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The hyperbolic polygons of type (ϵ, n) and Möbius transformations

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Abstract: An n-sided hyperbolic polygon of type (ϵ, n) is a hyperbolic polygon with ordered interior angles $\frac{\pi}{2}+\epsilon, \theta_1, \theta_2, \dots, \theta_{n-2}, \frac{\pi}{2}-\epsilon$, where $0<\epsilon<\frac{\pi}{2}$ and $0<\theta_i<\pi$ satisfying

$$\sum_{i=1}^{n-2}\theta_i + \left(\frac{\pi}{2}+\epsilon\right) + \left(\frac{\pi}{2}-\epsilon\right) < (n-2)\pi$$

and $\theta_i + \theta_{i+1} \neq \pi$ $(1 \le i \le n-3)$, $\theta_1 + (\frac{\pi}{2}+\epsilon) \neq \pi$, $\theta_{n-2} + (\frac{\pi}{2}-\epsilon) \neq \pi$. In this paper, we present a new characterization of Möbius transformations by using n-sided hyperbolic polygons of type (ϵ, n). Our proofs are based on a geometric approach.

Keywords: conformal mapping, Möbius transformations, hyperbolic polygons

MSC 2010: 30C20, 30C35, 30F45, 51M10, 51M15

1 Introduction

A Möbius transformation $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a map defined by $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. They are the automorphisms of extended complex plane $\overline{\mathbb{C}}$ and define the Möbius transformation group $M(\overline{\mathbb{C}})$ with respect to composition. Möbius transformations are also directly conformal homeomorphisms of $\overline{\mathbb{C}}$ onto itself and they have beautiful properties. For example, a map is Möbius if and only if it preserves cross ratios. As for geometric aspect, circle-preserving is another important characterization of Möbius transformations. The following result is one of the most famous theorems for Möbius transformations:

Theorem 1. *[1] If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a circle preserving map, then f is a Möbius transformation if and only if f is a bijection.*

The transformations $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ are known as conjugate Möbius transformations of $\overline{\mathbb{C}}$. It is easy to see that each conjugate Möbius transformation f is the composition of complex conjugation with a Möbius transformation. Since the complex conjugate transformation and Möbius transformations are homeomorphisms of $\overline{\mathbb{C}}$ onto itself (complex conjugation is given by reflection in the plane through $\mathbb{R}\cup\{\infty\}$), conjugate Möbius transformations are homeomorphisms of $\overline{\mathbb{C}}$ onto itself. Notice that the composition of a conjugate Möbius transformation with a Möbius transformation is a conjugate Möbius transformation and composition of two conjugate Möbius transformations is a Möbius transformation. There is topological distinction between Möbius transformations and conjugate Möbius transformations: Möbius transformations preserve the orientation of $\overline{\mathbb{C}}$, whereas conjugate Möbius transformations reverse it. To see

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more details about conjugate Möbius transformations, we refer the reader to [2]. The following definitions are well known and fundamental in hyperbolic geometry.

Definition 2. *[3] A Lambert quadrilateral is a hyperbolic quadrilateral with ordered interior angles* $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$ *and* θ, *where* $0 < \theta < \frac{\pi}{2}$.

Definition 3. *[3] A Saccheri quadrilateral is a hyperbolic quadrilateral with ordered interior angles* $\frac{\pi}{2}, \frac{\pi}{2}, \theta, \theta$, *where* $0 < \theta < \frac{\pi}{2}$.

A Möbius invariant property is naturally related to hyperbolic geometry. To see the characteristics of Möbius transformations involving Lambert quadrilaterals and Saccheri quadrilaterals, we refer the reader to [4]. Moreover, there are many characterizations of Möbius transformations by using various hyperbolic polygons; see, for instance, [5–7].

In [8, 9], O. Demirel presented some characterizations of Möbius transformations by using new classes of geometric hyperbolic objects called "degenerate Lambert quadrilaterals"and "degenerate Saccheri quadrilaterals", respectively.

Definition 4. *[9] A degenerate Lambert quadrilateral is a hyperbolic convex quadrilateral with ordered interior angles* $\frac{\pi}{2} + \epsilon, \frac{\pi}{2}, \frac{\pi}{2} - \epsilon, \theta$, *where* $0 < \theta < \frac{\pi}{2}$ *and* $0 < \epsilon < \frac{\pi}{2} - \frac{\theta}{2}$.

