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Research Article

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Tripled best proximity point in complete metric spaces

Abstract: In this paper, we introduce a new type of contraction to seek the existence of tripled best proximity point results. Here, using the new contraction and P-property, we generalize and extend results of W. Shatanawi and A. Pitea and prove the existence and uniqueness of some tripled best proximity point results. Examples are also given to support our results.

Keywords: best proximity point, almost contraction, best proximity coupled point, tripled best proximity point, metric space

MSC 2010: 47H10, 54H25

1 Introduction

Fixed point theory has become the focus of many researchers and that is due the fact that it has many applications in different fields, such as physics, engineering, computer sciences, ..., etc,... However, sometimes maps do not have a fixed point so the best we can do is to get the minimum "distance" of a input and its output, which it turns out to be very interesting and it has many applications such a point is called best proximity point. Introduction of coupled fixed point by Guo and Lakshmikantham [1] in the year 1987 leads to the introduction of tripled fixed point by Vasile Berinde and Marine Borcut [2]. After this we had seen many coupled and tripled fixed point results on different spaces and under different contractions. B. Samet [3] proved some best proximity points theorems endowed with P-property. In [4] W. Shantanawi et. al. proved best proximity point and coupled best proximity point theorems. For more results on best proximity point and its application, readers can see research papers [5–11] and references therein.

W. Shantanawi et. al. [4] motivated us to introduce tripled best proximity point. In this paper, we proved some tripled best proximity point theorems and examples are also given.

Let A and B be any two nonempty subsets of a metric space (X, d). Define

$$P_A(x) = \{y \in X : d(x, y) = d(x, A)\},$$
$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\},$$
$$A_0 = \{x \in A : d(x, y) = d(A, B), \text{ for some } y \in B\},$$
$$\text{and } B_0 = \{y \in B : d(x, y) = d(A, B), \text{ for some } x \in B\}.$$

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2 Preliminaries

Definition 2.1. *[3] Let (X, d) be a metric space and $A \neq \phi, B \neq \phi$ are subsets of X. Let $T : A \to B$ be a mapping. Then $a \in A$ is said to be a best proximity point if and only if $d(a, Ta) = d(A, B)$.*

Definition 2.2. *[2] Let $F : X \times X \times X \to X$. An element (a, b, c) is called a tripled fixed point of F if $F(a, b, c) = a$, $F(b, a, b) = b$ and $F(c, b, a) = c$.*

Definition 2.3. *[7] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \Phi$. Then, the pair (A, B) has P-property if and only if*

$$\left(\begin{array}{c} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array} \right) \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 2.4. *[5] A map $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a comparison function if*

1. *$x < y \Rightarrow \phi(x) \leq \phi(y) \; \forall \, x, y \in \mathbb{R}^+;$*
2. *$\lim_{n \to +\infty} \phi^n(t) = 0.$*

If ϕ is a comparison function, we have $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$. Here $[0, +\infty)^6$ denote $[0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty)$. Let Θ denote collection of continuous functions $\theta : [0, +\infty)^6 \to [0, +\infty)$ such that

$$\theta(0, t, s, u, v, w) = 0 \; for \; all \; t, s, u, v, w \in [0, +\infty);$$
$$\theta(t, s, 0, u, v, w) = 0 \; for \; all \; t, s, u, v, w \in [0, +\infty);$$
$$and \; \; \theta(t, s, u, v, 0, w) = 0 \; for \; all \; t, s, u, v, w \in [0, +\infty).$$

Definition 2.5. *[4] Let θ be a continuous function in Θ and ϕ be a comparison function. A mapping $T : A \to B$ is said to be a generalized almost (ϕ, θ)-contraction if*

$$d(Tx, Ty) = \phi(d(x, y)) + \theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), d(x, Tx) - d(A, B), d(y, Ty) - d(A, B))$$

$$for \; all \; x, y \in A.$$

Definition 2.6. *[4] Let (X, d) be a metric space with $A \neq \phi$ and $B \neq \phi$ are closed subsets. Let $F : X \times X \to X$ be a mapping such that $d(u, F(u, v)) = d(A, B)$ and $d(v, F(v, u)) = d(A, B)$. Then F has a coupled best proximity point (u, v).*

