DE GRUYTER Open Mathematics 2020; 18: 886–893

Research Article

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Asymptotic relations for the products of elements of some positive sequences

https://doi.org/10.1515/math-2020-0029
received December 2, 2019; accepted June 12, 2020

Abstract: The aim of this study was to present a simple method for finding the asymptotic relations for products of elements of some positive real sequences. The main reason to carry out this study was the result obtained by Alzer and Sandor concerning an estimation of a sequence of the product of the first k primes.

Keywords: prime numbers, special sequences, asymptotic relations, limits

MSC 2020: 41A60, 40A20, 40A25, 11A41

1 Introduction

Let p_i be the ith prime number. Let us denote $\prod p_k := \prod_{i=1}^{k} p_i$. Consider the sequence $b_k := \frac{\binom{k^2}{k}}{\prod p_k}$. Alzer and Sándor [1] proved the following inequality:

$$\exp[k(c_0 - \log\log k)] \le b_k \le \exp[k(c_1 - \log\log k)]$$

for every $k \ge 5$, where constants c_0 and c_1 are equal to

$$c_0 = \frac{1}{5}\log 23 + \log\log 5 \approx 1.10298,$$

$$c_1 = \frac{1}{192}\log\binom{36\,864}{192} + \log\log 192 - \frac{1}{192}\log\left(\prod p_{192}\right) \approx 2.04287,$$

respectively. In this study, another estimation of the sequence $\{b_k\}$ connected with the limit $\lim\limits_{k\to\infty} f(k) = 2$ will be shown, where

$$f(k) = \frac{1}{k}\log\binom{k^2}{k} + \log\log k - \frac{1}{k}\theta(p_k), \quad \theta(x) := \sum_{p\le x}\log p,$$

and p runs over all prime numbers less than or equal to x. The above limit was proven by Alzer and Sándor [1]. Nevertheless, we decided to obtain a new proof of this limit as the original proof was obtained in an over

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complicated way (in a certain sense). In consequence, we also found a new result (Theorem 1), which gives a simple and universal tool for generating asymptotic relations of many known sequences of real numbers, especially for the products of elements of certain sequences.

2 Main result

The discussion is based on the following well-known fact, which is connected to d' Alembert's ratio test.

Lemma 1. *Suppose that* $\{a_k\}_{k=1}^{\infty}$ *is a sequence of positive terms and there exists a finite limit* $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} = g$. *If* $g < 1$, *then* $\lim\limits_{k\to\infty} a_k = 0$, *and if* $g > 1$, *then* $\lim\limits_{k\to\infty} a_k = \infty$.

This result will be used for finding the estimation of sequence $\{b_k\}_{k=1}^{\infty}$ in the following way. If we can find sequences $\{m_k\}_{k=1}^{\infty}$, $\{M_k\}_{k=1}^{\infty}$ of positive real numbers such that

$$\lim_{k\to\infty} \frac{b_{k+1}}{m_{k+1}} \cdot \left(\frac{b_k}{m_k}\right)^{-1} > 1, \quad \lim_{k\to\infty} \frac{b_{k+1}}{M_{k+1}} \cdot \left(\frac{b_k}{M_k}\right)^{-1} < 1,$$

then $\lim\limits_{k\to\infty} \frac{b_k}{m_k} = \infty$, $\lim\limits_{k\to\infty} \frac{b_k}{M_k} = 0$, thereby $m_k \le b_k \le M_k$ for sufficiently large k.

