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On more general forms of proportional fractional operators

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Abstract: In this article, more general types of fractional proportional integrals and derivatives are proposed. Some properties of these operators are discussed.

Keywords: proportional derivative of a function with respect to another function, general fractional proportional integrals, general fractional proportional derivatives

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1 Introduction

The fractional calculus, which is engaged in integral and differential operators of arbitrary orders, is as old as the conceptional calculus that deals with integrals and derivatives of non-negative integer orders. Since not all of the real phenomena can be modeled using the operators in the traditional calculus, researchers searched for generalizations of these operators. It turned out that the fractional operators are excellent tools to use in modeling long-memory processes and many phenomena that appear in physics, chemistry, electricity, mechanics and many other disciplines. Here, we invite the readers to read [1–10] and the reference cited in these books. However, targeting the best understanding more accurate modeling real world problems, researchers were in need of other types of fractional operators that were confined to Riemann-Liouville fractional operators. In the literature, one can find many works that propose new fractional operators. We mention [11–16]. Nonetheless, the fractional integrals and derivatives which were proposed in these works were just particular cases of what so called fractional integrals/derivatives with dependence on a kernel function [2, 5, 17]. There are other types of fractional operators which were suggested in the literature.

On the other hand, due to the singularities found in the traditional fractional operators which are thought to make some difficulties in the modeling process, some researches recently proposed new types of non-singular fractional operators. Some of these operators contain exponential kernels and some of them involve the Mittag-Leffler functions. For such types of fractional operators we refer to [18–27].

All the fractional operators considered in the references in the first and the second paragraphs are non-local. However, there are many local operators found in the literature that allow differentiation to a non-integer order and these are called local fractional operators. In [28], the authors presented what they called conformable (fractional) derivative. The author in [29] proposed other basic concepts related to the

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conformable derivatives. We would like to mention that the fractional operators proposed in [12, 13] are the non-local fractional version of the local operators suggested in [28]. In addition, the non-local fractional version of the ones in [29] can be seen in [16].

It is customary that any derivative of order 0 when performed to a function should give the function itself. This essential property is dispossessed by the conformable derivatives. Notwithstanding, in [30, 31], for sake of overcoming this obstacle, the authors proposed a new definition of the conformal derivative that gives the function itself when the order of the local derivative approaches 0. In addition to this, the non-local fractional operators that emerge from iterating the above-mentioned derivative were held forth in [32].

In this article, we extend the work done in [32] to introduce a new fractional operators relying on the proportional derivatives of a function with respect to another function which can be defined in parallel with the definitions discussed in [30]. The kernel obtained in the fractional operators which will be proposed contains an exponential function and is function dependent. The semi-group properties will be discussed.

The article is organized as follows: Section 2 presents some essential definitions for fractional derivatives and integrals. In Section 3 we present the general forms of the fractional proportional integrals and derivatives. In section 4, we present the general form of Caputo fractional proportional derivatives. In the end, we conclude our results.

2 Preliminaries

In this section, we present some principal definitions of fractional operators. We first present the traditional fractional operators and then the fractional proportional operators.

2.1 The conventional fractional operators and their general forms

For $\omega \in \mathbb{C}$, $Re(\omega) > 0$, the forward (left) ω th order Riemann–Liouville fractional integral is defined by

$$({}_a I^\omega f)(x) = \frac{1}{\Gamma(\omega)} \int_a^x (x-u)^{\omega-1} f(u) du. \quad (2.1)$$

The backward (right) ω th Riemann–Liouville fractional integral reads

$$({}_b I^\omega f)(x) = \frac{1}{\Gamma(\omega)} \int_x^b (u-x)^{\omega-1} f(u) du. \quad (2.2)$$

The forward ω th order Riemann–Liouville fractional derivative, where $Re(\omega) \geq 0$ is given as

$$({}_a D^\omega f)(x) = \left(\frac{d}{dx} \right)^n ({}_a I^{n-\omega} f)(x), \quad n = [\omega] + 1. \quad (2.3)$$

The backward ω th order Riemann–Liouville fractional derivative, where $Re(\omega) \geq 0$ reads

$$({}_b D^\omega f)(t) = \left(-\frac{d}{dt} \right)^n ({}_b I^{n-\omega} f)(t). \quad (2.4)$$

The forward Caputo fractional derivative has the following form

$$({}_a^C D^\omega f)(x) = ({}_a I^{n-\omega} f^{(n)})(x), \quad n = [\omega] + 1. \quad (2.5)$$

