

Research Article

Qingdong Guo and Wenhua Wang*

The molecular characterization of anisotropic Herz-type Hardy spaces with two variable exponents

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Abstract: In this article, the authors establish the characterizations of a class of anisotropic Herz-type Hardy spaces with two variable exponents associated with a non-isotropic dilation on \mathbb{R}^n in terms of molecular decompositions. Using the molecular decompositions, the authors obtain the boundedness of the central δ -Calderón-Zygmund operators on the anisotropic Herz-type Hardy space with two variable exponents.

Keywords: anisotropic Herz-type Hardy space, two variable exponents, molecular decomposition, central δ -Calderón-Zygmund operator, boundedness

MSC 2010: Primary: 42B30, Secondary: 42B35, 46E30

1 Introduction

The theory of function spaces with variable exponents has rapidly made progress in the past 20 years since some elementary properties were established by Kováčik and Rákosník [1]. Lebesgue and Sobolev spaces with variable exponents have been extensively investigated, see [2] and the references therein. In 2012, Almeida and Drihem [3] introduced the Herz spaces with two variable exponents and obtain the boundedness of some sublinear operators on those spaces. In the same year, Wang and Liu [4] introduced the Herz-type Hardy spaces with variable exponents $HK_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and $HK_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$, which is a generalization of the classical Herz-type Hardy spaces. In 2015, Dong and Xu [5] introduced the Herz-type Hardy spaces with two variable exponents $\dot{HK}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and $HK_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$.

Recently, extending classical function spaces arising in harmonic analysis of Euclidean spaces to other domains and non-isotropic settings is an important topic. In 2003, Bownik [6] introduced the anisotropic Hardy spaces $H_A^p(\mathbb{R}^n)$ associated with very general discrete groups of dilations. This formulation includes the classical isotropic Hardy space theory established by Fefferman and Stein [7] and the parabolic Hardy space theory established by Calderón and Torchinsky [8,9]. In 2008, Ding et al. [10] introduced the anisotropic Herz-type Hardy spaces $HK_p^{\alpha,q}(A; \mathbb{R}^n)$ and $HK_p^{\alpha,q}(A; \mathbb{R}^n)$ and established their atomic and molecular decompositions. In 2018, Zhao and Zhou [11] introduced the variable anisotropic Herz-type Hardy spaces $\dot{HK}_{p(\cdot)}^{\alpha,q}(A; \mathbb{R}^n)$ and $HK_{p(\cdot)}^{\alpha,q}(A; \mathbb{R}^n)$ and established their atomic and molecular decompositions. Using these decompositions, they gave some applications. In 2019, Wang and Guo [12] introduced the variable anisotropic Herz-type Hardy spaces $\dot{HK}_{p(\cdot)}^{\alpha,q}(A; \mathbb{R}^n)$ and $HK_{p(\cdot)}^{\alpha,q}(A; \mathbb{R}^n)$ and established their atomic decomposition and some applications.

* **Corresponding author: Wenhua Wang**, College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, P. R. China, e-mail: wangwhmath@163.com

Qingdong Guo: College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, P. R. China, e-mail: guoqingdongshuxin@126.com

Inspired by the previous study, we would like to declare that the goal of this study is to establish the characterizations of a class of anisotropic Herz-type Hardy spaces with two variable exponents associated with a non-isotropic dilation on \mathbb{R}^n in terms of molecular decompositions and obtain the boundedness of the central δ -Calderón-Zygmund operators on the anisotropic Herz-type Hardy space with two variable exponents.

First, we recall some standard notations in variable function spaces. A measurable function $p(\cdot): \mathbb{R}^n \rightarrow (0, \infty)$ is called a variable exponent. Let f be a measurable function on \mathbb{R}^n and $p(\cdot) \in \mathcal{P}$. Then, the modular function (or, for simplicity, the modular) $\varrho_{p(\cdot)}$, associated with $p(\cdot)$, is defined by setting

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx$$

and the Luxemburg (also called Luxemburg–Nakano) quasi-norm $\|f\|_{L^{p(\cdot)}}$ by

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda \in (0, \infty): \varrho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

Moreover, the variable Lebesgue space $L^{p(\cdot)}$ is defined to the set of all measurable functions f satisfying that $\varrho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}}$. For any variable exponent $p(\cdot)$, let

$$p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) \text{ and } p_+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

Denote by \mathcal{P} the set of all variable exponents $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. We call $p'(\cdot)$ the conjugate exponent to $p(\cdot)$, that is, $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. Let \mathcal{B} is the set of $p(\cdot) \in \mathcal{P}$, such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}$. It is well known that if $p(\cdot) \in \mathcal{P}$ and satisfies the following global log-Hölder continuous, then $p(\cdot) \in \mathcal{B}$.

Definition 1.1. Let $\alpha(\cdot)$ be a real function on \mathbb{R}^n .

(i) $\alpha(\cdot)$ is called log-Hölder continuous on \mathbb{R}^n if there exists $C > 0$, such that

$$|\alpha(x) - \alpha(y)| \leq \frac{C}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$ and $|x - y| < \frac{1}{2}$.

(ii) $\alpha(\cdot)$ is called log-Hölder continuous at origin (or has a log decay at the origin), if there exists $C > 0$, such that

$$|\alpha(x) - \alpha(0)| \leq \frac{C}{\log(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$.

(iii) $\alpha(\cdot)$ is called log-Hölder continuous at infinity (or has a log decay at the infinity), if there exist some $\alpha_\infty \in \mathbb{R}^n$ and $C > 0$, such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{C}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$.

By $\mathcal{P}_0(\mathbb{R}^n)$ and $\mathcal{P}_\infty(\mathbb{R}^n)$, we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$, which are locally log-Hölder continuous at the origin and at the infinity, respectively.

