Therefore, $\varphi_1(S_{3p})$ has no zero-sum subsequence of length 2, whence $|\text{supp}(\varphi_1(S_{3p}))| \leq 4$. Suppose

$$\varphi_1(S_{3p}) = h_1^{3x_1+r_1}h_2^{3x_2+r_2}h_3^{3x_3+r_3}h_4^{3x_4+r_4},$$

where $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$, $r_1, r_2, r_3, r_4 \in [0, 2]$, and $h_1, h_2, h_3, h_4 \in H\backslash\{0\}$ are pairwise distinct elements. It follows by $|S_{3p}| = 3p + 1$ and inequalities (2) and (3) that $m = p - 1$. Then, $p - 1 \geq x_1 + x_2 + x_3 + x_4 \geq p - 2$, $r_1 + r_2 + r_3 + r_4 \equiv 1 \pmod 3$, and $|\{i \in [1, 4] \,|\, r_i = 0\}| \leq 2$. We distinguish three cases.

Suppose $|\{i \in [1, 4] \,|\, r_i = 0\}| = 0$. By Claim C, for every $i \in [1, 4]$ with $x_i \geq 1$, there exists $g_i \in G$ such that $\varphi_1(g_i) = h_i$ and $g_i^{3x_i+r_i} \mid S_{3p}$. If there exists $j \in [1, 4]$ such that $x_j = 0$, we choose an arbitrary $g_j \in G$. Let $W = g_1^{x_1}g_2^{x_2}g_3^{x_3}g_4^{x_4}$. Then, $W^3|S_{3p}$, a contradiction to Claim B.

Suppose $|\{i \in [1, 4] \,|\, r_i = 0\}| = 1$, say $r_4 = 0$. Then, $r_1 + r_2 + r_3 + r_4 = 4$. By Claim C, for every $i \in [1, 3]$ with $x_i \geq 1$, there exists $g_i \in G$ such that $\varphi_1(g_i) = h_i$ and $g_i^{3x_i+r_i} \mid S_{3p}$. If there exists $j \in [1, 3]$ such that $x_j = 0$, we choose an arbitrary $g_j \in G$. If $x_4 \leq 1$, then $x_1 + x_2 + x_3 \geq p - 2$. Let $W = g_1^{x_1}g_2^{x_2}g_3^{x_3}$. Then, $W^3|S_{3p}$, a contradiction to Claim B. If $x_4 \geq 2$, then Claim C implies that there exists $g_4 \in G$ such that $\varphi_1(g_4) = h_4$ and $g_4^{3x_4} \mid S_{3p}$. Note $x_1 + x_2 + x_3 + x_4 = p - 1$. Let $W = g_1^{x_1}g_2^{x_2}g_3^{x_3}g_4^{x_4}$. Then, $W^3|S_{3p}$, a contradiction to Claim A.

Suppose $|\{i \in [1, 4] \,|\, r_i = 0\}| = 2$, say $r_3 = r_4 = 0$. Then, $r_1 = r_2 = 2$. If $x_3 \neq 1$ or $x_4 \neq 1$, a similar argument to the above implies a contradiction. Thus, $x_3 = x_4 = 1$ and hence $\varphi_1(S_{3p}) = h_1^{3x_1+2}h_2^{3x_2+2}h_3^3h_4^3$. It follows by $\eta(H_1) = 7$ and $\varphi_1(S_{3p})$ has no zero-sum subsequence of length 2 that $h_1^2h_2^2h_3^2h_4^2$ has a zero-sum subsequence X of length 3. By symmetry, we may assume $h_3|X$. We claim that there exists $g_3 \in G$ such that $\varphi_1(g_3) = h_3$ and $g_3^3|S_{3p}$. Assume to the contrary that there exist distinct $g_4, g_5 \in \text{supp}(S_{3p})$ such that $\varphi_1(g_4) = \varphi_1(g_5) = h_3$. Let $W_1, \ldots, W_{x_1}, Y_1, \ldots, Y_{x_2}, V_1, V_2$ be disjoint subsequences of S_{3p} with $\varphi_1(W_i) = h_1^3$ for every $i \in [1, x_1]$, $\varphi_1(Y_i) = h_2^3$ for every $i \in [1, x_2]$, $\varphi_1(V_i) = X$ for every $i \in [1, 2]$, $g_4|V_1$, and $g_5|V_2$. Set $V_1' = V_1g_5g_4^{-1}$ and $V_2' = V_2g_4g_5^{-1}$. Then, both sequences $\sigma(W_1), \ldots, \sigma(W_{x_1})\sigma(Y_1), \ldots, \sigma(Y_{x_2})\sigma(V_1)\sigma(V_2)$ and $\sigma(W_1), \ldots, \sigma(W_{x_1})\sigma(Y_1), \ldots, \sigma(Y_{x_2})\sigma(V_1')\sigma(V_2')$ are zero-sum free over H_2 of length $p - 1$. It follows by Lemma 4 that $g_4 = g_5$, a contradiction. Therefore, there exists $g_3 \in G$ such that $\varphi_1(g_3) = h_3$ and $g_3^3|S_{3p}$. By Claim C, for every $i \in [1, 2]$ with $x_i \geq 1$, there exists $g_i \in G$ such that $\varphi_1(g_i) = h_i$ and $g_i^{3x_i+r_i} \mid S_{3p}$. If there exists $j \in [1, 2]$ such that $x_j = 0$, we choose an arbitrary $g_j \in G$. Let $W = g_1^{x_1}g_2^{x_2}g_3$. Then, $W^3|S_{3p}$, a contradiction to Claim B.