The backward Caputo fractional derivative reads

$$({}_b^C D^\omega f)(x) = ({}_b I^{n-\omega} (-1)^n f^{(n)})(x). \quad (2.6)$$

The generalized forward and backward fractional integrals Katugampola settings [12] are given respectively as

$$({}_a I^{\omega, \sigma} f)(x) = \frac{1}{\Gamma(\omega)} \int_a^x \left(\frac{x^\sigma - u^\sigma}{\sigma}\right)^{\omega-1} f(u) \frac{du}{u^{1-\sigma}} \tag{2.7}$$

and

$$(I_b^{\omega, \sigma} f)(x) = \frac{1}{\Gamma(\omega)} \int_t^b \left(\frac{u^\sigma - x^\sigma}{\sigma}\right)^{\omega-1} f(u) \frac{du}{u^{1-\sigma}}. \tag{2.8}$$

The generalized forward and backward fractional derivatives in the sense of Katugampola [13] are defined respectively as

$$({}_a D^{\omega, \sigma} f)(x) = \gamma^n ({}_a I^{n-\omega, \sigma} f)(t) = \frac{\gamma^n}{\Gamma(n-\omega)} \int_a^x \left(\frac{x^\sigma - u^\sigma}{\sigma}\right)^{n-\omega-1} f(u) \frac{du}{u^{1-\sigma}} \tag{2.9}$$

and

$$(D_b^{\omega, \sigma} f)(x) = (-\gamma)^n (I_b^{n-\omega, \sigma} f)(x) = \frac{(-\gamma)^n}{\Gamma(n-\omega)} \int_x^b \left(\frac{u^\sigma - x^\sigma}{\sigma}\right)^{n-\omega-1} f(u) \frac{du}{u^{1-\sigma}}, \tag{2.10}$$

where $\sigma > 0$ and $\gamma = x^{1-\sigma} \frac{d}{dx}$. The Caputo modification of the forward and backward generalized fractional derivatives are proposed in [14] in the following forms respectively

$$({}^C D_a^{\omega, \sigma} f)(x) = ({}_a I^{n-\omega, \sigma} \gamma^n f)(x) = \frac{1}{\Gamma(n-\omega)} \int_a^x \left(\frac{x^\sigma - u^\sigma}{\sigma}\right)^{n-\omega-1} \gamma^n f(u) \frac{du}{u^{1-\sigma}}, \tag{2.11}$$

and

$$({}^C D_b^{\omega, \sigma} f)(x) = ({}_a I^{n-\omega, \sigma} (-\gamma)^n f)(x) = \frac{1}{\Gamma(n-\omega)} \int_x^b \left(\frac{u^\sigma - x^\sigma}{\sigma}\right)^{n-\omega-1} (-\gamma)^n f(u) \frac{du}{u^{1-\sigma}}. \tag{2.12}$$

For $\omega \in \mathbb{C}$, $Re(\omega) > 0$ the forward Riemann-Liouville fractional integral of order ω of a function f with respect to a continuously differentiable and increasing function v has the following form [2, 5]

$${}_a I^{\omega, v} f(x) = \frac{1}{\Gamma(\omega)} \int_a^x (v(x) - v(u))^{\omega-1} f(u) v'(u) du. \tag{2.13}$$

For $\omega \in \mathbb{C}$, $Re(\omega) > 0$ the backward Riemann-Liouville fractional integral of order ω of f with respect to a continuously differentiable and increasing function v has the following form [2, 5]

$$I_b^{\omega, v} f(x) = \frac{1}{\Gamma(\omega)} \int_x^b (v(u) - v(x))^{\omega-1} f(u) v'(u) du. \tag{2.14}$$

For $\omega \in \mathbb{C}$, $Re(\omega) \geq 0$, the generalized forward and backward Riemann-Liouville fractional derivatives of order ω of f with respect to a continuously differentiable and increasing function v have respectively the forms [2, 6]

$${}_a D^{\omega, v} f(x) = \left(\frac{1}{v'(x)} \frac{d}{dx}\right)^n ({}_a I^{n-\omega, v} f)(x) = \frac{\left(\frac{1}{v'(x)} \frac{d}{dx}\right)^n}{\Gamma(n-\omega)} \int_a^x (v(x) - v(u))^{n-\omega-1} f(u) v'(u) du \tag{2.15}$$

and

$$D_b^{\omega, v} f(x) = \left(-\frac{1}{v'(x)} \frac{d}{dx}\right)^n (I_b^{n-\omega, v} f)(x) = \frac{\left(-\frac{1}{v'(x)} \frac{d}{dx}\right)^n}{\Gamma(n-\omega)} \int_x^b (v(x) - v(u))^{n-\omega-1} f(u) v'(u) du, \tag{2.16}$$

where $n = [\omega] + 1$. It is easy to observe that if we choose $\nu(x) = x$, the integrals in (2.13) and (2.14) becomes the left and right Riemann-Liouville fractional integrals respectively and (2.15) and (2.16) becomes the left and right Riemann-Liouville fractional derivatives. When $\nu(x) = \ln x$, the Hadamard fractional operators are obtained [2, 5]. While if one considers $\nu(x) = \frac{x^\sigma}{\sigma}$, the fractional operators in the settings of Katugampola [12, 13] are derived.