Using Stirling's approximation we can write

$$\binom{k^2}{k} = \frac{(k^2)!}{k!(k^2-k)!} \sim \frac{1}{\sqrt{2\pi}}(k-1)^{k-\frac{1}{2}} \cdot \left(1 - \frac{1}{k}\right)^{-k^2} \sim \frac{1}{\sqrt{2\pi}}(ke)^{k-\frac{1}{2}}.$$

From the prime number theorem, we get $p_k \sim k \log k$. Therefore, we have

$$\frac{b_{k+1}}{b_k} = \frac{\binom{(k+1)^2}{k+1}}{\prod p_k} \cdot \left(\frac{\binom{k^2}{k}}{\prod p_k}\right)^{-1} \sim \frac{1}{p_{k+1}} \cdot \frac{(k+1)^{k+\frac{1}{2}} \cdot e^{k+\frac{1}{2}}}{k^{k-\frac{1}{2}} \cdot e^{k-\frac{1}{2}}} \sim \frac{e^2}{\log(k+1)}.$$

If we denote $c_k(t) = e^{kt}(\log k)^{-k}$, then we get $\frac{c_k(t)}{c_{k+1}(t)} \sim e^{-t}\log(k+1)$, and thereby

$$\frac{b_{k+1}}{c_{k+1}(t)} \cdot \left(\frac{b_k}{c_k(t)}\right)^{-1} \sim e^{2-t},$$

where $e^{2-t} > 1$ if $t < 2$ and $e^{2-t} < 1$ if $t > 2$. Then, for every $t_1 < 2 < t_2$ and for sufficiently large $k \in \mathbb{N}$ the following inequalities hold

$$e^{kt_1}(\log k)^{-k} = c_k(t_1) \le b_k \le c_k(t_2) = e^{kt_2}(\log k)^{-k},$$

hence

$$\exp[k(t_1 - \log\log k)] \le \frac{\binom{k^2}{k}}{\prod p_k} \le \exp[k(t_2 - \log\log k)].$$

Corollary 1. *We have* (*see* [1])

$$\lim_{k\to\infty} \left(\frac{\log \binom{k^2}{k}}{k} - \frac{\log\left(\prod p_k\right)}{k} + \log\log k \right) = 2,$$

or equivalently

$$\lim_{k\to\infty}\sqrt[k]{\frac{\binom{k^2}{k}}{\prod p_k}}\log k = e^2.$$

In a similar way, we may find the estimation of sequence $\{\prod p_k\}_{k=1}^{\infty}$. We have

$$\frac{\prod p_{k+1}}{\prod p_k} = p_{k+1} \sim (k+1)\log(k+1).$$

Let us choose the sequence $d_k := (e^t k \log k)^k$. Then, $\frac{d_{k+1}(t)}{d_k(t)} \sim e^{t+1}(k+1)\log(k+1)$ and

$$\frac{\prod p_{k+1}}{d_{k+1}(t)}\cdot\left(\frac{\prod p_k}{d_k(t)}\right)^{-1} \sim e^{-t-1},$$

where $e^{-t-1} > 1$ if $t < -1$ and $e^{-t-1} < 1$ if $t > -1$. Therefore, we obtain

$$\exp[k(t_3 + \log k + \log\log k)] \le \prod p_k \le \exp[k(t_4 + \log k + \log\log k)]$$

for any $t_3 < -1 < t_4$ and for sufficiently large k. Moreover, we also have

$$\lim_{k\to\infty}\left(\frac{\log\left(\prod p_k\right)}{k} - \log k - \log\log k\right) = -1,$$

i.e.,

$$\lim_{k\to\infty}\frac{\sqrt[k]{\prod p_k}}{k\log k} = \frac{1}{e}.$$

The last result reminds another known limit

$$\lim_{k\to\infty}\frac{\sqrt[k]{k!}}{k} = \frac{1}{e}, \tag{1}$$

which is not an incident and comes from the general relationship presented in Theorem 1.

From now on we will use the symbol $\prod x_k$ to denote product $\prod_{i=1}^{k} x_i$, where $\{x_i\}_{i=1}^{\infty}$ is any real sequence. Now, we will prove our main result.

