

Research Article

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The second out-neighborhood for local tournaments

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Abstract: Sullivan stated the conjectures: (1) every oriented graph has a vertex x such that $d^{++}(x) \geq d^-(x)$ and (2) every oriented graph has a vertex x such that $d^{++}(x) + d^+(x) \geq 2d^-(x)$. In this paper, we prove that these conjectures hold for local tournaments. In particular, for a local tournament D , there are at least two vertices satisfying (1) and either there exist two vertices satisfying (2) or there exists a vertex v satisfying $d^{++}(v) + d^+(v) \geq 2d^-(v) + 2$ if D has no vertex of in-degree zero.

Keywords: local tournament, the second out-neighborhood, Sullivan's conjecture, round decomposable, nonround decomposable

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1 Terminology and introduction

In this paper, we consider finite digraphs without loops and multiple arcs. The main source for terminology and notation is ref. [1].

Let D be a digraph. We denote the vertex set and the arc set of D by $V(D)$ and $A(D)$, respectively. For a vertex subset X , we denote by $D\langle X \rangle$, the subdigraph of D induced by X . For convenience, we write $D - X$ instead of $D\langle V(D) - X \rangle$ and $D - x$ instead of $D - \{x\}$ for a vertex x of D . For a subgraph H of D , we define $D - H = D - V(H)$.

Let x, y be distinct vertices of D . If there is an arc from x to y , we say that x *dominates* y and denote it by $x \rightarrow y$ and call y (respectively, x) an *out-neighbor* (respectively, an *in-neighbor*) of x (respectively, y). If V_1 and V_2 are disjoint subsets of vertices of D such that there is no arc from V_2 to V_1 and $a \rightarrow b$ for each $a \in V_1$ and each $b \in V_2$, then we say that V_1 *completely dominates* V_2 and denote it by $V_1 \Rightarrow V_2$. We will use the same notation when V_1 or V_2 is the subdigraph of D . In particular, if V_1 contains only one vertex v , denote it by $v \Rightarrow V_2$.

For a subdigraph or simply a vertex subset H of D (possibly, $H = D$), we let $N_H^+(x)$ (respectively, $N_H^-(x)$) denote the set of out-neighbors (respectively, in-neighbors) of x in H and call it the *out-neighborhood* (respectively, *in-neighborhood*) of x in H . Furthermore, $d_H^+(x) = |N_H^+(x)|$ (respectively, $d_H^-(x) = |N_H^-(x)|$) is called the *out-degree* (respectively, *in-degree*) of x in H . Let

$$N_H^{++}(x) = \bigcup_{u \in N_H^+(x)} N_H^+(u) - N_H^+(x),$$

where $N_H^{++}(x)$ is called the *second out-neighborhood* of x in H . Furthermore, $d_H^{++}(x) = |N_H^{++}(x)|$. We will omit the subscript H if the digraph is known from the context. For a pair of vertex disjoint subdigraphs H and H' , we define

$$N_H^+(H') = \bigcup_{x \in V(H')} N_H^+(x) - V(H'), \quad N_H^-(H') = \bigcup_{x \in V(H')} N_H^-(x) - V(H').$$

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A vertex x is a *2-king* (for short, a *king*) of D , if for all $y \in V(D) - x$, there exists an (x, y) -path of length at most 2.

A digraph D is *strongly connected* (or just, *strong*) if, for every pair x, y of distinct vertices, D contains a path from x to y and a path from y to x . A strong component of a digraph D is a maximal-induced subdigraph of D which is strong. If D_1, D_2, \dots, D_t are the *strong components* of D , then clearly $V(D_1) \cup V(D_2) \cup \dots \cup V(D_t) = V(D)$ (note that a digraph with only one vertex is strong). Moreover, we must have $V(D_i) \cap V(D_j) = \emptyset$ for every $i \neq j$. The strong components of D can be labeled as D_1, D_2, \dots, D_t such that there is no arc from D_j to D_i unless $j < i$. We call such an ordering an *acyclic ordering of the strong components* of D .

For a vertex subset S of a strong digraph D , S is called a *separating set* of D if $D - S$ is not strong. A separating set S of D is *minimal* if for any proper subset S' of S , the subdigraph $D - S'$ is strong.

A digraph R on n vertices is *round* if we can label its vertices v_1, v_2, \dots, v_n so that for each i , we have $N^+(v_i) = \{v_{i+1}, v_{i+2}, \dots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \dots, v_{i-2}, v_{i-1}\}$ (all subscripts are taken modulo n). We will refer to the ordering v_1, v_2, \dots, v_n as a *round labeling* of R .

A digraph D is *semicomplete* if, for every pair x, y of distinct vertices in D , either x dominates y or y dominates x (or both). *Tournaments* are semicomplete digraphs with no 2-cycle.

A digraph D with no 2-cycle is an *oriented graph*.

In 1995, Dean [2] proposed the following conjecture which is one of the most interesting and challenging open questions concerning oriented graphs.

Conjecture 1.1. (Seymour's Second Neighborhood Conjecture (SSNC)) For any oriented graph D , there exists a vertex v in D such that $d^{++}(v) \geq d^+(v)$.

We call such a vertex v satisfying Conjecture 1.1 a *Seymour vertex*. The first nontrivial result for SSNC was obtained by Fisher [3] who proved Dean's conjecture [2], which is SSNC restricted to tournaments. Fisher used Farkas' Lemma and averaging arguments.

Theorem 1.2. [3] *In any tournament T , there exists a Seymour vertex.*

A more elementary proof of SSNC for tournaments was given by Havet and Thomassé [4] who introduced a median order approach. Their proof also yields the following stronger result.

Theorem 1.3. [4] *A tournament T with no vertex of out-degree zero has at least two Seymour vertices.*

Fidler and Yuster [5] further developed the median order approach and proved that SSNC holds for oriented graphs D with minimum degree $|V(D)| - 2$, tournaments minus a star and tournaments minus the arc set of a subtournament. The median order approach was also used by Ghazal [6] who proved a weighted version of SSNC for tournaments missing a generalized star. Kaneko and Locke [7] proved SSNC for oriented graphs with minimum out-degree at most 6. Cohn et al. [8] proved SSNC for random oriented graphs with probability $p < \frac{1}{2} - \delta$. Gutin and Li [9] proved SSNC for extended tournaments and quasi-transitive oriented graphs.

Another approach to SSNC is to determine the maximum value γ such that in every oriented graph D , there exists a vertex x such that $d^+(x) \leq \gamma d^{++}(x)$. SSNC asserts that $\gamma = 1$. Chen et al. [10] proved that $\gamma \geq r$ where $r = 0.657298\dots$ is the unique real root of $2x^3 + x^2 - 1 = 0$. Furthermore, they improve this bound to 0.67815... mentioned in the end of the article [10].

Sullivan [11] stated the following "compromise conjectures" on SSNC, where $d^-(v)$ is used instead of or together with $d^+(v)$.

Conjecture 1.4. [11]

- (1) Every oriented graph D has a vertex x such that $d^{++}(x) \geq d^-(x)$.
- (2) Every oriented graph D has a vertex x such that $d^{++}(x) + d^+(x) \geq 2d^-(x)$.

For convenience, a vertex x in D satisfying Conjecture 1.4 (i) is called a *Sullivan- i* vertex of D for $i \in \{1, 2\}$.

Li and Sheng [12,13] proved Sullivan's Conjectures for tournaments, extended tournaments, quasi-transitive oriented graphs, and bipartite tournaments. For tournaments, they obtained the following results:

Theorem 1.5. [12] *Every tournament has a Sullivan-1 vertex and a Sullivan-2 vertex. Every tournament with no vertex of in-degree zero has at least three Sullivan-1 vertices.*

\mathcal{T} is a special class of tournaments. $T \in \mathcal{T}$ if T is a tournament consisting of exactly two strong components T_1 and T_2 such that T_1 dominates T_2 , T_1 is a single vertex v , and T_2 is a tournament satisfying that $d_{T_2}^+(x) \leq d_{T_2}^-(x) + 1$ for any $x \in V(T_2)$. It is easy to check that v is the unique Sullivan-2 vertex of T .