In forward and backward generalized Caputo derivatives of a function with respect to another function are presented respectively as [17]

$${}^C_a D^{\omega, \nu} f(x) = \left({}_a I^{n-\omega, \nu} f^{[n]} \right)(x) \quad (2.17)$$

and

$${}^C_b D^{\omega, \nu} f(x) = \left({}_a I^{n-\omega, \nu} (-1)^n f^{[n]} \right)(x), \quad (2.18)$$

where $f^{[n]}(x) = \left(\frac{1}{\nu'(x)} \frac{d}{dx} \right)^n f(x)$.

2.2 The proportional derivatives and their fractional integrals and derivatives

In [28], the authors introduced The conformable derivative. More properties and a modified type of this derivative were explored in [29]. [30], Anderson et al. proposed a modified conformable derivative by utilizing proportional derivatives. In fact, they proposed the following definition.

Definition 2.1. (Modified conformable derivatives) For $\sigma \in [0, 1]$, let the functions $\mu_0, \mu_1 : [0, 1] \times \mathbb{R} \rightarrow [0, \infty)$ be continuous such that for all $t \in \mathbb{R}$ we have

$$\lim_{\sigma \rightarrow 0^+} \mu_1(\sigma, t) = 1, \quad \lim_{\sigma \rightarrow 0^+} \mu_0(\sigma, t) = 0, \quad \lim_{\sigma \rightarrow 1^-} \mu_1(\sigma, t) = 0, \quad \lim_{\sigma \rightarrow 1^-} \mu_0(\sigma, t) = 1,$$

and $\mu_1(\sigma, t) \neq 0, \sigma \in [0, 1), \mu_0(\sigma, t) \neq 0, \sigma \in (0, 1]$. Then, the modified conformable differential operator of order σ is defined by

$$D^\sigma f(t) = \mu_1(\sigma, t)f(t) + \mu_0(\sigma, t)f'(t). \quad (2.19)$$

For details about such derivatives we refer to [30, 31].

As a special case, we shall consider the simplest case and restrict our work to the case when $\mu_1(\sigma, t) = 1 - \sigma$ and $\mu_0(\sigma, t) = \sigma$. Therefore, (2.19) becomes

$$D^\sigma f(t) = (1 - \sigma)f(t) + \sigma f'(t). \quad (2.20)$$

Notice that $\lim_{\sigma \rightarrow 0^+} D^\sigma f(t) = f(t)$ and $\lim_{\sigma \rightarrow 1^-} D^\sigma f(t) = f'(t)$. It is obvious that the derivative (2.20) is generalizes the conformable derivative which does not yield the original function as σ approaches to 0. The associated fractional proportional integrals are defined as follows.

Definition 2.2. [32] For $\sigma > 0$ and $\omega \in \mathbb{C}, \operatorname{Re}(\omega) > 0$, the forward fractional proportional integral of f reads

$$({}_a I^{\omega, \sigma} f)(x) = \frac{1}{\sigma^\omega \Gamma(\omega)} \int_a^x e^{\frac{\sigma-1}{\sigma}(x-\xi)} (x-\xi)^{\omega-1} f(\xi) d\xi \quad (2.21)$$

and the backward one reads

$$({}_b I^{\omega, \sigma} f)(x) = \frac{1}{\sigma^\omega \Gamma(\omega)} \int_x^b e^{\frac{\sigma-1}{\sigma}(\xi-x)} (\xi-x)^{\omega-1} f(\xi) d\xi. \quad (2.22)$$

Definition 2.3. [32] For $\sigma > 0$ and $\omega \in \mathbb{C}$, $Re(\omega) \geq 0$, the forward fractional proportional derivative is defined as

$$({}_aD^{\omega, \sigma} f)(x) = D^{n, \sigma} {}_aI^{n-\omega, \sigma} f(x) = \frac{D_x^{n, \sigma}}{\sigma^{n-\omega} \Gamma(n-\omega)} \int_a^x e^{\frac{\sigma-1}{\sigma}(x-\xi)} (x-\xi)^{n-\omega-1} f(\xi) d\xi. \tag{2.23}$$