Theorem 1. *Let $\{f_k\}_{k=1}^{\infty}$ be an increasing sequence of positive reals. Suppose that there exists $r, w \in \mathbb{R}$ and a polynomial $p \in \mathbb{R}[k]$ such that*

$$f_k \sim e^r k^{p(k)} \log^w k.$$

Then, for every pair t_1, t_2 of real numbers satisfying the condition $t_1 < r - p(0) < t_2$, the following inequality holds

$$\left(e^{t_1} k^{p(0)} \log^w k\right)^k \le \prod \frac{f_k}{k^{p(k)-p(0)}} \le \left(e^{t_2} k^{p(0)} \log^w k\right)^k,$$

for all sufficiently large k, which implies the relation

$$\lim_{k\to\infty}\left(\frac{\log\left(\prod \frac{f_k}{k^{p(k)-p(0)}}\right)}{k} - p(0)\log k - \log(\log^w k)\right) = r - p(0),$$

or the equivalent one

$$\lim_{k\to\infty}\frac{\sqrt[k]{\prod \frac{f_k}{k^{p(k)-p(0)}}}}{k^{p(0)}\log^w k} = e^{r-p(0)}. \tag{2}$$

Proof. Let us set $g_k := e^{tk}k^{kp(0)}\log^{kw}k$. Then, we obtain

$$\frac{g_k}{g_{k+1}} = e^{-t}\frac{\left(1+\frac{1}{k}\right)^{-kp(0)}}{(k+1)^{p(0)}}\left(1+\frac{\log\left(1+\frac{1}{k}\right)}{\log k}\right)^{-kw}\log^w(k+1) \sim e^{-t-p(0)}(k+1)^{-p(0)}\log^w(k+1)$$

and

$$\left(\prod \frac{f_{k+1}}{(k+1)^{p(k+1)-p(0)}}\right)\cdot\left(\prod \frac{k^{p(k)-p(0)}}{f_k}\right)\cdot\frac{g_k}{g_{k+1}} = \frac{f_{k+1}}{(k+1)^{p(k+1)-p(0)}}\cdot\frac{g_k}{g_{k+1}} \sim e^{-t+r-p(0)},$$

where $e^{-t+r-p(0)} > 1$ if $t < r - p(0)$ and $e^{-t+r-p(0)} < 1$ if $t > r - p(0)$, which finishes the proof. □

Corollary 2. *Let $\{f_k\}_{k=1}^{\infty}$ be an increasing sequence of positive reals. Suppose that the sequence $\{\sqrt[k]{\prod f_k}\}_{k=1}^{\infty}$ is also increasing and there exist $r, s, w \in \mathbb{R}$ such that*

$$\sqrt[k]{\prod f_k} \sim e^{r-s}k^s\log^w k.$$

Then, we have

$$\lim_{k\to\infty}\frac{\sqrt[k]{\prod \sqrt[k]{\prod f_k}}}{k^s\log^w k} = e^{r-2s}. \tag{3}$$

Remark 1. Note that identity (2) can be n-times iterated for all $n \in \mathbb{N}$ if we make additional assumptions about monotonicity of the corresponding sequences.

3 Applications

Some applications of Theorem 1 are given as follows.

(3.1) Let $f_k = k$. Then, $\lim_{k\to\infty}\frac{\sqrt[k]{k!}}{k} = \frac{1}{e}$.

(3.2) Let $f_k = \frac{1}{\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)}$. Then, $f_k \sim e^{\gamma}\log k$ (see [2,3]) and

$$\lim_{k\to\infty}\frac{\sqrt[k]{\prod f_k}}{\log k} = \lim_{k\to\infty}\frac{\sqrt[k]{\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)^{-k+i-1}}}{\log k} = e^{\gamma},$$

where γ denotes Euler's constant. Let $\alpha > 0$. If we replace f_k by

$$f_k(\alpha) = \frac{1}{\prod_{i=i_0}^{k}\left(1-\frac{\alpha}{p_i}\right)}$$

for $k \geq i_0$, where $i_0 := \min\{i \in \mathbb{N} : p_i > \alpha\}$, then $f_k(\alpha) \sim e^{-\log(c(\alpha))}\log^{\alpha}k$ for some $c(\alpha) > 0$ (see [2,4,5]), and consequently

$$\lim_{k\to\infty}\frac{\sqrt[k]{\prod f_k(\alpha)}}{\log^{\alpha}k} = \frac{1}{c(\alpha)}.$$

For example, $c(2) \approx 0.832429065662$.