Theorem 5. *[9] Let* $f : B^2 \to B^2$ *be a surjective transformation. Then f is a Möbius transformation or a conjugate Möbius transformation if and only if f preserves all* ϵ*-Lambert quadrilaterals.*

Definition 6. *[8] A degenerate Saccheri quadrilateral is a hyperbolic convex quadrilateral with ordered angles* $\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon, \theta, \theta$, *where* $0 < \theta < \frac{\pi}{2}$ *and* $0 < \epsilon < \frac{\pi}{2} - \frac{\theta}{2}$.

Theorem 7. *[8] Let* $f : B^2 \to B^2$ *be a surjective transformation. Then f is a Möbius transformation or a conjugate Möbius transformation if and only if f preserves all* ϵ*-Saccheri quadrilaterals.*

In the theorems above, B^2 is the open unit disc in the complex plane. Naturally, one may wonder whether the counterpart of *Theorem 7* exists for hyperbolic polygons instead of using degenerate Saccheri quadrilaterals. Before giving the affirmative answer of this question let us state the following definition:

Definition 8. *Let n be a positive integer satisfying* $n \geq 5$. *An n-sided hyperbolic polygon of type* (ϵ, n) *is a convex hyperbolic polygon with ordered interior angles* $\frac{\pi}{2} + \epsilon, \theta_1, \theta_2, \ldots, \theta_{n-2}, \frac{\pi}{2} - \epsilon$ *or* $\frac{\pi}{2} - \epsilon, \theta_1, \theta_2, \ldots, \theta_{n-2}, \frac{\pi}{2} + \epsilon$, *where* $0 < \epsilon < \frac{\pi}{2}$ *and* $0 < \theta_i < \pi$ *satisfying*

$$\sum_{i=1}^{n-2} \theta_i + \left(\frac{\pi}{2} + \epsilon\right) + \left(\frac{\pi}{2} - \epsilon\right) < (n-2)\pi,$$

and $\theta_i + \theta_{i+1} \neq \pi$ $(1 \leq i \leq n-3)$, $\theta_1 + (\frac{\pi}{2} + \epsilon) \neq \pi$, $\theta_{n-2} + (\frac{\pi}{2} - \epsilon) \neq \pi$. *Notice that the sides of the n-sided hyperbolic polygon of type* (ϵ, n) *we mentioned here are hyperbolic line segments.*

The existence of n-sided hyperbolic polygons of type (ϵ, n) is clear by the following result:

Lemma 9. *[3] Let* $(\theta_1, \theta_2, \cdots, \theta_n)$ *be any ordered n-tuple with* $0 \leq \theta_j < \pi$, $j = 1, 2, \ldots, n$. *Then there exists a polygon P with interior angles* $\theta_1, \theta_2, \ldots, \theta_n$, *occurring in this order around* ∂P, *if and only if* $\sum_{i=1}^{n} \theta_i < (n-2)\pi$.

This paper presents a new characterization of Möbius transformations by use of mappings which preserve n-sided hyperbolic polygons of type (ϵ, n). To do so, we need Carathéodory's theorem which plays a major role in our results. C. Carathéodory [10] proved that every arbitrary one-to-one correspondence between the points of a circular disc C and a bounded point set C' such that which circles lying completely in C are transformed into circles lying in C' must always be either a Möbius transformation or a conjugate Möbius transformation.

Throughout the paper we denote by X' the image of X under f, by $[P, Q]$ the geodesic segment between points P and Q, by PQ the geodesic through points P and Q, by PQR the hyperbolic triangle with vertices P, Q and R, by $\angle PQR$ the angle between $[P, Q]$ and $[P, R]$ and by $d_H(P, Q)$ the hyperbolic distance between points P and Q. We consider the hyperbolic plane $B^2 = \{z \in \mathbb{C} : |z| < 1\}$ with length differential $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$. The Poincaré disc model of hyperbolic geometry is built on B^2; more precisely the points of this model are points of B^2 and the hyperbolic lines of this model are Euclidean semicircular arcs that intersect the boundary of B^2 orthogonally including diameters of B^2. Given two distinct hyperbolic lines that intersect at a point, the measure of the angle between these hyperbolic lines is defined by the Euclidean tangents at the common point.