Theorem 1.6. [12] *A tournament T has at least two Sullivan-2 vertices unless $T \in \mathcal{T}$.*

From Theorem 1.6, we obtain immediately the following result:

Corollary 1.7. *A strong tournament T with at least three vertices has at least two Sullivan-2 vertices.*

In ref. [14], Bang-Jensen introduced locally semicomplete digraphs. A digraph D is *locally semicomplete* if $D\langle N^+(x) \rangle$ and $D\langle N^-(x) \rangle$ are both semicomplete for every vertex x of D . Specifically, every round digraph is locally semicomplete [15]. A *local tournament* is a locally semicomplete digraph with no 2-cycle.

In ref. [16], we investigate SSNC for local tournaments. In this paper, we discuss Sullivan's conjectures for local tournaments. In Section 2, we introduce the structure of a local tournament. In Sections 3 and 4, we investigate the Sullivan- i vertex in a round decomposable local tournament and a nonround decomposable local tournament, respectively, for $i \in \{1, 2\}$.

2 Related works

In this section, all theorems are on the structure of locally semicomplete digraphs. Clearly, these theorems also hold if the digraph is restricted to a local tournament.

Let D be a digraph with vertex set $\{v_1, v_2, \dots, v_n\}$ and let G_1, G_2, \dots, G_n be digraphs which are pairwise vertex disjoint. The *composition* $D[G_1, G_2, \dots, G_n]$ is the digraph L with vertex set $V(G_1) \cup V(G_2) \cup \dots \cup V(G_n)$ and arc set $(\cup_{i=1}^n A(G_i)) \cup \{g_i g_j \mid g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\}$. If each G_i is an empty digraph for $i = 1, 2, \dots, n$, then D is an *extension* of H .

A digraph D is *round decomposable* if there exists a round local tournament R on $r \geq 2$ vertices such that $D = R[S_1, S_2, \dots, S_r]$, where each S_i is a strong semicomplete digraph. We call $R[S_1, S_2, \dots, S_r]$ a *round decomposition* of D . Clearly, a round decomposable digraph is locally semicomplete.

The following theorem, due to Bang-Jensen et al., stated a full classification of locally semicomplete digraphs.

Theorem 2.1. [15] *Let D be a connected locally semicomplete digraph. Then, exactly one of the following possibilities holds:*

- D is round decomposable with a unique round decomposition $R[S_1, S_2, \dots, S_r]$, where R is a round local tournament on $r \geq 2$ vertices and S_i is a strong semicomplete digraph for each $i \in \{1, 2, \dots, r\}$;
- D is nonround decomposable and not semicomplete and it has the structure as described in Theorem 2.6;
- D is a semicomplete digraph which is nonround decomposable.

If D is restricted to a local tournament, we have the following result:

Corollary 2.2. *Let D be a connected local tournament. Then, exactly one of the following possibilities holds:*

- D is round decomposable with a unique round decomposition $R[S_1, S_2, \dots, S_r]$, where R is a round local tournament on $r \geq 2$ vertices and S_i is a strong tournament for $i \in \{1, 2, \dots, r\}$;
- D is nonround decomposable and not a tournament and it has the structure as described in Theorem 2.6;
- D is a tournament which is nonround decomposable.

For a nonstrong locally semicomplete digraph, we can say more as follows:

Theorem 2.3. [17] *Let D be a connected, but not strong locally semicomplete digraph. Then, the following holds for D .*

- (a) *If A and B are distinct strong components of D with at least one arc between them, then either $A \Rightarrow B$ or $B \Rightarrow A$;*
- (b) *If A and B are strong components of D , such that $A \Rightarrow B$, then A and B are semicomplete digraphs;*
- (c) *The strong components of D can be ordered in a unique way D_1, D_2, \dots, D_p such that there is no arc from D_j to D_i for $j > i$, and D_i completely dominates D_{i+1} for $i \in \{1, 2, \dots, p - 1\}$.*

A kind of the decomposition of nonstrong locally semicomplete digraphs described in ref. [17] is the following:

Theorem 2.4. [17] *Let D be a connected, but not strong locally semicomplete digraph, and let D_1, D_2, \dots, D_p be the acyclic ordering of the strong components of D . Then, D can be decomposed into $r \geq 2$ induced subdigraphs D'_1, D'_2, \dots, D'_r as follows:*

$$D'_1 = D_p, \lambda_1 = p,$$

$$\lambda_{i+1} = \min \{j | N^+(D_j) \cap V(D'_i) \neq \emptyset\} \text{ for each } i \in \{1, 2, \dots, r - 1\},$$

and

$$D'_{i+1} = D \langle V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \dots \cup V(D_{\lambda_i}) \rangle.$$

The subdigraphs D'_1, D'_2, \dots, D'_r satisfy the properties below:

- (a) D'_i consists of some strong components of D and is semicomplete for each $i \in \{1, 2, \dots, r\}$;
- (b) D'_{i+1} completely dominates the initial component of D'_i , and there exists no arc from D'_i to D'_{i+1} for $i \in \{1, 2, \dots, r - 1\}$;
- (c) If $r \geq 3$, then there is no arc between D'_i and D'_j for i, j satisfying $|j - i| \geq 2$.

The unique sequence D'_1, D'_2, \dots, D'_r defined in Theorem 2.4 will be referred to as the *semicomplete decomposition* of D .

Theorem 2.5. [15] *If D is a round decomposable locally semicomplete digraph, then it has a unique round decomposition $D = R[S_1, S_2, \dots, S_r]$, where R is a round local tournament on $r \geq 2$ vertices and each S_i is a strong semicomplete digraph.*

Theorem 2.6. [15] *Let D be a strong locally semicomplete digraph which is not semicomplete. Then, D is nonround decomposable if and only if the following conditions are satisfied:*

- (a) *There is a minimal separating set S such that $D - S$ is not semicomplete, and for each such S , $D \langle S \rangle$ is semicomplete and the semicomplete decomposition of $D - S$ has exactly three components D'_1, D'_2, D'_3 ;*
- (b) *There are integers α, β, μ, ν with $\lambda \leq \alpha \leq \beta \leq p - 1$ and $p + 1 \leq \mu \leq \nu \leq p + q$ such that*

$$N^-(D_\alpha) \cap V(D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha) \cap V(D_\nu) \neq \emptyset,$$

or

$$N^-(D_\mu) \cap V(D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu) \cap V(D_\beta) \neq \emptyset,$$

where D_1, D_2, \dots, D_p and $D_{p+1}, D_{p+2}, \dots, D_{p+q}$ are the acyclic orderings of the strong components of $D - S$ and $D \langle S \rangle$, respectively, and D_λ is the initial component of D'_2 .

By Theorem 2.6, D is always strong if D is a nonround decomposable locally semicomplete digraph. An example of a nonround decomposable locally semicomplete digraph is shown in Figure 2.

Theorem 2.7. [15] *Let D be a strong nonround decomposable locally semicomplete digraph and let S be a minimal separating set of D such that $D - S$ is not semicomplete. Let D_1, D_2, \dots, D_p be the acyclic ordering of the strong components of $D - S$ and $D_{p+1}, D_{p+2}, \dots, D_{p+q}$ be the acyclic ordering of the strong components of $D \setminus S$. The following holds:*

- (a) $D_p \Rightarrow S \Rightarrow D_1$;
- (b) *Suppose that there is an arc $s \rightarrow v$ from S to D'_1 with $s \in V(D_i)$ and $v \in V(D_j)$. Then, $D_i \cup D_{i+1} \cup \dots \cup D_{p+q} \Rightarrow D'_3 \Rightarrow D_\lambda \cup \dots \cup D_j$;*
- (c) $D_{p+q} \Rightarrow D'_3$ and $D_f \Rightarrow D_{f+1}$ for $f \in \{1, 2, \dots, p+q\}$ where subscripts are modulo $p+q$.

3 In a round decomposable local tournament

In this section, D is always a round decomposable local tournament and let the unique round decomposition of D be $R[S_1, S_2, \dots, S_r]$, where R is a round local tournament on $r \geq 2$ vertices and each S_i is a strong tournament. For convenience, we use V_i to denote both the vertex set of S_i and an empty digraph with vertex set $V(S_i)$ for $i \in \{1, 2, \dots, r\}$.

We begin with a useful observation.