(3.3) Let $\alpha \in \mathbb{R}$, $0 \leq \alpha < 1$. Then (see [6])

$$f_k = \left(\frac{e}{k}\right)^k\prod_{i=1}^{k}(i-\alpha) \sim \frac{\sqrt{2\pi}}{\Gamma(1-\alpha)}k^{\frac{1}{2}-\alpha},$$

which implies the relation

$$\lim_{k\to\infty}\frac{\sqrt[k]{\prod f_k}}{k^{\frac{1}{2}-\alpha}}=\frac{\sqrt{2\pi}}{\Gamma(1-\alpha)}e^{\alpha-\frac{1}{2}}.$$

(3.4) Let us set $f_n=\sum_{i=1}^{n}i^n$. Then (see [6,14,15]), we have $f_n\sim\frac{e}{e-1}n^n$, which implies (see also final remark 2)

$$\lim_{n\to\infty}\frac{\sqrt[n]{\prod f_n}}{n^n}=\frac{e}{e-1},$$

i.e.,

$$\lim_{n\to\infty}\frac{\sqrt[n]{\prod f_n}}{(n+1)^n}=\frac{1}{e-1}.$$

(3.5) Next, we set $f_n=(\sqrt[n^2]{\prod_{k=1}^{n}k^k})^2$. By the definition of the Glaischer-Kinkelin constant (see [6–8])

$$A=\lim_{n\to\infty}\frac{\prod_{k=1}^{n}k^k}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}e^{-\frac{n^2}{4}}},$$

we also have

$$A=2^{-\frac{5}{36}}\pi^{-\frac{1}{6}}\exp\left(\frac{2}{3}\int_{0}^{\frac{1}{2}}\log\Gamma(t)\,\mathrm{d}t\right)=\exp\left[\frac{1}{12}-\zeta'(-1)\right]\approx 1.2824271291,$$

(see [7,9–11]) we obtain $f_n\sim e^{-\frac{1}{2}}n$ and

$$\prod f_n\sim\left(\prod_{k=1}^{n}k^{k\left(\sum_{i=k}^{n}\frac{1}{i^2}\right)}\right)^2,$$

which by Theorem 1 gives us the relation

$$\lim_{n\to\infty}\frac{\sqrt[n]{\prod f_n}}{n}=e^{-\frac{3}{2}}.$$

(3.6) At last, if we set

$$f_n=e^{\frac{n^2}{4}}\prod_{k=1}^{n}k^k,$$

then $f_n\sim An^{\frac{n(n+1)}{2}+\frac{1}{12}}$, where A denotes the Glaisher-Kinkelin constant. Therefore, we get

$$\lim_{n\to\infty}\frac{\sqrt[n]{\prod\frac{f_n}{n^{\frac{n(n+1)}{2}}}}}{\sqrt[12]{n}}=Ae^{-\frac{1}{12}}.$$

Now we present an application of Corollary 2.