The backward proportional fractional derivative is defined by [32]

$$(D_b^{\omega, \sigma} f)(x) = {}_{\ominus}D^{n, \sigma} I_b^{n-\omega, \sigma} f(x) = \frac{{}_{\ominus}D^{n, \sigma}}{\sigma^{n-\omega} \Gamma(n-\omega)} \int_x^b e^{\frac{\sigma-1}{\sigma}(\xi-x)} (\xi-x)^{n-\omega-1} f(\xi) d\xi, \tag{2.24}$$

where $n = [Re(\omega)] + 1$ and $({}_{\ominus}D^{\sigma} f)(t) = (1 - \sigma)f(t) - \sigma f'(t)$.

Lastly, the left and right fractional proportional derivatives in the Caputo settings respectively read [32]

$$({}_a^C D^{\omega, \sigma} f)(x) = ({}_aI^{n-\omega, \sigma} D^{n, \sigma} f)(x) = \frac{1}{\sigma^{n-\omega} \Gamma(n-\omega)} \int_a^x e^{\frac{\sigma-1}{\sigma}(x-\xi)} (x-\xi)^{n-\omega-1} (D^{n, \sigma} f)(\xi) d\xi \tag{2.25}$$

and

$$({}_b^C D^{\omega, \sigma} f)(x) = (I_b^{n-\omega, \sigma} {}_{\ominus}D^{n, \sigma} f)(x) = \frac{1}{\sigma^{n-\omega} \Gamma(n-\omega)} \int_x^b e^{\frac{\sigma-1}{\sigma}(\xi-x)} (\xi-x)^{n-\omega-1} ({}_{\ominus}D^{n, \sigma} f)(\xi) d\xi. \tag{2.26}$$

3 The fractional proportional derivative of a function with respect to another function

Definition 3.1. (The proportional derivative of a function with respect to another function)

For $\sigma \in [0, 1]$, let the functions $\mu_0, \mu_1 : [0, 1] \times \mathbb{R} \rightarrow [0, \infty)$ be continuous such that for all $t \in \mathbb{R}$ we have

$$\lim_{\sigma \rightarrow 0^+} \mu_1(\sigma, t) = 1, \quad \lim_{\sigma \rightarrow 0^+} \mu_0(\sigma, t) = 0, \quad \lim_{\sigma \rightarrow 1^-} \mu_1(\sigma, t) = 0, \quad \lim_{\sigma \rightarrow 1^-} \mu_0(\sigma, t) = 1,$$

and $\mu_1(\sigma, t) \neq 0$, $\sigma \in [0, 1)$, $\mu_0(\sigma, t) \neq 0$, $\sigma \in (0, 1]$. Let also $v(t)$ be a strictly increasing continuous function. Then, the proportional differential operator of order σ of f with respect to g is defined by

$$D^{\sigma, v} f(t) = \mu_1(\sigma, t) f(t) + \mu_0(\sigma, t) \frac{f'(t)}{v'(t)}. \tag{3.1}$$

We shall restrict ourselves to the case when $\mu_1(\sigma, t) = 1 - \sigma$ and $\mu_0(\sigma, t) = \sigma$. Therefore, (3.1) becomes

$$D^{\sigma, v} f(t) = (1 - \sigma) f(t) + \sigma \frac{f'(t)}{v'(t)}. \tag{3.2}$$

The corresponding integral of (3.2)

$${}_aI^{1, \sigma, v} f(t) = \frac{1}{\sigma} \int_a^t e^{\frac{\sigma-1}{\sigma}(v(t)-v(s))} f(s) v'(s) ds, \tag{3.3}$$

where we accept that ${}_aI^{0, \sigma} f(t) = f(t)$.

To generalize a more general class of fractional integral based on the proportional derivative, we use induction and changing the order of integrals to show that

$$({}_aI^{n, \sigma, v} f)(t) = \frac{1}{\sigma} \int_a^t e^{\frac{\sigma-1}{\sigma}(v(t)-v(\xi_1))} v'(\xi_1) d\xi_1 \frac{1}{\sigma} \int_a^{\xi_1} e^{\frac{\sigma-1}{\sigma}(v(\xi_1)-v(\xi_2))} v'(\xi_2) d\xi_2 \cdots \frac{1}{\sigma} \int_a^{\xi_{n-1}} e^{\frac{\sigma-1}{\sigma}(v(\xi_{n-1})-v(\xi_n))} f(\xi_n) v'(\xi_n) d\xi_n$$

$$= \frac{1}{\sigma^n \Gamma(n)} \int_a^t e^{\frac{\sigma-1}{\sigma}(v(t)-v(\xi))} (v(t) - v(\xi))^{n-1} f(\xi) v'(\xi) d\xi. \tag{3.4}$$

Based on (3.4), we can present the following general proportional fractional integral.