Next, we will recall the notion of expansive dilations on \mathbb{R}^n ; see [6, p. 5]. A real $n \times n$ matrix A is called an expansive dilation, if all eigenvalues λ of A satisfy $|\lambda| > 1$. Suppose $\lambda_1 \dots \lambda_n$ are eigenvalues of A (taken according to the multiplicity), so that $1 < |\lambda_1| \leq \dots \leq |\lambda_n|$. A set $\Delta \in \mathbb{R}^n$ is said to be an ellipsoid if $\Delta = \{x \in \mathbb{R}^n: |Px| < 1\}$, for some nondegenerate $n \times n$ matrix P , where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^n . For a dilation A , there exists an ellipsoid Δ and $r > 1$, such that $\Delta \subset r\Delta \subset A\Delta$, where $|\Delta|$, the Lebesgue measure of Δ , equals 1. Let $B_k = A^k\Delta$ for $k \in \mathbb{Z}$, then we have $B_k \subset rB_k \subset B_{k+1}$, and $B_k = b^k$, where $b = |\det A| > 1$. Let w be the smallest integer, so that $2B_0 \subset A^w B_0 = B_w$. A homogeneous quasi-norm associated with an expansive matrix A is a measurable mapping $\rho_A: \mathbb{R}^n \rightarrow [0, \infty)$ satisfying

$$\begin{aligned} \rho_A(x) &> 0 \text{ for } x \neq 0, \quad \rho_A(Ax) = |\det A| \rho_A(x) \text{ for } x \in \mathbb{R}^n, \\ \rho(x + y) &\leq C_A(\rho_A(x) + \rho_A(y)) \text{ for } x, y \in \mathbb{R}^n, \end{aligned}$$

where C_A is a positive constant.

It was proved, in [6, p. 6, Lemma 2.4], that all homogeneous quasi-norms associated with a given dilation A are equivalent. Define the step homogeneous quasi-norm ρ on \mathbb{R}^n induced by dilation A as

$$\rho(x) := \begin{cases} b^j, & \text{if } x \in B_{j+1} \setminus B_j, \\ 0, & \text{if } x = 0. \end{cases}$$

Then, for any $x, y \in \mathbb{R}^n$, $\rho(x + y) \leq b^w(\rho(x) + \rho(y))$.

In the following we denote $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$, where χ_{C_k} is the characteristic function of C_k . Throughout this paper, we denote by C a constant, which is independent of the main parameters and whose value may vary.

Definition 1.2. Let $\alpha(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < q \leq \infty$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous anisotropic Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)$ associated with the dilation A is defined by

$$\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n) = \left\{ f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}): \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}} = \left\{ \sum_{k=-\infty}^{\infty} \|b^{k\alpha(\cdot)} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q}.$$

The nonhomogeneous anisotropic Herz space $K_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)$ associated with the dilation A is defined by

$$K_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n) = \left\{ f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n): \|f\|_{K_{p(\cdot)}^{\alpha(\cdot), q}} < \infty \right\},$$

where

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot), q}} = \left\{ \sum_{k=0}^{\infty} \|b^{k\alpha(\cdot)} f \tilde{\chi}_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right\}^{1/q}.$$

Here, the usual modifications are made when $q = \infty$.

In variable L^p spaces, there are some important lemmas as follows.

Lemma 1.3. [1]. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on \mathbb{R}^n and

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $r_p = 1 + 1/p^- - 1/p^+$.

The following lemmas are from [13].

Lemma 1.4. Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then, there exists a constant $C > 0$, such that for all balls B in \mathbb{R}^n ,

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 1.5. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, there exist $0 < \delta_1, \delta_2 < 1$ depending only on $p(\cdot)$ and n , such that for all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_S\|_{L^{p(\cdot)}}} \leq \frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} \leq \left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}} \leq \left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

Next, we introduce the definition of homogeneous anisotropic Herz-type Hardy space with two variable exponents $\dot{HK}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ and the nonhomogeneous anisotropic Herz-type Hardy space with two variable exponents $HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ and the atomic characterization of $\dot{HK}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ and $HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$, which were obtained by Wang and Guo [12].

A C^∞ complex-valued function φ is said to belong to the Schwartz class \mathcal{S} , if for every integer $\ell \in \mathbb{Z}_+$ and multi-index α , $\|\varphi\|_{\alpha,\ell} := \sup_{x \in \mathbb{R}^n} |\rho(x)|^\ell |\partial^\alpha \varphi(x)| < \infty$. The dual space of \mathcal{S} , namely, the space of all tempered distributions on \mathbb{R}^n equipped with the weak- $*$ topology, is denoted by \mathcal{S}' . For any $N \in \mathbb{Z}_+$, let

$$\mathcal{S}_N := \{\varphi \in \mathcal{S} : \|\varphi\|_{\alpha,\ell} \leq 1, \quad |\alpha| \leq N, \ell \leq N\}.$$

For $\varphi \in \mathcal{S}$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_k(x) := b^{-k} \varphi(A^{-k}x)$.

Let $f \in \mathcal{S}'$. The non-tangential maximal function $M_\varphi(f)$ with respect to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\varphi(f)(x) := \sup\{|f * \varphi_k(y)| : x - y \in B_k, k \in \mathbb{Z}\}.$$

For any given $N \in \mathbb{N}$, the non-tangential grand maximal function $M_N(f)$ of $f \in \mathcal{S}'$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N(f)(x) := \sup_{\varphi \in \mathcal{S}_N} M_\varphi(f)(x).$$

For $0 < q < \infty$, we denote

$$N_q := \begin{cases} [(1/q - 1) \ln b / \ln \lambda_-] + 2, & 0 < q \leq 1, \\ 2, & q > 1. \end{cases}$$

Definition 1.6. Let $\alpha(\cdot) \in L^\infty$, $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}$, and $N > N_q$. The homogeneous anisotropic Herz-type Hardy space with variable exponents $\dot{HK}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ and the nonhomogeneous anisotropic Herz-type Hardy space with variable exponents $HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ are defined, respectively, by setting,

$$HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n) := \{f \in S' : M_N(f) \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)\}$$

and

$$HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n) := \{f \in S' : M_N(f) \in K_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)\},$$

where

$$\|f\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)} = \|M_N(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)} \text{ and } \|f\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)} = \|M_N(f)\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)}.$$

Definition 1.7. Let $p(\cdot) \in \mathcal{P}$, $\alpha(\cdot) \in L^\infty \cap \mathcal{P}_0^{\log} \cap \mathcal{P}_\infty^{\log}$, and nonnegative integer $s \in [(\alpha_l - \delta_2) \ln b / \ln \lambda_-, \infty)$ with δ_2 as in Lemma 1.5. Here, $\alpha_l = \alpha_0$, if $l < 0$, $\alpha_l = \alpha_\infty$, if $l > 0$.