Case 4. $|S_3| \le 2$, $|S_p| = 0$, and $|S_{3p}| \ge 3p + 2$.

Then, $S_{3p} = U_0 \cdot U_1 \cdot \ldots \cdot U_{p-1}$ and $|U_0| \ge 5$. Since the sequence $\varphi_1(U_0)$ does not contain short zero-sum subsequence over H_1, it follows from Lemma 6 that $|\mathrm{supp}(\varphi_1(U_0))| = 3$. Then, we may assume that $\mathrm{supp}(\varphi_1(U_0)) = \{h_1, h_2, h_3\}$. If there exists some $i \in [1, p-1]$ such that $|U_i| = 2$, then $|U_0| = 6$ and by Lemma 7 we see that $\varphi_1(U_0) = h_1^2h_2^2h_3^2$. Clearly, U_0 must contain a subsequence denoted by W of length 4 such that $\sigma(\varphi_1(W)) = 0$. Then, $\sigma(U_1) \cdot \ldots \cdot \sigma(U_{p-1}) \cdot \sigma(W)$ is a sequence of length p over H_2, and so it contains a zero-sum subsequence, whence the corresponding sequence over G is zero-sum of length less than $|U_1 \cdot \ldots \cdot U_{p-1} \cdot W| \le 3(p-2) + 2 + 4 = 3p$, i.e., we get a tiny zero-sum subsequence of S, a contradiction. Therefore, $|U_i| = 3$ for all $i \in [1, p-1]$.

If $|S_3| = 0$, then $|S_{3p}| = 3p + 4 = \eta(G)$, whence S contains a short zero-sum subsequence which is a tiny zero-sum subsequence. So, we suppose that $|S_3| \in \{1, 2\}$.

If $|S_3| = 1$, then $|S_{3p}| = 3p + 3$ and S_{3p} could be written by the form $S_{3p} = U_1 \cdot \ldots \cdot U_{p-1} \cdot U_0$ with $|U_i| = 3$ for $i \in [1, p-1]$ and $|U_0| = 6$. Furthermore, we see that $\varphi_1(U_0) = h_1^2h_2^2h_3^2$. We assert that $\mathrm{supp}(\varphi_1(S_{3p})) = \{h_1, h_2, h_3\}$. Assume to the contrary that there exist $h_4 \in H_1\backslash\{0, h_1, h_2, h_3\}$ and $i \in [1, p-1]$, say $i = 1$, such that $h_4 | \varphi_1(U_1)$. If there exists $h_5 \in H_1\backslash\{0, h_1, h_2, h_3\}$ such that $h_4h_5|\varphi_1(U_1)$. Then, $\eta(H_1) = 7$ implies that both $h_1^2h_2^2h_3^2h_4$ and $h_1^2h_2^2h_3^2h_5$ have short zero-sum subsequences, whence $h_1^2h_2^2h_3^2h_4h_5$ has two disjoint short zero-sum subsequences. Then, $\varphi_1(U_1U_0)$ has two disjoint short zero-sum subsequences, whence $m = p$, a contradiction. Since $h_1^2h_2^2h_3^2$ has no short zero-sum subsequence, we obtain $\varphi_1(U_i) \in \{h_1^3, h_2^3, h_3^3\}$. Then, we can assume $\varphi_1(S_{3p}) = h_1^{3x_1+2}h_2^{3x_2+2}h_3^{3x_3+2}$, where $x_1, x_2, x_3 \in \mathbb{N}_0$ and $x_1 + x_2 + x_3 = p - 1$. It follows by Claim C that for every $i \in [1, 3]$ with $x_i \ge 1$, there exists $g_i \in G$ such that $\varphi_1(g_i) = h_i$ and $g_i^{3x_i+2} \mid S_{3p}$. If there exists $j \in [1, 3]$ such that $x_j = 0$, then we choose an arbitrary $g_j \in G$. Let $W = g_1^{x_1}g_2^{x_2}g_3^{x_3}$. Then, $W^3|S_{3p}$, a contradiction to Claim A.

