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Research Article

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On the Malcev products of some classes of epigroups, I

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Abstract: A semigroup is called an epigroup if some power of each element lies in a subgroup. Under the universal of epigroups, the aim of the paper is devoted to presenting elements in the groupoid together with the multiplication of Malcev products generated by classes of completely simple semigroups, nil-semigroups and semilattices. The information about the set inclusion relations among them is also provided.

Keywords: epigroup, Malcev product, relational morphism

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1 Introduction

A semigroup S is called an epigroup if for each a in S there exists some power a^n of a such that a^n is a member of some subgroup of S. The identity of the subgroup is denoted by a^ω and this subgroup is denoted by G_{a^ω}. The least positive integer m such that $a^m \in G_{a^\omega}$ is called the index of a and is denoted by ind(a). It is well known that $aa^\omega = a^\omega a \in G_{a^\omega}$. We denote the group inverse of aa^ω in G_{a^ω} by $\bar{a}$ and call it the pseudo-inverse to a. Then, epigroups can be regarded as (2,1) – algebras with two operations, namely multiplication and taking pseudo-inverse. Thus, in speaking of a system of identities Σ of epigroups, we remind that they are written in the language of unary semigroups, i.e., for an identity $u = v$ in Σ, in the terms u and v both multiplication and pseudo-inversion may appear (as well as the derivative operation $a \mapsto a^\omega$ from them: $a^\omega = a\bar{a}$). We denote by $[\Sigma]$ the class of epigroups satisfying the identities in Σ and call the class $[\Sigma]$ an equational class of epigroups. It is clear that the equational class $[\Sigma]$ of epigroups is a pseudovariety of epigroups (a class $\mathbf{V}$ of epigroups is a pseudovariety of epigroups if $\mathbf{V}$ is closed under formation of finite direct products, subepigroups and homomorphic images). We also write $[\{u_\alpha = v_\alpha\}_{\alpha\in A} \Rightarrow u = v]$ for the class of epigroups whose membership is determined by satisfaction of a certain set $\{u_\alpha = v_\alpha\}_{\alpha\in A} \Rightarrow u = v$ of implications.

The class of all epigroups is denoted by $\mathbf{E}$, and it is not a variety (see [1, Subsection 2.3]). Remind that the class $[\Sigma]$ is also not obligatory to be a variety of epigroups because it may not be closed under the taking of infinite direct products (for example, from Proposition 4.1, the class $\mathbf{N}$ of nil-semigroups is an equational class of epigroups, while from [2, Example 2.14] it is not a variety). For which identity systems the class of epigroups satisfying a given system is a variety, the reader is referred to [2, Proposition 2.15] for details.

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For two classes **U** and **V** of epigroups,

$$\begin{aligned}\mathbf{U}\circ\mathbf{V} &= \{S \in \mathbf{E} \mid \text{There exists a congruence } \rho \text{ on } S \text{ such that } S/\rho \in \mathbf{V} \\ &\quad \text{and } e\rho \in \mathbf{U} \text{ for each idempotent } e \text{ in } S\}; \\ \mathbf{U}\textcircled{m}\mathbf{V} &= \{S \in \mathbf{E} \mid \text{There exists a surjective relational morphism } \tau\colon S \to T \text{ such that } T \in \mathbf{V} \\ &\quad \text{and } e\tau^{-1} \in \mathbf{U} \text{ for each idempotent } e \text{ in } T\}.\end{aligned}$$

Notation $\mathbf{U}\circ\mathbf{V}$ will denote the Malcev product of **U** and **V**, and in such a case, we also say that ρ is a **V**-congruence over semigroups from **U**. If **V** is a subclass of the class **N**, then $\mathbf{U}\circ\mathbf{V}$ is a class of all epigroups which are nil-extensions of semigroups from **U** by semigroups from **V**. Also, by $\mathbf{U}\circledast\mathbf{V}$ we denote a class of epigroups which are subdirect of semigroups from **U** and semigroups from **V**.

Under the universal of completely regular semigroups, the miscellany of Malcev products of varieties of completely regular semigroups was presented extensively in [3, Chapter 9]. For the Malcev products of pseudovarieties of finite semigroups and the related topics, the reader is referred to [4]. Under the universal of epigroups, this work focuses on the presentations of elements in the groupoid under the multiplication of Malcev products generated by equational classes of epigroups. It should be pointed out that some results on the Malcev products of well-known and even trivial varieties in the category of completely regular semigroups are no longer true in that of epigroups. For example, let **CR** and **T** denote the classes of completely regular semigroups and trivial semigroups, respectively. Under the universal of completely regular semigroups, $\mathbf{T}\circ\mathbf{CR} = \mathbf{CR}$, while under the universal of epigroups, $\mathbf{T}\circ\mathbf{CR} \neq \mathbf{CR}$ (the claim will be explained in Part II). Therefore, as one of the impetuses for the study, in this paper it is necessitous to reinspect the Malcev products of some familiar equational classes under the universal of epigroups. In Shevrin [1] (or [5]) one of the principal theorems is the structural characteristics of semilattices of archimedean epigroups (that is, semilattices of nil-extensions of completely simple semigroups), which is the Malcev product $(\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$. The main results of this paper, none the less, give the characterizations of the members of the groupoid under the multiplication of Malcev products generated by classes of completely simple semigroups, nil-semigroups and semilattices. We are to investigate elements or subsets of the groupoid Ω under the multiplication $\circ$ generated by the class **CS** of completely simple semigroups, the class **N** of nil-semigroups and the class **S** of semilattices. It is divided into two parts. Part II will appear in our forthcoming work. By performing multiplication $\circ$ of {**CS**, **N**, **S**} with words of length equal to 2, we exhibit some equational classes of epigroups and meanwhile subsemigroups of Ω. We see that, in this work, the generators {**N**, **S**} of Ω are both idempotents; while in Part II we will see that **CS** is not an idempotent of Ω. The descriptions of the Malcev products $\mathbf{CS}\circ\mathbf{CS}$, $\mathbf{S}\circ\mathbf{CS}$ and their derivatives will appear in Part II. Since many of them are no longer equational classes, their descriptions will be more complex and challenging. For the remaining Malcev products, the work mainly initiates the problem of characterizing some classes of epigroups by performing Malcev products of {**CS**, **N**, **S**} with words of length equal to 3. With limitation of semigroup techniques and methods, we also put more emphasis on semigroup presentations with examples to illustrate these statements for the general result.

From [6, Fact 2.1], under the universal of finite semigroups, for two pseudovarieties **U** and **V** of finite semigroups, the pseudovariety $\langle\mathbf{U}\circ\mathbf{V}\rangle$ of finite semigroups generated by the Malcev product of **U** and **V** is equal to $\mathbf{U}\textcircled{m}\mathbf{V}$. A similar statement holds for the varieties of completely regular semigroups. That is, for two varieties **U** and **V** of completely regular semigroups, the variety $\langle\mathbf{U}\circ\mathbf{V}\rangle$ of completely regular semigroups generated by the Malcev product of **U** and **V** is equal to $\mathbf{U}\textcircled{m}\mathbf{V}$ (for example, see [3, Theorem IX.6.12]). We are not going to present or show the analogous statement for epigroups in general, since we do not intend to search for generalizations which are from the sources of notions of varieties of completely regular semigroups, or pseudovarieties of finite semigroups. In this work, some frequently mentioned Malcev products $\mathbf{U}\circ\mathbf{V}$ for two equational classes **U** and **V** of epigroups can be characterized by identities, and in this case, it reveals, as an equational class of epigroups, $\mathbf{U}\circ\mathbf{V} = \mathbf{U}\textcircled{m}\mathbf{V}$. Quite often, in this paper, we present that for two equational classes **U** and **V** of epigroups, if $\mathbf{U}\circ\mathbf{V}$ is an equational class of epigroups, then $\mathbf{U}\circ\mathbf{V} = \mathbf{U}\textcircled{m}\mathbf{V}$; for example, we show that the equality $(\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S} = (\mathbf{CS}\textcircled{m}\mathbf{N})\textcircled{m}\mathbf{S}$ holds in epigroups.

In addition to this introduction, the paper is organized as follows. Section 2 contains preliminaries. In this section, some definitions, notations and auxiliary results are given. In Section 3, we recall some descriptions of Malcev products that will be used in the sequel. In Section 4, we calculate the derived groupoid with the operation of Malcev product generated by **S** and **N**, and in Section 5 the groupoid generated by **N** and **N**∘**CS**. In Section 6, we recall some characterizations of (**CS**∘**N**)∘**S** and also present some new material on this class of epigroups and its subclasses. In Section 7, we characterize the class **CS**∘(**N**∘**S**) in several ways. In Section 8, we consider relations among classes of epigroups presented in this paper with respect to the set inclusion. Many examples illustrate the corresponding problems that we consider. Meanwhile, for the varied generalizations of Malcev products, the results of products of **S**, **N** and **CS** under the operation of ⓜ within epigroups, together with their proofs, also appear in the paper.

2 Preliminaries

We summarize several basic notions and terminologies needed in the paper. The reader is referred to the books in [4,7–10] for background information on the semigroup theory. For further details of epigroups we refer to Shevrin [5,11] and his survey article [1].

The equality and the universal relations on any set are denoted by ∇ and Δ, respectively.

Let S be a groupoid (semigroup), and, for a subset $X \subseteq S$, let $\langle X\rangle$ be the subgroupoid (subsemigroup) of S generated by X. The element a in S is called to be an idempotent if $a^2 = a$ and the set of idempotents of S is denoted by E_S.

Let S be a semigroup. We write $S = S^1$, if S is a monoid; otherwise, S^1 stands for the semigroup with an identity adjoined. S is called archimedean if for any $a, b \in S$ there exists a natural number n such that $b \in S^1a^nS^1$. A semigroup S with zero 0 is called a nil-semigroup if for any $a \in S$, there exists $n \in \mathbb{Z}^+$ such that $a^n = 0$. An element a in S is regular, if there exists some $x \in S$ such that $a = axa$. The set of regular elements of S is denoted by $\operatorname{Reg} S$, and S is called regular if $\operatorname{Reg} S = S$. An element a in S is completely regular if there exists $x \in S$ such that $a = axa$, $x = xax$, $ax = xa$, and S is called completely regular if all its elements enjoy this property. Clearly, the element a is completely regular if and only if it is a group element of S. The set of all completely regular elements of a semigroup S (i.e., the group part of S) is denoted by $\operatorname{Gr} S$.

For an epigroup S, $a \in \operatorname{Gr} S$ (in this case $\operatorname{ind}(a) = 1$), if and only if $aa^\omega = a$. Formally, we sometimes write $a^{\omega+k}$ instead of a^ka^ω; for example, $a^{\omega+1} = a^\omega a = \overline{\overline{a}}$. Recall that for any $a \in S$ and any $n \geq \operatorname{ind}(a)$, $a^n \in \operatorname{Gr} S$ (see [5, Lemma 1]).

The congruence lattice of a semigroup S is denoted by $(C(S), \subseteq, \cap, \vee)$. From [8, Proposition 1.5.11], for $a, b \in S$ and $\rho, \sigma \in C(S)$,

$$a(\rho \vee \sigma)b \Leftrightarrow a\rho x_1\sigma x_2\rho x_3\sigma \ldots x_{2n-1}\sigma b, \tag{1}$$

where $n \in \mathbb{Z}^+$, $x_1, x_2, \ldots, x_{2n-1} \in S$. Let I be an ideal of S. Then, $\rho_I = \Delta \cup (I \times I)$ is a congruence on S known as a Rees congruence.

Let ρ be a binary relation on a semigroup S. The radical of ρ, in notation $\sqrt{\rho}$, which is due to Shevrin, is a relation on S defined by the following condition:

$$a\sqrt{\rho}\,b \Leftrightarrow \text{there exist } m, n \in \mathbb{Z}^+ \text{ such that } a^m \;\rho\; b^n (a, b \in S).$$

Let ρ^* denote the congruence on S generated by ρ. Explicitly, for $a, b \in S$,

$$a\,\rho^*\,b \Leftrightarrow a = b \text{ or } a = x_1u_1y_1,\, x_1v_1y_1 = x_2u_2y_2,\; \ldots, x_nv_ny_n = b \tag{2}$$

for some $x_i, y_i \in S^1$, $u_i, v_i \in S$ such that either $u_i(\rho \cup \Delta)v_i$ or $v_i(\rho \cup \Delta)u_i$, $i = 1, 2, \ldots, n$. For a subset K of S, define π_K by

$$a\,\pi_K\,b \Leftrightarrow (\forall\, x, y \in S^1)\ [xay \in K \Leftrightarrow xby \in K];$$

and π_K is called the principal congruence on S induced by K.

The next lemma was mentioned in [12, Lemma 1.1].

Lemma 2.1. *Let S be an epigroup and* $\rho \in C(S)$. *For* $a, b \in S$,
(i) *if* $a\,\rho\,b$, *then* $\bar{a}\rho\bar{b}$ *and* $a^{\omega}\,\rho\,b^{\omega}$;
(ii) *if* $a\rho \in E_{S/\rho}$, *then* $a\,\rho\,a^{\omega}$.

If ρ is an equivalence on a semigroup S and $a \in S$, then $a\rho$ denotes the ρ class containing a; if $A \subseteq S$, then $\rho|_A$ denotes the restriction of ρ to A. The kernel of ρ is the set

$$\ker\rho = \{a \in S \mid a\,\rho\,e \text{ for some } e \in E_S\};$$

and the trace of ρ is

$$\mathrm{tr}\rho = \rho|_{E_S}.$$

If S is an epigroup and $\rho \in C(S)$, then, from Lemma 2.1, it is easy to see that

$$\ker\rho = \{a \in S \mid a\,\rho\,a^{\omega}\}.$$

Lemma 2.2. *Let S be an epigroup and* $\rho \in C(S)$. *If* $e \in E_S$, *then* $e\rho$ *is a subepigroup of S.*

Proof. Since $e^2 = e$, we must have $e\rho\;\, e\rho \subseteq e\rho$ so that $e\rho$ is a subsemigroup of S. If $a \in e\rho$, then $a\,\rho\,e$ whence by Lemma 2.1 $\bar{a}\rho e$, that is, $\bar{a} \in e\rho$. Therefore, $e\rho$ is a subepigroup of S, as required. □

It is known that for $\rho, \lambda \in C(S)$ such that $\rho \subseteq \lambda$,

$$\lambda/\rho = \{(a\rho, b\rho) \in S/\rho \times S/\rho \mid (a, b) \in \lambda\}$$

is a congruence on S/ρ, and $S/\lambda \cong (S/\rho)/(\lambda/\rho)$ (see [8, Theorem 1.5.4]).

For an epigroup S, we define an equivalence relation $\mathcal{P}$ as follows:

$$a\,\mathcal{P}\,b \Leftrightarrow \bar{a} = \bar{b} \Leftrightarrow aa^{\omega} = bb^{\omega}\,(a, b \in S).$$

Notice that for any $a \in S$, $\bar{a} = \bar{\bar{\bar{a}}}$, and so $a\,\mathcal{P}\,\bar{\bar{a}}$, which implies that the restriction of $\mathcal{P}$ to the set $\mathrm{Gr}\,S$ is the equality relation on this set. Then, $\mathcal{P} = \Delta$ in S if and only if S is a completely regular semigroup.

The following lemma will be frequently used throughout the paper.

Lemma 2.3. ([13, Lemma 2.6]) *Let S be an epigroup. For any* $a, b \in S$,
(i) $(a^{\omega}ba^{\omega})^{\omega} = (a^{\omega}b)^{\omega}a^{\omega} = a^{\omega}(ba^{\omega})^{\omega}$;
(ii) $a\overline{ba} = \overline{ab}a$;
(iii) $a(ba)^{\omega} = (ab)^{\omega}a$.

On an epigroup S define the following relation: for $a, b \in S$,

$$a \leqslant b \Leftrightarrow (\exists\, e, f \in E_{S^1})\, a = eb = bf,$$

and the relation is called a natural partial order on any epigroup (see [7, Corollary 1.4.6]). It is easy to see that the restriction of the relation to E_S is given by

$$e \leqslant f \Leftrightarrow e = ef = fe.$$

For $a \in S$, $e \in E_S$, if $a \leqslant e$, that is, there exist $f, g \in E_{S^1}$ such that $a = fe = eg$, then $a^2 = fe{\cdot}eg = feg = ag = a$, that is, $a \in E_S$.

Lemma 2.4. [14, Lemma 2.3] *In an epigroup,* $\mathcal{D} \cap \leqslant\, = \Delta$.

The class of all epigroups is closed under the taking of homomorphic images (see [1, Observation 2.1]). A homomorphic image of a subepigroup of an epigroup is called an epidivisor. The relation "is an epidivisor of" is transitive for epigroups.