(1) An anisotropic central $(\alpha(\cdot), p(\cdot), s)$ -atom is a measurable function a on \mathbb{R}^n satisfying

- (i) $\text{supp } a \subset B_l$, for some $l \in \mathbb{Z}$;
- (ii) $\|a\|_{L^{p(\cdot)}} \leq |b|^{-k\alpha_l}$;
- (iii) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$ for any $\beta \in \mathbb{Z}_+^n$ with $|\beta| \leq s$.

(2) An anisotropic central $(\alpha(\cdot), p(\cdot), s)$ -atom of restricted type is a measurable function a on \mathbb{R}^n satisfying

- (i) $\text{supp } a \subset B_l$, $l \geq 0$, for some $l \in \mathbb{Z}$;
- (ii) $\|a\|_{L^{p(\cdot)}} \leq |b|^{-k\alpha_\infty}$;
- (iii) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$ for any $\beta \in \mathbb{Z}_+^n$ with $|\beta| \leq s$.

2 Molecular decompositions of $HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$

In this section, we first give the definitions of the molecules of the anisotropic Herz-type Hardy spaces with variable exponents. Before stating our results, we first give the notations of molecules.

Definition 2.1. Let $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty$, and nonnegative integer $s \geq \max\{[(\alpha(0) - \delta_1) \ln b / \ln \lambda_-], [(\alpha_\infty - \delta_1) \ln b / \ln \lambda_-]\}$, where δ_1 as in lemma 1.5. Set $\varepsilon > \max\{s, (\alpha(0) + \delta_1 - 1) \ln b / \ln \lambda_-, (\alpha_\infty + \delta_1 - 1) \ln b / \ln \lambda_-\}$ and $d = 1 - \delta_1 + \varepsilon$. Moreover, for any $l \in \mathbb{Z}$, when $l < 0$, $\alpha_l := \alpha(0)$ and $a := 1 - \delta_1 - \alpha(0) + \varepsilon$; when $l \geq 0$, $\alpha_l := \alpha_\infty$ and $a := 1 - \delta_1 - \alpha_\infty + \varepsilon$.

(1) A function $M_l \in L^{p(\cdot)}$ with $l \in \mathbb{Z}$ is said to be a dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$ -molecule if it satisfies

- (i) $\|M_l\|_{L^{p(\cdot)}} \leq b^{-l\alpha_l}$;
- (ii) $\mathcal{R}_{p(\cdot)}(M_l) = \|M_l\|_{L^{p(\cdot)}}^{\alpha/d} \|(\rho(\cdot))^d M_l\|_{L^{p(\cdot)}}^{1-\alpha/d} < \infty$;
- (iii) $\int_{\mathbb{R}^n} M_l(x)x^\beta dx = 0$, for any β with $|\beta| \leq s$.

(2) A function $M_l \in L^{p(\cdot)}$ with $l \in \mathbb{N}_0$ is said to be a dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$ -molecule of restricted type if it satisfies (ii), (iii) and

- (i') $\|M_l\|_{L^{p(\cdot)}} \leq b^{-l\alpha_l}$.

Definition 2.2. Let $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty$, and nonnegative integer $s \geq \max\{[(\alpha(0) - \delta_1) \ln b / \ln \lambda_-], [(\alpha_\infty - \delta_1) \ln b / \ln \lambda_-]\}$, where δ_1 as in Lemma 1.5. Set $\varepsilon > \max\{s, (\alpha(0) + \delta_1 - 1) \ln b / \ln \lambda_-, (\alpha_\infty + \delta_1 - 1) \ln b / \ln \lambda_-\}$ and $d = 1 - \delta_1 + \varepsilon$. Moreover, for any function $M \in L^{p(\cdot)}$, when $\|M\|_{L^{p(\cdot)}} > 1$, $a := 1 - \delta_1 - \alpha(0) + \varepsilon$; when $\|M\|_{L^{p(\cdot)}} \leq 1$, $a := 1 - \delta_1 - \alpha_\infty + \varepsilon$.

- (1) A function $M \in L^{p(\cdot)}$ is said to be a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule if it satisfies
- (i) $\mathcal{R}_{p(\cdot)}(M) = \|M\|_{L^{p(\cdot)}}^{a/d} \|(\rho(\cdot))^d M\|_{L^{p(\cdot)}}^{1-a/d} < \infty$;
 - (ii) $\int_{\mathbb{R}^n} M(x)x^\beta dx = 0$, for any β with $|\beta| \leq s$.
- (2) A function $M \in L^{p(\cdot)}$ is said to be a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule of restricted type if it satisfies (i), (ii) and
- (i') $\|M\|_{L^{p(\cdot)}} \leq 1$.

The following lemma shows that a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule is a generalization of the central $(\alpha(\cdot), p(\cdot), s)$ -atom.