If $|S_3| = 2$, then $|S_{3p}| = 3p + 2$. Since $\varphi_1(S_{3p})$ has no zero-sum subsequence of length 2, we may assume that

$$\varphi_1(S_{3p}) = h_1^{3x_1+r_1}h_2^{3x_2+r_2}h_3^{3x_3+r_3}h_4^{3x_4+r_4},$$

where $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$, $r_1, r_2, r_3, r_4 \in [0, 2]$, and $h_1, h_2, h_3, h_4 \in H\backslash\{0\}$ are pairwise distinct elements. Then, $p - 1 \ge x_1 + x_2 + x_3 + x_4 \ge p - 2$ and $r_1 + r_2 + r_3 + r_4 \in \{5, 8\}$. If $r_1 + r_2 + r_3 + r_4 = 8$, then $h_1^2h_2^2h_3^2h_4^2$ has two disjoint short zero-sum subsequences, a contradiction to $m \le p - 1$. Thus, $x_1 + x_2 + x_3 + x_4 = p - 1$ and $r_1 + r_2 + r_3 + r_4 = 5$. Since $h_1^{r_1}h_2^{r_2}h_3^{r_3}h_4^{r_4}$ has no short zero-sum subsequence, it follows by Lemma 6 that there exists $i \in [1, 4]$, say $i = 4$, such that $r_4 = 0$. By symmetry, we can assume that $r_1 = r_2 = 2$ and $r_3 = 1$. It follows by Claim C that for every $i \in [1, 3]$ with $x_i \ge 1$, there exists $g_i \in G$ such that $\varphi_1(g_i) = h_i$ and $g_i^{3x_i+r_i} \mid S_{3p}$. If there exists $j \in [1, 3]$ such that $x_j = 0$, then we choose an arbitrary $g_j \in G$. If $x_4 = 0$, then let $W = g_1^{x_1}g_2^{x_2}g_3^{x_3}$ and hence $W^3|S_{3p}$, a contradiction to Claim A. Suppose $x_4 = 1$. Note that $h_1^2h_2^2h_3h_4^2$ has a short zero-sum subsequence X with $h_4|X$. A similar argument to that at the end of Case 3 shows that there exists $g_4 \in G$ such that $\varphi_1(g_4) = h_4$ and $g_4^3|S_{3p}$. Let $W = g_1^{x_1}g_2^{x_2}g_3^{x_3}g_4^{x_4}$ and hence $W^3|S_{3p}$, a contradiction to Claim A.

Now the proof of Theorem 2 is completed.

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References

[1] B. Girard, *On a combinatorial problem of Erdös, Kleitman and Lemke*, Adv. Math. **231** (2012), 1843–1857, DOI: 10.1016/j.aim.2012.06.025.

[2] W. D. Gao, Q. H. Hou, W. A. Schmid, and R. Thangadurai, *On short zero-sum subsequences II*, Integers **7** (2007), A21, DOI: 10.1.1.86.2348.

[3] Yushuang Fan, W. D. Gao, Jiangtao Peng, L. L. Wang, and Qinghai Zhong, *Remarks on tiny zero-sum sequences*, Integers **13** (2013), A52, DOI: 10.1515/9783110298161.752.

[4] F. Chung, *Pebbling in hypercubes*, SIAM J. Discrete Math. **2** (1989), 467–472, DOI: 10.1137/0402041.

[5] S. T. Chapman and A. Geroldinger, *On cross numbers of minimal zero sequences*, Australas. J. Combin. **14** (1996), 85–92.

[6] T. Denley, *On a result of Lemke and Kleitman*, Combin. Probab. Comput. **6** (1997), 39–43, DOI: 10.1017/s0963548396002726.

[7] G. Hurlbert, *A survey of graph pebbling*, in: Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory and Computing, vol. 139, 1999, pp. 41–64, DOI: 10.1007/s003730200015.

[8] G. Hurlbert, *Recent progress in graph pebbling*, Graph Theory Notes N. Y. **49** (2005), 25–37, DOI: 10.1.1.235.7205.

[9] W. D. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. **24** (2006), 337–369, DOI: 10.1016/j.exmath.2006.07.002.

[10] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematics*, vol. 278, Chapman & Hall/CRC, 2006.

[11] A. Geroldinger and D. J. Grynkiewicz, *On the structure of minimal zero-sum sequences with maximal cross number*, J. Comb. Number Theory **1** (2009), no. 2, 9–26, DOI: 10.1.1.717.8748.

[12] Xiaoyu He, *Cross number invariants of finite abelian groups*, J. Number Theory **136** (2014), 100–117, DOI: 10.1016/j.jnt.2013.09.006.

[13] B. Kim, *The cross number of minimal zero-sum sequences over finite abelian groups*, J. Number Theory **157** (2015), 99–122, DOI: 10.1016/j.jnt.2015.04.012.

[14] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin, and L. Rackham, *Zero-sum problems in finite abelian groups and affine caps*, Q. J. Math. **58** (2007), 159–186, DOI: 10.1093/qmath/ham003.

[15] W. D. Gao and A. Geroldinger, *On long minimal zero sequences in finite abelian groups*, Period Math. Hung. **38** (1999), 179–211, DOI: 10.1023/A:1004854508165.