2 A characterization of Möbius transformations by use of hyperbolic polygons of type (ϵ, n)

The assertion f preserves n-sided hyperbolic polygons $A_1A_2\cdots A_n$ of type (ϵ, n), $n \geq 5$, with ordered interior angles $\frac{\pi}{2}-\epsilon, \theta_1, \theta_2, \ldots, \theta_{n-2}, \frac{\pi}{2}+\epsilon$ means that the image of $A_1A_2\cdots A_n$ under f is again an n-sided hyperbolic polygon $A_1'A_2'\cdots A_n'$ with ordered interior angles $\frac{\pi}{2}-\epsilon, \theta_1', \theta_2', \ldots, \theta_{n-2}', \frac{\pi}{2}+\epsilon$ and if P is a point on any side of $A_1A_2\cdots A_n$, then P' is a point on any side of $A_1'A_2'\cdots A_n'$.

Lemma 10. *Let $f : B^2 \to B^2$ be a mapping which preserves n-sided hyperbolic polygons of type (ϵ, n) for all $0 < \epsilon < \frac{\pi}{2}$. Then f is injective.*

Proof. Let A_1 and A_2 be two distinct points in B^2. It is clear that there exists an $(2n-4)$-sided hyperbolic regular polygon ($n > 4$, $n \in \mathbb{N}$), say $A_1A_2\cdots A_{2n-4}$. By β denote the interior angles of $A_1A_2\cdots A_{2n-4}$. Let M and N be the midpoints of $[A_{2n-4}, A_1]$ and $[A_{n-2}, A_{n-1}]$, respectively. Then the hyperbolic polygons $MA_1A_2\cdots A_{n-2}N$ and $NA_{n-1}A_n\cdots A_{2n-4}M$ are n-sided hyperbolic polygons satisfying $MN \perp A_1A_{2n-4}$ and $MN \perp A_{n-2}A_{n-1}$. Let P be a point on $[M, A_1]$ and let Q be a point on $[N, A_{n-1}]$ satisfying $d_H(P, A_1) = d_H(Q, A_{n-1})$. By ψ denote the angle $\angle QPA_1$. Since $A_1A_2A_3\cdots A_{2n-4}$ is an $(2n-4)$-sided hyperbolic regular polygon, we immediately get $\angle PQA_{n-1} = \psi$. If $\psi > \frac{\pi}{2}$ let's denote $\psi = \frac{\pi}{2}+\alpha$ and if $\psi < \frac{\pi}{2}$ let's denote $\psi = \frac{\pi}{2}-\alpha$. Hence we see that $PA_1A_2\cdots A_{n-2}Q$ is an n-sided hyperbolic polygon of type (α, n). By assumption, we obtain that $P'A_1'A_2'\cdots A_{n-2}'Q'$ is also an n-sided hyperbolic polygon of type (α, n), which implies $A_1' \neq A_2'$. Thus f is injective. □

Lemma 11. *Let $f : B^2 \to B^2$ be a mapping which preserves n-sided hyperbolic polygons of type (ϵ, n) for all $0 < \epsilon < \frac{\pi}{2}$. Then f preserves the collinearity and betweenness property of the points.*

Proof. Let P and Q be two distinct points in B^2 and assume that S is an interior point of $[P, Q]$. Let Δ be the set of all n-sided hyperbolic polygons of type (ϵ, n) such that the points P and Q are two adjacent vertices of these hyperbolic polygons. Then S belongs to all elements of Δ. By the property of f, the images of the elements of Δ are n-sided hyperbolic polygons of type (ϵ, n) whose vertices contain P' and Q'. Moreover, the images of the elements of Δ must contain S'. Since f is injective by *Lemma 10*, we get $P' \neq S' \neq Q'$. Therefore, S' must be an interior point of $[P', Q']$, which implies that f preserves the collinearity and betweenness of the points. □