Lemma 2.3. *Let $0 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty$, and nonnegative integer $s \geq \max\{[(\alpha(0) - \delta_1) \ln b/\ln \lambda_-], [(\alpha_\infty - \delta_1) \ln b/\ln \lambda_-], [(\alpha(0) - \delta_2) \ln b/\ln \lambda_-], [(\alpha_\infty - \delta_2) \ln b/\ln \lambda_-]\}$, where $\max\{\delta_1, \delta_2\} \leq \alpha(0)$, $\alpha_\infty < \infty$ and δ_1, δ_2 as in Lemma 1.5. Set $\varepsilon > \max\{s, (\alpha(0) + \delta_1 - 1) \ln b/\ln \lambda_-, (\alpha_\infty + \delta_1 - 1) \ln b/\ln \lambda_-\}$ and $d = 1 - \delta_1 + \varepsilon$. Moreover, for any function $M \in L^{p(\cdot)}$, when $\|M\|_{L^{p(\cdot)}} > 1$, $a := 1 - \delta_1 - \alpha(0) + \varepsilon$; when $\|M\|_{L^{p(\cdot)}} \leq 1$, $a := 1 - \delta_1 - \alpha_\infty + \varepsilon$.*

- (i) *If M is a central $(\alpha(\cdot), p(\cdot), s)$ -atom, then M is a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule, such that $\mathcal{R}_{p(\cdot)}(M) < C$ with C independent of M .*
- (ii) *If M is a central $(\alpha(\cdot), p(\cdot), s)$ -atom of restricted type, then M is a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule of restricted type, such that $\mathcal{R}_{p(\cdot)}(M) < C$ with C independent of M .*

Proof. We only prove (i). (ii) can be proved in the similar way.

Let M be a $(\alpha(\cdot), p(\cdot), s)$ -atom with support on a ball B_k , then we get

$$\|M\|_{L^{p(\cdot)}}^{a/d} \|(\rho(\cdot))^d M\|_{L^{p(\cdot)}}^{1-a/d} \leq b^{(k-1)d(1-a/d)} \|M\|_{L^{p(\cdot)}} \leq C. \quad \square$$

Now, we give the molecular decompositions of anisotropic Herz-type Hardy spaces with two variable exponents.

Theorem 2.4. *Let $0 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty$, and nonnegative integer $s \geq \max\{[(\alpha(0) - \delta_1) \ln b/\ln \lambda_-], [(\alpha_\infty - \delta_1) \ln b/\ln \lambda_-], [(\alpha(0) - \delta_2) \ln b/\ln \lambda_-], [(\alpha_\infty - \delta_2) \ln b/\ln \lambda_-]\}$, where $\max\{\delta_1, \delta_2\} \leq \alpha(0)$, $\alpha_\infty < \infty$ and δ_1, δ_2 as in Lemma 1.5. Set $\varepsilon > \max\{s, (\alpha(0) + \delta_1 - 1) \ln b/\ln \lambda_-, (\alpha_\infty + \delta_1 - 1) \ln b/\ln \lambda_-\}$ and $d = 1 - \delta_1 + \varepsilon$. Moreover, for any function $M \in L^{p(\cdot)}$, when $\|M\|_{L^{p(\cdot)}} > 1$, $a := 1 - \delta_1 - \alpha(0) + \varepsilon$; when $\|M\|_{L^{p(\cdot)}} \leq 1$, $a := 1 - \delta_1 - \alpha_\infty + \varepsilon$.*

- (i) $f \in \dot{HK}_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)$ if and only if can be represented as

$$f = \sum_{k=-\infty}^{\infty} \lambda_k M_k, \text{ in } S',$$

where each M_k is a dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_k$ -molecule and $\sum_{k=-\infty}^{\infty} |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{\dot{HK}_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \sim \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^q \right)^{1/q},$$

where the infimum is taken over all above decompositions of f .

- (ii) $f \in HK_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)$ if and only if can be represented as

$$f = \sum_{k=0}^{\infty} \lambda_k M_k, \text{ in } S',$$

where each M_k is a dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_k$ -molecule of restricted type and $\sum_{k=0}^{\infty} |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A;\mathbb{R}^n)} \sim \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^q \right)^{1/q},$$

where the infimum is taken over all above decompositions of f .

Theorem 2.5. Let $0 < q \leq 1$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty$, and nonnegative integer $s \geq \max\{[(\alpha(0) - \delta_1) \ln b / \ln \lambda_-], [(\alpha_\infty - \delta_1) \ln b / \ln \lambda_-], [(\alpha(0) - \delta_2) \ln b / \ln \lambda_-], [(\alpha_\infty - \delta_2) \ln b / \ln \lambda_-]\}$, where $\max\{\delta_1, \delta_2\} \leq \alpha(0)$, $\alpha_\infty < \infty$ and δ_1, δ_2 as in Lemma 1.5. Set $\varepsilon > \max\{s, (\alpha(0) + \delta_1 - 1) \ln b / \ln \lambda_-, (\alpha_\infty + \delta_1 - 1) \ln b / \ln \lambda_-\}$ and $d = 1 - \delta_1 + \varepsilon$. Moreover, for any function $M \in L^{p(\cdot)}$, when $\|M\|_{L^{p(\cdot)}} > 1$, $a := 1 - \delta_1 - \alpha(0) + \varepsilon$; when $\|M\|_{L^{p(\cdot)}} \leq 1$, $a := 1 - \delta_1 - \alpha_\infty + \varepsilon$.

(i) $f \in HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ if and only if can be represented as

$$f = \sum_{k=-\infty}^{\infty} \lambda_k M_k, \text{ in } S',$$

where each M_k is a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule and $\sum_{k=-\infty}^{\infty} |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A;\mathbb{R}^n)} \sim \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^q \right)^{1/q},$$

where the infimum is taken over all above decompositions of f .

(ii) $f \in HK_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ if and only if can be represented as

$$f = \sum_{k=0}^{\infty} \lambda_k M_k, \text{ in } S',$$

where each M_k is a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule of restricted type and $\sum_{k=0}^{\infty} |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A;\mathbb{R}^n)} \sim \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^q \right)^{1/q},$$

where the infimum is taken over all above decompositions of f .

By theorem 3.2 of [12] and Lemma 2.3, we see that Theorems 2.4 and 2.5 can be obtained from the following lemma.