In this paper, by L_2 and R_2 we denote the two-element left zero and right zero semigroups, respectively. Y_2 is the two-element semilattice {0,1}. We will use several other semigroups given by the following presentations:

$$\begin{aligned}
A_2 &= \langle c, d \mid c^2 = 0, d^2 = d, cdc = c, dcd = d\rangle = \{0, c, d, cd, dc\};\\
B_2 &= \langle c, d \mid c^2 = d^2 = 0, cdc = c, dcd = d\rangle = \{0, c, d, cd, dc\};\\
V &= \langle e, f \mid e^2 = e, f^2 = f, fe = 0\rangle = \{0, e, f, ef\};\\
P &= \langle a, e \mid e^2 = e, ea = a, ae = 0\rangle = \{0, a, e\};\\
L_{3,2} &= \langle a, f \mid a^3 = a^2, fa = f^2 = f\rangle = \{a, f, af, a^2, a^2f\};\\
L_{3,1} &= \langle a, f \mid a^2f = a^2, fa = f^2 = f\rangle = \{a, f, af, a^2\};
\end{aligned}$$

and $\overleftarrow{P}$ $[R_{3,1}, R_{3,2}]$ is the dual semigroup of P $[L_{3,1}, L_{3,2}]$. We observe that among the subsemigroups of A_2 (similarly, of B_2), $\{0, c, cd\} \cong P$, $\{0, c, dc\} \cong \overleftarrow{P}$. Then, both A_2 and B_2 have epidivisors P and $\overleftarrow{P}$. We also see that the semigroups A_2, B_2, P and $\overleftarrow{P}$ have a common epidivisor Y_2.

The monogenic semigroup with index m and period r is denoted by $C_{m,r}$. Notice that $C_{1,r}$ is the cyclic group of order r and $C^1_{m,1}$ (obtained from $C_{m,1}$ by adjoining an identity 1) can be given by the following:

$$C^1_{m,1} = \langle a, e \mid e^2 = e, ae = ea = a, a^m = 0\rangle.$$

It is known that for $r_1, r_2 \geqslant 2$, if r_1 divides r_2, then the group C_{1,r_1} is a homomorphic image of the group C_{1,r_2}. In particular, for any $r \in \mathbb{Z}^+$, the group $C_{1,r}$ is a homomorphic image of the infinite cyclic group $C_{1,\infty}$. From [14, Lemma 2.6] for all positive integers $m \geqslant 2$, the semigroups $C^1_{m,1}$ have a common epidivisor $C^1_{2,1}$.

Let S, T be epigroups. A relational morphism τ: $S \to T$ is a function τ: $S \to 2^T$ satisfying the following conditions:

- $s\tau \neq \varnothing$, for all $s \in S$ (i.e., τ is fully defined);
- $s_1\tau s_2\tau \subseteq (s_1s_2)\tau$, for $s_1, s_2 \in S$;
- $t \in s\tau \Rightarrow \bar{t} \in \bar{s}\tau$, for $s \in S$, $t \in T$.

The graph of τ is

$$\#\tau = \{(s, t) \in S \times T \mid t \in s\tau\}.$$

Alternatively, a relational morphism τ: $S \to T$ is a relation from S to T such that the graph $\#\tau$ is a subepigroup of $S \times T$ projecting onto S. Now for $s \in S$, $t \in T$, if $t \in s\tau$, then $t^\omega \in s^\omega\tau$. It is easy to check that the composition of two relation morphisms is again a relational morphism.

Let τ: $S \to T$ be a relational morphism. Let p_S and p_T be the projections from $\#\tau$ to S and T, respectively. Then, $p_S^{-1}p_T$ is termed the canonical factorization of τ. The inverse of τ is the relation τ^{-1}: $T \to S$ defined by $t\tau^{-1} = \{s \in S \mid t \in s\tau\}$. If $t\tau^{-1} \neq \varnothing$ for every $t \in T$, then τ is called surjective.

Lemma 2.5. *Let S, T be epigroups. Let* τ: $S \to T$ *be a surjective relational morphism.*
(i) τ^{-1}: $T \to S$ *is also a surjective relation morphism.*

(ii) *If S' is a subepigroup of S, then $S'\tau = \cup\{s\tau \mid s \in S'\}$ is a subepigroup of T.*
(iii) *If T' is a subepigroup of T, then $T'\tau^{-1} = \cup\{t\tau^{-1} \mid t \in T'\}$ is a subepigroup of S.*

Proof. Here, we only present the proof of (i), and the statements of (ii) and (iii) can be proved similarly.
(i) Clearly τ^{-1} is fully defined, since τ is surjective. Now for $t_1, t_2 \in T$ and $s_1 \in t_1\tau^{-1}$, $s_2 \in t_2\tau^{-1}$, that is, $t_1 \in s_1\tau$, $t_2 \in s_2\tau$, we have $t_1t_2 \in s_1\tau s_2\tau \subseteq (s_1s_2)\tau$, that is, $s_1s_2 \in (t_1t_2)\tau^{-1}$. We also get $\overline{s_1} \in \overline{t_1}\tau^{-1}$. Therefore, $\tau^{-1}\colon T \to S$ is also a surjective relation morphism. □

The relational morphism τ is injective (pseudo-injective, respectively) if the implication

$$s\tau \cap t\tau \neq \varnothing \Rightarrow s = t\,(s\tau \cap t\tau \neq \varnothing \Rightarrow s\tau = t\tau)$$

holds.

Lemma 2.6. *Let $\tau\colon S \to T$ be a homomorphism between epigroups. Then, τ is an injective (hence a pseudo-injective) relational morphism from S to T.*

Proof. From [1, Observation 2.1], it is easy to check that τ is an injective relational morphism from S to T. □

By $\mathcal{J}, \mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{H}$ we denote Green's relations on a semigroup S. We denote the $\mathcal{L}$ class containing the element a in S by L_a, and in the same way we define J_a, R_a, H_a and D_a.

In the next lemma, statements (i)–(iii) are corollaries of [10, Theorem 6.45], and (iv) comes from [5, Lemma 5].

Lemma 2.7. *Let S be an epigroup.*
(i) *$\mathcal{J} = \mathcal{D}$ in S.*
(ii) *For $a \in S$ and $x \in S^1$, if $a\,\mathcal{D}\,xa$, then $a\,\mathcal{L}\,xa$.*
(iii) *For $a \in S$ and $y \in S^1$, if $a\,\mathcal{D}\,ay$, then $a\,\mathcal{R}\,ay$.*
(iv) *For $a, b \in S$, $(ab)^\omega \mathcal{D} (ba)^\omega$.*

Corollary 2.8. *A simple epigroup S is a completely simple semigroup.*

Proof. It is known that S is simple if and only if $\mathcal{J} = S \times S$. Then, from Lemma 2.7(i) for any $a \in S$, $a\,\mathcal{D}\,a^2$ and so from Lemma 2.7(ii and iii) we have $a\,\mathcal{H}\,a^2$, that is, $a \in \operatorname{Gr} S$ (see [8, Theorem 2.2.5]). Thus, S is a completely regular simple semigroup, and so from [3, Proposition III.1.1] S is a completely simple semigroup. □

From [3, Theorem VI.5.1], for a completely regular semigroup S, letting $\rho \in C(S)$ and $\mathcal{G}$ any of Green's relations, then for $a, b \in S$, $a(\rho \vee \mathcal{G})b \Leftrightarrow a\rho\,\mathcal{G}\,b\rho$. In the next lemma, we shall show that for some Green's relations the statement is also true for an epigroup.

Lemma 2.9. *Let S be an epigroup and $\rho \in C(S)$. For $\mathcal{G} \in \{\mathcal{D}, \mathcal{L}, \mathcal{R}\}$,*

$$a(\rho \vee \mathcal{G})b \Leftrightarrow a\rho\,\mathcal{G}\,b\rho\,(a, b \in S).$$

Proof necessity. Notice that Green's relations are preserved under homomorphisms (see [3, Lemma I.7.5]). Then,

$$\begin{aligned} a(\rho \vee \mathcal{G})b &\Leftrightarrow a\,\rho\,x_1\,\mathcal{G}\,x_2\,\rho\,x_3\,\mathcal{G} \ldots x_{2n-1}\,\mathcal{G}\,b \text{ for some } n \in \mathbb{Z}^+ \text{ and } x_1, x_2, \ldots, x_{2n-1} \in S \\ &\Rightarrow a\rho = x_1\,\rho\,\mathcal{G}\,x_2\rho = x_3\rho\,\mathcal{G} \ldots = x_{2n-1}\rho\,\mathcal{G}\,b\rho \\ &\Rightarrow a\rho\,\mathcal{G}\,b\rho. \end{aligned}$$

□

Proof sufficiency. For the converse, we consider several cases. For the case of $\mathcal{D}$, we have

$$
\begin{aligned}
a\rho\ \mathcal{D}\ b\rho &\Leftrightarrow a\ \rho\ xby,\ b\ \rho\ uav \text{ for some } x, y, u, v \in S^1, \text{ where } 1\rho \text{ is the (adjoined) identity of } S/\rho \\
&\Rightarrow a\ \rho\ xu\cdot a\cdot vy,\ b\ \rho\ ux\cdot b\cdot yv \\
&\Rightarrow a\rho\ (xu)^k a\,(vy)^k,\ b\ \rho\,(ux)^k b\,(yv)^k,\ k = \max\{\mathrm{ind}(xu), \mathrm{ind}(ux), \mathrm{ind}(vy), \mathrm{ind}(yv)\} \\
&\Rightarrow a\rho\ (xu)^\omega a\,(vy)^\omega,\ b\ \rho\,(ux)^\omega b\,(yv)^\omega \\
&\Rightarrow a\ \rho\ (xu)^\omega xby\,(vy)^\omega,\ b\ \rho\,(ux)^\omega b\,(yv)^\omega \\
&\Rightarrow a\ \rho\ x\ (ux)^\omega b\,(yv)^\omega y,\ b\ \rho\,(ux)^\omega b\,(yv)^\omega \text{ by Lemma 2.3 (iii)} \\
&\Rightarrow a\ \rho\ x\,(ux)^\omega b\,(yv)^\omega y\ \mathcal{D}\ (ux)^\omega b\,(yv)^\omega \rho\ b \text{ since } (ux)^\omega b\,(yv)^\omega = \overline{uxu}\cdot x\,(ux)^\omega b\,(yv)^\omega y\cdot y\overline{vy} \\
&\Rightarrow a\ \rho\mathcal{D}\ \rho b \\
&\Rightarrow a\,(\rho \vee \mathcal{D})\ b.
\end{aligned}
$$

For the case of $\mathcal{L}$, we have

$$
\begin{aligned}
a\ \rho\ \mathcal{L}\ b\rho &\Leftrightarrow a\ \rho\ xb,\ b\ \rho\ ya \text{ for some } x, y \in S^1 \\
&\Rightarrow a\ \rho\ xy\cdot a,\ b\rho\ yx\cdot b \\
&\Rightarrow a\ \rho\ (xy)^k a,\ b\ \rho\ (yx)^k b,\ k = \max\{\mathrm{ind}(xy), \mathrm{ind}(yx)\} \\
&\Rightarrow a\ \rho\ (xy)^\omega a,\ b\ \rho\ (yx)^\omega b \\
&\Rightarrow a\ \rho\,(xy)^\omega xb,\ b\ \rho\ (yx)^\omega b \\
&\Rightarrow a\ \rho\ x\,(yx)^\omega b,\ b\ \rho\,(yx)^\omega b \text{ by Lemma 2.3 (iii)} \\
&\Rightarrow a\ \rho\ x\,(yx)^\omega b\ \mathcal{L}\,(yx)^\omega b\ \rho\ b \text{ since } (yx)^\omega b = \overline{yxy}\cdot x\,(yx)^\omega b \\
&\Rightarrow a\ \rho\mathcal{L}\rho\ b \\
&\Rightarrow a\ (\rho \vee \mathcal{L})\ b.
\end{aligned}
$$

The proof of the case of $\mathcal{R}$ is dual to that of $\mathcal{L}$ and here we omit it. □

From the proof of Lemma 2.9, we also get the next result.

Corollary 2.10. *Let S be an epigroup and $\rho \in C(S)$. For $\mathcal{G} \in \{\mathcal{D}, \mathcal{L}, \mathcal{R}\}$, we have $\rho \vee \mathcal{G} = \rho\ \mathcal{G}\ \rho$.*

It was mentioned in [3, Lemma IV.3.5] that for a congruence ρ on a completely regular semigroup, ρ is a completely simple congruence if and only if $\rho \vee \mathcal{D} = \nabla$. We claim that the statement is also true for an epigroup.

Lemma 2.11. *Let S be an epigroup and $\rho \in C(S)$. Then, ρ is a completely simple congruence if and only if $\rho \vee \mathcal{D} = \nabla$.*

Proof. This follows from Lemma 2.9 and Corollary 2.8. □

Let S be an epigroup and ρ a congruence on S. It is not true that for $a, b \in S$, $a\,(\rho \vee \mathcal{H})\ b \Leftrightarrow a\rho\ \mathcal{H}\ b\rho$. The following example illustrates the problem. Let S_1 be the semigroup defined by the presentation (see [11, Example 4])

$$S_1 = \langle a, g \mid a^3 = a^2, g^5 = g, g^2 a = ag^4 = a, ga^2 = a^2, a^2 g^2 = aga\rangle.$$

Here $a^2 = a^\omega$, $g^4 = g^\omega$ and a^ω is a right zero, so that $a^\omega g$, $a^\omega g^2$, $a^\omega g^3$ are also right zeros. The semigroup S_1 consisting of 16 elements is a chain of two archimedean epigroups. Its eggbox picture is presented in Figure 1. The family of subsets $\{ga, ag^2\}$, $\{a, gag^2\}$, $\{ag, gag^3\}$, $\{ag^3, gag\}$ and all the singleton sets which are not contained in the four former subsets of S_1 is a partition π of S_1. Let ρ be the equivalence on S_1

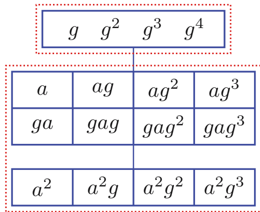

Figure 1: The eggbox picture of S_1 and its archimedean components.

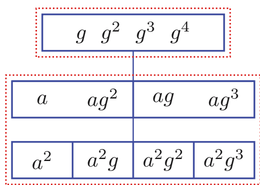

Figure 2: The eggbox picture of S_2 and its archimedean components.

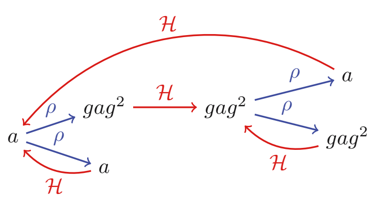

Figure 3: The sequence of transitions of a under the relation ρ on S_1.

induced by π. It is easy to check that ρ is a congruence on S_1 and $S_1/\rho \cong S_2$, where S_2 is the semigroup defined by the presentation (see [11, Example 5])

$$S_2 = \langle a, g \mid a^3 = a^2, g^5 = g, ga = ag^2, ag^4 = a, ga^2 = a^2 \rangle.$$

Its eggbox picture is presented in Figure 2. It is easy to check that, in S_1/ρ, $a\rho\ \mathcal{H}\ (ag^2)\,\rho$, while from (1) and Figure 3 $(a, ag^2) \notin \rho \vee \mathcal{H}$.

Lemma 2.12. *For pseudovarieties* $\mathbf{U}$, $\mathbf{V}$ *of* $\mathbf{E}$,
(i) $\mathbf{U} \circ \mathbf{V} \subseteq \mathbf{U} \circledm \mathbf{V}$;
(ii) $\mathbf{U} \circledm \mathbf{V}$ *is pseudovariety of* $\mathbf{E}$.

Proof.
(i) This is clear, since any congruence ρ on an epigroup can induce a natural homomorphism $\rho^{\#}$ (see [8, Theorem 1.5.2]), and from Lemma 2.6 a homomorphism is a relational morphism.
(ii) The proof is analogous to the one of [15, Lemma 6.2] or [3, Proposition IX.6.11]. □

Let $\mathbf{V}$ be a subclass of epigroups. If there exists the least congruence ρ on S such that $S/\rho \in \mathbf{V}$, then ρ is called the least $\mathbf{V}$-congruence and is denoted by $\rho_{\mathbf{V}}$. It is known that for any semigroup and a class $\mathbf{V}$ of semigroups containing the trivial semigroup and closed under the formation of subdirect products (hence for any variety $\mathbf{V}$ of epigroups), $\rho_{\mathbf{V}}$ exists (see [3, Lemma I.8.12]).

It is known that under universe of finite semigroups, for pseudovarieties $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ of finite semigroups, $\mathbf{U} \circledm (\mathbf{V} \circledm \mathbf{W}) \subseteq (\mathbf{U} \circledm \mathbf{V}) \circledm \mathbf{W}$ (for example, see [4, Exercise 2.3.20]). In the next lemma, we can obtain the analogous result under universe of epigroups.

Lemma 2.13. *Let* **U**, **V**, **W** *be equational classes of epigroups. Then,* $\mathbf{U}Ⓜ(\mathbf{V}Ⓜ\mathbf{W}) \subseteq (\mathbf{U}Ⓜ\mathbf{V})Ⓜ\mathbf{W}$.