(3.7) In [7, Problem 1.5], it was proved that

$$f_k=e^{-\frac{k}{2}}\sqrt[k]{\prod_{i=1}^{k}\binom{k}{i}}\sim e^{1-\ln\sqrt{2\pi}}k^{-\frac{1}{2}}. \tag{4}$$

Hence, by Theorem 1, we get

$$\lim_{k\to\infty}\sqrt{k}\sqrt[k]{\prod f_k}=e^{\frac{3}{2}-\ln\sqrt{2\pi}}$$

and by Corollary 2

$$\lim_{k\to\infty} \sqrt{k}\sqrt[k]{\prod \sqrt[k]{\prod f_k}} = e^{2-\ln\sqrt{2\pi}}.$$

We note that from (4) we obtain the solution of Problem 51, p. 45 from Pólya and Szegő [12]:

$$\lim_{k\to\infty} \sqrt[k^2]{\prod_{i=1}^{k} \binom{k}{i}} = \sqrt{e}\,.$$

Note also that, applying Corollary 2, monotonicity of sequences $\{f_k\}_{k=1}^{\infty}$ and $\{\sqrt[k]{\prod f_k}\}_{k=1}^{\infty}$ is not important at all, because the following lemma holds.

Lemma 2. *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive reals. If $\lim_{k\to\infty} a_k = \alpha$, then there exists an increasing sequence $\{A_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} A_k = \alpha$ and $\lim_{k\to\infty} \frac{A_k}{a_k} = 1$. We may assume that the sequence $\{\sqrt[k]{\prod (A_k)}\}_{k=1}^{\infty}$ is increasing as well.*

(3.8) Also from [7, Problem 1.14] we get

$$\lim_{k\to\infty} \frac{\sqrt[k]{\prod \Gamma\left(\frac{1}{k}\right)}}{k} = e^{-1},$$

which by Corollary 2 implies

$$\lim_{k\to\infty} \frac{1}{k}\sqrt[k]{\prod \sqrt[k]{\prod \Gamma\left(\frac{1}{k}\right)}} = e^{-2}.$$

4 Final remarks

(1) Using the other asymptotic expansions, especially the ones given in the study of Kellner [6] (e.g., for product of Bernoulli numbers, for products of the special values of gamma function, etc.), we can generate many new relations that are omitted here. Other relations were published recently in [13] as well.

(2) In the history of the following asymptotic relation (see [14–16])

$$n^{-n}\sum_{k=1}^{n} k^n = \frac{e}{e-1} - \frac{e(e+1)}{2n(e-1)^3} + o(n^{-1}),$$

one thread – Dutch connection is missing. The above relationship was found by the last author as one of the problems in the problem section of the known Dutch journal *Nieuw Archief voor Wiskunde* but in the equivalent form:

$$(n+1)^{-n}\sum_{k=1}^{n} k^n = \frac{1}{e-1} - \frac{e(3-e)}{2n(e-1)^3} + o(n^{-1}).$$

It is interesting that in both proofs of these equalities (from *Nieuw Archief voor Wiskunde* and by Lampret – see [15]) Tannery's theorem was used.

(3) Another estimation for a number of primes is discussed by Meštrović in [17].

5 Problems

The natural question about asymptoticity of the following expressions:

$$\frac{\sqrt[k]{\prod \frac{f_k}{k^{p(k)-p(0)}}}}{k^{p(0)} \log^w k} - e^{r-p(0)}$$

and

$$\frac{\sqrt[k]{\prod \sqrt[k]{\prod f_k}}}{k^s \log^w k} - e^{r-2s}$$

arises from (1) and (2). For now, this question remains unanswered. The following formula (see [18,19])

$$\left(e^{\frac{(-1)^n}{n}}\left(\dots\left(e^{\frac{1}{4}}\left(e^{-\frac{1}{3}}\left(e^{\frac{1}{2}}\left(e^{-1}(1+x)^{\frac{1}{x}}\right)^{\frac{1}{x}}\right)^{\frac{1}{x}}\right)^{\frac{1}{x}}\right)^{\frac{1}{x}}\dots\right)^{\frac{1}{x}}\right)^{\frac{1}{x}} \xrightarrow[x\to 0]{} e^{\frac{(-1)^n}{n+1}}$$

is an inspiration for solving this problem.

Acknowledgments: The authors would like to thank reviewers for invaluable comments which help to improve the presentation of this study.

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