Lemma 2.6. Let $0 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty$, and nonnegative integer $s \geq \max\{[(\alpha(0) - \delta_1) \ln b / \ln \lambda_-], [(\alpha_\infty - \delta_1) \ln b / \ln \lambda_-], [(\alpha(0) - \delta_2) \ln b / \ln \lambda_-], [(\alpha_\infty - \delta_2) \ln b / \ln \lambda_-]\}$, where $\max\{\delta_1, \delta_2\} \leq \alpha(0)$, $\alpha_\infty < \infty$ and δ_1, δ_2 as in Lemma 1.5. Set $\varepsilon > \max\{s, (\alpha(0) + \delta_1 - 1) \ln b / \ln \lambda_-, (\alpha_\infty + \delta_1 - 1) \ln b / \ln \lambda_-\}$ and $d = 1 - \delta_1 + \varepsilon$. Moreover, for any function $M \in L^{p(\cdot)}$ and $l \in \mathbb{Z}$, when $\|M\|_{L^{p(\cdot)}} > 1$ or $l < 0$, let $\alpha_l := \alpha(0)$ and $a := 1 - \delta_1 - \alpha(0) + \varepsilon$; when $\|M\|_{L^{p(\cdot)}} \leq 1$ or $l \geq 0$, let $\alpha_l := \alpha_\infty$ and $a := 1 - \delta_1 - \alpha_\infty + \varepsilon$.

(i) If $0 < q \leq 1$, there exists a constant C , such that for any central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule M and any central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule of restricted type M_l ,

$$\|M\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A;\mathbb{R}^n)} \leq C \text{ and } \|M_l\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A;\mathbb{R}^n)} \leq C,$$

respectively.

(ii) There exists a constant C , such that for any dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$ -molecule M_l , $l \in \mathbb{Z}$, and any dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$ -molecule of restricted type M_l , $l \in \mathbb{N}_0$,

$$\|M_l\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A;\mathbb{R}^n)} \leq C \text{ and } \|M_l\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}(A;\mathbb{R}^n)} \leq C,$$

respectively.

Proof. We only prove (i) for the homogeneous case, the proof of the nonhomogeneous case and (ii) are similar.

Suppose that M is a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$ -molecule. Taking

$$r := \begin{cases} \|M\|_{L^{p(\cdot)}}^{-1/\alpha(0)}, & \|M\|_{L^{p(\cdot)}} > 1, \\ \|M\|_{L^{p(\cdot)}}^{-1/\alpha_{\infty}}, & \|M\|_{L^{p(\cdot)}} \leq 1, \end{cases}$$

and denote by σ_r , the unique integer satisfying $b^{\sigma_r} < r \leq b^{\sigma_r+1}$. Denote $E_0 = B_{\sigma_r}$ and $E_k = B_{\sigma_r+k} \setminus B_{\sigma_r+k-1}$ for $k \in \mathbb{N}$. Set

$$M(x)\chi_{E_k(x)} - \frac{\chi_{E_k(x)}}{|E_k|} \int_{\mathbb{R}^n} M(y)\chi_{E_k}(y) dy := H_k(x) - F_k(x).$$

It follows that

$$M(x) = \sum_{k=0}^{\infty} (H_k(x) - F_k(x)) + \sum_{k=0}^{\infty} \frac{\chi_{E_k(x)}}{|E_k|} \int_{\mathbb{R}^n} M(y)\chi_{E_k}(y) dy.$$

Obviously, $\text{supp}(H_k(x) - F_k(x)) \subset B_{\sigma_r+k}$ and $\int_{\mathbb{R}^n} (H_k(x) - F_k(x)) dx = 0$. We claim that

(a) There is a positive constant C and a sequence of numbers $\{\lambda_k\}_k$, such that

$$\sum_{k=0}^{\infty} |\lambda_k|^q < \infty, \quad H_k - F_k = \lambda_k a_k,$$

where each a_k is a $(\alpha(\cdot), p(\cdot), 0)$ -atom;

(b) $\sum_{k=0}^{\infty} F_k$ has a $(\alpha(\cdot), p(\cdot), 0)$ -atom decomposition,

then our desired conclusion can be deduced directly.

We first show (a). Without loss of generality, we can suppose that $\mathcal{R}_{p(\cdot)}(M) = 1$, which implies that

$$\|(\rho(\cdot))^d M\|_{L^{p(\cdot)}} = \|M\|_{L^{p(\cdot)}}^{-a/(d-a)} = r^a.$$

For $k = 0$, we have

$$\begin{aligned} \|H_0(x) - F_0(x)\|_{L^{p(\cdot)}} &\leq \|M\|_{L^{p(\cdot)}} + \frac{\|\chi_{B_{\sigma_r}}\|_{L^{p(\cdot)}}}{|B_{\sigma_r}|} \int_{\mathbb{R}^n} \left| M(y)\chi_{B_{\sigma_r}} \right| dy \\ &\leq C\|M\|_{L^{p(\cdot)}} = Cr^{-\alpha} \leq C|B_{\sigma_r}|^{-\alpha_1}, \end{aligned}$$

and for $k \in \mathbb{N}$,

$$\begin{aligned} \|H_k(x) - F_k(x)\|_{L^{p(\cdot)}} &\leq \|H_k(x)\|_{L^{p(\cdot)}} + \|F_k(x)\|_{L^{p(\cdot)}} \\ &\leq \|H_k(x)\|_{L^{p(\cdot)}} + \frac{C}{|E_k|} \|H_k(x)\|_{L^{p(\cdot)}} \|\chi_{E_k}\|_{L^{p'(\cdot)}} \|\chi_{E_k}\|_{L^{p(\cdot)}} \\ &\leq C\|H_k(x)\|_{L^{p(\cdot)}} \\ &\leq C\|(\rho(\cdot))^d M\|_{L^{p(\cdot)}} (b^{\sigma_r+k-1})^{-d} \\ &= Cr^a (b^{\sigma_r+k-1})^{-d} \leq Cb^{-ka} |B_{\sigma_r+k}|^{-\alpha_1}. \end{aligned}$$

Thus, for any $k \in \mathbb{N} \cup \{0\}$, there is a constant C independent of k , such that

$$\|H_k(x) - F_k(x)\|_{L^{p(\cdot)}} \leq Cb^{-ka}|B_{\sigma_r+k}|^{-\alpha_l}.$$

If we denote $\lambda_{1,k} = Cb^{-ka}$ and $a_{1,k} = (H_k(x) - F_k(x))/\lambda_{1,k}$, then the $a_{1,k}$ are central $(\alpha(\cdot), p(\cdot), 0)$ -atoms and $\sum_{k=0}^{\infty} (H_k(x) - F_k(x))(x) = \sum_{k=0}^{\infty} \lambda_{1,k} a_{1,k}(x)$. Moreover,

$$\sum_{k=0}^{\infty} |\lambda_k|^q \leq C \sum_{k=0}^{\infty} b^{-kaq} \leq C,$$

where C is independent of M .