Lemma 3.1. *Let D be a round decomposable local tournament and $R[S_1, S_2, \dots, S_r]$ be the unique round decomposition of D . Let $D^* = R[V_1, V_2, \dots, V_r]$ and $v_i \in V_i$ be arbitrary, where V_i is the vertex set of S_i for $i \in \{1, 2, \dots, r\}$. If $N_D^+(v_j) = V_{j+1} \cup \dots \cup V_k$, then $d_D^{++}(v_j) \geq d_D^+(v_k)$.*

Proof. Let $v \in N_D^+(v_k)$. We claim that $v \notin N_D^+(v_j)$. In fact, if $v_j \rightarrow v$, then v_j, v, v_k are in the order of the round labeling of R . Then, $v_j \rightarrow v_j$ since $v_k \rightarrow v$. Note that $v_j \rightarrow v_k$. This contradicts the fact that D has no 2-cycle. So, $v \notin N_D^+(v_j)$. Thus, $v \in N_D^{++}(v_j)$, i.e., $N_D^+(v_k) \subseteq N_D^{++}(v_j)$. Then, $d_D^{++}(v_j) \geq d_D^+(v_k)$. \square

First, we consider the existence of a Sullivan- i vertex in D for $i \in \{1, 2\}$.

Lemma 3.2. *Let D be a round decomposable local tournament and $R[S_1, S_2, \dots, S_r]$ be the unique round decomposition of D . Let $D = D^* = R[V_1, V_2, \dots, V_r]$ and $v_j \in V_j$ be arbitrary, where V_j is the vertex set of S_j for $j \in \{1, 2, \dots, r\}$. If there is a vertex $v \in V_j$ such that v is a Sullivan- i vertex of S_j and a Sullivan- i vertex of D^* , then v is a Sullivan- i vertex of D for $i \in \{1, 2\}$.*

Proof. For the case when $i = 1$, since v is a Sullivan-1 vertex of S_j and a Sullivan-1 vertex of D^* , we have $d_{S_j}^{++}(v) \geq d_{S_j}^-(v)$, $d_D^{++}(v) \geq d_D^-(v)$. Clearly

$$d_D^{++}(v) = d_{S_j}^{++}(v) + d_D^{++}(v), \quad d_D^-(v) = d_{S_j}^-(v) + d_D^-(v).$$

Thus, $d_D^{++}(v) \geq d_D^-(v)$ and v is a Sullivan-1 vertex of D .

For the case when $i = 2$, it can be proved similarly. \square

Theorem 3.3. *Let D be a round decomposable local tournament. Then, D has a Sullivan- i vertex for $i \in \{1, 2\}$.*

Proof. Let $R[S_1, S_2, \dots, S_r]$ be the unique round decomposition of D . Let $D^* = R[V_1, V_2, \dots, V_r]$ and $v_j \in V_j$ be arbitrary, where V_j is the vertex set of S_j for $j \in \{1, 2, \dots, r\}$. W.l.o.g., assume that $v_1 \in V_1$ is a vertex of D^* with minimum out-degree, i.e., $d_D^+(v_1) = \delta^+(D^*)$. Let $N_D^+(v_1) = V_2 \cup \dots \cup V_t$. Since $v_1 \rightarrow v_{t+1}$, we have $N_D^-(v_{t+1}) \subseteq V_2 \cup \dots \cup V_t = N_D^+(v_1)$. Then,

$$d_D^-(v_{t+1}) \leq d_D^+(v_1) = \delta^+(D^*).$$

Let $N_D^+(v_{t+1}) = V_{t+2} \cup \dots \cup V_h$. By Lemma 3.1, we have

$$d_D^{++}(v_{t+1}) \geq d_D^+(v_h) \geq \delta^+(D^*).$$

Thus, $d_D^{++}(v_{t+1}) \geq \delta^+(D^*) \geq d_D^-(v_{t+1})$ and v_{t+1} is a Sullivan-1 vertex of D^* . Note that $d_D^+(v_{t+1}) \geq \delta^+(D^*)$. Then, $d_D^{++}(v_{t+1}) + d_D^+(v_{t+1}) \geq 2\delta^+(D^*) \geq 2d_D^-(v_{t+1})$ and v_{t+1} is also a Sullivan-2 vertex of D^* .

Clearly, all the vertices of V_{t+1} are Sullivan- i vertices of D^* for $i \in \{1, 2\}$. By Corollary 1.5, the tournament S_{t+1} always has a Sullivan-1 vertex and a Sullivan-2 vertex, say v_{t+1} and v'_{t+1} , respectively. By Lemma 3.2, v_{t+1} is a Sullivan-1 vertex of D and v'_{t+1} is a Sullivan-2 vertex of D . \square

Next, we consider the number of Sullivan- i vertices in a connected round decomposable local tournament with no vertex of in-degree zero for $i \in \{1, 2\}$. Note that every nonstrong local tournament is round decomposable. We consider two cases: (1) a connected, but not strong local tournament with no vertex of in-degree zero and (2) a strong round decomposable local tournament.

Theorem 3.4. *Let D be a connected, but not strong local tournament with no vertex of in-degree zero. Then, D has at least three Sullivan-1 vertices and two Sullivan-2 vertices.*

Proof. Let D_1, D_2, \dots, D_p be the acyclic ordering of the strong components of D . Let v be a Sullivan-1 vertex of D_1 and v' be a Sullivan-2 vertex of D_1 , i.e.,

$$d_{D_1}^{++}(v) \geq d_{D_1}^-(v), d_{D_1}^{++}(v') + d_{D_1}^+(v') \geq 2d_{D_1}^-(v').$$

Clearly,

$$\begin{aligned} d_D^{++}(v) &\geq d_{D_1}^{++}(v), d_D^-(v) = d_{D_1}^-(v), \\ d_D^{++}(v') &\geq d_{D_1}^{++}(v'), d_D^+(v') \geq d_{D_1}^+(v'), d_D^-(v) = d_{D_1}^-(v'). \end{aligned}$$

Thus, $d_D^{++}(v) \geq d_D^-(v)$ and $d_D^{++}(v') + d_D^+(v') \geq 2d_D^-(v')$, i.e., v is a Sullivan-1 vertex of D and v' is a Sullivan-2 vertex of D . So, the Sullivan- i vertex of D_1 is always the Sullivan- i vertex of D for $i \in \{1, 2\}$.

Since D has no vertex of in-degree zero, we see that D_1 has no vertex of in-degree zero and D_1 has at least three vertices. By Corollaries 1.5 and 1.7, D_1 has at least three Sullivan-1 vertices and two Sullivan-2 vertices. Then, these three vertices (respectively, two vertices) are Sullivan-1 (respectively, Sullivan-2) vertices of D . \square

Theorem 3.5. *Let D be a strong round decomposable local tournament. Then, D has at least two Sullivan- i vertices for $i \in \{1, 2\}$.*

Proof. Let $D = R[S_1, S_2, \dots, S_r]$ be the unique round decomposition. Let $D^* = R[V_1, V_2, \dots, V_r]$ and $v_j \in V_j$ be arbitrary, where V_j is the vertex set of S_j for $j \in \{1, 2, \dots, r\}$. W.l.o.g., assume that $v_1 \in V_1$ is a vertex of D^* with minimum out-degree, i.e., $d_D^+(v_1) = \delta^+(D^*)$. Let $N_D^+(v_1) = V_2 \cup \dots \cup V_r$. According to the proof of Theorem 3.3, a Sullivan- i vertex of S_{t+1} is a Sullivan- i vertex of D for $i \in \{1, 2\}$. \square

For the case when $|V_{t+1}| \geq 2$, by Corollaries 1.5 and 1.7, the strong tournament S_{t+1} has at least three Sullivan-1 vertices and two Sullivan-2 vertices. Then, these three vertices (respectively, two vertices) are the Sullivan-1 (respectively, Sullivan-2) vertices of D .

For the case when $|V_{t+1}| = 1$ and there exists $v_h \notin V_1$ such that $d_D^+(v_h) = \delta^+(D^*)$, we can repeat the proof of Theorem 3.3 and obtain a different “ v_{t+1} ”. Now the so-called “ v_{t+1} ” is another Sullivan- i vertex for $i \in \{1, 2\}$.