Lemma 12. *Let $f : B^2 \to B^2$ be a mapping which preserves n-sided hyperbolic polygons of type (ϵ, n) for all $0 < \epsilon < \frac{\pi}{2}$. Then f preserves the angles $\frac{\pi}{2}+\epsilon$ and $\frac{\pi}{2}-\epsilon$.*

Proof. Let $A_1A_2\cdots A_n$ be an n-sided hyperbolic polygon of type (ϵ, n) such that $\angle A_nA_1A_2 = \frac{\pi}{2} - \epsilon$, $\angle A_{i-1}A_iA_{i+1} = \theta_i$ $(2 \leq i \leq n-1)$ and $\angle A_{n-1}A_nA_1 = \frac{\pi}{2}+\epsilon$. Let us denote the midpoint of A_1 and A_n by M. Clearly, the hyperbolic line A_1A_2 and the complex unit disc B^2 meet at two points, say P and Q. Assume $A_1 \in [P, A_2]$ and $\angle MA_2Q = \rho$. Let X be a point on $[P, A_2]$ moving from P to A_2. It is easy to see that if X moves from P to A_2, the angle $\angle MXQ$ increases continuously from 0 to ρ. Let H be a point on $[A_1, A_2]$ such that $\angle MHA_2 < \frac{\pi}{2}$. Now take a point on the hyperbolic line A_nA_{n-1}, say S, satisfying $A_n \in [S, A_{n-1}]$ and $d_H(S, A_n) = d_H(H, A_1)$. It is easy to see that $d_H(M, A_1) = d_H(M, A_n)$, $\angle SA_nM = \angle HA_1M$, and $d_H(A_1, H) = d_H(A_n, S)$ hold, which implies that the hyperbolic triangles HA_1M and SA_nM are congruent triangles by the *hyperbolic side-angle-side theorem*. Hence we get $\angle A_1MH = \angle A_nMS$ and this yields that the points H, M and S must be collinear and $\angle MHA_1 = \angle MSA_n$. Since $\angle MHA_2 < \frac{\pi}{2}$, we may assume the representation $\angle MHA_2 = \frac{\pi}{2} - \alpha$, where $0 < \alpha < \frac{\pi}{2}$, which implies $\angle MSA_n = \frac{\pi}{2} + \alpha$. Notice that α must be less than ϵ since $\frac{\pi}{2} - \epsilon < \frac{\pi}{2} - \alpha$. Therefore, one can easily see that $HA_2 \cdots A_{n-1}S$ is an n-sided hyperbolic polygon of type (α, n). By the property of f, the images of the hyperbolic polygons $A_1A_2 \cdots A_n$ and $HA_2 \cdots A_{n-1}S$ are n-sided hyperbolic polygons of type (ϵ, n) and type (α, n), respectively. The hyperbolic polygons $A_1'A_2' \cdots A_n'$ and $H'A_2' \cdots A_{n-1}'S'$ have $n - 2$ common angles, which implies $\angle A_n'A_1'A_2' = \frac{\pi}{2} \pm \epsilon$ and $\angle S'H'A_2' = \frac{\pi}{2} \pm \alpha$. Now assume $\angle A_n'A_1'A_2' = \frac{\pi}{2} + \epsilon$. By *Lemma 11*, we get $H' \in [A_1', A_2']$, $M' \in [A_1', A_n']$ and $M' \in [H', S']$. Since $\angle S'H'A_2' = \frac{\pi}{2} \pm \alpha$ and $\alpha < \epsilon$ hold, we get that the sum of the measures of interior angles of the hyperbolic triangle $M'A_1'H'$ is $\angle A_1'M'H' + (\frac{\pi}{2} + \epsilon) + (\frac{\pi}{2} \pm \alpha)$, which is greater then π. This is not possible in hyperbolic geometry and so we get $\angle A_n'A_1'A_2' = \frac{\pi}{2} - \epsilon$, which yields $\angle A_{n-1}'A_n'A_1' = \frac{\pi}{2} + \epsilon$. □

Lemma 13. *Let $f : B^2 \to B^2$ be a mapping which preserves n-sided hyperbolic polygons of type (ϵ, n) for all $0 < \epsilon < \frac{\pi}{2}$. Then f preserves the hyperbolic distance.*