Definition 2.7. *Let (X, d) be a complete metric space and $A \neq \phi$, $B \neq \phi$ are closed subsets. An element $(u, v, w) \in X \times X \times X$ is said to be a tripled best proximity point of $F : X \times X \times X \to X$ if $u, w \in A$ and $y \in B$ such that $d(u, F(u, v, w)) = d(A, B)$, $d(v, F(v, u, w)) = d(A, B)$ and $d(w, F(w, v, u)) = d(A, B)$.*

3 Main results

We prove the following theorem

Theorem 3.1. *Let (X, d) be a complete metric space. Let $A \neq \phi$, $B \neq \phi$ are closed subsets such that A_0 and B_0 are nonempty. Let $F : X \times X \times X \to X$ be a continuous mapping which satisfies*

(a) *$F(A_0 \times B_0 \times A_0) \subseteq B_0;$*
(b) *$F(B_0 \times A_0 \times B_0) \subseteq A_0;$*
(c) *Pair (A, B) has the (P)-property.*

Let θ be a continuous function in Θ and ϕ be a comparison function satisfying

$$d(F(x, y, z), F(u, v, w)) \leq \phi(\max\{d(x, u), d(y, v), d(z, w)\}) + \theta[d(u, F(x, y, z)) - d(A, B),$$
$$d(v, F(y, x, y)) - d(A, B), d(w, F(z, y, x)) - d(A, B),$$
$$d(x, F(x, y, z)) - d(A, B), d(y, F(y, x, y)) - d(A, B),$$
$$d(z, F(z, y, x)) - d(A, B)] \tag{3.1}$$

for all $x, y, z, u, v, w \in X$.

  Then (u, u, u) is the unique tripled best proximity point of F.

Proof. Choose $x_0, z_0 \in A_0$ and $y_0 \in B_0$. Since $F(x_0, y_0, z_0), F(z_0, y_0, x_0) \in B_0, F(y_0, x_0, y_0) \in A_0$, there exists $x_1, z_1 \in A$ and $y_1 \in B$ such that $d(x_1, F(x_0, y_0, z_0)) = d(y_1, F(y_0, x_0, y_0)) = d(z_1, F(z_0, y_0, x_0)) = d(A, B)$.

  Continuing this way, there exist sequences $\{x_n\}, \{z_n\}$ in A and $\{y_n\}$ in B such that

$$d(x_{n+1}, F(x_n, y_n, z_n)) = d(A, B); \tag{3.2}$$
$$d(y_{n+1}, F(y_n, x_n, y_n)) = d(A, B); \tag{3.3}$$
$$\text{and } d(z_{n+1}, F(z_n, y_n, x_n)) = d(A, B) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

If $d(x_n, x_{n+1}) = d(y_n, y_{n+1}) = d(z_n, z_{n+1}) = 0$ for all $n \in \mathbb{N}$, then we are done.

Suppose $d(x_n, x_{n+1}) > 0$ or $d(y_n, y_{n+1}) > 0$ or $d(z_n, z_{n+1}) > 0$.

  Now, by condition (c), $d(x_n, F(x_{n-1}, y_{n-1}, z_{n-1})) = d(A, B)$, $d(x_{n+1}, F(x_n, y_n, z_n)) = d(A, B)$, and using (3.1), we have

$$d(x_n, x_{n+1}) = d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n))$$
$$\leq \phi[\max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n)\}]$$
$$+ \theta[d(x_n, F(x_{n-1}, y_{n-1}, z_{n-1})) - d(A, B), d(y_n, F(y_{n-1}, x_{n-1}, y_{n-1})) - d(A, B)$$
$$d(z_n, F(z_{n-1}, y_{n-1}, x_{n-1})) - d(A, B), d(x_{n-1}, F(x_{n-1}, y_{n-1}, z_{n-1})) - d(A, B),$$
$$d(y_{n-1}, F(y_{n-1}, x_{n-1}, y_{n-1})) - d(A, B), d(z_{n-1}, F(z_{n-1}, y_{n-1}, x_{n-1})) - d(A, B)]$$
$$= \phi[\max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(z_{n-1}, z_n)\}]. \tag{3.5}$$