Now we consider the case when $|V_{t+1}| = 1$ and there exists no $v_h \notin V_1$ such that $d_D^+(v_h) = \delta^+(D^*)$. Then, $d_D^+(v_j) > \delta^+(D^*) \geq 1$ for any $j \neq 1$. According to the proof of Theorem 3.3, the only vertex v_{t+1} of V_{t+1} is a Sullivan- i vertex of D for $i \in \{1, 2\}$. It is sufficient to find another Sullivan- i vertex of D for $i \in \{1, 2\}$.

We claim that $v_2 \rightarrow v_{t+2}$ and $v_{t+2} \notin V_1$. In fact, we have v_{t+2} must be in the set $N_D^+(v_2)$ since $d_D^+(v_2) > d_D^+(v_1)$, $|V_2| \geq |V_{t+1}| = 1$ and $N_D^+(v_1) = V_2 \cup V_3 \cup \dots \cup V_r$, $N_D^+(v_2) \supseteq V_3 \cup \dots \cup V_r \cup V_{t+1}$. So, $v_2 \rightarrow v_{t+2}$. Note that $v_1 \rightarrow v_2$. Furthermore, $v_{t+2} \notin V_1$ since D has no 2-cycle.

Note $v_2 \rightarrow v_{t+2}$ and $v_1 \rightarrow v_{t+2}$. Then, $N_D^-(v_{t+2}) = V_2 \cup V_3 \cup \dots \cup V_r \cup V_{t+1}$. Since $|V_{t+1}| = 1$, we have

$$d_D^-(v_{t+2}) = |V_2 \cup V_3 \cup \dots \cup V_t \cup V_{t+1}| = d_D^+(v_1) + 1 = \delta^+(D^*) + 1.$$

Let $N_D^+(v_{t+2}) = V_{t+3} \cup \dots \cup V_g$. By Lemma 3.1, we have

$$d_D^{++}(v_{t+2}) \geq d_D^+(v_g).$$

Case 1. $v_g \neq v_1$.

We see that $d_D^{++}(v_{t+2}) \geq d_D^+(v_g) \geq \delta^+(D^*) + 1 \geq d_D^-(v_{t+2})$, and v_{t+2} is a Sullivan-1 vertex of D^* .

Note $v_{t+2} \notin V_1$. Then, $d_D^+(v_{t+2}) \geq \delta^+(D^*) + 1 \geq d_D^-(v_{t+2})$. Thus, $d_D^{++}(v_{t+2}) + d_D^+(v_{t+2}) \geq 2d_D^-(v_{t+2})$ and v_{t+2} is a Sullivan-2 vertex of D^* .

Clearly, all the vertices of V_{t+2} are Sullivan- i vertices of D^* for $i \in \{1,2\}$. By Corollary 1.5, the tournament S_{t+2} has a Sullivan-1 vertex and a Sullivan-2 vertex, say v_{t+2} and v'_{t+2} , respectively. By Lemma 3.2, v_{t+2} is another Sullivan-1 vertex of D and v'_{t+2} is another Sullivan-2 vertex of D .

Case 2. $v_g = v_1$.

Now $N_D^+(v_{t+2}) = V_{t+3} \cup \dots \cup V_r \cup V_1$.

First, we show that v_2 is another Sullivan-1 vertex of D^* . Recall $v_2 \rightarrow v_{t+2}$. Then, $V_2 \Rightarrow V_3 \cup V_4 \dots \cup V_{t+2}$. Also, $V_{t+2} \Rightarrow V_{t+3} \cup \dots \cup V_r \cup V_1$. Then, $v \in N^+(v_2) \cup N^{++}(v_2)$ for any $v \in V(D^* - V_2)$. Since $N_{D^*-V_2}^+(v_2) \cap N_{D^*-V_2}^-(v_2) = \emptyset$, we have $N_{D^*-V_2}^-(v_2) \subseteq N_{D^*-V_2}^{++}(v_2)$. Then, $d_{D^*-V_2}^{++}(v_2) \geq d_{D^*-V_2}^-(v_2)$. This means that all vertices of V_2 are Sullivan-1 vertices of D^* . By Corollary 1.5, the tournament S_2 has a Sullivan-1 vertex, say also v_2 . By Lemma 3.2, then v_2 is another Sullivan-1 vertex of D .

To find another Sullivan-2 vertex of D , we consider the following two cases:

If $d_D^+(v_{t+2}) \geq \delta^+(D^*) + 2$, we have $d_D^{++}(v_{t+2}) \geq d_D^+(v_g) = d_D^+(v_1) = \delta^+(D^*)$. Thus, $d_D^{++}(v_{t+2}) + d_D^+(v_{t+2}) \geq \delta^+(D^*) + \delta^+(D^*) + 2 \geq 2(\delta^+(D^*) + 1) \geq 2d_D^-(v_{t+2})$, and hence, all vertices of V_{t+2} are Sullivan-2 vertices of D^* . By Corollary 1.5, the tournament S_{t+2} has a Sullivan-2 vertex, say also v_{t+2} . By Lemma 3.2, v_{t+2} is another Sullivan-2 vertex of D .

If $d_D^+(v_{t+2}) \leq \delta^+(D^*) + 1$, we have $d_D^+(v_{t+2}) = \delta^+(D^*) + 1$ since we note $v_{t+2} \notin V_1$. Also note $v_2 \rightarrow v_{t+2}$. Then, $N_D^+(v_2) \supseteq V_3 \cup \dots \cup V_{t+2}$, and hence, $N_D^-(v_2) \subseteq V_{t+3} \cup \dots \cup V_r \cup V_1$. So,

$$d_D^-(v_2) \leq |V_{t+3} \cup \dots \cup V_r \cup V_1| = d_D^+(v_{t+2}) = \delta^+(D^*) + 1.$$

Let $N_D^+(v_2) = V_3 \cup \dots \cup V_k$. By Lemma 3.1, we have $d_D^{++}(v_2) \geq d_D^+(v_k)$. Note $v_2 \rightarrow v_k$ and $v_1 \rightarrow v_2$. Then, $v_k \notin V_1$ since D has no 2-cycle. Note that

$$d_D^{++}(v_2) \geq d_D^+(v_k) \geq \delta^+(D^*) + 1, d_D^+(v_2) \geq \delta^+(D^*) + 1.$$

Then, $d_D^{++}(v_2) + d_D^+(v_2) \geq 2(\delta^+(D^*) + 1) \geq 2d_D^-(v_2)$, and hence, all vertices of V_2 are Sullivan-2 vertices of D^* . By Corollary 1.5, the tournament S_2 always has a Sullivan-2 vertex, say also v_2 . By Lemma 3.2, v_2 is another Sullivan-2 vertex of D .

Corollary 3.6.

- (a) Every round decomposable local tournament has a Sullivan- i vertex for $i \in \{1, 2\}$.
- (b) Every round decomposable local tournament with no vertex of in-degree zero has at least two Sullivan- i vertices for $i \in \{1, 2\}$.

Let D_n be a round local tournament on $n \geq 5$ vertices with the round labeling v_1, v_2, \dots, v_n and the arc set $v_i \rightarrow v_j$ for all $1 \leq i < j \leq n - 1$, $v_{n-1} \rightarrow v_n$, $v_n \rightarrow v_1$, and $v_n \rightarrow v_2$. See an example in Figure 1. It is easy to check that D_n has no vertex of in-degree zero and exactly two Sullivan-1 vertices v_1, v_n .

Also, let D'_n be a round local tournament on $n \geq 5$ vertices with the round labeling v_1, v_2, \dots, v_n and the arc set $v_i \rightarrow v_{i+1}$ for all $i = 1, 2, \dots, n - 1$, $v_n \rightarrow v_1$, $v_n \rightarrow v_2$, and $v_1 \rightarrow v_i$ for all $i = 3, 4, \dots, n - 1$. See an example in Figure 1. It is easy to check that D'_n has no vertex of in-degree zero and exactly two Sullivan-2 vertices v_1, v_n .