Definition 3.2. For $\sigma \in (0, 1]$, $\omega \in \mathbb{C}$, $Re(\omega) > 0$, we define the left fractional integral of f with respect to g by

$$({}_a I^{\omega, \sigma, v} f)(t) = \frac{1}{\sigma^\omega \Gamma(\omega)} \int_a^t e^{\frac{\sigma-1}{\sigma}(v(t)-v(\xi))} (v(t) - v(\xi))^{\omega-1} f(\xi) v'(\xi) d\xi. \tag{3.5}$$

The right fractional proportional integral ending at b can be defined by

$$({}_b I^{\omega, \sigma, v} f)(t) = \frac{1}{\sigma^\omega \Gamma(\omega)} \int_t^b e^{\frac{\sigma-1}{\sigma}(v(\xi)-v(t))} (v(\xi) - v(t))^{\omega-1} f(\xi) v'(\xi) d\xi. \tag{3.6}$$

Remark 3.1. To deal with the right proportional fractional case we shall use the notation

$$({}_\ominus D^{\sigma, v} f)(t) := (1 - \sigma)f(t) - \sigma \frac{f'(t)}{v'(t)}. \tag{3.7}$$

We shall also write

$$({}_\ominus D^{n, \sigma, v} f)(t) = \underbrace{({}_\ominus D^{\sigma, v} \ominus D^{\sigma, v} \dots \ominus D^{\sigma, v} f)(t)}_{n \text{ times}}. \tag{3.8}$$

Remark 3.2. The integrals in (3.5) and (3.6) coincide with the integrals (2) and (3) in [33] and the integrals in (6) and (7) in [34]. If one sets $v(t) = \ln t$ (3.5) and (3.6) coincide with the integrals (2.5) and (2.6) in [35].

Definition 3.3. For $\sigma > 0$, $\omega \in \mathbb{C}$, $Re(\omega) \geq 0$ and $v \in C[a, b]$, where $v'(t) > 0$, we define the general left fractional derivative of f with respect to v as

$$({}_a D^{\omega, \sigma, v} f)(t) = D^{n, \sigma, v} {}_a I^{n-\omega, \sigma, v} f(t) = \frac{D_t^{n, \sigma, v}}{\sigma^{n-\omega} \Gamma(n-\omega)} \int_a^t e^{\frac{\sigma-1}{\sigma}(v(t)-v(\xi))} (v(t) - v(\xi))^{n-\omega-1} f(\xi) v'(\xi) d\xi \tag{3.9}$$

and the general right fractional derivative of f with respect to g as

$$({}_b D^{\omega, \sigma, v} f)(t) = \ominus D^{n, \sigma, v} {}_b I^{n-\omega, \sigma, v} f(t) = \frac{\ominus D_t^{n, \sigma, v}}{\sigma^{n-\omega} \Gamma(n-\omega)} \int_t^b e^{\frac{\sigma-1}{\sigma}(v(\xi)-v(t))} (v(\xi) - v(t))^{n-\omega-1} f(\xi) v'(\xi) d\xi, \tag{3.10}$$

where $n = [Re(\omega)] + 1$.

Remark 3.3. Clearly, if we let $\sigma = 1$ in Definition 3.2 and Definition 3.3, we obtain

- the Riemann-Liouville fractional operators (2.1), (2.2), (2.3) and (2.4) if $v(t) = t$;
- the fractional operators in the Katugampola setting (2.7), (2.8), (2.9) and (2.10) if $v(t) = \frac{t^\mu}{\mu}$;
- the Hadamard fractional operators if $v(t) = \ln t$ [2, 5];
- the fractional operators mentioned in [16] if $v(t) = \frac{(t-a)^\mu}{\mu}$.