  Similarly, from (c), $d(y_n, F(y_{n-1}, x_{n-1}, y_{n-1})) = d(A, B)$, $d(y_{n+1}, F(y_n, x_n, y_n)) = d(A, B)$, and $d(z_n, F(z_{n-1}, y_{n-1}, x_{n-1})) = d(A, B)$, $d(z_{n+1}, F(z_n, y_n, x_n)) = d(A, B)$ respectively and using (3.1), we obtain

$$d(y_n, y_{n+1}) = d(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n))$$
$$\leq \phi[\max\{d(y_{n-1}, y_n), d(x_{n-1}, x_n), d(y_{n-1}, y_n)\}] \text{ and} \tag{3.6}$$

$$d(z_n, z_{n+1}) = d(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_n, y_n, x_n))$$
$$= \phi[\max\{d(z_{n-1}, z_n), d(y_{n-1}, y_n), d(x_{n-1}, x_n)\}]. \tag{3.7}$$

From (3.5), (3.6) and (3.7), we get

$$\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(z_n, z_{n+1})\} \leq \phi[\max\{d(z_{n-1}, z_n), d(y_{n-1}, y_n), d(x_{n-1}, x_n)\}] \tag{3.8}$$

Repeating (3.8) n-times, we obtain

$$\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}), d(z_n, z_{n+1})\} \leq \phi^n[\max\{d(x_0, x_1), d(y_0, y_1), d(z_0, z_0)\}].$$

Hence

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = \lim_{n \to +\infty} d(y_n, y_{n+1}) = \lim_{n \to +\infty} d(z_n, z_{n+1}) = 0.$$

Now,

$$d(A, B) \leq d(x_n, F(x_n, y_n, z_n))$$

$$\leq d(x_n, x_{n+1}) + d(x_{n+1}, F(x_n, y_n, z_n))$$
$$= d(x_n, x_{n+1}) + d(A, B)$$

which gives

$$\lim_{n \to +\infty} d(x_n, F(x_n, y_n, z_n)) = d(A, B).$$

Similarly,

$$\lim_{n \to +\infty} d(y_n, F(y_n, x_n, y_n)) = \lim_{n \to +\infty} d(z_n, F(z_n, y_n, x_n)) = d(A, B).$$

Let $\epsilon > 0$. When $n \to +\infty$, $\phi^n(\max\{d(x_0, x_1), d(y_0, y_1), d(z_0, z_1)\}) \to 0$ then there exists $n \in \mathbb{N}$, such that

$$d(x_n, x_{n+1}) < \frac{1}{2}(\epsilon - \phi(\epsilon)), d(y_n, y_{n+1}) < \frac{1}{2}(\epsilon - \phi(\epsilon)) \text{ and } d(z_n, z_{n+1}) < \frac{1}{2}(\epsilon - \phi(\epsilon)) \text{ for all } n \geq n_0.$$

Now, we have to prove

$$\max\{d(x_n, x_m), d(y_n, y_m), d(z_n, z_m)\} < \epsilon \text{ for all } m > n \geq n_0. \tag{3.9}$$

Suppose (3.9) is true for $m = k$. Now,

$$d(x_n, x_{k+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{k+1}). \tag{3.10}$$

From (c), $d(x_{n+1}, F(x_n, y_n, z_n)) = d(x_{k+1}, F(x_k, y_k, z_k)) = d(A, B)$, and using (3.1), we have

$$d(x_{n+1}, x_{k+1}) = d(F(x_n, y_n, z_n), F(x_k, y_k, z_k))$$
$$\leq \phi(\max\{d(x_n, x_k), d(y_n, y_k), d(z_n, z_k)\}) + \theta[d(x_k, F(x_n, y_n, z_n)) - d(A, B),$$
$$d(y_k, F(y_n, x_n, y_n)) - d(A, B), d(z_k, F(z_n, y_n, x_n)) - d(A, B), d(x_n, F(x_n, y_n, z_n)) - d(A, B),$$
$$d(y_n, F(y_n, x_n, y_n)) - d(A, B), d(z_n, F(z_n, y_n, x_n)) - d(A, B)] \tag{3.11}$$