Next, we will show (b). Set

$$m_k = \sum_{i=k}^{\infty} \int_{\mathbb{R}^n} M(x)\chi_{E_i}(x) dx, \quad \varphi_k(x) = \frac{\chi_{E_k}(x)}{|E_k|}.$$

Noting that $m_0 = 0$, summing by parts, we have

$$\sum_{i=k}^{\infty} F_k(x) = \sum_{i=k}^{\infty} (m_k - m_{k+1})\varphi_k(x) = \sum_{i=k}^{\infty} m_{k+1}(\varphi_{k+1}(x) - \varphi_k(x)).$$

Clearly,

$$\int_{\mathbb{R}^n} m_{k+1}(\varphi_{k+1}(x) - \varphi_k(x)) dx = 0, \quad \text{supp}\{m_{k+1}(\varphi_{k+1} - \varphi_k)\} \subset B_{\sigma_r+k+1},$$

and

$$m_k = m_0 - \int_{B_{\sigma_r+k-1}} M(x) dx.$$

Hence, we obtain

$$\begin{aligned} \|m_{k+1}(\varphi_{k+1} - \varphi_k)\|_{L^{p(\cdot)}} &= \left\| - \int_{B_{\sigma_r+k}} M(x)\chi_{E_i}(x) dx (\varphi_{k+1} - \varphi_k) \right\|_{L^{p(\cdot)}} \\ &\leq \left(\frac{\|\chi_{E_k}\|_{L^{p(\cdot)}}}{|E_{k+1}|} \frac{\|\chi_{E_{k+1}}\|_{L^{p(\cdot)}}}{\|\chi_{E_k}\|_{L^{p(\cdot)}}} + \frac{\|\chi_{E_k}\|_{L^{p(\cdot)}}}{|E_{k+1}|} \frac{|E_{k+1}|}{|E_k|} \right) \int_{B_{\sigma_r+k}} |M(x)| dx \\ &\leq C \frac{b\|\chi_{E_k}\|_{L^{p(\cdot)}}}{|E_{k+1}|} \|(\rho(\cdot))^d M(\cdot)\|_{L^{p(\cdot)}} \|(\rho(\cdot))^{-d} \chi_{B_{\sigma_r+k}}\|_{L^{p'(\cdot)}} \\ &\leq C \frac{b}{(b-1)/b} r^a b^{-(\sigma_r+k-1)d} |B_{\sigma_r+k}|^{-1} \|\chi_{E_k}\|_{L^{p(\cdot)}} \|\chi_{B_{\sigma_r+k}}\|_{L^{p'(\cdot)}} \\ &\leq C \frac{b}{(b-1)/b} b^{\sigma_r+1} b^{-(\sigma_r+k-1)d} \\ &\leq C \frac{b^{2(1-\delta_1+\varepsilon)} + 1}{(b-1)/b} b^{-ka} b^{-(\sigma_r+k+1)\alpha_l} \\ &\leq Cb^{-ka} |B_{\sigma_r+k+1}|^{-\alpha_l}, \end{aligned}$$

where C is independent of k . Setting

$$\lambda_{2,k} = Cb^{-ka} \quad \text{and} \quad a_{2,k} = m_{k+1}(\varphi_{k+1} - \varphi_k)/\lambda_{2,k},$$

we have

$$\sum_{k=0}^{\infty} \frac{\chi_{E_k(x)}}{|E_k|} \int_{\mathbb{R}^n} M(y)\chi_{E_k}(y) dy = \sum_{k=0}^{\infty} \lambda_{2,k} a_{2,k},$$

where the $a_{2,k}$ are central $(\alpha(\cdot), p(\cdot), 0)$ -atoms. Furthermore,

$$\sum_{k=0}^{\infty} |\lambda_{2,k}|^q \leq C \sum_{k=0}^{\infty} b^{-kaq} \leq C,$$

where C is independent of M . The conclusion (b) then holds. Hence, the proof of Lemma 2.6 is completed. \square

3 Applications

In this section, we give an application of the molecular decomposition theory established in Section 2. We study the boundedness of the central δ -Calderón-Zygmund operators from $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ to $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$. The following condition is necessary for our discussion on the central δ -Calderón-Zygmund operators on the $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ spaces:

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } S' \Rightarrow Tf = \sum_{i \in \mathbb{N}} \lambda_i T a_i \text{ in } S'. \tag{3.1}$$

The central δ -Calderón-Zygmund operators, which are more general than the classical Calderón-Zygmund operators, were introduced by Lu and Yang [14] in the isotropic setting of \mathbb{R}^n . Moreover, Ding et al. [10] extended them to the following non-isotropic setting of \mathbb{R}^n associated with the dilation A .

Definition 3.1. Let $0 < \delta < 1$ and $1 < p < \infty$. Let $T: S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ be a linear continuous operator. If there exists $K(x, y) \in S'(\mathbb{R}^n \times \mathbb{R}^n)$, being continuous away from the diagonal in \mathbb{R}^{2n} and satisfying:

- (i) $|K(x, 0)| + |K(0, x)| \leq C(\rho(x))^{-1}$, for all $x \neq 0$;
- (ii) $|K(x, 0) - K(x, y)| + |K(0, x) - K(y, x)| \leq C(\rho(y))^\delta(\rho(x))^{-1-\delta}$, when $\rho(x) \geq b^{2w}\rho(y)$;
- (iii) $\langle T(f), g \rangle = \int \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y)f(y)g(x) dy dx$, for $f, g \in S(\mathbb{R}^n)$ with disjoint supports, and if T can be extended to a bounded operator on $L^p(\mathbb{R}^n)$, then we say T is a central δ -Calderón-Zygmund operator in $L^p(\mathbb{R}^n)$.