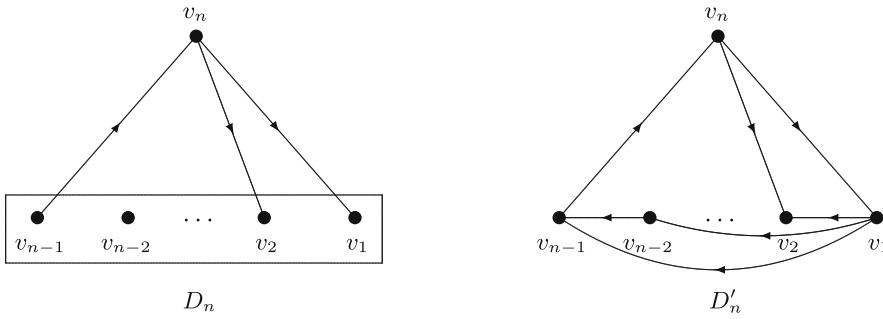


Figure 1: D_n has exactly two Sullivan-1 vertices v_1, v_n , where the box denotes a tournament with the vertex set $\{v_1, v_2, \dots, v_{n-1}\}$ and the arc set $\{v_i \rightarrow v_j : 1 \leq i < j \leq n - 1\}$. D'_n has exactly two Sullivan-2 vertices v_1, v_n .

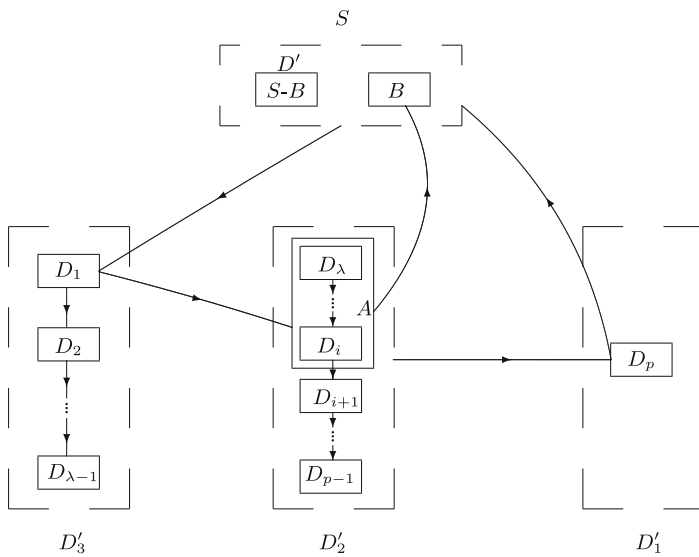


Figure 2: The structure of a nonround decomposable local tournament D in Section 4.

4 In a nonround decomposable local tournament

In this section, D is always a nonround decomposable local tournament, which is not a tournament. According to Theorems 2.4 and 2.6, we assume that S is chosen with minimum cardinality among all minimal separating sets of D satisfying that $D - S$ is not a tournament, D_1, D_2, \dots, D_p is the acyclic ordering of the strong components of $D - S$, $D_{p+1}, D_{p+2}, \dots, D_{p+q}$ is the acyclic ordering of the strong components of $D \langle S \rangle$, D'_1, D'_2, D'_3 is the semicomplete decomposition of $D - S$, and D_λ is the initial component of D'_2 . Clearly, in a local tournament D , the subdigraphs S, D'_1, D'_2 , and D'_3 are all tournaments. See Figure 2. Let

$$\begin{aligned}
 A &= N_{D'_2}^+(D_1) = V(D_\lambda) \cup V(D_{\lambda+1}) \cup \dots \cup V(D_i), & B &= N_S^+(A), \\
 X &= N_{D-A}^+(A) = B \cup (V(D'_2) - A) \cup V(D_p), \\
 D' &= D \langle S - B \rangle, \text{ when } |X| \leq |S|.
 \end{aligned}$$

Lemma 4.1. For any strong component D_j of D'_3 , $d_{D-D_j}^+(D_j) \geq |S|$.

Proof. Let $D_j \subseteq D'_3$ be arbitrary. Since $N_{D-D_j}^+(D_j)$ is a separating set of D and $D - N_{D-D_j}^+(D_j)$ is not a tournament, we have $d_{D-D_j}^+(D_j) \geq |S|$ by the choice of S . □

Lemma 4.2. *If $|X| < |S|$, then $S - B \Rightarrow A$ and $S - B \Rightarrow D'_3$.*

Proof. Note that X is also a separating set of D . By the choice of S , combining with $|X| < |S|$, we see that $D - X = D \langle V(D'_3) \cup A \cup (S - B) \rangle$ is a tournament. So, any vertex of $S - B$ is adjacent to the vertices of A and D'_3 . Since $N_{S-B}^+(A) = \emptyset$, we have $S - B \Rightarrow A$. Also, an arc from D'_3 to S implies that D'_3 and D'_1 are adjacent, which contradicts Theorem 2.4 (c). So $S - B \Rightarrow D'_3$. \square

Lemma 4.3. *If $|X| < |S|$, then for any $v \in V(D') = S - B$,*

- (a) $X \subseteq N_{D-D'}^+(v) \cup N_{D-D'}^{++}(v)$;
- (b) $N_{D-D'}^-(v) \subseteq N_{D-D'}^{++}(v) \subseteq X$;
- (c) $d_{D-D'}^+(v) \geq d_{D-D'}^-(v) + 2$.

Proof.

- (a) Recall that $X = B \cup (V(D'_2) - A) \cup V(D_p)$. Since $|X| < |S|$, we have $|B| \leq |X| - 1 \leq |S| - 2$, and hence, $|S - B| \geq 2$. By Lemma 4.2, for any $v \in V(D') = S - B$, we have $v \Rightarrow A \Rightarrow D'_2 - A$ and $v \Rightarrow A \Rightarrow D_p$. Since $B = N_S^+(A)$, for any $y \in B$, there exists a vertex $x \in A$ such that $x \rightarrow y$ and hence $v \rightarrow x \rightarrow y$. So, any vertex of X either belongs to $N_{D-D'}^+(v)$ or belongs to $N_{D-D'}^{++}(v)$, i.e., $X \subseteq N_{D-D'}^+(v) \cup N_{D-D'}^{++}(v)$.
- (b) Recall that $D' = D \langle S - B \rangle$. By the definition of A , B , and X , we have $V(D - D') - X = A \cup V(D'_3)$. For any $v \in V(D')$, by Lemma 4.2, we have $v \Rightarrow A$ and $v \Rightarrow D'_3$. Then,

$$N_{D-D'}^+(v) \supseteq A \cup V(D'_3) = V(D - D') - X$$

and hence, $N_{D-D'}^-(v) \subseteq X$ and $N_{D-D'}^{++}(v) \subseteq X$. By (a), $N_{D-D'}^-(v) \subseteq X \subseteq N_{D-D'}^+(v) \cup N_{D-D'}^{++}(v)$. Since $N_{D-D'}^+(v) \cap N_{D-D'}^-(v) = \emptyset$, we have $N_{D-D'}^-(v) \subseteq N_{D-D'}^{++}(v) \subseteq X$.

- (c) By Lemma 4.1, we have $d_{D-D_1}^+(D_1) \geq |S|$. Then, $|A \cup V(D'_3)| = |D_1| + d_{D-D_1}^+(D_1) \geq 1 + |S| \geq |X| + 2$ since $|X| < |S|$. Note that $A \cup V(D'_3) \subseteq N_{D-D'}^+(v)$ and $N_{D-D'}^-(v) \subseteq X$. Then,

$$|X| + 2 \leq |A \cup V(D'_3)| \leq d_{D-D'}^+(v), d_{D-D'}^-(v) \leq |X|.$$

So, $d_{D-D'}^+(v) \geq d_{D-D'}^-(v) + 2$. \square

Lemma 4.4. *If $|X| < |S|$, then a Sullivan- i vertex of D' is a Sullivan- i vertex of D for $i \in \{1, 2\}$.*

Proof. By Lemma 4.3 (c), for any $v \in V(D')$, $d_{D-D'}^+(v) \geq d_{D-D'}^-(v) + 2$. Combining with Lemma 4.3 (b), we have $d_{D-D'}^{++}(v) + d_{D-D'}^+(v) \geq 2d_{D-D'}^-(v) + 2$. By Corollary 1.5, the tournament D' always has a Sullivan-1 vertex and a Sullivan-2 vertex, say v' and v'' , respectively. Then, v' is a Sullivan-1 vertex of D and v'' is a Sullivan-2 vertex of D . \square

Lemma 4.5. *If $|X| \geq |S|$, then a Sullivan- i vertex of D_1 is a Sullivan- i vertex of D for $i \in \{1, 2\}$.*

Proof. Let v be a Sullivan- i vertex of D_1 for $i \in \{1, 2\}$. We will prove that v is a Sullivan- i vertex of D for $i \in \{1, 2\}$.