Similarly,

$$d(y_{n+1}, y_{k+1}) = d(F(y_n, x_n, y_n), F(y_k, x_k, y_k))$$
$$\leq \phi(\max\{d(y_n, y_k), d(x_n, x_k), d(y_n, y_k)\}) + \theta[d(y_k, F(y_n, x_n, y_n)) - d(A, B),$$
$$d(x_k, F(x_n, y_n, x_n)) - d(A, B), d(y_k, F(y_n, x_n, y_n)) - d(A, B), d(y_n, F(y_n, x_n, y_n)) - d(A, B),$$
$$d(x_n, F(x_n, y_n, x_n)) - d(A, B), d(y_n, F(y_n, x_n, y_n)) - d(A, B)] \tag{3.12}$$

and

$$d(z_{n+1}, z_{k+1}) = d(F(z_n, y_n, x_n), F(z_k, y_k, x_k))$$
$$\leq \phi(\max\{d(z_n, z_k), d(y_n, y_k), d(x_n, x_k)\}) + \theta[d(z_k, F(z_n, y_n, x_n)) - d(A, B),$$
$$d(y_k, F(y_n, z_n, y_n)) - d(A, B), d(x_k, F(x_n, y_n, z_n)) - d(A, B), d(z_n, F(z_n, y_n, x_n)) - d(A, B),$$
$$d(y_n, F(y_n, z_n, y_n)) - d(A, B), d(x_n, F(x_n, y_n, z_n)) - d(A, B)]. \tag{3.13}$$

By using the properties of θ, $\lim_{n \to +\infty} d(x_n, F(x_n, y_n, z_n)) = d(A, B)$, $\lim_{n \to +\infty} d(y_n, F(y_n, x_n, y_n)) = d(A, B)$ and $\lim_{n \to +\infty} d(z_n, F(z_n, y_n, x_n)) = d(A, B)$, we have

$$\limsup_{n \to +\infty} \theta[d(x_k, F(x_n, y_n, z_n)) - d(A, B), d(y_k, F(y_n, x_n, y_n)) - d(A, B), d(z_k, F(z_n, y_n, x_n)) - d(A, B),$$
$$d(x_n, F(x_n, y_n, z_n)) - d(A, B), d(y_n, F(y_n, x_n, y_n)) - d(A, B), d(z_n, F(z_n, y_n, x_n)) - d(A, B)] = 0;$$

$$\limsup_{n \to +\infty} \theta[d(y_k, F(y_n, x_n, y_n)) - d(A, B), d(x_k, F(x_n, y_n, x_n)) - d(A, B), d(y_k, F(y_n, x_n, y_n)) - d(A, B),$$
$$d(y_n, F(y_n, x_n, y_n)) - d(A, B), d(x_n, F(x_n, y_n, x_n)) - d(A, B), d(y_n, F(y_n, x_n, y_n)) - d(A, B)] = 0$$

and

$$\limsup_{n\to+\infty} \theta[d(z_k, F(z_n, y_n, x_n)) - d(A, B), d(y_k, F(y_n, z_n, y_n)) - d(A, B), d(x_k, F(x_n, y_n, z_n)) - d(A, B),$$
$$d(z_n, F(z_n, y_n, x_n)) - d(A, B), d(y_n, F(y_n, z_n, y_n)) - d(A, B), d(x_n, F(x_n, y_n, z_n)) - d(A, B)] = 0.$$