Proof. Let $S \in \mathbf{U}Ⓜ(\mathbf{V}Ⓜ\mathbf{W})$. Then, there exist surjective relational morphisms $\tau\colon S \to A$ with $A \in \mathbf{V}Ⓜ\mathbf{W}$ and for each idempotent $a \in A$, $a\tau^{-1} \in \mathbf{U}$ and $\sigma\colon A \to W$ with $W \in \mathbf{W}$ and for each idempotent $w \in W$, $w\sigma^{-1} \in \mathbf{V}$. We consider the composition $\tau\sigma\colon S \to W$ of τ and σ, which is also a surjective relational morphism. For idempotent $w \in W$, set $w(\tau\sigma)^{-1} = K$, that is, $w\sigma^{-1}\tau^{-1} = K$. Then, $K\tau = w\sigma^{-1}$, and so from Lemma 2.5(ii and iii) K is a subepigroup of S and $K\tau$ is a subepigroup of A with $K\tau \in \mathbf{V}$. Now the relation $\tau|_K\colon K \to K\tau$ is a surjective relational morphism with $K\tau \in \mathbf{V}$. For any $e \in K\tau$, since $e \in A$, we have $e(\tau|_K)^{-1}$ is a subepigroup of $e\tau^{-1} \in \mathbf{U}$ (actually $e(\tau|_K)^{-1} = e\tau^{-1}$)). Thus, $K \in \mathbf{U}Ⓜ\mathbf{V}$, so that $S \in (\mathbf{U}Ⓜ\mathbf{V})Ⓜ\mathbf{W}$. □

From the proof of Lemma 2.13, we remark if **U**, **V**, **W** are classes of epigroups closed under the taking subepigroups, then $\mathbf{U}Ⓜ(\mathbf{V}Ⓜ\mathbf{W}) \subseteq (\mathbf{U}Ⓜ\mathbf{V})Ⓜ\mathbf{W}$.

3 Some results

In this section, we recall some descriptions of Malcev products that will be used; some new characterizations of them will also be presented below.

Proposition 3.1. *The following equalities hold for epigroups*:
(i) $\mathbf{CR} = [x^{\omega+1} = x]$;
(ii) $\mathbf{CS} = [x(yx)^{\omega} = x]$.

Proof.
(i) Clearly the equality holds since an epigroup is completely regular if and only if it is a union of its subgroups.
(ii) From [3, Proposition III.1.1], $\mathbf{CS} \subseteq [x(yx)^{\omega} = x]$. For the converse, substituting $y \to x^{\omega}$ in the given identity yields $x^{\omega+1} = x$, so that $[x(yx)^{\omega} = x] \subseteq \mathbf{CR}$. Again from [3, Proposition III.1.1], we have $[x(yx)^{\omega} = x] \subseteq \mathbf{CS}$. □

It is well known that a semigroup S is completely regular if and only if S is a semilattice of completely simple semigroups (for example, see [3, Theorem II.1.4]). Now we present the following result.

Proposition 3.2. *For epigroups*, $\mathbf{CR} = \mathbf{CS}\circ\mathbf{S} = \mathbf{CS}Ⓜ\mathbf{S}$.

Proof. As mentioned above $\mathbf{CR} = \mathbf{CS}\circ\mathbf{S}$, we remain to show that $\mathbf{CS}Ⓜ\mathbf{S} \subseteq \mathbf{CR}$. Let $S \in \mathbf{CS}Ⓜ\mathbf{S}$ be such that there is a surjective relational morphism $\tau\colon S \to Y$ with $Y \in \mathbf{S}$ and for each $\alpha \in Y$, $\alpha\tau^{-1}$ belongs to **CS**. For arbitrary $x \in S$, there exists $\alpha \in Y$ such that $\alpha \in x\tau$. Then, $x \in \alpha\tau^{-1} = \alpha^{\omega}\tau^{-1} \in \mathbf{CS}$ and so $x^{\omega+1} = x$. Consequently, we have $S \in \mathbf{CR}$. □

The next result comes from [16, Theorem 3.1].

Theorem 3.3. *The following conditions on an epigroup S are equivalent:*
(i) $S \in \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$;
(ii) $\mathcal{P}$ *is a congruence on* S;
(iii) *there are no semigroups* A_2, B_2. $L_{3,1}$ and $R_{3,1}$ *among the epidivisors of* S;
(iv) *S satisfies the identity* $(xy)^{\omega} = (x^{\omega}xyy^{\omega})^{\omega}$.

Corollary 3.4. *For epigroups,* $\mathbf{N}ⓜ(\mathbf{CS}ⓜ\mathbf{S}) = \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$.

Proof. From Proposition 3.2, we only need to show that $\mathbf{N}ⓜ(\mathbf{CS}\circ\mathbf{S}) \subseteq \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$. Let $S \in \mathbf{N}ⓜ(\mathbf{CS}\circ\mathbf{S})$ be such that there is a surjective relational morphism $\tau: S \to A$ with $A \in \mathbf{CS}\circ\mathbf{S}$ and for each idempotent $e \in A$, $e\tau^{-1} \in \mathbf{N}$. For arbitrary $x, y \in S$, there exist $a, b \in A$ such that $a \in x\tau$, $b \in y\tau$. Since $(ab)^\omega = (a^\omega abb^\omega)^\omega$ in A, we have $(xy)^\omega, (x^\omega xyy^\omega)^\omega \in (ab)^\omega\tau^{-1} \in \mathbf{N}$, so that $(xy)^\omega = (x^\omega xyy^\omega)^\omega$ since a nil-semigroup has only one idempotent (completely regular element) as its zero. Thus, from Theorem 3.3, $S \in \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$. □

Lemma 3.5. *For epigroups,* $\mathbf{T}\circ(\mathbf{CS}\circ\mathbf{S}) = \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S}) \cap [xx^\omega = x^\omega \Rightarrow x = x^\omega]$.

Proof. Let $S \in \mathbf{T}\circ(\mathbf{CS}\circ\mathbf{S})$. It is trivial that $S \in \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$. Let $\rho \in C(S)$ be such that $S/\rho \in \mathbf{CS}\circ\mathbf{S}$ and over $\mathbf{T}$. Now if $xx^\omega = x^\omega$, then $x \,\rho\, xx^\omega = x^\omega$. Since $x^\omega\rho$ is trivial, we have $x = x^\omega$.

For the opposite inclusion, set $S \in \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S}) \cap [xx^\omega = x^\omega \Rightarrow x = x^\omega]$. Then, from Theorem 3.3, $\mathcal{P}$ is a congruence on S and $S/\mathcal{P} \in \mathbf{CS}\circ\mathbf{S}$. Now for any $e \in E_S$, if $a \,\mathcal{P}\, e$, then $aa^\omega = e = a^\omega$, so that by hypothesis we have $a = e$. Thus, $e\mathcal{P} \in \mathbf{T}$ and so $S \in \mathbf{T}\circ(\mathbf{CS}\circ\mathbf{S})$. □

The class of nil-extensions of completely regular semigroups is denoted by $\mathbf{CR}\circ\mathbf{N}$. In the following proposition, we give some characterizations of nil-extensions of completely regular semigroups (see [14, Proposition 3.3]).

Proposition 3.6. *The following conditions on an epigroup S are equivalent:*
(i) $S \in (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$, *that is,* $S \in \mathbf{CR}\circ\mathbf{N}$;
(ii) *$Gr\,S$ is an ideal of S;*
(iii) *the semigroups P, $\overrightarrow{P}$ and $C^1_{2,1}$ are not epidivisors of S;*
(iv) *S satisfies the identity* $(xy^\omega z)^{\omega+1} = xy^\omega z$.

Corollary 3.7. *For epigroups,* $(\mathbf{CS}ⓜ\mathbf{S})ⓜ\mathbf{N} = (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$.

Proof. From Proposition 3.2, we only need to show that $(\mathbf{CS}\circ\mathbf{S})ⓜ\mathbf{N} \subseteq (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$. Let $S \in (\mathbf{CS}\circ\mathbf{S})ⓜ\mathbf{N}$ be such that there is a surjective relational morphism $\tau: S \to N$ with $N \in \mathbf{N}$ and for each idempotent $i \in N$, the semigroup $i\tau^{-1}$ belongs to $\mathbf{CS}\circ\mathbf{S}$. For arbitrary $x, y, z \in S$, there exist $a, b, c \in N$ such that $a \in x\tau$, $b \in y\tau$, $c \in z\tau$. Since the nil-semigroup N has the unique idempotent as its zero, we have $b^\omega = ab^\omega c$ in N. Then, $xy^\omega z \in b^\omega\tau^{-1} \in \mathbf{CS}\circ\mathbf{S}$, so that $(xy^\omega z)^{\omega+1} = xy^\omega z$. Hence, from Theorem 3.6, $S \in \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$. □

4 The semigroup generated by S and N with the operation of Malcev product

In this section, we shall calculate the derived groupoid with the operation of Malcev product generated by $\mathbf{S}$ and $\mathbf{N}$.

Proposition 4.1. *The following equalities hold for epigroups*:
(i) $\mathbf{S}ⓜ\mathbf{S} = \mathbf{S}\circ\mathbf{S} = \mathbf{S} = [x = x^\omega, xy = yx]$;
(ii) $\mathbf{N}ⓜ\mathbf{N} = \mathbf{N}\circ\mathbf{N} = \mathbf{N} = [x^\omega = x^\omega y = yx^\omega] = [x^\omega x = y^\omega y]$.

Proof.
(i) Clearly the equality $\mathbf{S} = [x = x^\omega, xy = yx]$ holds.

It suffices to show that $\mathbf{S}ⓜ\mathbf{S} \subseteq \mathbf{S}$. Let $S \in \mathbf{S}ⓜ\mathbf{S}$ be such that there is a surjective relational morphism $\tau: S \to Y$ with $Y \in \mathbf{S}$ and for each $\alpha \in Y$ the semigroup $\alpha\tau^{-1}$ belongs to $\mathbf{S}$. For arbitrary

$x, y \in S$, there exist $\alpha, \beta \in Y$ such that $\alpha \in x\tau$, $\beta \in y\tau$. On one hand, $x \in \alpha\tau^{-1} \in \mathbf{S}$, so that $x = x^{\omega}$, which means that S is a band (i.e., a semigroup in which each element is an idempotent). On the other hand, $xy, yx \in (\alpha\beta)\tau^{-1} = (\alpha\beta)^{\omega}\tau^{-1} \in \mathbf{S}$, which means idempotents xy, yx are commutative, so that $xyx = xyyx = yxxy = yxy$. Thus, $xy = xyxy = yxy{\cdot}y = yxy = yx{\cdot}yxy = yx{\cdot}xyx = yx$. Therefore, we get $S \in \mathbf{S}$.

(ii) Denote the five classes by **A**, **B**, **C**, **D**, **E** in order.

$\mathbf{B} \subseteq \mathbf{E}$. Let $S \in \mathbf{B}$ and $\rho \in C(S)$ be such that $S/\rho \in \mathbf{N}$ and for $e \in E_S$, $e\rho \in \mathbf{N}$. Then, $x^{\omega}\ \rho\ x^{\omega}\ x\ \rho\ y^{\omega}y\ \rho\ y^{\omega}$, since S/ρ has one unique idempotent (group element) as its zero. Now $x^{\omega}x, y^{\omega}y \in x^{\omega}\rho \in \mathbf{N}$, then $x^{\omega}x = y^{\omega}y$ since $x^{\omega}x, y^{\omega}y \in \operatorname{Gr} S$.

$\mathbf{E} \subseteq \mathbf{D}$. Let $S \in \mathbf{E}$. Substituting $y \to x^{\omega}$ in the given identity yields $x^{\omega}x = x^{\omega}$. Also, the given identity implies $x^{\omega} = y^{\omega}$. We obtain $x^{\omega} = x^{\omega}x = y^{\omega}y = x^{\omega}y$, $y^{\omega}y = yy^{\omega} = yx^{\omega}$, and so $x^{\omega} = x^{\omega}y = yx^{\omega}$.

$\mathbf{D} \subseteq \mathbf{C}$. Let $S \in \mathbf{D}$. For any $a, b \in S$, the given identity yields $a^{\omega} = a^{\omega}b^{\omega} = b^{\omega}a^{\omega} = b^{\omega}$. Then, S has only one idempotent, playing the role of zero. Also, substituting $y \to x$ in the given identity yields $x^{\omega}x = x^{\omega}$. Thus, $a^{n} = a^{n}a^{\omega} = (aa^{\omega})^{n} = (a^{\omega})^{n} = a^{\omega}$ for $n = \operatorname{ind}(a)$, so that $S \in \mathbf{N}$.

$\mathbf{C} \subseteq \mathbf{B}$. It is clear.

Since $\mathbf{A} \supseteq \mathbf{C}$ and $\mathbf{C} = \mathbf{D}$ as shown above, we remain to show

$\mathbf{A} \subseteq \mathbf{D}$. Let $S \in \mathbf{A}$ and $\tau\colon S \to N$ be a surjective relational morphism with $N \in \mathbf{N}$ and for each $\alpha \in N$, $\alpha\tau^{-1} \in \mathbf{N}$. For arbitrary $x, y \in S$, there exist $a, b \in N$ such that $a \in x\tau$, $b \in y\tau$. As shown above $\mathbf{C} = \mathbf{D}$, we have $a^{\omega} = a^{\omega}b = ba^{\omega}$. Then, $x^{\omega}, x^{\omega}y, yx^{\omega} \in a^{\omega}\tau^{-1} \in \mathbf{N}$, and, again from $\mathbf{C} = \mathbf{D}$, we have $x^{\omega} = x^{\omega}x^{\omega}y = yx^{\omega}x^{\omega}$, so that $S \in [x^{\omega} = x^{\omega}y = yx^{\omega}]$. ☐

The class of $\mathcal{D}$ trivial epigroups (i.e., epigroups whose $\mathcal{D}$ classes are singletons) is denoted by **J**, which is characterized in the next result as semilattices of nil-semigroups, Malcev products and in terms of identities.

Proposition 4.2. *The following equalities hold for epigroups:*
(i) $\mathbf{N} \textcircled{m} \mathbf{S} = \mathbf{J} = \mathbf{N} \circ \mathbf{S} = [(xy)^{\omega+1} = (yx)^{\omega}] = [(xy)^{\omega}x = (xy)^{\omega} = y(xy)^{\omega}]$;
(ii) $\mathbf{J} \textcircled{m} \mathbf{J} = \mathbf{J} \circ \mathbf{J} = \mathbf{S} \circ \mathbf{J} = \mathbf{J} \circ \mathbf{S} = \mathbf{J}$.

Proof.
(i) We will show the first equality and for the other equalities, see [16, Proposition 4.13] and [14, Lemma 4.6].
From [16, Proposition 4.13] $\mathbf{J} = \mathbf{N} \circ \mathbf{S}$, then by Lemma 2.12 we have $\mathbf{J} \subseteq \mathbf{N} \textcircled{m} \mathbf{S}$. For the reverse inclusion, take $\mathbf{S} \in \mathbf{N} \textcircled{m} \mathbf{S}$. Then, there is a surjective relational morphism $\tau\colon S \to Y$ with $Y \in \mathbf{S}$ and for each $\alpha \in Y$, the semigroup $\alpha\tau^{-1}$ belongs to **N**. For arbitrary $x, y \in S$, there exist $\alpha, \beta \in Y$ such that $\alpha \in x\tau$, $\beta \in y\tau$. Then, $\alpha\beta \in x\tau y\tau \subseteq (xy)\tau$, $\alpha\beta = \beta\alpha \in y\tau x\tau \subseteq (yx)\tau$ and $\alpha\beta = (\alpha\beta)^{\omega} \in (xy)^{\omega}\tau$. Therefore, $xy, yx, (xy)^{\omega} \in (\alpha\beta)\tau^{-1} \in \mathbf{N}$, and so from Proposition 4.1(ii) $(xy)^{\omega+1} = (yx)^{\omega}$. Consequently, from the equality $\mathbf{J} = [(xy)^{\omega+1} = (yx)^{\omega}]$, we have $S \in \mathbf{J}$.
(ii) It suffices to show that $\mathbf{J} \textcircled{m} \mathbf{J} \subseteq \mathbf{J}$. Let $S \in \mathbf{J} \textcircled{m} \mathbf{J}$ be such that there is a surjective relational morphism $\tau\colon S \to J$ with $J \in \mathbf{J}$ and for each $j \in E_J$ the semigroup $j\tau^{-1}$ belongs to **J**. For arbitrary $x, y \in S$, there exist $a, b \in J$ such that $a \in x\tau$, $b \in y\tau$. On one hand, from Lemma 2.3

$$(xy)^{\omega+1} = xy(xy)^{\omega} = x(yx)^{\omega}{\cdot}(yx)^{\omega}{\cdot}(yx)^{\omega}y,$$
$$(yx)^{\omega} = yx\overline{yx} = y\overline{xy}x = y\,\overline{(xy)^{2}}{\cdot}(xy)^{\omega+1}{\cdot}(xy)^{\omega}x;$$

so that $(xy)^{\omega+1}\mathcal{D}(yx)^{\omega}$. On the other hand, analogous to the proof of (i), we can show that $(xy)^{\omega+1}$, $x(yx)^{\omega}$ (notice that $x(yx)^{\omega} = (xy)^{\omega}x$), $(yx)^{\omega}y$ and $y\,\overline{(xy)^{2}}$ are all among $(ba)^{\omega}\tau^{-1}$. Thus, $(xy)^{\omega+1}$ and $(yx)^{\omega}$ are $\mathcal{D}$ related in $(ba)^{\omega}\tau^{-1}$, whereas $(ba)^{\omega}\tau^{-1} \in \mathbf{J}$, so that $(xy)^{\omega+1} = (yx)^{\omega}$. Therefore, by (i) we have $S \in \mathbf{J}$. ☐

Lemma 4.3. *For epigroups,* $\mathbf{S}\circ\mathbf{N}\subsetneq\mathbf{N}\circ\mathbf{S}$.