Proposition 3.1. Let $\omega, \eta \in \mathbb{C}$ be such that $Re(\omega) \geq 0$ and $Re(\eta) > 0$. Then, for any $\sigma > 0$ we have

- (a) $({}_a I^{\omega, \sigma, v} e^{\frac{\sigma-1}{\sigma} v(x)} (v(x) - v(a))^{\eta-1})(t) = \frac{\Gamma(\eta)}{\Gamma(\eta+\omega)\sigma^\omega} e^{\frac{\sigma-1}{\sigma} v(t)} (v(t) - v(a))^{\omega+\eta-1}$, $Re(\omega) > 0$;
- (b) $({}_b I^{\omega, \sigma, v} e^{-\frac{\sigma-1}{\sigma} v(x)} (v(b) - v(x))^{\eta-1})(t) = \frac{\Gamma(\eta)}{\Gamma(\eta+\omega)\sigma^\omega} e^{-\frac{\sigma-1}{\sigma} v(t)} (v(b) - v(t))^{\omega+\eta-1}$, $Re(\omega) > 0$;
- (c) $({}_a D^{\omega, \sigma} e^{\frac{\sigma-1}{\sigma} v(x)} (v(x) - v(a))^{\eta-1})(t) = \frac{\sigma^\omega \Gamma(\eta)}{\Gamma(\eta-\omega)} e^{\frac{\sigma-1}{\sigma} v(t)} (v(t) - v(a))^{\eta-1-\omega}$, $Re(\omega) \geq 0$;

$$- (d) \left(D_b^{\omega, \sigma, \nu} e^{-\frac{\sigma-1}{\sigma} \nu(x)} (\nu(b) - \nu(x))^{\eta-1} \right) (t) = \frac{\sigma^\omega \Gamma(\eta)}{\Gamma(\eta-\omega)} e^{-\frac{\sigma-1}{\sigma} \nu(t)} (\nu(b) - \nu(t))^{\eta-1-\omega}, \quad \operatorname{Re}(\omega) \geq 0.$$

Proof. The proofs of relations (a) and (b) are very easy to handle. We will prove (c) while the proof of (d) is analogous.

By the definition of the left proportional fractional derivative and relation (a), we have

$$\begin{aligned} \left(D_a^{\omega, \sigma, \nu} e^{\frac{\sigma-1}{\sigma} \nu(x)} (\nu(x) - \nu(a))^{\eta-1} \right) (t) &= D^{n, \sigma, \nu} \left(I_a^{n-\omega, \sigma, \nu} e^{\frac{\sigma-1}{\sigma} \nu(x)} (\nu(x) - \nu(a))^{\eta-1} \right) (t) \\ &= D^{n, \sigma, \nu} \frac{\Gamma(\eta)}{\Gamma(\eta + n - \omega) \sigma^{n-\omega}} e^{\frac{\sigma-1}{\sigma} \nu(t)} (\nu(t) - \nu(a))^{n-\omega+\eta-1} \\ &= \frac{\sigma^n \Gamma(\eta) (n - \omega + \eta - 1) (n - \omega + \eta - 1) \cdots (\eta - \omega)}{\sigma^{n-\omega} \Gamma(n - \omega + \eta)} \times e^{\frac{\sigma-1}{\sigma} \nu(t)} (\nu(t) - \nu(a))^{\eta-1-\omega} \\ &= \frac{\sigma^\omega \Gamma(\eta)}{\Gamma(\eta - \omega)} e^{\frac{\sigma-1}{\sigma} \nu(t)} (\nu(t) - \nu(a))^{\eta-1-\omega}. \end{aligned}$$

Here, we have used the fact that $D^{\sigma, \nu} \left(h(t) e^{\frac{\sigma-1}{\sigma} \nu(t)} \right) = \sigma \frac{h'(t)}{g'(t)} e^{\frac{\sigma-1}{\sigma} \nu(t)}$. □

Below we present the semi-group property for the general fractional proportional integrals of a function with respect to another function.

Theorem 3.1. [33] *Let $\sigma \in (0, 1]$, $\operatorname{Re}(\omega) > 0$ and $\operatorname{Re}(\eta) > 0$. Then, if f is continuous and defined for $t \geq a$ or $t \leq b$, we have*

$${}_a I^{\omega, \sigma, \nu} ({}_a I^{\eta, \sigma, \nu} f)(t) = {}_a I^{\eta, \sigma, \nu} ({}_a I^{\omega, \sigma, \nu} f)(t) = ({}_a I^{\omega+\eta, \sigma, \nu} f)(t) \tag{3.11}$$

and

$$I_b^{\omega, \sigma, \nu} (I_b^{\eta, \sigma, \nu} f)(t) = I_b^{\eta, \sigma, \nu} (I_b^{\omega, \sigma, \nu} f)(t) = (I_b^{\omega+\eta, \sigma, \nu} f)(t). \tag{3.12}$$