When taking n_0 large enough, we have

$$\theta[d(x_k, F(x_n, y_n, z_n)) - d(A, B), d(y_k, F(y_n, x_n, y_n)) - d(A, B),$$
$$d(z_k, F(z_n, y_n, x_n)) - d(A, B), d(x_n, F(x_n, y_n, z_n)) - d(A, B),$$
$$d(y_n, F(y_n, x_n, y_n)) - d(A, B), d(z_n, F(z_n, y_n, x_n)) - d(A, B)] < \frac{1}{2}(\varepsilon - \phi(\varepsilon)), \tag{3.14}$$

$$\theta[d(y_k, F(y_n, x_n, y_n)) - d(A, B), d(x_k, F(x_n, y_n, x_n)) - d(A, B),$$
$$d(y_k, F(y_n, x_n, y_n)) - d(A, B), d(y_n, F(y_n, x_n, y_n)) - d(A, B),$$
$$d(x_n, F(x_n, y_n, x_n)) - d(A, B), d(y_n, F(y_n, x_n, y_n)) - d(A, B)] < \frac{1}{2}(\varepsilon - \phi(\varepsilon)) \tag{3.15}$$

and

$$\theta[d(z_k, F(z_n, x_n, z_n)) - d(A, B), d(y_k, F(y_n, z_n, y_n)) - d(A, B),$$
$$d(x_k, F(x_n, y_n, z_n)) - d(A, B), d(z_n, F(z_n, y_n, x_n)) - d(A, B),$$
$$d(y_n, F(y_n, z_n, y_n)) - d(A, B), d(x_n, F(x_n, y_n, z_n)) - d(A, B)] < \frac{1}{2}(\varepsilon - \phi(\varepsilon)) \tag{3.16}$$

From the relations (3.9)-(3.16), we get

$$\max\{d(x_n, x_{k+1}), d(y_n, y_{k+1}), d(z_n, z_{k+1})\} < \varepsilon.$$

Thus (3.9) is true for all $m \geq n \geq n_0$. Hence, $\{x_n\}$ and $\{z_n\}$ are Cauchy sequences in A and $\{y_n\}$ in B. Since (X, d) is complete, there exist $u, v, w \in X$ such that

$$\lim_{n\to+\infty} x_n = u, \ \lim_{n\to+\infty} z_n = w \text{ and } \lim_{n\to+\infty} y_n = v.$$

Since A and B are closed, we get $u, w \in A$ and $v \in B$. Since F is continuous,

$$\lim_{n\to+\infty} d(x_{n+1}, F(x_n, y_n, z_n)) = d(A, B) \Rightarrow d(u, F(u, v, w)) = d(A, B).$$

Similarly, $d(v, F(v, u, v)) = d(A, B)$ and $d(w, F(w, v, u)) = d(A, B)$.

Thus, (u, v, w) is a tripled best proximity point of F. Now, we show that $u = v = w$.
Lastly, from (c) and using (3.1), we have

$$d(u, w) = d(F(u, v, w), F(w, v, u)) \leq \phi(d(u, w)) \Rightarrow u = w. \tag{3.17}$$

Therefore, $u = v = w$.

To prove the uniqueness, let t be another tripled best proximity point. Now,

$$d(u, t) = d(F(u, u, u), F(t, t, t)) \leq \phi(d(u, t)) \Rightarrow u = t.$$

This completes the proof. $\square$

Theorem 3.2. *Let (X, d) be a complete metric space. Let $A \neq \phi$, $B \neq \phi$ are closed subsets such that A_0 and B_0 are nonempty. Let $F : X \times X \times X \to X$ be a continuous mapping which satisfies*

1. *$F(A_0 \times A_0 \times A_0) \subseteq B_0$ or $F(B_0 \times B_0 \times B_0) \subseteq A_0$;*
2. *Pair (A, B) has the (P)-property.*

Let θ be a continuous function in Θ and ϕ be a comparison function satisfying

$$d(F(x, y, z), F(u, v, w)) \le \phi(\max\{d(x, u), d(y, v), d(z, w)\}) + \theta[d(u, F(x, y, z)) - d(A, B),$$
$$d(v, F(y, x, y)) - d(A, B), d(w, F(z, y, x)) - d(A, B),$$
$$d(x, F(x, y, z)) - d(A, B), d(y, F(y, x, y)) - d(A, B),$$
$$d(z, F(z, y, x)) - d(A, B)]$$

for all $x, y, z, u, v, w \in X$.