Proof. If $S\in\mathbf{S}\circ\mathbf{N}$, then from [14, Corollary] $\mathrm{Gr}\,S$ is an ideal of S and $\mathrm{Gr}\,S\in\mathbf{S}$. We show that in this case $S\in\mathbf{N}\circ\mathbf{S}$. Let $a, b\in S$ be such that $a\,\mathcal{D}\,b$. Then, there exist some $x, y, u, v\in S^1$ such that $a = xby$, $b = uav$. Thus, $a = (xu)^{\omega}x\cdot b\cdot y(vy)^{\omega}$, $b = (ux)^{\omega}u\cdot a\cdot v(yv)^{\omega}$ (see the proof of Lemma 2.9). If $x = y = 1$, then $a = b$. If either of x, y is not equal to 1, say, $x\neq 1$, then $a, b\in E_S$ and $a\,\mathcal{D}\,b$ in $\mathrm{Gr}\,S$, so that $a = b$ since $\mathrm{Gr}\,S$ is a semilattice. Therefore, we obtain the $\mathcal{D}$ is trivial on S, that is, $S\in\mathbf{N}\circ\mathbf{S}$.

For the proper containment, we only consider the semigroup P. It is observed that $P\in\mathbf{N}\circ\mathbf{S}$ while $P\notin\mathbf{S}\circ\mathbf{N}$. □

We remark that in [14, Proposition 4.8] the authors were unaware of the result of Lemma 4.3 at that time, and now we rewrite [14, Proposition 4.8] as follows.

Proposition 4.4. *The following conditions on an epigroup S are equivalent:*

(i) $S\in\mathbf{S}\circ\mathbf{N}$;
(ii) $S\in\mathbf{CR}\circ\mathbf{N}\cap\mathbf{J}$;
(iii) $S\in\mathbf{N}\circledast\mathbf{S}$;
(iv) the semigroups P, $\overleftarrow{P}$, $C_{2,1}^1$, L_2, R_2 and $\mathrm{C}_{1,\mathrm{p}}$ for any prime p are not epidivisors of S;
(v) *S satisfies the identities* $(xy)^{\omega+1} = (yx)^{\omega}$, $(xy^{\omega}z)^{\omega} = xy^{\omega}z$;
(vi) *S satisfies the identity* $(xy^{\omega}z)^{\omega} = zy^{\omega}x$;
(vii) *S satisfies the identities* $(xy^{\omega})^{\omega} = y^{\omega}x$, $(x^{\omega}y)^{\omega} = yx^{\omega}$.

Proof. The equivalences of (i)–(vi) follow from [14, Proposition 4.8] and Lemma 4.3.

(i)⇒(vii). If $S\in\mathbf{S}\circ\mathbf{N}$, then from [14, Corollary 3.4] $\mathrm{Gr}\,S$ is an ideal of S and $\mathrm{Gr}\,S = E_S\in\mathbf{S}$. Then, for any $a, b\in S$, we have ab^{ω}, $b^{\omega}a\in E_S$ and, from Lemma 2.7(iv), $(ab^{\omega})^{\omega}\mathcal{D}(b^{\omega}a)^{\omega} = b^{\omega}a$. Since from Lemma 4.3 $S\in\mathbf{J}$, we have $(ab^{\omega})^{\omega} = b^{\omega}a$. Similarly we have $(a^{\omega}b)^{\omega} = ba^{\omega}$.

(vii)⇒(iv). Since identities are inherited by epidivisors, it remains to show that none of the semigroups listed in (iv) satisfies the identities in (vii). It is clear that none of the semigroups L_2, R_2 and $C_{1,p}$ satisfies the identities. Also, we can see that the identities fail in $C_{2,1}^1$ if one sets $x = a$, $y = e$; the former identity fails in P if $x = a$, $y = e$; the latter identity fails in $\overrightarrow{P}$ if $x = e$, $y = a$. □

Corollary 4.5. *For epigroups,* $\mathbf{S}\,ⓜ\,\mathbf{N} = \mathbf{S}\circ\mathbf{N}$.

Proof. We only need to show that $\mathbf{S}\,ⓜ\,\mathbf{N}\subseteq\mathbf{S}\circ\mathbf{N}$. Let $S\in\mathbf{S}\,ⓜ\,\mathbf{N}$ be such that there is a surjective relational morphism $\tau\colon S\to N$ with $N\in\mathbf{N}$ and for each idempotent $a\in N$, the semigroup $a\tau^{-1}$ belongs to $\mathbf{S}$. For arbitrary $x, y\in S$, there exist $a, b\in N$ such that $a\in x\tau$, $b\in y\tau$. Since $b^{\omega} = ab^{\omega} = b^{\omega}a$ in N, we obtain y^{ω}, xy^{ω}, $y^{\omega}x\in b^{\omega}\tau^{-1}\in\mathbf{S}$, so that $(xy^{\omega})^{\omega} = xy^{\omega} = xy^{\omega}\cdot y^{\omega} = y^{\omega}x\cdot y^{\omega} = y^{\omega}\cdot y^{\omega}x = y^{\omega}x$. Similarly, we have $(x^{\omega}y)^{\omega} = yx^{\omega}$. Thus, from Proposition 4.4, $S\in\mathbf{S}\circ\mathbf{N}$. □

Corollary 4.6. *For epigroups,*

(i) $\mathbf{J} = \mathbf{N}\circ\mathbf{J} = \mathbf{S}\circ\mathbf{J} = (\mathbf{S}\circ\mathbf{N})\circ\mathbf{J}$;
(ii) $\mathbf{J} = \mathbf{J}\circ\mathbf{N} = \mathbf{J}\circ\mathbf{S} = \mathbf{J}\circ(\mathbf{S}\circ\mathbf{N})$;
(iii) $\mathbf{J} = \mathbf{N}\circ(\mathbf{S}\circ\mathbf{N}) = (\mathbf{S}\circ\mathbf{N})\circ(\mathbf{S}\circ\mathbf{N})$;
(iv) $\mathbf{S}\circ(\mathbf{S}\circ\mathbf{N}) = (\mathbf{S}\circ\mathbf{N})\circ\mathbf{N} = \mathbf{S}\circ\mathbf{N}$.

Proof. From Proposition 4.2, Lemma 4.3, as $\mathbf{J}\circ\mathbf{J} = \mathbf{J}$ and $\mathbf{S}\circ\mathbf{N}\subseteq\mathbf{J}$, the equalities in (i) and (ii) hold in epigroups.

(iii) From Proposition 4.2, Lemma 4.3 and the equalities in (i) or (ii), we have

$$\mathbf{J} = \mathbf{N}\circ\mathbf{S}\subseteq\mathbf{N}\circ(\mathbf{S}\circ\mathbf{N})\subseteq(\mathbf{S}\circ\mathbf{N})\circ(\mathbf{S}\circ\mathbf{N})\subseteq\mathbf{J}\circ\mathbf{J} = \mathbf{J},$$

so that the equalities in (iii) hold.

(iv) Clearly $\mathbf{S}\circ\mathbf{N} \subseteq \mathbf{S}\circ(\mathbf{S}\circ\mathbf{N})$, $\mathbf{S}\circ\mathbf{N} \subseteq (\mathbf{S}\circ\mathbf{N})\circ\mathbf{N}$.

Let $S \in \mathbf{S}\circ(\mathbf{S}\circ\mathbf{N})$. Then, there exists $\rho \in C(S)$ such that $S/\rho \in \mathbf{S}\circ\mathbf{N}$, $e\rho \in \mathbf{S}$ for any $e \in E_S$. Then, from Proposition 4.4 $(xy^{\omega})^{\omega}\ \rho\ y^{\omega}x$, and so $y^{\omega}x = (y^{\omega}x)^{\omega}$, since $(xy^{\omega})^{\omega}\rho \in \mathbf{S}$. Now from Lemma 2.7(iv) $(xy^{\omega})^{\omega}\mathcal{D}(y^{\omega}x)^{\omega} = y^{\omega}x$ and from (i) $S \in \mathbf{J}$ (which means $\mathcal{D} = \Delta$ in S), thus $(xy^{\omega})^{\omega} = y^{\omega}x$. Similarly, we have $(x^{\omega}y)^{\omega} = yx^{\omega}$. Therefore, $S \in \mathbf{S}\circ\mathbf{N}$.

Let $S \in (\mathbf{S}\circ\mathbf{N})\circ\mathbf{N}$. Then, there exists $\rho \in C(S)$ such that $S/\rho \in \mathbf{N}$, $e\rho \in \mathbf{S}\circ\mathbf{N}$ $(e \in E_S)$. Then, from Proposition 4.1 $(xy^{\omega})^{\omega}\,\rho\, y^{\omega}x\,\rho\, y^{\omega}$. Now $y^{\omega}, (xy^{\omega})^{\omega}, y^{\omega}x \in y^{\omega}\rho \in \mathbf{S}\circ\mathbf{N}$ and, from Proposition 4.4, we have

$$y^{\omega}x = y^{\omega}\cdot y^{\omega}x = (y^{\omega}xy^{\omega})^{\omega} = (xy^{\omega}\cdot y^{\omega})^{\omega} = (xy^{\omega})^{\omega}.$$

Similarly, we have $(x^{\omega}y)^{\omega} = yx^{\omega}$. Therefore, $S \in \mathbf{S}\circ\mathbf{N}$. □

Proposition 4.7. *The semigroup* $\langle \mathbf{S},\mathbf{N}\rangle$ *with multiplication* $\circ$ *is given by the Cayley table in* Figure 4, *and* $\langle \mathbf{S},\mathbf{N}\rangle \cong V$.

$\circ$	S	N	S ∘ N	J
S	S	S ∘ N	J	J
N	J	N	J	J
S ∘ N	J	S ∘ N	J	J
J	J	J	J	J

Figure 4: The Cayley table of $\langle \mathbf{S},\mathbf{N}\rangle$.

Proof. From Proposition 4.2, Corollary 4.6, we observe that the underlying set of $\langle \mathbf{S}, \mathbf{N}\rangle$ is equal to $\{\mathbf{S}, \mathbf{N}, \mathbf{S}\circ\mathbf{N}, \mathbf{J}\}$ and the semigroup has the Cayley table in Figure 4. It is easy to show that $\mathbf{S} \cap \mathbf{N} = \mathbf{T}$; and also from Lemma 4.3 elements $\mathbf{S}, \mathbf{N}, \mathbf{S}\circ\mathbf{N}$ and $\mathbf{J}$ are all distinct. Define φ: $\langle \mathbf{S}, \mathbf{N}\rangle \to V$ by $\mathbf{S}\varphi = e$, $\mathbf{N}\varphi = f$, $(\mathbf{S}\circ\mathbf{N})\varphi = ef$, $\mathbf{J}\varphi = 0$; it is easy to check that φ is an isomorphism. □

We add a result that will be used in the rest of the paper for closing the section.

Lemma 4.8. *Let S be an epigroup and $\rho \in C(S)$.*
(i) *ρ is over completely simple semigroups if and only if for any $e \in E_S$, $e\rho \subseteq D_e$.*
(ii) *$S/\rho \in \mathbf{N}\circ\mathbf{S} \Leftrightarrow \mathcal{D} \subseteq \rho$.*

Proof.
(i) If ρ is over completely simple semigroups, then for any $a \in e\rho$, $a\ \mathcal{D}^{e\rho}\ e$, so that $a\ \mathcal{D}^{S}\ e$, that is, $a \in D_e$. If $e\rho \subseteq D_e$, then for any $a \in e\rho$, from Lemma 2.1, a^{ω}, $a \in e\rho$, so that a^{ω}, $a \in D_e$, which yields $a \in \operatorname{Gr} S$ since $a^{\omega}\ \mathcal{D}\ a$. Thus, from Lemma 2.2, $e\rho$ is a completely regular subsemigroup of S. Now for any a, $b \in e\rho$, we have $ab \in e\rho$, so that $a\ \mathcal{D}\ ab\ \mathcal{D}\ b$. Then, from Lemma 2.7 $a\ \mathcal{R}\ ab\ \mathcal{L}\ b$ and so from [8, Proposition 2.4.2] $a\ \mathcal{R}^{e\rho}\ ab\ \mathcal{L}^{e\rho}\ b$, since $e\rho$ is a regular subsemigroup of S. Hence, $a\ \mathcal{D}^{e\rho}\ b$, so that from Corollary 2.8 $e\rho$ is a completely simple semigroup. Therefore, ρ is over completely simple semigroups.
(ii) If $S/\rho \in \mathbf{N}\circ\mathbf{S}$, then for any a, $b \in S$, we have

$$\begin{aligned} a\ \mathcal{D}\ b &\Leftrightarrow a = xby,\ b = uav, \text{ for some } x, y, u, v \in S^1 \\ &\Rightarrow a = xu\cdot a\cdot vy,\ b = ux\cdot b\cdot yv \\ &\Rightarrow a = (xu)^k a\,(vy)^k,\ b = (ux)^k b\,(yv)^k,\ k = \max\{\operatorname{ind}(xu), \operatorname{ind}(ux), \operatorname{ind}(vy), \operatorname{ind}(yv)\} \\ &\Rightarrow a = (xu)^{\omega} a\,(vy)^{\omega},\ b = (ux)^{\omega} b\,(yv)^{\omega} \\ &\Rightarrow a = (xu)^{\omega} xby\,(vy)^{\omega},\ b = (ux)^{\omega} b\,(yv)^{\omega} \\ &\Rightarrow a = x\,(ux)^{\omega} b\,(yv)^{\omega} y\ \rho\,(ux)^{\omega} b\,(yv)^{\omega} = b \text{ by Proposition 4.2} \\ &\Rightarrow a\ \rho\ b. \end{aligned}$$

Consequently, we obtain $\mathcal{D} \subseteq \rho$.

For the converse, let a, $b \in S$ be such that $a\rho\ \mathcal{D}^{S/\rho}\ b\rho$. Then, from Lemma 2.9 $a\,(\rho \vee \mathcal{D})b$. If $\mathcal{D} \subseteq \rho$, then $a\,\rho\,b$, so that the Green relation $\mathcal{D}$ is trivial on S/ρ, that is, $S/\rho \in \mathbf{N}\circ\mathbf{S}$ (notice that from Proposition 4.2 $\mathbf{N}\circ\mathbf{S} = \mathbf{J}$). □

5 The semigroup generated by N and N∘CS with the operation of Malcev products

Our goal in this section is to calculate the derived groupoid with the operation of Malcev products generated by **N** and **N∘CS**. We begin with Malcev products **CS∘N** and **N∘CS**.

A semigroup S either has no minimal ideals or possesses a unique minimal ideal denoted by $K(S)$, which is called the kernel of S. We remark that, from [5, Corollary of Proposition 1], the kernel of an epigroup is a completely simple semigroup. Let $e, f \in E_S$. Define

$$M(e, f) = \{g \in E_S | ge = g = fg\}.$$

Proposition 5.1. *The following conditions on an epigroup S are equivalent:*

(i) $S \in \mathbf{CS}\circ\mathbf{N}$;
(ii) *E_S is an antichain*;
(iii) *$S \in \mathbf{CR}\circ\mathbf{N}$ and $Gr\,S$ is the (completely simple) kernel of S*;
(iv) $\sqrt{\mathcal{D}} = \nabla$;
(v) *the semigroup Y_2 is not among the epidivisors of S*;
(vi) *$M(e,f)$ is a singleton for all $e, f \in E_S$*;
(vii) *S satisfies the identity $(x^\omega y^\omega x^\omega)^\omega = x^\omega$*;
(viii) *S satisfies the identity $(x^\omega y x^\omega)^\omega = x^\omega$*;
(ix) *S satisfies the identity $(x^\omega z y^\omega)^\omega = (x^\omega y^\omega)^\omega$.*

Proof. The equivalences (i)–(iii), (v) and (vii)–(viii) come from [5, Propositions 1, 3 and 3′, Observation 6] (or [14, Proposition 3.6]) and their corollaries.

(iv)⇒(i). If (iv) holds, then from [1, Theorem 3.16] (or see Theorem 6.1 below) $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{T}$, that is, $S \in \mathbf{CS}\circ\mathbf{N}$.