Theorem 3.2. *Let $0 \leq m < [\operatorname{Re}(\omega)] + 1$. Then, we have*

$$D^{m, \sigma, \nu} ({}_a I^{\omega, \sigma, \nu} f)(t) = ({}_a I^{\omega-m, \sigma, \nu} f)(t) \tag{3.13}$$

and

$$\ominus D^{m, \sigma, \nu} (I_b^{\omega, \sigma, \nu} f)(t) = (I_b^{\omega-m, \sigma, \nu} f)(t) \tag{3.14}$$

Proof. Here we prove (3.13), while one can prove (3.14) likewise. Using the fact that $D_t^{\sigma, \nu} e^{\frac{\sigma-1}{\sigma} (\nu(t) - \nu(\xi))} = 0$, we have

$$D^{m, \sigma, \nu} ({}_a I^{\omega, \sigma, \nu} f)(t) D^{m-1, \sigma, \nu} (D^{\sigma, \nu} {}_a I^{\omega, \sigma, \nu} f)(t) = D^{m-1, \sigma, \nu} \frac{1}{\sigma^{\omega-1} \Gamma(\omega-1)} \int_a^t e^{\frac{\sigma-1}{\sigma} (\nu(t) - \nu(\xi))} (\nu(t) - \nu(\xi))^{\omega-2} f(\xi) \nu'(\xi) d\xi.$$

Proceeding m -times in the same manner we obtain (3.13). □

Corollary 3.1. *Let $0 < \operatorname{Re}(\eta) < \operatorname{Re}(\omega)$ and $m - 1 < \operatorname{Re}(\eta) \leq m$. Then, we have*

$${}_a D^{\eta, \sigma, \nu} {}_a I^{\omega, \sigma, \nu} f(t) = {}_a I^{\omega-\eta, \sigma, \nu} f(t) \tag{3.15}$$

and

$$D_b^{\eta, \sigma, \nu} I_b^{\omega, \sigma, \nu} f(t) = I_b^{\omega-\eta, \sigma, \nu} f(t). \tag{3.16}$$

Proof. By the help of Theorem 3.1 and Theorem 3.2, we have

$$\begin{aligned} {}_a D^{\eta, \sigma, \nu} {}_a I^{\omega, \sigma, \nu} f(t) &= D_a^{m, \sigma, \nu} I_a^{m-\eta, \sigma, \nu} I_a^{\omega, \sigma, \nu} f(t) \\ &= D^{m, \sigma, \nu} {}_a I^{m-\eta+\omega, \sigma, \nu} f(t) = {}_a I^{\omega-\eta, \sigma, \nu} f(t). \end{aligned}$$

This was the proof of (3.15). One can prove (3.16) in a similar way. □

Theorem 3.3. Let f be integrable on $t \geq a$ or $t \leq b$ and $\operatorname{Re}[\omega] > 0$, $\sigma \in (0, 1]$, $n = [\operatorname{Re}(\omega)] + 1$. Then, we have

$${}_a D^{\omega, \sigma, \nu} {}_a I^{\omega, \sigma, \nu} f(t) = f(t) \quad (3.17)$$

and

$$D_b^{\omega, \sigma, \nu} I_b^{\omega, \sigma, \nu} f(t) = f(t). \quad (3.18)$$

Proof. By the definition and Theorem 3.1, we have

$${}_a D^{\omega, \sigma, \nu} {}_a I^{\omega, \sigma, \nu} f(t) = D^{n, \sigma, \nu} {}_a I^{n-\omega, \sigma, \nu} {}_a I^{\omega, \sigma, \nu} f(t) = D^{n, \sigma, \nu} {}_a I^{n, \sigma, \nu} f(t) = f(t). \quad \square$$

4 The Caputo fractional proportional derivative of a function with respect to another function

Definition 4.1. For $\sigma \in (0, 1]$ and $\omega \in \mathbb{C}$ with $\operatorname{Re}(\omega) \geq 0$ we define the left derivative of Caputo type as

$$({}^C D_a^{\omega, \sigma, \nu} f)(t) = {}_a I^{n-\omega, \sigma, \nu} (D^{n, \sigma, \nu} f)(t) = \frac{1}{\sigma^{n-\omega} \Gamma(n-\omega)} \int_a^t e^{\frac{\sigma-1}{\sigma}(v(t)-v(s))} (v(t)-v(s))^{n-\omega-1} (D^{n, \sigma, \nu} f)(s) v'(s) ds. \quad (4.1)$$

Similarly, the right derivative of Caputo type ending is defined by

$$({}^C D_b^{\omega, \sigma, \nu} f)(t) = I_b^{n-\omega, \sigma, \nu} ({}_{\ominus} D^{n, \sigma, \nu} f)(t) = \frac{1}{\sigma^{n-\omega} \Gamma(n-\omega)} \int_t^b e^{\frac{\sigma-1}{\sigma}(v(s)-v(t))} (v(s)-v(t))^{n-\omega-1} ({}_{\ominus} D^{n, \sigma, \nu} f)(s) v'(s) ds, \quad (4.2)$$

where $n = [\operatorname{Re}(\omega)] + 1$.