Using the molecular theory of $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$, we can prove the following theorem:

Theorem 3.2. Let $0 < \delta < 1$, $0 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty$ and $\max\{\delta_2, \delta_1\} \leq \alpha_1 < \min\{(\delta_1 + \delta)\ln \lambda_- / \ln b, (\delta_2 + \delta)\ln \lambda_- / \ln b\}$ with α_1 as in Definition 1.7. Suppose that T is a central δ -Calderón-Zygmund operator and is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. If T satisfies (3.1) for every central atomic decomposition and $\int_{\mathbb{R}^n} T a(x) dx = 0$ for each central $(\alpha(\cdot), p(\cdot), 0)$ -atom $a(x)$, then T can be extended to a bounded operator from $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$ to $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(A; \mathbb{R}^n)$.

Remark 3.3. The boundedness of the central δ -Calderón-Zygmund operators on the homogeneous anisotropic Herz-type Hardy space $HK_{p(\cdot)}^{\alpha, q}(A; \mathbb{R}^n)$ is contained by Theorem 3.2, and the results also hold for the nonhomogeneous anisotropic Herz-type Hardy space $HK_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)$.

Proof. Case 1. For $0 < q \leq 1$. Let a be a central $(\alpha(\cdot), p(\cdot), 0)$ -atom with its support in B_k for some $k \in \mathbb{Z}$. If Ta is a central $(\alpha(\cdot), p(\cdot), 0, \varepsilon)$ -molecule for some $\delta + \delta_1 > \varepsilon > \delta_1 + \alpha_1 - 1$ and by condition (3.1) and Lemma 2.6, we have $\|Ta\|_{HK_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \leq C$. Then, for homogeneous case

$$\|Tf\|_{HK_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)}^q \leq \sum_{k=-\infty}^{\infty} |\lambda_k|^q \|Ta\|_{HK_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)}^q \leq \infty.$$

Thus, T is a bounded operator on $HK_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)$ by taking supremum of the above formula. It suffices to show Ta is a central $(\alpha(\cdot), p(\cdot), 0, \varepsilon)$ -molecule for some $\delta + \delta_1 > \varepsilon > \delta_1 + \alpha_1 - 1$. To this aim, let $a = 1 - \delta_1 - \alpha_1 + \varepsilon$, $d = 1 - \delta_1 + \varepsilon$. Obviously, we only need to verify the size condition of molecules, that is,

$$\mathcal{R}_{p(\cdot)}(Ta) = \|Ta\|_{L^{p(\cdot)}(B_k)}^{a/d} \|(\rho(\cdot))^d Ta(\cdot)\|_{L^{p(\cdot)}(B_k)}^{1-a/d} \leq C, \tag{3.2}$$

with C independent of a . From the hypothesis of the theorem, we need to show only (3.2). To do this, we first estimate $\|(\rho(\cdot))^d Ta(\cdot)\|_{L^{p(\cdot)}(B_k)}$. By the boundedness of T on the $L^{p(\cdot)}$, we have

$$\|(\rho(\cdot))^d Ta(\cdot)\|_{L^{p(\cdot)}(B_{k+2w})} \leq Cb^{kd} \|Ta\|_{L^{p(\cdot)}(B_{k+2w})} \leq Cb^{kd} \|a\|_{L^{p(\cdot)}(B_{k+2w})} \leq Cb^{k(d-\alpha_1)}.$$

On the other hand, if $x \in B_{k+2w}^c$, from the condition(ii) of definition 3.1 and $\int_{B_k} a(x) dx = 0$, we can get

$$\begin{aligned} |T(a)| &= \left| \int_{B_k} (K(x, y) - K(x, 0)) a(y) dy \right| \leq C \int_{B_k} \frac{\rho(y)^\delta}{\rho(x)^{1+\delta}} |a(y)| dy \\ &\leq Cb^{k(1+\delta)} \rho(x)^{-(1+\delta)} \frac{1}{|B_k|} \int_{B_k} |a(y)| dy \leq Cb^{k(1+\delta)} \rho(x)^{-(1+\delta)} M(a)(x). \end{aligned}$$

Therefore, noting that $\varepsilon < \delta + \delta_1$, we have

$$\begin{aligned} \|(\rho(\cdot))^d Ta(\cdot)\|_{L^{p(\cdot)}(B_{k+2w}^c)} &\leq Cb^{k(1+\delta)} \|(\rho(\cdot))^{d-1-\delta} Ma(\cdot)\|_{L^{p(\cdot)}(B_{k+2w}^c)} \\ &\leq Cb^{k(1+\delta)+k(d-1-\delta)} \|Ma(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq Cb^{kd} \|a(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Cb^{k(d-\alpha_1)}. \end{aligned}$$

That is, $\|(\rho(\cdot))^d Ta(\cdot)\|_{L^{p(\cdot)}} \leq Cb^{k(d-\alpha_1)}$. Thus,

$$\begin{aligned} \mathcal{R}_{p(\cdot)}(Ta) &= \|Ta\|_{L^{p(\cdot)}(B_k)}^{a/d} \|(\rho(\cdot))^d Ta(\cdot)\|_{L^{p(\cdot)}(B_k)}^{1-a/d} \\ &\leq C \|a\|_{L^{p(\cdot)}(B_k)}^{a/d} b^{k(d-\alpha_1)(1-a/d)} \\ &\leq Cb^{-(k\alpha_1)a/d} b^{k(d-\alpha_1)(1-a/d)} \\ &= C < \infty, \end{aligned}$$

where C is independent of a .