   Then (u, u, u) is the unique tripled best proximity point of F.

Proof. Choose $x_0, y_0, z_0 \in A_0$. Since $F(A_0 \times A_0 \times A_0) \subseteq B_0$, we get $F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0) \in B_0$. Then by following Theorem 3.1, we get that (u, u, u) is the unique tripled best proximity point. $\qquad\square$

   Taking $A = B$ in Theorem 3.1, we can get a triple fixed point which is given below:

Theorem 3.3. *Let (X, d) be a complete metric space. Let $A \ne \phi$ be a closed subset. Let $F : X \times X \times X \to X$ be a continuous mapping such that $F(A \times A \times A) \subseteq A$, θ be a continuous function in Θ and ϕ be a comparison function satisfying*

$$d(F(x, y, z), F(u, v, w)) \le \phi(\max\{d(x, u), d(y, v), d(z, w)\}) + \theta[d(u, F(x, y, z)),$$
$$d(v, F(y, x, y)), d(w, F(z, y, x)), d(x, F(x, y, z)),$$
$$d(y, F(y, x, y)), d(z, F(z, y, x))] \;\; \text{for all } x, y, z, u, v, w \in X.$$

Then (u, u, u) is the unique tripled point of F.

Example 3.4. *Consider $X = \{0, 2, 3, 4, 5\}$ and $d(x, y) = \frac{x-y}{2}$ for all $x, y \in X$. Let $U = \{2, 5\}$ and $V = \{2, 4\}$ be subsets of X. Let $F : X \times X \times X \to X$ be a continuous mapping given by $F(x, y, z) = x + y - z$ for all $x, y, z \in X$, $\theta : [0; +\infty)^6 \to [0, +\infty)$ given by $\theta(r, s, t, u, v, w) = \min\{r, s, t, u, v, w\}$ and $\phi : [0, +\infty) \to [0, +\infty)$ be given by $\phi(t) = \frac{t}{1+t}$.*

Proof. Here, $A_0 = \{2\}$, $B_0 = \{2\}$, $d(A, B) = 0$. Take $x, z \in A_0$ and $y \in B_0$, then clearly $F(A_0 \times B_0 \times A_0) \subseteq B_0$, $F(B_0 \times A_0 \times B_0) \subseteq A_0$, condition (c) of Theorem 3.1 is true, satisfying (3.1) and also all the conditions of Theorem 3.2. Hence, from Theorem 3.1 and Theorem 3.2, (2,2,2) is the unique tripled best proximity point. $\qquad\square$

Example 3.5. *Consider $(X, d) = \mathbb{R}$, $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Let $U = [1, 2]$ and $V = [-2, -1]$ be subsets of X. Let $F : X \times X \times X \to X$ be a continuous mapping given by $F(x, y, z) = \frac{x+y-z}{3}$ for all $x, y, z \in X$, $\theta : [0; +\infty)^6 \to [0, +\infty)$ given by $\theta(r, s, t, u, v, w) = \min\{r, s, t, u, v, w\}$ and $\phi : [0, +\infty) \to [0, +\infty)$ be given by $\phi(t) = \frac{t}{1+t}$.*

Proof. Here, $A_0 = \{1\}$, $B_0 = \{-1\}$, $d(A, B) = 2$. Take $x, z \in A_0$ and $y \in B_0$, then clearly $F(A_0 \times B_0 \times A_0) \subseteq B_0$, $F(B_0 \times A_0 \times B_0) \subseteq A_0$, the pair (A, B) has the (P)-property and (1,-1,1) is the unique tripled best proximity point but not of the form (u, u, u). This is because (3.1) is not satisfied. Therefore, by Theorem 3.1, we cannot get the results. $\qquad\square$

4 Conclusion

In closing, we would like to bring to the readers' attention that our results were proven in metric spaces. So, we can prove these results in partial metric spaces, metric like spaces, or M-metric spaces.

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