(ii)⇒(ix). If (ii) holds, then $(x^\omega y^\omega x^\omega)^\omega = x^\omega$, $(y^\omega x^\omega y^\omega)^\omega = y^\omega$, since $(x^\omega y^\omega x^\omega)^\omega \leqslant x^\omega$, $(y^\omega x^\omega y^\omega)^\omega \leqslant y^\omega$ and E_S is an antichain. Thus, from Lemma 2.3(i)

$$\begin{aligned}(x^\omega z y^\omega)^\omega\cdot(x^\omega y^\omega)^\omega &= (x^\omega z y^\omega)^\omega\cdot y^\omega\,(x^\omega y^\omega)^\omega = (x^\omega z y^\omega)^\omega y^\omega = (x^\omega z y^\omega)^\omega,\\ (x^\omega y^\omega)^\omega\cdot(x^\omega z y^\omega)^\omega &= (x^\omega y^\omega)^\omega x^\omega\cdot(x^\omega z y^\omega)^\omega = x^\omega\,(x^\omega z y^\omega)^\omega = (x^\omega z y^\omega)^\omega,\end{aligned}$$

that is, $(x^\omega z y^\omega)^\omega \leqslant (x^\omega y^\omega)^\omega$, so that $(x^\omega z y^\omega)^\omega = (x^\omega y^\omega)^\omega$.

(ix)⇒(vi). Let $g \in E_S$ be such that $ge = g = fg$. Then, the identity in (ix) yields $g = g^\omega = (fge)^\omega = (fe)^\omega$ and so $M(e,f) = \{(fe)^\omega\}$.

(vi)⇒(vii). It is easy to check that $(x^\omega y^\omega x^\omega)^\omega$, $x^\omega \in M\,(x^\omega, x^\omega)$. If (vi) holds, then $(x^\omega y^\omega x^\omega)^\omega = x^\omega$.

(vii)⇒(iv). For any a, $b \in S$, if (vii) holds, then from Lemma 2.7(iv) $a^\omega = (a^\omega b^\omega a^\omega)\mathcal{D}(b^\omega a^\omega b^\omega) = b^\omega$. Thus, $a^{\mathrm{ind}(a)}\mathcal{H}a^\omega\mathcal{D}b^\omega\mathcal{H}a^{\mathrm{ind}(a)}$, so that $a\,\sqrt{\mathcal{D}}\,b$. Therefore, $\sqrt{\mathcal{D}} = \nabla$, as required. □

Corollary 5.2. *For epigroups*, **CSⓜN** = **CS∘N**.

Proof. We only need to show that **CSⓜN** ⊆ **CS∘N**. Let $S \in$ **CSⓜN** be such that there is a surjective relational morphism $\tau\colon S \to N$ with $N \in \mathbf{N}$ and for each idempotent $i \in N$, the semigroup $i\tau^{-1}$ belongs to **CS**. For arbitrary

$x, y \in S$, there exist $a, b \in N$ such that $a \in x\tau$, $b \in y\tau$. Since $a^\omega = b^\omega$ in N, we have $y^\omega, x^\omega \in b^\omega\tau^{-1} \in \mathbf{CS} \subseteq \mathbf{CS}\circ\mathbf{N}$, so that from Proposition 5.1 $(x^\omega y^\omega x^\omega)^\omega = x^\omega$. Thus, again from Proposition 5.1, $S \in \mathbf{CS}\circ\mathbf{N}$. □

Lemma 5.3. *Let $S \in \mathbf{CS}\circ\mathbf{N}$. Then, $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ are congruences on S.*

Proof. As known, $\mathcal{D}$ is equal to the Rees Congruence $\rho_{\mathrm{Gr}\,S}$ (see Proposition 5.1), and of course a congruence on S.

By duality we only need to consider $\mathcal{R}$. Since $\mathcal{R}$ is left compatible, we remain to show that $\mathcal{R}$ is right compatible. Now let $a, b \in S$ be such that $a\,\mathcal{R}\,b$. Then, $a = b$ or $a, b \in \mathrm{Gr}\,S$, since $\mathcal{R} \subseteq \mathcal{D} = \rho_{\mathrm{Gr}\,S}$. If $a = b$, there is nothing to say; if $a, b \in \mathrm{Gr}\,S$, then $a^\omega \mathcal{R} b^\omega$ and $a^\omega, a^\omega ac, b^\omega, b^\omega bc \in \mathrm{Gr}\,S$ (since $\mathrm{Gr}\,S$ is an ideal of S), that is, $a^\omega ac\,\mathcal{D}\,a^\omega\,\mathcal{R}\,b^\omega\,\mathcal{D}\,b^\omega bc$, so that from Lemma 2.7(ii and iii) $ac = a^\omega ac\,\mathcal{R}\,a^\omega\,\mathcal{R}\,b^\omega\,\mathcal{R}\,b^\omega bc = bc$. Consequently, $ac\,\mathcal{R}\,bc$, establishing that $\mathcal{R}$ is right compatible. □

In a regular semigroup, $\mathcal{J}^\star(=\mathcal{D}^\star)$ is the least semilattice congruence (see [7, Theorem 1.4.17]). In the following result, we say about $\mathcal{D}^\star, \mathcal{P}^\star$ for an epigroup.

Proposition 5.4. *Let S be an epigroup.*
(i) $\mathcal{D}^\star = \rho_{\mathbf{N}\circ\mathbf{S}}$.
(ii) $\mathcal{P}^\star = \rho_{\mathbf{CR}}$.
(iii) $(\leqslant)^\star = \{(a, (ab)^\omega a)\}^\star = \rho_{\mathbf{CS}}$.
(iv) $(\leqslant|_{E_S})^\star = \rho_{\mathbf{CS}\circ\mathbf{N}}$.

Proof.
(i) As $\mathcal{D} \subseteq \mathcal{D}^\star$, from Lemma 4.8 $S/\mathcal{D}^\star \in \mathbf{S}\circ\mathbf{N}$, so that by the minimality of $\rho_{\mathbf{N}\circ\mathbf{S}}$, we have $\rho_{\mathbf{N}\circ\mathbf{S}} \subseteq \mathcal{D}^\star$. Also, $S/\rho_{\mathbf{N}\circ\mathbf{S}} \in \mathbf{N}\circ\mathbf{S}$, again from Lemma 4.8, $\mathcal{D} \subseteq \rho_{\mathbf{N}\circ\mathbf{S}}$ and so $\mathcal{D}^\star \subseteq \rho_{\mathbf{N}\circ\mathbf{S}}$.
(ii) Since $\overline{\overline{a}}\,\mathcal{P}\,a$, we have $\overline{\overline{a}}\,\mathcal{P}^\star a$, so that from Proposition 3.1 $S/\mathcal{P}^\star \in \mathbf{CR}$. Therefore, $\mathcal{P}^\star \subseteq \rho_{\mathbf{CR}}$.
Conversely, let $a\,\mathcal{P}^\star\,b$, $a, b \in S$ and ρ be a completely regular congruence. Then,

$$a = x_1u_1y_1, x_1v_1y_1 = x_2u_2y_2, \;\ldots, x_nv_ny_n = b,$$

for some $x_i, y_i \in S^1$, $u_i, v_i \in S$, $u_i\mathcal{P}v_i$, $i = 1, 2, \ldots, n$. Notice that, from Proposition 3.1, $u_i\,\rho\,\overline{\overline{u_i}} = \overline{\overline{v_i}}\,\rho\,v_i$, which means $u_i\,\rho\,v_i$. Then,

$$a = x_1u_1y_1\rho x_1v_1y_1\rho x_2u_2y_2\ldots\rho x_nv_ny_n = b,$$

and so $a\,\rho\,b$. Consequently, $\mathcal{P}^\star \subseteq \rho$. Thus, the minimality of $\mathcal{P}^\star$ is proved.
(iii) We denote $(\leqslant)^\star$ by λ and show first that $\lambda = \rho_{\mathbf{CS}}$. As $a^\omega a \leqslant a$, we have $aa^\omega\lambda a$ and so from Proposition 3.1 $S/\lambda \in \mathbf{CR}$. For two idempotents in S/λ, from Lemma 2.1, we take $e\lambda, f\lambda$ for $e, f \in E_S$. If $e\lambda \leqslant f\lambda$, then $ef\lambda fe\lambda e$, so that $fef\lambda e$, which means $(fef)^\omega\lambda e$. As $(fef)^\omega \leqslant f$, we have $(fef)^\omega\lambda f$. Consequently, $e\lambda f$, that is, $e\lambda = f\lambda$. Thus, every idempotent of S/λ is primitive, and so from [3, Theorem III.1.3] $S/\lambda \in \mathbf{CS}$.
Conversely, let $a\,\lambda\,b$, $a, b \in S$ and ρ be a completely simple congruence. We have

$$a = x_1u_1y_1, x_1v_1y_1 = x_2u_2y_2, \;\ldots, x_nv_ny_n = b,$$

for some $x_i, y_i \in S^1$, $u_i, v_i \in S$, $u_i \leqslant v_i$ or $v_i \leqslant u_i$, $i = 1, 2, \ldots, n$. It is easy to check that $u_i\rho \leqslant v_i\rho$, or $v_i\rho \leqslant u_i\rho$, so that $u_i\,\rho\,v_i$, since $S/\rho \in \mathbf{CS}$ and the natural partial order on S/ρ is trivial (see [3, Theorem II.4.2]). Now

$$a = x_1u_1y_1\rho x_1v_1y_1\rho x_2u_2y_2\ldots\rho x_nv_ny_n = b,$$

so that $a\,\rho\,b$. Consequently, $\lambda \subseteq \rho$ and the minimality of λ is proved.

Let σ be the congruence $\{(a, (ab)^{\omega}a) | a, b \in S\}^{\star}$. From Lemma 2.3(iii) $(ab)^{\omega}a = (ab)^{\omega}\cdot a = a\cdot(ba)^{\omega}$, then $(ab)^{\omega}a \leqslant a$, so that $\sigma \subseteq \rho_{\mathbf{CS}}$. As $a \, \sigma \, (ab)^{\omega}a$, from Proposition 3.1, we have $S/\sigma \in \mathbf{CS}$ and so $\rho_{\mathbf{CS}} \subseteq \sigma$.

(iv) We denote $(\leqslant|_{E_S})^{*}$ by λ. For any $a, b, c \in S$, since $(a^{\omega}b^{\omega}a^{\omega})^{\omega} \leqslant a^{\omega}$, we have $(a^{\omega}b^{\omega}a^{\omega})^{\omega} \, \lambda \, a^{\omega}$, and so by Proposition 5.1 $S/\lambda \in \mathbf{CS}{\circ}\mathbf{N}$.

Conversely, for $a, b \in S$, let $a \lambda b$ and ρ be a completely simple congruence on S. We have

$$a = x_1 e_1 y_1, x_1 f_1 y_1 = x_2 e_2 y_2, \ \ldots, x_n f_n y_n = b,$$

for some $x_i, y_i \in S^1$, $e_i, f_i \in E_S$, $e_i \leqslant f_i$ or $f_i \leqslant e_i$, $i = 1, 2, \ldots, n$. It is easy to check that $e_i\rho \leqslant f_i\rho$, or $f_i\rho \leqslant e_i\rho$, so that $e_i \rho f_i$, since $S/\rho \in \mathbf{CS}{\circ}\mathbf{N}$ and $E_{S/\rho}$ is an antichain (see Proposition 5.1). Now

$$a = x_1 e_1 y_1 \rho x_1 f_1 y_1 \rho x_2 e_2 y_2 \ldots \rho x_n f_n y_n = b,$$

so that $a \rho b$. Consequently, $\lambda \subseteq \rho$ and the minimality of λ is proved. □

Proposition 5.5. *The following conditions on an epigroup S are equivalent:*

(i) $S \in \mathbf{N}{\circ}\mathbf{CS}$;

(ii) $S \in \mathbf{CS} \circledast \mathbf{N}$;

(iii) $S \in \mathbf{N}{\circ}(\mathbf{CS}{\circ}\mathbf{S}) \cap \mathbf{CS}{\circ}\mathbf{N}$;

(iv) *the semigroups $L_{3,1}$, $R_{3,1}$ and Y_2 are not epidivisors of S*;

(v) *S satisfies the identity $(xy)^{\omega} = (xzy)^{\omega}$*;

(vi) *S satisfies the identity $x^{\omega} = (xyx)^{\omega}$.*

Proof. The equivalences of (ii), (iv) and (vi) were mentioned in [11, Theorem 9] or [1, Theorem 3.39].

(iii)⇔(iv). Notice that semigroups A_2 and B_2 have a common epidivisor Y_2. Then, from Proposition 5.1 and Theorem 3.3, the equivalence holds.

(i)⇒(ii). If (i) holds, then there exists $\rho \in C(S)$ such that $S/\rho \in \mathbf{CS}$ with ρ being over $\mathbf{N}$. From Proposition 5.4 $\mathcal{P} \subseteq \rho$ and from Theorem 3.3, $\mathcal{P} \in C(S)$. Now for any $a, b \in S$, since $\mathbf{CS} \subseteq \mathbf{CS}{\circ}\mathbf{N}$, from Proposition 5.1, we have $(a^{\omega}b^{\omega}a^{\omega})^{\omega}\rho a^{\omega}$, so that $(a^{\omega}b^{\omega}a^{\omega})^{\omega} = a^{\omega}$, since $a^{\omega}\rho \in \mathbf{N}$. Then, again from Proposition 5.1, we have $S \in \mathbf{CS}{\circ}\mathbf{N}$. Therefore, $S \in \mathbf{CS}{\circ}\mathbf{N} \cap \mathbf{N}{\circ}\mathbf{CS}$ and so from [16, Proposition 4.1] $S \in \mathbf{CS} \circledast \mathbf{N}$ (we remark here that $\mathcal{P} \cap \rho_{\mathrm{Gr}\,S} = \Delta$).

(ii)⇒(iii). This is clear from [16, Proposition 4.1].

(iii)⇒(i). From Theorem 3.3, the relation $\mathcal{P}$ is a congruence on S and $S/\mathcal{P} \in \mathbf{CR}$ with $\mathcal{P}$ being over $\mathbf{N}$. Whereas also by hypothesis $S \in \mathbf{CS}{\circ}\mathbf{N}$, $S/\mathcal{P} \in \mathbf{CS}{\circ}\mathbf{N}$, so that $S/\mathcal{P} \in \mathbf{CR} \cap \mathbf{CS}{\circ}\mathbf{N} = \mathbf{CS}$. Hence, $S \in \mathbf{N}{\circ}\mathbf{CS}$, as required.

(iii)⇒(v). On one hand, by hypothesis and Proposition 5.1, $\mathrm{Gr}\,S$ is a completely simple subsemigroup of S. Then, for any $a, b, c \in S$, as aa^{ω}, $a^{\omega}cb^{\omega}$, $b^{\omega}b \in \mathrm{Gr}\,S$, from [3, Proposition III.1.1], we have $(aa^{\omega}b^{\omega}b)^{\omega} = (aa^{\omega}cb^{\omega}b)^{\omega}$. On the other hand, by hypothesis and Theorem 3.3, $(aa^{\omega}b^{\omega}b)^{\omega} = (ab)^{\omega}$. Also, by Theorem 3.3, $\mathcal{P}$ is a congruence on S, then $acb \, \mathcal{P} \, aa^{\omega}cb^{\omega}b$, so that $(acb)^{\omega} = (aa^{\omega}cb^{\omega}b)^{\omega}$. Consequently, $(acb)^{\omega} = (ab)^{\omega}$, as required.

(v)⇒(vi). This is clear. □

Corollary 5.6. *For epigroups*, $\mathbf{N} \circledcirc \mathbf{CS} = \mathbf{N}{\circ}\mathbf{CS}$.

Proof. We only need to show that $\mathbf{N} \circledcirc \mathbf{CS} \subseteq \mathbf{N}{\circ}\mathbf{CS}$. Let $S \in \mathbf{N} \circledcirc \mathbf{CS}$ be such that there is a surjective relational morphism $\tau \colon S \to A$ with $A \in \mathbf{CS}$ and for each idempotent $a \in A$, the semigroup $a\tau^{-1}$ belongs to $\mathbf{N}$. For arbitrary $x, y \in S$, there exist $a, b \in A$ such that $a \in x\tau$, $b \in y\tau$. Since $\mathbf{CS} \subseteq \mathbf{N}{\circ}\mathbf{CS}$, from Proposition 5.5, $(aba)^{\omega} = a^{\omega}$ in A. Then, $(xyx)^{\omega}$, $x\omega \in a^{\omega}\tau^{-1} \in \mathbf{N}$, and so $(xyx)^{\omega} = x^{\omega}$ since a nil-semigroup has only one idempotent. Thus, from Proposition 5.5, we have $S \in \mathbf{CS}{\circ}\mathbf{N}$. □

In the following proposition, we give some observations pertaining to the properties of the natural partial order in **CS**∘**N** and **N**∘**CS**.