Proposition 4.1. Let $\omega, \eta \in \mathbb{C}$ be such that $\operatorname{Re}(\omega) > 0$ and $\operatorname{Re}(\eta) > 0$. Then, for any $\sigma \in (0, 1]$ and $n = [\operatorname{Re}(\omega)] + 1$ we have

1. $({}^C D_a^{\omega, \sigma, \nu} e^{\frac{\sigma-1}{\sigma} v(x)} (v(x) - v(a))^{\eta-1})(t) = \frac{\sigma^\omega \Gamma(\eta)}{\Gamma(\eta-\omega)} e^{\frac{\sigma-1}{\sigma} v(t)} (v(t) - v(a))^{\eta-1-\omega}$, $\operatorname{Re}(\eta) > n$;
2. $({}^C D_b^{\omega, \sigma, \nu} e^{-\frac{\sigma-1}{\sigma} v(x)} (v(b) - v(x))^{\eta-1})(t) = \frac{\sigma^\omega \Gamma(\eta)}{\Gamma(\eta-\omega)} e^{-\frac{\sigma-1}{\sigma} v(t)} (v(b) - v(t))^{\eta-1-\omega}$, $\operatorname{Re}(\eta) > n$.

For $k = 0, 1, \dots, n-1$, we have

$$({}^C D_a^{\omega, \sigma, \nu} e^{\frac{\sigma-1}{\sigma} v(x)} (v(x) - v(a))^k)(t) = 0 \quad \text{and} \quad ({}^C D_b^{\omega, \sigma, \nu} e^{-\frac{\sigma-1}{\sigma} v(x)} (v(b) - v(x))^k)(t) = 0.$$

In particular, $({}^C D_a^{\omega, \sigma} e^{\frac{\sigma-1}{\sigma} v(x)})(t) = 0$ and $({}^C D_b^{\omega, \sigma} e^{-\frac{\sigma-1}{\sigma} v(x)})(t) = 0$.

Proof. We only prove the first relation. The proof of the second relation is similar. We have

$$\begin{aligned} ({}^C D_a^{\omega, \sigma, \nu} e^{\frac{\sigma-1}{\sigma} v(x)} (v(x) - v(a))^{\eta-1})(t) &= {}_a I^{n-\omega, \sigma, \nu} D^{n, \sigma, \nu} \left[e^{\frac{\sigma-1}{\sigma} v(t)} (v(t) - v(a))^{\eta-1} \right] \\ &= {}_a I^{n-\omega, \sigma, \nu} \left[\sigma^n (\eta-1)(\eta-2) \dots (\eta-1-n) (v(t) - v(a))^{\eta-n-1} e^{\frac{\sigma-1}{\sigma} v(t)} \right] \\ &= \frac{\sigma^n (\eta-1)(\eta-2) \dots (\eta-1-n) \Gamma(\eta-n)}{\Gamma(\eta-\omega) \sigma^{n-\omega}} (v(t) - v(a))^{\eta-\omega-1} e^{\frac{\sigma-1}{\sigma} v(t)} \\ &= \frac{\sigma^\omega \Gamma(\eta)}{\Gamma(\eta-\omega)} e^{\frac{\sigma-1}{\sigma} v(t)} (v(t) - v(a))^{\eta-1-\omega}. \quad \square \end{aligned}$$

5 Conclusions

We have used the proportional derivatives of a function with respect to another function to obtain left and right generalized type fractional integrals and derivatives involving two parameters ω and σ and depending

on a kernel function. The Riemann-Liouville and Caputo fractional derivatives in the classical fractional calculus can be obtained as σ tends to 1 and by choosing $v(t) = t$. The integrals have the semi-group property and together with their corresponding derivatives have exponential functions as part of their kernels. It should be noted that other properties of these new operators can be obtained by using the Laplace transform proposed in [17]. Moreover, for a specific choice of v , the proportional fractional operators in the settings of Hadamard and Katugampola can be obtained.

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