Case 2. For $1 < q < \infty$. By a proof similar to that of [3, Proposition 3.8], we easily obtain an important lemma as follows. □

Lemma 3.4. Let $\alpha(\cdot) \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty$, $p(\cdot) \in \mathcal{P}$ and $q \in (0, \infty)$, then

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \approx \left(\sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \|f \chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q} + \left(\sum_{k=0}^{+\infty} b^{\alpha_\infty k q} \|f \chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q}.$$

We now proceed with the proof of Theorem 3.2. Let a be a central $(\alpha(\cdot), p(\cdot), 0)$ -atom with its support in B_k for some $k \in \mathbb{Z}$. We write

$$\begin{aligned} \|M_N T(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)}^q &\approx \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \|M_N T(f) \chi_k\|_{L^{p(\cdot)}}^q + \sum_{k=0}^{+\infty} b^{\alpha_\infty k q} \|M_N T(f) \chi_k\|_{L^{p(\cdot)}}^q \\ &\leq C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=-\infty}^{+\infty} |\lambda_j| \|M_N T(a_j) \chi_k\|_{L^{p(\cdot)}} \right)^q + C \sum_{k=0}^{+\infty} b^{\alpha_\infty k q} \left(\sum_{j=-\infty}^{+\infty} |\lambda_j| \|M_N T(a_j) \chi_k\|_{L^{p(\cdot)}} \right)^q \\ &=: I_3 + I_4. \end{aligned}$$

For I_3 , it is easy to see that

$$\begin{aligned} I_3 &\leq C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=-\infty}^{k-\sigma-1} |\lambda_j| \|M_N T(a_j) \chi_k\|_{L^{p(\cdot)}} \right)^q + C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=k-\sigma}^{+\infty} |\lambda_j| \|M_N T(a_j) \chi_k\|_{L^{p(\cdot)}} \right)^q \\ &=: I_{31} + I_{32}. \end{aligned}$$

We first estimate $M_N T(a)$ on C_k for $k \geq k_0 + \sigma + 1$. For any $x \in C_k$, $\varphi \in \mathcal{S}_N$, $j \in \mathbb{Z}$ and a polynomial P_s of degree $\leq s$, by a proof similar to those of [10, p. 1454], we have

$$M_N T(a)(x) \leq C b^{-k_0 \alpha_0 - k_0} \|\chi_{B_{k_0}}\|_{L^{p(\cdot)}} (b \lambda_-^{s+1})^{-m}, \quad (3.3)$$

where $m = k - k_0 - 1 - \sigma$. By the Hölder inequality, $|b^{\alpha_0 - \delta_2} \lambda_-^{-s-1}| < 1$ and (3.3), we obtain

$$\begin{aligned} I_{31} &\leq C \sum_{k=-\infty}^{-1} b^{k q \alpha_0} \left(\sum_{j=-\infty}^{k-\sigma-1} |\lambda_j| b^{-j(\alpha_0+1)+k} b^{(j-k)\delta_2} (b \lambda_-^{s+1})^{j+\sigma+1-k} \right)^q \\ &\leq C \sum_{k=-\infty}^{-1} b^{k q (\alpha_0 - \delta_2)} \lambda_-^{-k q (s+1)} \left[\sum_{j=-\infty}^{k-\sigma-1} |\lambda_j| (b^{\alpha_0 - \delta_2} \lambda_-^{-s-1})^{-j} \right]^q \\ &\leq C \sum_{k=-\infty}^{-1} b^{k q (\alpha_0 - \delta_2)} \lambda_-^{-k q (s+1)} \sum_{j=-\infty}^{k-\sigma-1} |\lambda_j|^q (b^{\alpha_0 - \delta_2} \lambda_-^{-s-1})^{-q j / 2} \left[\sum_{j=-\infty}^{k-\sigma-1} (b^{\alpha_0 - \delta_2} \lambda_-^{-s-1})^{-q j / 2} \right]^{q/q'} \\ &\leq C \sum_{k=-\infty}^{-1} (b^{\alpha_0 - \delta_2} \lambda_-^{-s-1})^{k q / 2} \sum_{j=-\infty}^{k-\sigma-1} |\lambda_j|^q (b^{\alpha_0 - \delta_2} \lambda_-^{-s-1})^{-q j / 2} \\ &\leq C \sum_{j=-\infty}^{+\infty} |\lambda_j|^q (b^{\alpha_0 - \delta_2} \lambda_-^{-s-1})^{-j q / 2} \sum_{k=j+\sigma+1}^{+\infty} (b^{\alpha_0 - \delta_2} \lambda_-^{-s-1})^{k q / 2} \\ &\leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^q. \end{aligned}$$

From the $L^{p(\cdot)}$ boundedness of M_N , the size condition of a , and the Hölder inequality, we conclude that

$$\begin{aligned}
 I_{32} &\leq C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=k-\sigma}^{+\infty} |\lambda_j| \|T(a_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
 &\leq C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=k-\sigma}^{+\infty} |\lambda_j| |B_j|^{-\alpha_j} \right)^q \\
 &\leq C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=k-\sigma}^{-1} |\lambda_j| |B_j|^{-\alpha_0} \right)^q + C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=0}^{+\infty} |\lambda_j| |B_j|^{-\alpha_\infty} \right)^q \\
 &\leq C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=k-\sigma}^{-1} |\lambda_j|^q |B_j|^{-\alpha_0 q/2} \right) \left(\sum_{j=k-\sigma}^{-1} |B_j|^{-\alpha_0 q'/2} \right)^{q/q'} \\
 &\quad + C \sum_{k=-\infty}^{-1} b^{\alpha_0 k q} \left(\sum_{j=0}^{+\infty} |\lambda_j|^q |B_j|^{-\alpha_\infty q/2} \right) \left(\sum_{j=0}^{+\infty} |B_j|^{-\alpha_\infty q'/2} \right)^{q/q'} \\
 &\leq C \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^j |\lambda_j|^q b^{(k-j)\alpha_0 q/2} + C \sum_{j=0}^{+\infty} |\lambda_j|^q \\
 &\leq C \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sum_{j=0}^{+\infty} |\lambda_j|^q \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^q.
 \end{aligned}$$

The proof of I_4 is similar to I_3 , we are omitting it there. From the I_3, I_4 , we can get

$$\|M_N T(f)\|_{K_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)}^q \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^q.$$

Thus,

$$\|T(f)\|_{HK_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \leq C \|f\|_{HK_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)}.$$

Therefore, we finish the proof of Theorem 3.2. □

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