Proposition 5.7. *The following statements hold for an epigroup S:*
(i) $S \in \mathbf{CS}\circ\mathbf{N} \Leftrightarrow [(a \in S, b \in \mathrm{Gr}S)a \leq b \Rightarrow a = b]$;
(ii) $S \in \mathbf{N}\circ\mathbf{CS} \Rightarrow [(a, b \in S)a \leq b \Rightarrow a^\omega = b^\omega]$.

Proof.
(i) Sufficiency. Let $e, f \in E_{S^1}$ be such that $a = eb = bf$. If $e = 1$ or $f = 1$, trivially $a = b$; if $e, f \in E_S$, then $a \in \mathrm{Gr}\,S$, since from Proposition 5.1 $\mathrm{Gr}\,S$ is an ideal of S. Now again from Proposition 5.1 we have

$$a\mathcal{H}a^\omega = (b^\omega bfebb^\omega) = b^\omega\mathcal{H}b,$$

and then $a(\mathcal{H} \cap \leqslant)b$, so that from Lemma 2.4 $a = b$.

Necessity. From Proposition 5.1, this is an immediate consequence of the fact that E_S is an antichain.

(ii) Let $e, f \in E_{S^1}$ be such that $a = eb = bf$. If $f = 1$ or $g = 1$, in this case $a = b$; if $e, f \in E_S$, then from Proposition 5.5 $a^\omega = (bfeb)^\omega = b^\omega$. □

Proposition 5.8. *For epigroups,*
(i) $(\mathbf{CS}\circ\mathbf{N})\circ(\mathbf{CS}\circ\mathbf{N}) = \mathbf{CS}\circ\mathbf{N}$;
(ii) $(\mathbf{N}\circ\mathbf{CS})\circ(\mathbf{N}\circ\mathbf{CS}) = \mathbf{CS}\circ\mathbf{N}$;
(iii) $\mathbf{N}\circ(\mathbf{N}\circ\mathbf{CS}) = \mathbf{N}\circ\mathbf{CS}$;
(iv) $\mathbf{CS}\circ(\mathbf{N}\circ\mathbf{CS}) = \mathbf{CS}\circ\mathbf{N}$.

Proof.
(i) Let $S \in (\mathbf{CS}\circ\mathbf{N})\circ(\mathbf{CS}\circ\mathbf{N})$. Let $\rho \in C(S)$ be such that $S/\rho \in \mathbf{CS}\circ\mathbf{N}$ and $e\rho \in \mathbf{CS}\circ\mathbf{N}$ for $e \in E_S$. Taking $e, f \in E_S$, if $e \leqslant f$, then from Proposition 5.4(iv) $e\rho f$, that is, $e, f \in e\rho \in \mathbf{CS}\circ\mathbf{N}$. Thus, from Proposition 5.1, $E_{e\rho}$ is an antichain, and so $e = f$. Now we obtain that E_S is an antichain and again from Proposition 5.1 $S \in \mathbf{CS}\circ\mathbf{N}$.
(ii) Since $\mathbf{CS}, \mathbf{N} \subseteq \mathbf{N}\circ\mathbf{CS}$, we have $\mathbf{CS}\circ\mathbf{N} \subseteq (\mathbf{N}\circ\mathbf{CS})\circ(\mathbf{N}\circ\mathbf{CS})$. On the other hand, from (i) and Proposition 5.5, $(\mathbf{N}\circ\mathbf{CS})\circ(\mathbf{N}\circ\mathbf{CS}) \subseteq (\mathbf{CS}\circ\mathbf{N})\circ(\mathbf{CS}\circ\mathbf{N}) \subseteq \mathbf{CS}\circ\mathbf{N}$.
(iii) Let $S \in \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{N})$ and $\rho \in C(S)$ be such that $S/\rho \in \mathbf{CS}\circ\mathbf{N}$ and $e\rho \in \mathbf{N}$ for $e \in E_S$. Taking $x, y \in S$, from Proposition 5.5, we have $x^\omega\rho(xyx)^\omega$, that is, $x^\omega, (xyx)^\omega \in (xyx)^\omega\rho \in \mathbf{N}$, so that $x^\omega = (xyx)^\omega$ since in a nil-semigroup there exists only one idempotent. Thus, again from Proposition 5.5, $S \in \mathbf{N}\circ\mathbf{CS}$.
(iv) It is clear that $\mathbf{CS}\circ\mathbf{N} \subseteq \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{CS})$. By (ii), we have $\mathbf{CS}\circ(\mathbf{N}\circ\mathbf{CS}) \subseteq \mathbf{CS}\circ\mathbf{N}$. □

In the following result, we shall calculate the derived groupoid with the operation of Malcev products generated by **N** and **N**∘**CS**. Here, we deliberately avoid mentioning the class **CS**∘**CS** and its derivatives, since we will describe them in Part II as mentioned before.

Proposition 5.9. *The semigroup* $\langle \mathbf{N}, \mathbf{N}\circ\mathbf{CS}\rangle$ *with multiplication* ∘ *is given by the Cayley table in* Figure 5, *and* $\langle \mathbf{N}, \mathbf{N}\circ\mathbf{CS}\rangle \cong P$.

Proof. From Propositions 4.1, 5.8, it is routine to check that $\langle \mathbf{N}, \mathbf{N}\circ\mathbf{CS}\rangle$ has the Cayley table in Figure 5. We observe that elements **N**, **N**∘**CS** and **CS**∘**N** are all distinct. For example, from Proposition 5.5, $L_{3,1} \in \mathbf{CS}\circ\mathbf{N}$, while $L_{3,1} \notin \mathbf{N}\circ\mathbf{CS}$, which means $\mathbf{N}\circ\mathbf{CS} \subsetneq \mathbf{CS}\circ\mathbf{N}$. Let $N_2 = \{a, 0\}$ be the two element nil-semigroup, in which $a^2 = 0a = a0 = 0$ (we remark that $N_2 \cong C_{2,1}$), and $C_{1,2} = \{g, e\}$ the cyclic group of order 2, in which $g^2 = e$, $eg = ge = g$; they are disjoint. Considering the direct product $N_2 \times C_{1,2}$, from Proposition 5.5, $N_2 \times C_{1,2} \in \mathbf{N}\circ\mathbf{CS}$ (Figure 6). Since the unique idempotent $(0, e)$ is not zero of $N_2 \times C_{1,2}$, we see $N_2 \times C_{1,2} \notin \mathbf{N}$. Therefore, $\mathbf{N} \subsetneq \mathbf{N}\circ\mathbf{CS}$.

Define φ: $\langle \mathbf{N}, \mathbf{N}\circ\mathbf{CS}\rangle \to P$ by $\mathbf{N}\varphi = e$, $(\mathbf{N}\circ\mathbf{CS})\varphi = a$; it is clear that φ is an isomorphism. □

∘	$\mathbf{N}$	$\mathbf{N}\circ\mathbf{CS}$	$\mathbf{CS}\circ\mathbf{N}$
$\mathbf{N}$	$\mathbf{N}$	$\mathbf{N}\circ\mathbf{CS}$	$\mathbf{CS}\circ\mathbf{N}$
$\mathbf{N}\circ\mathbf{CS}$	$\mathbf{CS}\circ\mathbf{N}$	$\mathbf{CS}\circ\mathbf{N}$	$\mathbf{CS}\circ\mathbf{N}$
$\mathbf{CS}\circ\mathbf{N}$	$\mathbf{CS}\circ\mathbf{N}$	$\mathbf{CS}\circ\mathbf{N}$	$\mathbf{CS}\circ\mathbf{N}$

Figure 5: The Cayley table of $\langle\mathbf{N}, \mathbf{N}\circ\mathbf{CS}\rangle$.

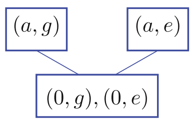

Figure 6: The eggbox of $N_2 \times C_{1,2}$.

6 $(\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$

By far, we do not consider the case $\mathbf{V}\circ\mathbf{CS}$ for the class $\mathbf{V} \in \{\mathbf{S}, \mathbf{CS}\}$ (which will appear in Part II). The universe will be $(\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$ in the rest of the paper, since the given classes of epigroups below are contained in it. We gather here some characterizations about $(\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$ (see [1, Theorem 3.16] for more characterizations about such epigroups), including some new conclusions.

Theorem 6.1. *The following conditions on an epigroup S are equivalent:*
- (i) $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$;
- (ii) *$Reg\, S = Gr\, S$*;
- (iii) *there are no semigroups A_2, B_2 among the epidivisors of S*;
- (iv) *$\sqrt{\mathcal{D}}$ is the least semilattice congruence*;
- (v) $\mathrm{tr}\mathcal{D} = \mathrm{tr}\mathcal{D}^\star$;
- (vi) *S satisfies either (hence both) of the identities $((xy)^\omega x(xy)^\omega)^\omega = (xy)^\omega$, $((xy)^\omega y(xy)^\omega)^\omega = (xy)^\omega$*;
- (vii) *S satisfies the identity $((xy)^\omega x)^{\omega+1} = (xy)^\omega x$*;
- (viii) *S satisfies either (hence both) of the identities $((xy)^\omega(yx)^\omega(xy)^\omega)^\omega = (xy)^\omega$, $((xy)^\omega yx(xy)^\omega)^\omega = (xy)^\omega$.*

Proof. The equivalences of (i), (ii), (iii), (iv) and (viii) were mentioned in [1, Theorem 3.16] or [5, Theorem 3]. For a periodic semigroup, the equivalence of (iii) and (vii) was mentioned in [17, Lemma 2.1].

(iv)⇒(v). If $\sqrt{\mathcal{D}}$ is the least semilattice congruence, then $\mathcal{D}^\star \subseteq \sqrt{\mathcal{D}}$ since $\mathcal{D} \subseteq \sqrt{\mathcal{D}}$. Now let $e, f \in E_S$ be such that $e\, \mathcal{D}^\star f$. Then, $e\sqrt{\mathcal{D}}f$ and so $e\, \mathcal{D}\, f$. Now we have shown that $\mathrm{tr}\mathcal{D}^\star \subseteq \mathrm{tr}\mathcal{D}$; the reverse inclusion holds trivially.

(v)⇒(vi). It is easy to verify that for any $a, b \in S$, $(ab)^\omega a\, \mathcal{D}\, (ab)^\omega$ and so $(ab)^\omega a\, \mathcal{D}^\star (ab)^\omega$. Thus, $(ab)^\omega a(ab)^\omega \mathcal{D}^\star (ab)^\omega$. Then, from Lemma 2.1 $((ab)^\omega a(ab)^\omega)^\omega\, \mathcal{D}^\star (ab)^\omega$, so that $((ab)^\omega a(ab)^\omega)^\omega \mathrm{tr}\mathcal{D}^\star (ab)^\omega$. If (v) holds, then $((ab)^\omega a(ab)^\omega)^\omega \mathrm{tr}\, \mathcal{D}\, (ab)^\omega$ and so $((ab)^\omega a(ab)^\omega)^\omega\, \mathcal{D}\, (ab)^\omega$. Since $((ab)^\omega a(ab)^\omega)^\omega \leqslant (ab)^\omega$, from Lemma 2.4, we obtain $((ab)^\omega a(ab)^\omega)^\omega = (ab)^\omega$. The latter identity can be obtained similarly.

(vi)⇒(vii). If the former identity in (vi) holds, then

$$\begin{aligned} ((xy)^\omega x)^{\omega+1} &= (xy)^\omega x((xy)^\omega x)^\omega = (xy)^\omega (x(xy)^\omega)^\omega x \text{ by Lemma 2.3(iii)} \\ &= ((xy)^\omega x(xy)^\omega)^\omega x \text{ by Lemma 2.3(i)} \\ &= (xy)^\omega x \text{ by the former identity in (vi).} \end{aligned}$$

(vii)⇒(ii). If $a \in \mathrm{Reg}\, S$, then there exists $b \in S$ such that $aba = a$. Since ab is an idempotent of S, we have $a = (ab)^\omega a$. If (vii) holds, then $(ab)^\omega a$ is a group element and so $a \in \mathrm{Gr}\, S$. □

Corollary 6.2. *For epigroups,* $(\mathbf{CS}ⓜ\mathbf{N})ⓜ\mathbf{S} = (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$.

Proof. From Corollary 5.2, we only need to show that $(\mathbf{CS}\circ\mathbf{N})ⓜ\mathbf{S} \subseteq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$. Let $S \in (\mathbf{CS}\circ\mathbf{N})ⓜ\mathbf{S}$ be such that there is a surjective relational morphism $\tau\colon S \to Y$ with $Y \in \mathbf{S}$ and for each idempotent $\alpha \in Y$, the semigroup $\alpha\tau^{-1} \in \mathbf{CS}\circ\mathbf{N}$. For arbitrary $x, y \in S$, there exist $\alpha, \beta \in Y$ such that $\alpha \in x\tau$, $\beta \in y\tau$. Since $\alpha\beta = (\alpha\beta)^\omega = (\beta\alpha)^\omega$ in Y. Then, $(xy)^\omega, (yx)^\omega \in (\alpha\beta)\tau^{-1} \in \mathbf{CS}\circ\mathbf{N}$, so that from Proposition 5.1 $((xy)^\omega(yx)^\omega(xy)^\omega)^\omega = (xy)^\omega$. Thus, from Theorem 6.1, $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$. □

The basic characterization of $(\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$ is that they are semilattices of archimedean epigroups. For $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$, we say $S = (Y;S_\alpha)$, and mean that $Y \cong S/\sqrt{\mathcal{D}}$ and the subepigroups S_α $(\alpha \in Y)$ are the archimedean components of S.

Proposition 6.3. *Let S be a semilattice Y of archimedean epigroups S_α, that is, $S = (Y; S_\alpha) \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$. For $a, b \in S$, the following statements are equivalent*:

(i) $a\,\mathcal{D}\,b$;

(ii) *$a, b \in K(S_\alpha)$ for some $\alpha \in Y$, or $a, b \in S_\alpha \backslash K(S_\alpha)$ and for some $\beta, \gamma \in Y$ with $\beta, \gamma > \alpha$,*

$$(K(S_\beta))^1 a (K(S_\gamma))^1 = (K(S_\beta))^1 b (K(S_\gamma))^1,$$

where, say, $K(S_\alpha)$ is the kernel of S_α and $K(S_\alpha) = \operatorname{Gr} S_\alpha$;

(iii) *there exist $\alpha, \beta, \gamma \in Y$ such that $\beta, \gamma \geqslant \alpha$ and $a, b \in S_\alpha$,*

$$(K(S_\beta))^1 a (K(S_\gamma))^1 = (K(S_\beta))^1 b (K(S_\gamma))^1.$$

Proof. (i)⇒(ii). If $a\,\mathcal{D}\,b$, then $a, b \in S_\alpha$ for some $\alpha \in Y$. Since the regular $\mathcal{D}$ class of S_α (of S) is a completely simple kernel of S_α (see [5, Theorem 3]), either $a, b \in \operatorname{Gr} S$ (in this case $a, b \in K(S_\alpha)$), or $a, b \notin \operatorname{Gr} S$ (notice that neither the case $a \in \operatorname{Gr} S$, $b \notin \operatorname{Gr} S$ nor the case $a \notin \operatorname{Gr} S$, $b \in \operatorname{Gr} S$ occurs).

Now we consider the latter case and clearly $a, b \in S_\alpha \backslash K(S_\alpha)$. Since $a\,\mathcal{D}\,b$, there exist some $x, y, u, v \in S^1$ such that $a = xby$, $b = uav$. Thus, $a = xu{\cdot}a{\cdot}vy$, $b = ux{\cdot}b{\cdot}yv$ and so $a = (xu)^n{\cdot}a{\cdot}(vy)^n$, $b = (ux)^n{\cdot}b{\cdot}(yv)^n$ for all positive integers n. Now S is an epigroup, and so we can choose n such that $n = \max\{\operatorname{ind}(xu), \operatorname{ind}(ux), \operatorname{ind}(vy), \operatorname{ind}(yv)\}$. Then, $a = (xu)^\omega{\cdot}a{\cdot}(vy)^\omega$, $b = (ux)^\omega{\cdot}b{\cdot}(yv)^\omega$ and so

$$a = (xu)^\omega x{\cdot}b{\cdot}y\,(vy)^\omega,\; b = (ux)^\omega u{\cdot}a{\cdot}v\,(yv)^\omega.$$

As above, it is easy to verify that $(xu)^\omega x$, $(ux)^\omega u$ fall into the same archimedean component possibly with an identity 1 adjoined, say, $(S_\beta)^1$, and $y(vy)^\omega$, $v(yv)^\omega \in (S_\gamma)^1$, where $\beta, \gamma > \alpha$, $\beta, \gamma \in Y$. Also, $(xu)^\omega x$, $(ux)^\omega u \in (K(S_\beta))^1$, $y(vy)^\omega$, $v(yv)^\omega \in (K(S_\gamma))^1$, and so

$$\begin{aligned}(K(S_\beta))^1 a (K(S_\gamma))^1 &= (K(S_\beta))^1 (xu)^\omega xby\,(vy)^\omega (K(S_\gamma))^1\\ &\subseteq (K(S_\beta))^1 b (K(S_\gamma))^1;\end{aligned}$$

similarly, $(K(S_\beta))^1 b (K(S_\gamma))^1 \subseteq (K(S_\beta))^1 a (K(S_\gamma))^1$.