- Note that there is no arc from D_1 to S . Otherwise D_1 and D'_1 are adjacent which contradicts Theorem 2.4
- (c). So, $N_{D-D_1}^+(v) = N_{D-S-D_1}^+(v)$. Combining with Theorem 2.3 (a), we have

$$N_{D-D_1}^+(v) = N_{D-S-D_1}^+(v) = N_{D-S-D_1}^+(D_1) = N_{D-D_1}^+(D_1).$$

Similarly, we have $N_{D-D_1}^-(v) = N_{D-D_1}^-(D_1)$. By Lemma 4.1,

$$d_{D-D_1}^+(v) = d_{D-D_1}^+(D_1) \geq |S|.$$

By the structure of D described in Theorems 2.6 and 2.7, we have $N_{D-D_1}^-(D_1) \subseteq S$ and $X \subseteq N_{D-D_1}^{++}(D_1) = N_{D-D_1}^{++}(v)$. Now

$$d_{D-D_1}^{++}(v) \geq |X| \geq |S|, d_{D-D_1}^-(v) \leq |S|.$$

So, $d_{D-D_1}^{++}(v) \geq |S| \geq d_{D-D_1}^-(v)$ and $d_{D-D_1}^{++}(v) + d_{D-D_1}^+(v) \geq 2|S| \geq 2d_{D-D_1}^-(v)$. Since v is also a Sullivan- i vertex of D_1 , we see that v is a Sullivan- i vertex of D . \square

Now, we consider the existence of a Sullivan-1 vertex in D and the number of Sullivan-1 vertices of D . In fact, the existence can be directly obtained from the following two results, which were proved by Wang, Yang, and Wang [18], Li and Sheng [12], respectively.

Lemma 4.6. [18] *Let D be a nonround decomposable locally semicomplete digraph. Then, D has a king.*

Proposition 4.7. [12] *Let D be an oriented graph. A king of D is a Sullivan-1 vertex.*

Corollary 4.8. *Let D be a nonround decomposable local tournament, which is not a tournament. Then, D has a Sullivan-1 vertex.*

We consider primarily the number of Sullivan-1 vertices.

Theorem 4.9. *Let D be a nonround decomposable local tournament, which is not a tournament. Then, D has at least two Sullivan-1 vertices.*

Proof. Recall that $A = N_{D_2}^+(D_1) = V(D_\lambda) \cup V(D_{\lambda+1}) \cup \dots \cup V(D_i)$, $B = N_S^+(A)$, $X = N_{D-A}^+(A) = B \cup (V(D_2') - A) \cup V(D_p)$. The structure of D is illustrated in Figure 2. \square

For the case when $|X| \geq |S|$, let v be a Sullivan-1 vertex of D_1 . By Lemma 4.5, v is a Sullivan-1 vertex of D . By the proof of Lemma 4.6 (see reference [18]), there exists a king either belonging to D_2' or belonging to S in D , say v' . By Proposition 4.7, v' is a Sullivan-1 vertex of D . Clearly, $v' \neq v$. Then, v' is another Sullivan-1 vertex of D .

For the case when $|X| < |S|$, note that $|B| \leq |S| - 2$. Recall that $D' = D \langle S - B \rangle$. Let v be a Sullivan-1 vertex of D' . By Lemma 4.4, v is Sullivan-1 vertex of D . Next, we will find another Sullivan-1 vertex of D .

If $A = D_2'$, we will show that a king of D_p , say v' , is another Sullivan-1 vertex of D . By Theorem 2.7 (a), we have $D_p \Rightarrow S$. Combining with Lemma 4.2, $D_p \Rightarrow D' \Rightarrow A = D_2'$ and $D_p \Rightarrow D' \Rightarrow D_3'$. Then, v' is a king of D . Clearly, $v' \neq v$. By Proposition 4.7, v' is another Sullivan-1 vertex of D .

Now, $A \not\subseteq D_2'$. Let $D'' = D' - v$ and u be a Sullivan-1 vertex of D'' . Assume $u \in V(D_j)$, where D_j is a strong component of S . We consider the following two cases:

Case 1. There exists no arc between u and D_{i+1} .

We will prove that u is another Sullivan-1 vertex of D .

We claim that $N_{D-D'}^-(u) \not\subseteq N_{D-D'}^{++}(u)$. By Lemma 4.3 (b), we have $N_{D-D'}^-(u) \subseteq N_{D-D'}^{++}(u)$. We only need to prove that $N_{D-D'}^-(u) \neq N_{D-D'}^{++}(u)$. By Lemma 4.2, $u \Rightarrow A \Rightarrow D_{i+1}$. Combining with the fact that there exists no arc between u and D_{i+1} , we have $D_{i+1} \subseteq N_{D-D'}^{++}(u)$ and $D_{i+1} \subseteq N_{D-D'}^-(u)$. Then, $N_{D-D'}^-(u) \neq N_{D-D'}^{++}(u)$.

Now $d_{D-D''}^{++}(u) \geq d_{D-D'}^{++}(u) \geq d_{D-D'}^-(u) + 1 \geq d_{D-D''}^-(u)$ since $N_{D-D'}^-(u) \not\subseteq N_{D-D'}^{++}(u)$. Clearly, $u \neq v$. Since u is also a Sullivan-1 vertex of D'' , we see that u is another Sullivan-1 vertex of D .

Case 2. There exists at least one arc between u and D_{i+1} .

Let g be a king of D_{i+1} . We will show that g is another Sullivan-1 vertex.

We claim that $D_{i+1} \Rightarrow D_j$. Since A is a separating set, we see that $D_{i+1}, \dots, D_p, D_{p+1}, \dots, D_{p+q}, D_1, \dots, D_{\lambda-1}$ is the acyclic ordering of the strong components of $D - A$. Since D_{i+1} and D_j are distinct strong components of $D - A$, by Theorem 2.3 (a), we only need to prove that there exists at least one arc from D_{i+1} to D_j . Now there exists at least one arc between u and D_{i+1} , and hence, there exists one arc between D_j and D_{i+1} . However, by Theorem 2.7 (b), an arc from D_j to D_{i+1} implies that $D_1 \Rightarrow D_{i+1}$ which contradicts the definition of A . Then, there exists no arc from D_j to D_{i+1} , and hence, there exists at least one arc from D_{i+1} to D_j .

Note that $g \rightarrow u$ since $g \in V(D_{i+1})$ and $u \in V(D_j)$. By Lemma 4.2, $u \Rightarrow A$ and $u \Rightarrow D'_3$. Hence, $g \rightarrow u \Rightarrow A$ and $g \rightarrow u \Rightarrow D'_3$. By the structure of D described in Theorems 2.6 and 2.7, we see that $g \Rightarrow D'_2 - A - D_{i+1}$ and $g \Rightarrow D_p \Rightarrow S$. So, g is a king of D . Clearly, $g \neq v$. By Proposition 4.7, g is another Sullivan-1 vertex of D .

Next, we consider the existence of a Sullivan-2 vertex and the number of Sullivan-2 vertices in D .

Lemma 4.10. *If $|X| = |S|$, then D has at least two Sullivan-2 vertices.*

Proof. Let v be a Sullivan-2 vertex of D_1 . By Lemma 4.5, v is a Sullivan-2 of D . To find another Sullivan-2 vertex of D , we consider the following two cases.

For the case when $|X| = |S|$ and $D - X$ is not a tournament, let X be a minimal separating set of D instead of S . By Lemmas 4.4 and 4.5, there exists a Sullivan-2 vertex in D , say u . We can check $u \in S - B$ or $u \in X$ due to the new separating set X . Then, $u \neq v$ since $v \in V(D_1)$. Thus, u is another Sullivan-2 vertex of D .