Proof. Let P and Q be two distinct points in B^2. Take a point S such that PQS forms a hyperbolic equilateral triangle. By β denote its angles $\angle PQS = \angle QSP = \angle SPQ := \beta$. Since $\beta < \frac{\pi}{2}$ let's use the representation $\beta = \frac{\pi}{2} - \alpha$ with $0 < \alpha < \frac{\pi}{2}$. *By Lemma 9*, there exists an n-sided hyperbolic polygon of type (α, n), say $A_1A_2 \cdots A_n$, such that $\angle A_1A_2A_3 = \gamma_2, \angle A_2A_3A_4 = \gamma_3, \ldots, \angle A_{n-2}A_{n-1}A_n = \gamma_{n-1}$, $\angle A_{n-1}A_nA_1 = \frac{\pi}{2} + \alpha$ and $\angle A_nA_1A_2 = \frac{\pi}{2} - \alpha$. Then the angle $\angle A_nA_1A_2$ of the hyperbolic polygon $A_1A_2 \cdots A_n$ can be moved to the point P by using an appropriate hyperbolic isometry g such that $g(A_2) \in [P, S]$ (or $S \in [P, g(A_2)]$) and $g(A_n) \in [P, Q]$ (or $Q \in [P, g(A_n)]$). Since f preserves the angles $\frac{\pi}{2} + \epsilon$ and $\frac{\pi}{2} - \epsilon$ of n-sided hyperbolic polygons of type (ϵ, n) for all $0 < \epsilon < \frac{\pi}{2}$ by *Lemma 12*, we get

$$\frac{\pi}{2} - \alpha = \angle SPQ = \angle g(A_n)g(A_1)g(A_2) = \angle g(A_n)'g(A_1)'g(A_2)' = \angle S'P'Q',$$

which implies $\angle PQS = \angle P'Q'S'$ and $\angle QSP = \angle Q'S'P'$. Because of the fact that the angles at the vertices of a hyperbolic triangle determine its lengths, we get $d_H(P, Q) = d_H(P', Q')$; see [11, 12]. □

Corollary 14. *Let $f : B^2 \to B^2$ be a mapping which preserves n-sided hyperbolic polygons of type (ϵ, n) for all $0 < \epsilon < \frac{\pi}{2}$. Then f preserves the measures of all angles.*

Theorem 15. *Let $f : B^2 \to B^2$ be a surjective transformation. Then f is a Möbius transformation or a conjugate Möbius transformation if and only if f preserves n-sided hyperbolic polygons of type (ϵ, n) for all $0 < \epsilon < \frac{\pi}{2}$.*

Proof. The "only if" part is clear since f is an isometry. Conversely, we may assume that f preserves all n-sided hyperbolic polygons of type (ϵ, n) in B^2 and $f(O) = O$ by composing an hyperbolic isometry if necessary. Let us take two distinct points in B^2 and denote them by x, y. By *Lemma 13*, we immediately get $d_H(O, x) = d_H(O, x')$ and $d_H(O, y) = d_H(O, y')$, namely $|x| = |x'|$ and $|y| = |y'|$, where $|\cdot|$ denotes the Euclidean norm. Therefore, we get $|x - y| = |x' - y'|$ since f preserves angular sizes by *Corollary 14*. As

$$2\langle x, y\rangle = |x|^2 + |y|^2 - |x - y|^2 = |x'|^2 + |y'|^2 - |x' - y'|^2 = 2\langle x', y'\rangle,$$

f preserves the Euclidean inner-product this implies that f is a restriction of an orthogonal transformation on B^2, that is, f is a Möbius transformation or a conjugate Möbius transformation by *Carathéodory's theorem*. □

Corollary 16. *Let $f : B^2 \to B^2$ be a conformal (angle preserving with sign) surjective transformation. Then f is a Möbius transformation if and only if f preserves n-sided hyperbolic polygons of type (ϵ, n) for all $0 < \epsilon < \frac{\pi}{2}$.*

Corollary 17. *Let $f : B^2 \to B^2$ be an angle reversing surjective transformation. Then f is a conjugate Möbius transformation if and only if f preserves n-sided hyperbolic polygons of type (ϵ, n) for all* $0 < \epsilon < \frac{\pi}{2}$.

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