(ii)⇒(iii). It is clear.

(iii)⇒(i). It is clear. □

We now give one characterization of the class $(\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$. In Section 8, we can see that it is not an equational class of epigroups and is contained in $(\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$.

Theorem 6.4. *The following conditions on an epigroup S are equivalent:*

(i) $S \in (\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$;

(ii) $(\forall\, a, b \in S)\, a^\omega \mathcal{D} b^\omega \Rightarrow a^\omega = (aba)^\omega$.

Proof. (i)⇒(ii). On one hand, from Proposition 5.5 and Theorem 6.1, $\sqrt{\mathcal{D}}$ is the least semilattice congruence. Let $\rho \in C(S)$ be such that $S/\rho \in \mathbf{S}$ and $e\rho \in \mathbf{N} \circ \mathbf{CS}$ for $e \in E_S$. Then, we have $\sqrt{\mathcal{D}} \subseteq \rho$, so that $e\sqrt{\mathcal{D}} \subseteq e\rho \in \mathbf{N}\circ\mathbf{CS}$. Taking $a, b \in S$, if $a^\omega \mathcal{D} b^\omega$, then $a^{\mathrm{ind}(a)}\mathcal{H}a^\omega\mathcal{D}b^\omega\mathcal{H}b^{\mathrm{ind}(b)}$, so that $a^\omega\sqrt{\mathcal{D}}\, a \sqrt{\mathcal{D}}\, b$. Therefore, $a, b \in a^\omega\rho \in \mathbf{N}\circ\mathbf{CS}$, and again from Proposition 5.5 $a^\omega = (aba)^\omega$.

(ii)⇒(i). From Lemma 2.7(iv), $(xy)^\omega \mathcal{D} (yx)^\omega$, and so by hypothesis $(xy)^\omega = ((xy)^\omega(yx)^\omega(xy)^\omega)^\omega$. Thus, from Theorem 6.1, $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$ and $\sqrt{\mathcal{D}}$ is the least semilattice congruence. Now for any $a, b \in S$, if $a \sqrt{\mathcal{D}}\, b$, which means $a^\omega \mathcal{D} b^\omega$, then by hypothesis $a^\omega = (aba)^\omega$. Now from Proposition 5.5 each $\sqrt{\mathcal{D}}$ class belongs to $\mathbf{N}\circ\mathbf{CS}$. Therefore, $S \in (\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$, as required. □

7 CS∘(N∘S)

In this section, we characterize the class $\mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$ (that is, $\mathbf{CS}\circ\mathbf{J}$) in several ways and consider one special case in which the Green relation $\mathcal{D}$ is a congruence. We begin with an auxiliary result of independent interest.

Lemma 7.1. *Let ρ be a congruence on an epigroup S. The following conditions on S are equivalent:*
(i) $\mathrm{tr}\rho \subseteq \mathrm{tr}\mathcal{D}$;
(ii) $\rho \subseteq \sqrt{\mathcal{D}}$;
(iii) *for any $e \in E_S$, $e\rho \in \mathbf{CS}\circ\mathbf{N}$.*

Proof. (i)⇒(ii). Let $a, b \in S$ be such that $a \rho b$. Then, from Lemma 2.1 $a^\omega \rho b^\omega$. If (i) holds, then $a^\omega \mathcal{D} b^\omega$, so that $a^{\mathrm{ind}(a)} \mathcal{H} a^\omega \mathcal{D} b^\omega \mathcal{H} b^{\mathrm{ind}(b)}$. Hence, $a \sqrt{\mathcal{D}}\, b$.

(ii)⇒(iii). For any $a, b \in e\rho$, that is, $a\rho e\rho b$, from Lemma 2.1, we have $a^\omega\rho e\rho b^\omega$. Thus, $a^\omega b^\omega a^\omega \rho a^\omega$ and so $(a^\omega b^\omega a^\omega)^\omega \rho a^\omega$. If $\rho \subseteq \sqrt{\mathcal{D}}$, then $(a^\omega b^\omega a^\omega)^\omega \sqrt{\mathcal{D}}\, a^\omega$ and so $(a^\omega b^\omega a^\omega)^\omega \mathcal{D} a^\omega$. Whereas $(a^\omega b^\omega a^\omega)^\omega \leqslant a^\omega$, then from Lemma 2.4 $(a^\omega b^\omega a^\omega)^\omega = a^\omega$. Now from Lemma 2.2, $e\rho$ is a subepigroup of S, and, from Proposition 5.1, we have $e\rho \in \mathbf{CS}\circ\mathbf{N}$.

(iii)⇒(i). Let $e, f \in E_S$ be such that $e \rho f$. If (iii) holds, then from Proposition 5.1 $e\, \mathcal{D}^{e\rho} f$, so that $e \mathcal{D} f$. □

Now we give some characterizations of the class $\mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$.

Theorem 7.2. *The following conditions on an epigroup S are equivalent:*
(i) $S \in \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$;
(ii) $S \in \mathbf{CS}$ⓜ$(\mathbf{N}\circ\mathbf{S})$ *and the corresponding relational morphism is pseudo-injective*;
(iii) *there exists $\rho \in C(S)$ such that $\mathcal{D} \subseteq \rho$, and $e\rho = D_e$ for any $e \in E_S$*;
(iv) *there exists $\rho \in C(S)$ such that $\mathcal{D} \subseteq \rho \subseteq \sqrt{\mathcal{D}}$ and $\ker \rho = \mathrm{Gr}\, S$*;
(v) *there exists $\rho \in C(S)$ such that $\mathcal{D} \subseteq \rho$, $\ker\rho = \ker\mathcal{D}$ and $\mathrm{tr}\rho = \mathrm{tr}\mathcal{D}$*;
(vi) $\ker\mathcal{D}^\star = \ker\mathcal{D}$, $\mathrm{tr}\mathcal{D}^\star = \mathrm{tr}\mathcal{D}$;
(vii) *$\mathcal{D}^\star$ is over completely simple semigroups*;
(viii) *$\mathcal{D}^\star$ is over completely regular semigroups*;
(ix) $\ker\mathcal{D}^\star = \mathrm{Gr}S$;
(x) *$\mathcal{D} \subseteq \pi_K$, where $K = \mathrm{Gr}\, S$.*

Proof. The equivalence (i)⇔(iii) follows immediately from Lemma 4.8.

(i)⇒(ii). If (i) holds, then there exists $\rho \in C(S)$ such that $S/\rho \in \mathbf{N}\circ\mathbf{S}$ with ρ being over $\mathbf{CS}$. Then, from [8, Theorem 1.5.2], the induced natural map $\rho^\#$ from S to the quotient set S/ρ is a homomorphism, and so from Lemma 2.6, it is easy to show that (ii) holds.

(ii)⇒(x). If (ii) holds, then there exists a pseudo-injective relational morphism $\tau \colon S \to J$ such that $J \in \mathbf{N}\circ\mathbf{S}$ and $e\tau^{-1} \in \mathbf{CS}$ for any $e \in E_S$. For any $x \in S$, let x^τ denote an element in J such that $x^\tau \in x\tau$ (since τ is fully defined); and if $1 \in S^1\backslash S$, $1^\tau = 1$, $1^\omega = 1$ (we remark that these special conventions have no harm to our proof).

Let $a, b \in S$ be such that $a\,\mathcal{D}\,b$. Then, there exist $s, t, u, v \in S^1$ such that $a = sbt$, $b = uav$. Analogous to the proof of Lemma 2.9, we have $a = (su)^{\omega}s{\cdot}b{\cdot}t(vt)^{\omega}$, $b = (us)^{\omega}{\cdot}b{\cdot}(tv)^{\omega}$. It is a routine matter to show that $(s^{\tau}u^{\tau})^{\omega}s^{\tau}{\cdot}b^{\tau}{\cdot}t^{\tau}(v^{\tau}t^{\tau})^{\omega} \in a\tau$, $(u^{\tau}s^{\tau})^{\omega}{\cdot}b^{\tau}{\cdot}(t^{\tau}v^{\tau})^{\omega} \in b\tau$. As $J \in \mathbf{N}\circ\mathbf{S}$, from Proposition 4.2, $(s^{\tau}u^{\tau})^{\omega}s^{\tau}{\cdot}b^{\tau}{\cdot}t^{\tau}(v^{\tau}t^{\tau})^{\omega} = (u^{\tau}s^{\tau})^{\omega}{\cdot}b^{\tau}{\cdot}(t^{\tau}v^{\tau})^{\omega}$. Set $(s^{\tau}u^{\tau})^{\omega}s^{\tau}{\cdot}b^{\tau}{\cdot}t^{\tau}(v^{\tau}t^{\tau})^{\omega} = \alpha$; and so $\alpha \in a\tau \cap b\tau$ (note that here α may be not an idempotent). Then, for any $x, y \in S^1$, $x^{\tau}\alpha y^{\tau} \in (xay)\tau \cap (xby)\tau$, which implies $(xay)\tau \cap (xby)\tau \neq \varnothing$ so that $(xay)\tau = (xby)\tau$, since τ is pseudo-injective. Now if $xay \in \operatorname{Gr} S$, then

$$\begin{aligned}(x^{\tau}\alpha y^{\tau})^{\omega} &= (x^{\tau}\alpha y^{\tau})^{\omega+1} \text{ since } J \in \mathbf{N}\circ\mathbf{S}\\ &= x^{\tau}\alpha y^{\tau}(x^{\tau}\alpha y^{\tau})^{\omega} \in (xay)\tau(xay)^{\omega}\tau\\ &\subseteq (xay)^{\omega+1}\tau = (xay)\tau \text{ since } xay \in \operatorname{Gr}S\\ &= (xby)\tau \quad \text{as proved above.}\end{aligned}$$

Then, $xby \in (x^{\tau}\alpha y^{\tau})^{\omega}\tau^{-1} \in \mathbf{CS}$ and so $xby \in \operatorname{Gr} S$. Similarly, the implication $xby \in \operatorname{Gr} S \Rightarrow xay \in \operatorname{Gr} S$ holds. Thus, $a\pi_K b$ and so $\mathcal{D} \subseteq \pi_K$.

(x)⇒(iii). If (x) holds, we will show that $\mathcal{D}^{\star}$ is the congruence ρ in (iii) which we are seeking. Clearly $\mathcal{D} \subseteq \mathcal{D}^{\star}$, and we remain to show that for any $e \in E_S$, $e\mathcal{D}^{\star} \subseteq D_e$.

First, if $b \in \operatorname{Reg} S$, that is, $b\,\mathcal{D}\,f$ for some $f \in E_S$, then $b\pi_K f$, so that $b \in \operatorname{Gr} S$ since $f \in \operatorname{Gr} S$. Therefore, $\operatorname{Reg}S \subseteq \operatorname{Gr}S$, and so $\operatorname{Reg} S = \operatorname{Gr} S$, since the inverse inclusion is clear. Now from Theorem 6.1, we have $\operatorname{tr}\mathcal{D}^{\star} = \operatorname{tr}\mathcal{D}$.

Second, let $a \in S$ be such that $a\,\mathcal{D}^{\star}\,e$. On one hand, we have $a^{\omega}\,\mathcal{D}^{\star}\,e$, so that, from the fact that $\operatorname{tr}\mathcal{D}^{\star} = \operatorname{tr}\mathcal{D}$ just proved above, we have $a^{\omega}\,\mathcal{D}\,e$. On the other hand, from (2), $a = e$, or

$$e = x_1u_1y_1,\ x_1v_1y_1 = x_2u_2y_2,\ \ldots, x_nv_ny_n = a,$$

for some $x_i, y_i \in S^1$, $u_i, v_i \in S$, $u_i\,\mathcal{D}\,v_i$, $i = 1, 2, \ldots, n$. For the case $a = e$, there is nothing to prove. For the latter case, since $\mathcal{D} \subseteq \pi_K$, it is easy to observe that $a\pi_K e$, so that $a \in \operatorname{Gr} S$ since $e \in \operatorname{Gr} S$, which means $a\,\mathcal{D}\,a^{\omega}$. Therefore, $a\,\mathcal{D}\,e$, that is, $a \in D_e$.

(iii)⇒(iv). Let (iii) hold. From Lemmas 4.8, $e\rho \in \mathbf{CS}$ for any $e \in E_S$, so that by Lemma 7.1 $\rho \subseteq \sqrt{\mathcal{D}}$. Also, $\ker\rho = \cup_{e\in E_S} e\rho \subseteq \operatorname{Gr}S$; conversely, if $a \in \operatorname{Gr} S$, then $a\,\mathcal{D}\,a^{\omega}$ and so by hypothesis $a\rho a^{\omega}$, which implies $a \in \ker\rho$. Therefore, we have $\ker\rho = \operatorname{Gr} S$.

(iv)⇒(v). If (iv) holds, we need only to show that $\ker\rho \subseteq \ker\mathcal{D}$ and $\operatorname{tr}\rho \subseteq \operatorname{tr}\mathcal{D}$. To this end, take $a \in S$ with $a \in \ker\rho$, that is, $a\rho a^{\omega}$. Since $\ker\rho = \operatorname{Gr} S$, we have $a\,\mathcal{D}\,a^{\omega}$, that is, $a \in \ker\mathcal{D}$. Taking $e, f \in E_S$ with $e\rho f$, since $\rho \subseteq \sqrt{\mathcal{D}}$, we have $e\sqrt{\mathcal{D}}f$, that is, $e\,\mathcal{D}\,f$.

(v)⇒(vi). If (v) holds, then $\mathcal{D} \subseteq \mathcal{D}^{\star} \subseteq \rho$ and so the implication holds.

(vi)⇒(vii). For any $e \in E_S$, let $a \in S$ be such that $a \in e\mathcal{D}^{\star}$. On one hand, from Lemma 2.1, $a^{\omega}\mathcal{D}^{\star}\,e$ and so $a^{\omega}\,\mathcal{D}\,e$, sine $\operatorname{tr}\mathcal{D}^{\star} = \operatorname{tr}\mathcal{D}$. On the other hand, clearly $a \in \ker\mathcal{D}^{\star}$ and then $a \in \ker\mathcal{D}$ since $\ker\mathcal{D}^{\star} = \ker\mathcal{D}$, so that $a\,\mathcal{D}\,f$ for some $f \in E_S$, which implies $a \in \operatorname{Reg} S$. Now from Theorem 6.1, we have $a \in \operatorname{Gr} S$. Thus, $a\,\mathcal{H}\,a^{\omega}\,\mathcal{D}\,e$, so that $a \in D_e$. By far, we have shown that for any $e \in E_S$, $e\mathcal{D}^{\star} \subseteq D_e$, then, from Lemma 4.8, $\mathcal{D}^{\star}$ is over completely simple semigroups.

(vii)⇒(viii). It is clear.

(viii)⇒(ix). It is clear.

(ix)⇒(x). Let $a, b \in S$ be such that $a\,\mathcal{D}\,b$. For any $x, y \in S^1$, we have

$$\begin{aligned}xay \in \operatorname{Gr}S &\Rightarrow xby\ \mathcal{D}^{\star}xay\ \mathcal{D}\ (xay)^{\omega}\\ &\Rightarrow xby\ \mathcal{D}^{\star}(xay)^{\omega} \text{ since } \mathcal{D} \subseteq \mathcal{D}^{\star} \text{ and } \mathcal{D}^{\star} \text{ is transitive}\\ &\Rightarrow xby \in \operatorname{Gr}S \text{ by the condition in (ix).}\end{aligned}$$

Similarly, the implication $xby \in \operatorname{Gr} S \Rightarrow xay \in \operatorname{Gr} S$ holds. Thus, $a\pi_K b$ and so $\mathcal{D} \subseteq \pi_K$. □

From Theorem 7.2 and Proposition 5.4(i), we obtain the statement (i) in the following result.

Proposition 7.3. *The following equalities hold for epigroups:*
(i) $\mathbf{CR}\circ(\mathbf{N}\circ\mathbf{S}) = \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$;
(ii) $\mathbf{CR}ⓜ(\mathbf{N}ⓜ\mathbf{S}) = \mathbf{CS}ⓜ(\mathbf{N}\circ\mathbf{S})$.