For the case when $|X| = |S|$ and $D - X$ is a tournament, note that $|B| \leq |X| - 1 = |S| - 1$ since $X = B \cup (V(D'_2) - A) \cup V(D_p)$. Recall that $D' = D \langle S - B \rangle$. Let u be a Sullivan-2 vertex of the tournament D' . We will show that u is another Sullivan-2 vertex of D .

We claim $D' \Rightarrow A$ and $D' \Rightarrow D'_3$. Since $D - X = D \langle V(D'_3) \cup A \cup V(D') \rangle$ is a tournament, we see that any vertex of D' is adjacent to the vertices of A and D'_3 . Since $N_{D'}^+(A) = \emptyset$, we have $D' \Rightarrow A$. Also, an arc from D'_3 to S implies that D'_3 and D'_1 are adjacent, which contradicts Theorem 2.4 (c). So, $D' \Rightarrow D'_3$.

By the definition of A , B , and X , we have $V(D - D') - X = A \cup V(D'_3)$. Combining with $u \Rightarrow A$ and $u \Rightarrow D'_3$, we see that

$$N_{D-D'}^+(u) \supseteq A \cup V(D'_3) = V(D - D') - X$$

and hence, $N_{D-D'}^-(u) \subseteq X$, $N_{D-D'}^{++}(u) \subseteq X$.

Also, any vertex of $X = B \cup (V(D'_2) - A) \cup V(D_p)$ either belongs to $N_{D-D'}^+(u)$ or belongs to $N_{D-D'}^{++}(u)$ since $u \Rightarrow A \Rightarrow D'_2 - A$, $u \Rightarrow A \Rightarrow D_p$ and for any $g \in B$, there exists a vertex $h \in A$ such that $h \rightarrow g$. So, $X \subseteq N_{D-D'}^+(u) \cup N_{D-D'}^{++}(u)$. Since $N_{D-D'}^+(u) \cap N_{D-D'}^-(u) = \emptyset$, we have $N_{D-D'}^-(u) \subseteq N_{D-D'}^{++}(u) \subseteq X$, and hence,

$$d_{D-D'}^-(u) \leq d_{D-D'}^{++}(u) \leq |X|.$$

By Lemma 4.1, $d_{D-D_1}^+(D_1) \geq |S|$. Then, $|A \cup V(D'_3)| \geq |D_1| + d_{D-D_1}^+(D_1) \geq 1 + |S|$. Combining with $A \cup V(D'_3) \subseteq N_{D-D'}^+(u)$, we have

$$d_{D-D'}^+(u) \geq |A \cup V(D'_3)| \geq 1 + |S| \geq 1 + |X|.$$

So, $d_{D-D'}^+(u) + d_{D-D'}^-(u) \geq 2d_{D-D'}^-(u) + 1$. Clearly, $u \neq v$ since $u \in S - B$ and $v \in V(D_1)$. The fact that u is a Sullivan-2 vertex of D' implies that u is another Sullivan-2 vertex of D . \square

Theorem 4.11. *Let D be a nonround decomposable local tournament, which is not a tournament. Then, D either has at least two Sullivan-2 vertices or has a Sullivan-2 vertex v satisfying $d^{++}(v) + d^+(v) \geq 2d^-(v) + 2$.*

Proof. Recall that $A = N_{D_2}^+(D_1) = V(D_\lambda) \cup V(D_{\lambda+1}) \cup \dots \cup V(D_i)$, $B = N_S^+(A)$, and $X = B \cup (V(D'_2) - A) \cup V(D_p)$. We consider the following cases:

Case 1. $|X| < |S|$.

Note that $|B| \leq |S| - 2$. Recall that $D' = D \langle S - B \rangle$. Let v be a Sullivan-2 vertex of D' . By Lemma 4.4, v is a Sullivan-2 vertex of D . Let $D'' = D' - v$ and u be a Sullivan-2 vertex of the tournament D'' . We will prove that u is another Sullivan-2 vertex of D .

By Lemma 4.3 (b) and (c), we see that $d_{D-D'}^{++}(u) + d_{D-D'}^-(u) \geq 2d_{D-D'}^-(u) + 2$, and hence, $d_{D-D'}^{++}(u) + d_{D-D'}^-(u) \geq 2d_{D-D'}^-(u)$. Since u is also a Sullivan-2 vertex of D'' , we have $d_{D''}^{++}(u) + d_{D''}^-(u) \geq 2d_{D''}^-(u)$. Clearly, $u \neq v$. Thus, $d_{D'}^{++}(u) + d_{D'}^-(u) \geq 2d_{D'}^-(u)$ and u is another Sullivan-2 vertex of D .

Case 2. $|X| = |S|$.

By Lemma 4.10, D has at least two Sullivan-2 vertices.

Case 3. $|X| > |S|$ and $|D_1| \geq 2$.

By Lemma 4.5, a Sullivan-2 vertex of D_1 is a Sullivan-2 vertex of D . Since D_1 is strong and $|D_1| \geq 2$, we see that D_1 has at least three vertices. By Corollary 1.7, D_1 has at least two Sullivan-2 vertices. These two vertices are Sullivan-2 vertices of D .

Case 4. $|X| > |S|$, $|D_1| = 1$ and $D_2 \subseteq D'_2$.

By Lemma 4.5, the only vertex v of D_1 is a Sullivan-2 vertex of D . Note that $A = N_{D-D_1}^+(D_1)$. By Lemma 4.1, $d_{D-D_1}^+(D_1) = |A| \geq |S|$. We consider the following two cases. In the first case, we can find two Sullivan-2 vertices in D . In the second case, we can find a vertex v satisfying $d^{++}(v) + d^+(v) \geq 2d^-(v) + 2$ in D .

If $|A| = |S|$, let A be a minimal separating set of D instead of S . By Lemmas 4.4 and 4.5, there exists a Sullivan-2 vertex in D , say u . We can check $u \in A$ or $u \in V(D_{i+1})$ due to the new separating set A . Then, $u \neq v$ since $v \in V(D_1)$. Thus, u is another Sullivan-2 vertex of D .

If $|A| > |S|$, we see that $N_{D-D_1}^+(v) = A$, $X \subseteq N_{D-D_1}^{++}(v)$ and $N_{D-D_1}^-(v) \subseteq S$. Then, $d_{D-D_1}^+(v) = |A| \geq |S| + 1$, $d_{D-D_1}^{++}(v) \geq |X| \geq |S| + 1$ and $d_{D-D_1}^-(v) \leq |S|$, and hence, $d_{D-D_1}^{++}(v) + d_{D-D_1}^+(v) \geq 2d_{D-D_1}^-(v) + 2$. So, v is the desired vertex.

Case 5. $|X| > |S|$, $|D_1| = 1$ and $D_2 \subseteq D'_3$.

By Lemma 4.5, the only vertex v of D_1 is a Sullivan-2 vertex of D . Let

$$\begin{aligned} A' &= N_{D_2}^+(D_2) = V(D_\lambda) \cup V(D_{\lambda+1}) \cup \cdots \cup V(D_{i'}) , & B' &= N_S^+(A') , \\ X' &= N_{D-A'}^+(A') = B' \cup (V(D'_2) - A') \cup V(D_p) . \end{aligned}$$

The structure of D is illustrated in Figure 3. By Lemma 4.1, we have

$$d_{D-D_2}^+(D_2) \geq |S| .$$

Let $C = N_{D-D_2}^+(D_2) = (D_3 \cup \cdots \cup D_{\lambda-1}) \cup A'$. Then, $|C| \geq |S|$.

If $|X'| < |S|$, let $D'' = D \langle S - B' \rangle$ and u be a Sullivan-2 vertex of D'' . Note that $u \neq v$ since $v \in V(D_1)$. Similarly to the proof of Lemma 4.4, we can show that u is another Sullivan-2 vertices of D .

If $|X'| = |S|$, similarly to the proof of Lemma 4.10, we can prove that D has two Sullivan-2 vertices.

If $|C| = |S|$, let C be a minimal separating set of D instead of S . By Lemmas 4.4 and 4.5, there exists a Sullivan-2 vertex in D , say u . We can check $u \in C$ or $u \in V(D_{i'+1})$ due to the new separating set C . Then, $u \neq v$ since $v \in V(D_1)$. Then, u is another Sullivan-2 vertex of D .

If $|C| > |S|$ and $|X'| > |S|$, let u be a Sullivan-2 vertex of D_2 . We will show that u is another Sullivan-2 vertex of D . By the structure of D described in Theorems 2.6 and 2.7, we have $N_{D-D_2}^-(u) \subseteq S \cup D_1$, $N_{D-D_2}^+(u) = C$, and $X' \subseteq N_{D-D_2}^{++}(u)$. Then, $d_{D-D_2}^-(u) \leq |S| + |D_1| = |S| + 1$, $d_{D-D_2}^+(u) = |C| \geq |S| + 1$, $d_{D-D_2}^{++}(u) \geq |X'| \geq |S| + 1$. So, $d_{D-D_2}^{++}(u) + d_{D-D_2}^+(u) \geq 2(|S| + 1) \geq 2d_{D-D_2}^-(u)$. Clearly, $u \neq v$. Since u is also a Sullivan-2 vertex of D_2 , we see that u is another Sullivan-2 vertex of D .

In any case, we find either two Sullivan-2 vertices or a vertex v satisfying $d^{++}(v) + d^+(v) \geq 2d^-(v) + 2$ in D . The proof of the theorem is complete. \square

Corollary 4.12.

- Every nonround decomposable local tournament which is not a tournament has a Sullivan- i vertex for $i \in \{1, 2\}$;
- Every nonround decomposable local tournament which is not a tournament has at least two Sullivan-1 vertices;

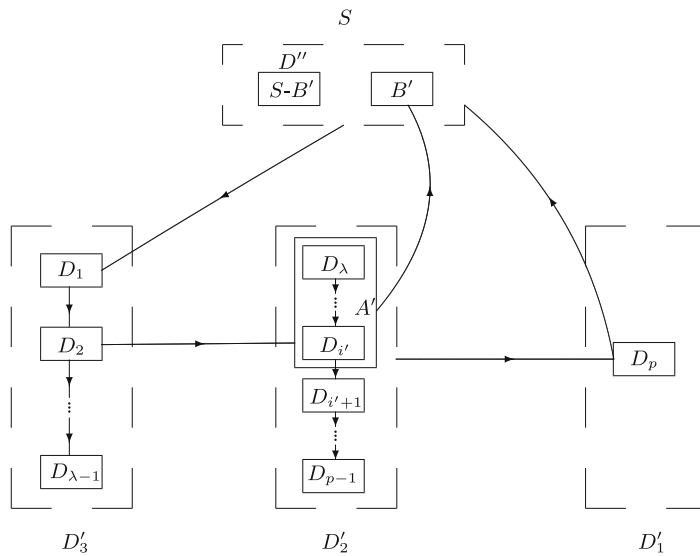


Figure 3: The structure of a nonround decomposable local tournament D in Case 5 of the proof of Theorem 4.11.

- (c) Every nonround decomposable local tournament which is not a tournament either has at least two Sullivan-2 vertices or has a Sullivan-2 vertex v satisfying $d^{++}(v) + d^+(v) \geq 2d^-(v) + 2$.

5 Conclusion

According to a full classification of local tournaments in Corollary 2.2, Corollary 1.5, Theorems 1.6, Corollary 3.6, and Corollary 4.12 imply the following theorem:

Theorem 5.1. *Let D be a local tournament. Then, the following holds for D .*

- (a) D has vertices u and v such that $d^{++}(u) \geq d^-(u)$ and $d^{++}(v) + d^+(v) \geq 2d^-(v)$;
- (b) If D has no vertex of in-degree zero, then D has at least two vertices u such that $d^{++}(u) \geq d^-(u)$;
- (c) If D has no vertex of in-degree zero, then D either has at least two vertices u satisfying $d^{++}(u) + d^+(u) \geq 2d^-(u)$ or has a vertex v satisfying $d^{++}(v) + d^+(v) \geq 2d^-(v) + 2$.

Finally, we list some related unsolved problems as follows.

Problem 5.2. Are there at least two vertices u such that $d^{++}(u) + d^+(u) \geq 2d^-(u)$ in a local tournament with no vertex of in-degree zero, in particular, in a nonround decomposable local tournament with no vertex of in-degree zero?

Problem 5.3. Characterize the local tournaments that contain exactly one vertex u such that $d^{++}(u) \geq d^-(u)$.

Problem 5.4. Characterize the round local tournaments that contain exactly k vertices u such that $d^{++}(u) + d^+(u) \geq 2d^-(u)$ for $1 \leq k \leq n$.

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References

- [1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, 2nd edn, Springer-Verlag, London, 2009.
- [2] N. Dean and B. J. Latka, *Squaring the tournament – an open problem*, *Congr. Numer.* **109** (1995), 73–80.
- [3] David C. Fisher, *Squaring a tournament: a proof of Dean’s conjecture*, *J. Graph Theory* **23** (1996), no. 1, 43–48, DOI: 10.1002/(sici)1097-0118(199609)23:13.0.co;2-k.
- [4] F. Havet and S. Thomassé, *Median orders of tournaments: a tool for the second neighbourhood problem and Sumner’s conjecture*, *J. Graph Theory* **35** (2000), no. 4, 244–256, DOI: 10.1002/1097-0118(200012)35:4<244::AID-JGT2>3.0.CO;2-H.
- [5] D. Fidler and R. Yuster, *Remarks on the second neighbourhood problem*, *J. Graph Theory* **55** (2007), no. 3, 208–220, DOI: 10.1002/jgt.20229.
- [6] S. Ghazal, *Seymour’s second neighborhood conjecture for tournaments missing a generalized star*, *J. Graph Theory* **71** (2012), no. 1, 89–94, DOI: 10.1002/jgt.20634.
- [7] Y. Kaneko and S. C. Locke, *The minimum degree approach for Paul Seymour’s distance 2 conjecture*, *Congr. Numer.* **148** (2001), 201–206.
- [8] Z. Cohn, A. Godbole, E. W. Harkness, and Y. Zhang, *The number of Seymour vertices in random tournaments and digraphs*, *Graphs Combin.* **32** (2016), no. 5, 1805–1816, DOI: 10.1007/s00373-015-1672-9.
- [9] G. Gutin and R. Li, *Seymour’s second neighbourhood conjecture for quasi-transitive oriented graphs*, arxiv.org/abs/1704.01389.
- [10] G. Chen, J. Shen, and R. Yuster, *Second neighbourhood via first neighbourhood in digraphs*, *Ann. Combin.* **7** (2003), no. 1, 15–20, DOI: 10.1007/s000260300001.
- [11] B. D. Sullivan, *A summary of results and problems related to the Caccetta-Häggkvist conjecture*, arxiv.org/abs/math/0605646.
- [12] R. Li and B. Sheng, *The second neighbourhood for quasi-transitive oriented graph*, *Acta Math. Sinica, English Series* **34** (2018), no. 9, 1391–1402, DOI: 10.1007/s10114-018-7287-3.
- [13] R. Li and B. Sheng, *The second neighbourhood for bipartite tournaments*, *Discuss. Math. Graph Theory* **39** (2019), no. 2, 555–565, DOI: 10.7151/dmgt.2018.
- [14] J. Bang-Jensen, *Locally semicomplete digraph: a generalization of tournaments*, *J. Graph Theory* **14** (1990), no. 3, 371–390, DOI: 10.1002/jgt.3190140310.
- [15] J. Bang-Jensen, Y. Guo, G. Guin, and L. Volkmann, *A classification of locally semicomplete digraphs*, *Discrete Math.* **167** (1997), 101–114, DOI: 10.1016/S0012-365X(96)00219-1.
- [16] R. Li and J. Liang, *Seymour’s second neighbourhood conjecture for local tournaments*, submitted.
- [17] Y. Guo and L. Volkmann, *Connectivity properties of locally semicomplete digraphs*, *J. Graph Theory* **18** (1994), no. 3, 269–280, DOI: 10.1002/jgt.3190180306.
- [18] R. Wang, A. Yang, and S. Wang, *Kings in locally semicomplete digraphs*, *J. Graph Theory* **63** (2010), no. 4, 279–287, DOI: 10.1002/jgt.20426.