Proof.
(i) It suffices to prove that $\mathbf{CR}\circ(\mathbf{N}\circ\mathbf{S}) \subseteq \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$. Let $S \in \mathbf{CR}\circ(\mathbf{N}\circ\mathbf{S})$ be such that there exists $\rho \in C(S)$ such that $S/\rho \in \mathbf{N}\circ\mathbf{S}$, $e\rho \in \mathbf{CR}$ for $e \in E_S$. Then, $\ker\rho \subseteq \operatorname{Gr} S$, and from Lemma 4.8 $\mathcal{D} \subseteq \rho$, which yields $\mathcal{D} \subseteq \mathcal{D}^\star \subseteq \rho$, so that $\ker\mathcal{D}^\star \subseteq \operatorname{Gr}S$. Conversely, if $a \in \operatorname{Gr} S$, then $a\,\mathcal{D}\,a^\omega$ and so $a\,\mathcal{D}^\star\,a^\omega$, which implies $a \in \ker\mathcal{D}^\star$. Now $\ker\mathcal{D}^\star = \operatorname{Gr}S$; and so from Theorem 7.2, we have $S \in \mathbf{C}S\circ(\mathbf{N}\circ\mathbf{S})$.
(ii) From Proposition 4.2, it suffices to prove that $\mathbf{CR}ⓜ(\mathbf{N}\circ\mathbf{S}) \subseteq \mathbf{CS}ⓜ(\mathbf{N}\circ\mathbf{S})$. Let $S \in \mathbf{CR}ⓜ(\mathbf{N}\circ\mathbf{S})$ be such that there exists a surjective relational morphism $\tau\colon S \to J$ such that $J \in \mathbf{N}\circ\mathbf{S}$ and $i\tau^{-1} \in \mathbf{CR}$ for all $i \in E_J$. Now $\#\tau p_J = J \in \mathbf{N}\circ\mathbf{S}$ and for any $i \in E_J$

$$ip_J^{-1} = \{(s, i) \in S \times J \mid i \in s\tau\} \cong i\tau^{-1} \in \mathbf{CR}.$$

Then, $\#\tau \in \mathbf{CR}\circ(\mathbf{N}\circ\mathbf{S})$ and so by (i) $\#\tau \in \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$, which obviously yields $\#\tau \in \mathbf{CS}ⓜ(\mathbf{N}\circ\mathbf{S})$. Now $S = \#\tau p_S$, so that $S \in \mathbf{CS}ⓜ(\mathbf{N}\circ\mathbf{S})$, since from Lemma 2.12 $\mathbf{CS}ⓜ(\mathbf{N}\circ\mathbf{S})$ is closed under the taking of homomorphic images. □

Lemma 7.4. *In epigroups*, $\mathbf{CS}ⓜ(\mathbf{N}\circ\mathbf{S}) \subseteq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$.

Proof. Let $S \in \mathbf{CS}ⓜ(\mathbf{N}\circ\mathbf{S})$ be such that there is a surjective relational morphism $\tau\colon S \to J$ with $J \in \mathbf{N}\circ\mathbf{S}$ and for each $i \in J$, the semigroup $i\tau^{-1}$ belongs to $\mathbf{CS}$. Now for any $x, y \in S$, there exist $a, b \in J$ such that $a \in x\tau$, $b \in y\tau$. Then, $(ab)^\omega a \in ((xy)^\omega x)\tau$. Since from Proposition 4.2 $(ab)^\omega a = (ba)^\omega$ in J, we have $(xy)^\omega x \in (ba)^\omega\tau^{-1} \in \mathbf{CS}$. It follows that $(xy)^\omega x \in \operatorname{Gr} S$ and so S satisfies the identity $((xy)^\omega x)^{\omega+1} = (xy)^\omega x$. Thus, from Theorem 6.1, $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$. □

The following result implies that $\mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$ is not an equational class of epigroups.

Lemma 7.5. *The class* $\mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$ *of epigroups is not closed under the taking of homomorphic images.*

Proof. Consider the semigroup Θ with presentation

$$\Theta = \langle e, f, c | ef = f = f^2, fe = e = e^2, c^2 = ce = cf = 0\rangle.$$

It is easy to check that the collection of $\mathcal{D}$ classes of Θ is $\{\{e, f\}, \{ec\}, \{fc\}, \{c\}, \{0\}\}$ (Figure 7). Now consider the partition $\{\{e, f\}, \{ec, fc, c\}, \{0\}\}$ of Θ. It is clear that the partition induces a congruence ρ on Θ, and $\mathcal{D} \subseteq \rho$, in which $\Theta/\rho \cong P \in \mathbf{N}\circ\mathbf{S}$ (see Lemma 4.3), $e\rho(= f\rho)$, $0 \in \mathbf{CS}$. Thus, from Theorem 7.2, $\Theta \in \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$.

If we consider the partition $\{\{e\}, \{f\}, \{ec, c\}, \{fc, 0\}\}$ of Θ. It is clear that the partition induces a congruence σ on Θ, and $\Theta/\sigma \cong Q$, where

$$\Theta = \langle e, f, c | ef = f = f^2, fe = e = e^2, c^2 = ce = cf = fc = 0\rangle$$

(the semigroup Q was studied in [18]; also see [19]). We claim that $Q \notin \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$. Otherwise, from Theorem 7.2, there exists $\rho \in C(Q)$ such that $\mathcal{D} \subseteq \rho$ and $0\rho = D_0 = 0$. While from the fact that $e\,\mathcal{D}\,f$, we have $e\,\rho\,f$, so that $ec\,\rho\,fc = 0$, a contradiction. □

A semigroup S is said to be $\mathcal{D}$ compatible if $\mathcal{D}$ is a congruence on S. The class of $\mathcal{D}$-compatible epigroups is denoted by $\mathbf{DC}$. From Theorem 7.2, we arrive at $\mathbf{DC} \subseteq \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$. The next result, together with Lemma 7.5, implies that the inclusion is proper.

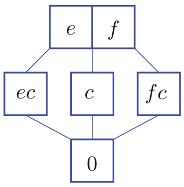

Figure 7: The eggbox of Θ.

Lemma 7.6. *The class of $\mathcal{D}$-compatible epigroups S is closed under the taking of homomorphic images.*

Proof. Since $\mathcal{D}$ is a congruence on S, for any $\rho \in C(S)$, from Lemma 2.9 $\mathcal{D}^{S/\rho} = (\rho \vee \mathcal{D})/\rho$ is also a congruence on S/ρ. Thus, S is closed under the taking of homomorphic images. ☐

Lemma 7.7. *For epigroups,* $(\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N} \subseteq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$.

Proof. If $S \in (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$ and $a, b \in S$, then from Proposition 3.6 $(ab)^\omega\, a \in \operatorname{Gr} S$, that is, $((ab)^\omega a)^{\omega+1} = (ab)^\omega a$. Thus, from Theorem 7.2, $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$. ☐

Lemma 7.8. *Let S be an epigroup such that $S \in (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$ (that is, $\operatorname{Gr} S$ is an ideal of S, which implies $\operatorname{Gr} S = K \in \mathbf{CR}$).*
(i) $(\forall\, a, b \in S)\, a\, \mathcal{D}\, b \Leftrightarrow a = b \in S \backslash K$ *or* $a\, \mathcal{D}^K\, b$.
(ii) $\mathcal{D} \subseteq \rho_K$.

Proof.
(i) This holds directly from Lemma 7.7 and Proposition 6.3.
(ii) This is true from the fact of (i). ☐

8 Some further set inclusion relations

In this section, we study the set inclusion relations among the classes of epigroups mentioned in the previous sections. Along with some illustrative examples, the proper containments among them are discussed.

Proposition 8.1. *For epigroups,*
(i) $\mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S}) \subsetneqq (\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S} \subsetneqq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$;
(ii) $(\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N} \subsetneqq \mathbf{DC} \subsetneqq \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S}) \subsetneqq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$;
(iii) $\mathbf{CS}\circ(\mathbf{S}\circ\mathbf{N}) = (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$.

Proof.
(i) From the proof of Theorem 6.4, we observe that $(\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S} \subseteq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$. It is easy to see that $L_{3,1} \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$; while from Theorem 6.4, $L_{3,1} \notin (\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$, since in $L_{3,1}$ we see that $a^\omega \mathcal{D} f \not\Rightarrow a^\omega = (afa)^\omega$. Therefore, $(\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$ is properly contained in $(\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$.

For the first inclusion, let $S \in \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$. Then, from Theorem 3.3, $\mathcal{P}$ is a congruence on S. Observing the forbidden epidivisors of the two class of epigrouips, from Theorems 6.1 and 3.3, we also get $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$ and $\sqrt{\mathcal{D}}$ is the least semilattice congruence on S. Now for any $e \in E_S$, $e\sqrt{\mathcal{D}} \in \mathbf{CS}\circ\mathbf{N}$. On the other hand, from Proposition 5.4, $\mathcal{P} \subseteq \sqrt{\mathcal{D}}$ and then $\mathcal{P}|_{e\sqrt{\mathcal{D}}}$ is a congruence on $e\sqrt{\mathcal{D}}$, so that $e\sqrt{\mathcal{D}} \in \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S}) \cap \mathbf{CS}\circ\mathbf{N} = \mathbf{N}\circ\mathbf{CS}$ (see Proposition 5.5). Therefore, $S \in (\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$.

Considering the semigroups $L_{3,2}$, from Theorem 6.4, it is easy to check that $L_{3,2} \in (\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$; while from Theorem 3.3, $L_{3,2} \notin \mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$, since $a\,\mathcal{P}\,a^2 \not\Rightarrow af\,\mathcal{P}\,a^2f$ (which means that $\mathcal{P}$ is not a congruence on S). Therefore, $\mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S})$ is properly contained in $(\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$.

(ii) For the first inclusion, let $S \in (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$. Then, from Lemma 7.7, $S \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$. We shall show that $\mathcal{D}$ is right compatible.

Let $a, b \in S$ be such that $a\,\mathcal{D}\,b$. From Lemma 7.8, either $a, b \notin \operatorname{Gr} S$, in this case $a = b$, or $a, b \in \operatorname{Gr} S$. For any $c \in S$, if $a = b$, then trivially $ac\,\mathcal{D}\,bc$; if $a, b \in \operatorname{Gr} S$, then from Proposition 3.6, $ac, bc \in \operatorname{Gr} S$, so that

$$\begin{aligned} a\,\mathcal{D}\,b &\Rightarrow a\,\mathcal{D}^\star b \Rightarrow ac\,\mathcal{D}^\star\,bc \quad \text{since } \mathcal{D}^\star \text{ is a congruence}\\ &\Rightarrow (ac)^\omega \mathcal{D}^\star (bc)^\omega \quad \text{from Lemma 2.1}\\ &\Rightarrow (ac)^\omega \mathcal{D} (bc)^\omega \quad \text{from Theorem 6.1}\\ &\Rightarrow ac\,\mathcal{D}\,(ac)^\omega\,\mathcal{D}\,(bc)^\omega\,\mathcal{D}\,bc\\ &\Rightarrow ac\,\mathcal{D}\,bc. \end{aligned}$$

Thus, $\mathcal{D}$ is right compatible. The left compatibility is proved dually and so $\mathcal{D}$ is a congruence on S, that is, $S \in \mathbf{DC}$.

We claim that the class $(\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$ is properly contained in $\mathbf{DC}$. For example, if we consider the semigroup P, we see that from Proposition 3.6 $P \notin (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$, whereas $P \in \mathbf{DC}$.

For the second proper inclusion, from Theorem 7.2, we observe that $\mathbf{DC} \subseteq \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$. We declare that the inclusion is proper when we consider the semigroup Θ in the proof of Lemma 7.5. It is easy to see that $\Theta^1 fc \Theta^1 = \{fc, 0\}$, so that $(ec, fc) \notin \mathcal{D}$, while $(e, f) \in \mathcal{D}$ (Figure 7). Hence, Θ is not $\mathcal{D}$ compatible, while from the proof of Lemma 7.5, $\Theta \in \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$.

The third inclusion is an immediate consequence of Lemma 7.4. Considering the semigroup Q in the proof of Lemma 7.5, we have observed that $Q \notin \mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S})$, whereas from Theorem 6.1, $Q \in (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$, since $\operatorname{Reg} Q = E_Q$. Therefore, $\mathbf{CS}\circ(\mathbf{N}\circ\mathbf{S}) \subsetneq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$, as required.

(iii) Let $S \in (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$. Let $\rho \in C(S)$ be such that $S/\rho \in \mathbf{S}\circ\mathbf{N}$ and ρ is over $\mathbf{CS}$. Then, for any $x, y \in S$, from Proposition 4.4, $xy^\omega z\,\rho\,(xy^\omega z)^\omega$. As $(xy^\omega z)^\omega \rho \in \mathbf{CS}$, we have $xy^\omega z \in \operatorname{Gr} S$, that is, $(xy^\omega z)^{\omega+1} = xy^\omega z$. Thus, from Theorem 3.6, $S \in \mathbf{CS}\circ(\mathbf{S}\circ\mathbf{N})$.

Let $S \in \mathbf{CS}\circ(\mathbf{S}\circ\mathbf{N})$. Then, for any $x, y \in S$, from Theorem 3.6, $(xy^\omega z)^{\omega+1} = xy^\omega z$, and so $(xy^\omega z)^\omega \mathcal{D} xy^\omega z$. Now from (ii), $\mathcal{D}$ is a congruence on S; and so from Proposition 4.4, we have $S/\mathcal{D} \in \mathbf{S}\circ\mathbf{N}$. Also, from Proposition 4.8, $\mathcal{D}$ is over $\mathbf{CS}$. Therefore, we have $S \in \mathbf{CS}\circ(\mathbf{S}\circ\mathbf{N})$. □

Note that the semigroup $L_{3,1}$ is a homomorphic image of the semigroup $L_{3,2}$. Then, from the proof of Proposition 8.1(i), we arrive at the next fact.

Corollary 8.2. *The class* $(\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$ *is not closed under the taking of homomorphic images.*

We observe that the semigroups $P, \overrightarrow{P}$ and the cyclic groups $C_{1,p}$, for any prime p, are $\mathcal{D}$ compatible. Considering the direct product $\overrightarrow{P} \times C_{1,p} \times P$, from [19, Lemma 2.1], it is not $\mathcal{D}$ compatible, which reminds us that the class of all $\mathcal{D}$-compatible epigroups is not closed for taking direct products. Thus, we cannot characterize $\mathcal{D}$-compatible epigroups in terms of identities. From Theorem 6.4, we can check that $\overrightarrow{P} \times C_{1,p} \times P \in (\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$. Therefore, we claim that $\mathbf{DC} \subsetneq (\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$. In fact, we have the next result.

Proposition 8.3. *For epigroups,* $\mathbf{DC}$ *and* $(\mathbf{N}\circ\mathbf{CS})\circ\mathbf{S}$ *are incomparable with respect to the partial order of set inclusion.*

Proof. It follows from Corollary 8.2 and Lemma 7.6. □

From Proposition 3.6, $(\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$ is an equational class of epigroups, and so from Theorem 8.1(iv), we get the next corollary.

Corollary 8.4. $\mathbf{CS}\textcircled{m}(\mathbf{S}\circ\mathbf{N}) = (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$.

Proof. From Theorem 8.1(iv), we only need to show that $\mathbf{CS}\textcircled{m}(\mathbf{S}\circ\mathbf{N}) \subseteq (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$. Let $S \in \mathbf{CS}\textcircled{m}(\mathbf{S}\circ\mathbf{N})$ be such that there is a surjective relational morphism $\tau\colon S \to A$ with $A \in \mathbf{S}\circ\mathbf{N}$ and for each idempotent $a \in A$, $a\tau^{-1} \in \mathbf{CS}$. For arbitrary $x, y, z \in S$, there exist $a, b, c \in A$ such that $a \in x\tau$, $b \in y\tau$, $c \in z\tau$. Now from Proposition 4.4, $(cb^{\omega}a)^{\omega} = ab^{\omega}c \in xy^{\omega}z\tau$, that is, $xy^{\omega}z \in (cb^{\omega}a)^{\omega}\tau^{-1} \in \mathbf{CS}$, so that $xy^{\omega}z \in \operatorname{Gr} S$, that is, $(xy^{\omega}z)^{\omega+1} = xy^{\omega}z$. Thus, from Theorem 3.6, $S \in (\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N}$. □

Theorem 8.5. *For epigroups,*
(i) $\mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S}) \subseteq (\mathbf{N}\circ\mathbf{CS})\textcircled{m}\mathbf{S} \subseteq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$;
(ii) $(\mathbf{CS}\circ\mathbf{S})\circ\mathbf{N} \subseteq \mathbf{CS}\textcircled{m}(\mathbf{N}\circ\mathbf{S}) \subseteq (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S}$.

Proof.
(i) From Corollaries 3.4, 5.6 and Lemma 2.13,

$$\mathbf{N}\circ(\mathbf{CS}\circ\mathbf{S}) = \mathbf{N}\textcircled{m}(\mathbf{CS}\textcircled{m}\mathbf{S}) \subseteq (\mathbf{N}\textcircled{m}\mathbf{CS})\textcircled{m}\mathbf{S} = (\mathbf{N}\circ\mathbf{CS})\textcircled{m}\mathbf{S},$$

and so the first inclusion holds. The second inclusion holds immediately from Corollaries 6.2 and 5.5.
(ii) The first inclusion holds from Theorem 8.1(iv). For the second inclusion, from Corollaries 8.4, 6.2 and Lemma 2.13,

$$\mathbf{CS}\textcircled{m}(\mathbf{N}\circ\mathbf{S}) = \mathbf{CS}\textcircled{m}(\mathbf{N}\textcircled{m}\mathbf{S}) \subseteq (\mathbf{CS}\textcircled{m}\mathbf{N})\textcircled{m}\mathbf{S} = (\mathbf{CS}\circ\mathbf{N})\circ\mathbf{S},$$

and so the second inclusion holds. □

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