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Yongwei Yang, Kuanyun Zhu*, and Xiaolong Xin

Rough sets based on fuzzy ideals in distributive lattices

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Abstract: In this paper, we present a rough set model based on fuzzy ideals of distributive lattices. In fact, we consider a distributive lattice as a universal set and we apply the concept of a fuzzy ideal for definitions of the lower and upper approximations in a distributive lattice. A novel congruence relation induced by a fuzzy ideal of a distributive lattice is introduced. Moreover, we study the special properties of rough sets which can be constructed by means of the congruence relations determined by fuzzy ideals in distributive lattices. Finally, the properties of the generalized rough sets with respect to fuzzy ideals in distributive lattices are also investigated.

Keywords: rough set, distributive lattice, fuzzy ideals, set-valued mapping, generalized rough set

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1 Introduction

It is well known that the real world problems under consideration are full of indeterminacy and vagueness. In fact, most of the problems that we deal with are vague rather than precise. In the face of so many uncertain data, classical methods are not always successful in dealing with them, because of various types of uncertainties presented in these problems. As far as known, there are several theories to describe uncertainty, for example, fuzzy set theory, rough set theory and other mathematical tools. Over the years, many experts and scholars are looking for some different ways to solve the problem of uncertainty.

Rough set theory was first introduced by Pawlak [1] which is an extension of set theory, as a new mathematical approach to deal with uncertain knowledge and has attracted the interest of researchers and practitioners in various fields of science and technology. In rough set theory, rough sets can be described by a pair of ordinary sets called the lower and upper approximations. However, these equivalence relations in Pawlak rough sets are restrictive in some areas of applications. To solve this issue, some more general models have been proposed, such as quantitative rough sets based on subsethood measure, generalized rough sets based on relations and so on [2, 3]. Nowadays, rough set theory has been applied to many areas, such as knowledge discovery, machine learning, approximate classification and so on [4–6]. In particular, many researchers applied this theory to algebraic structures. Wang [7] investigated the topological characterizations of generalized fuzzy rough sets. Zhu and Hu [8] introduced the notion of Z-soft rough fuzzy BCI-algebras (ideals), which is an extended notion of soft rough BCI-algebras (ideals) and rough fuzzy BCI-algebras (ideals), and investigated roughness in BCI-algebras with respects to a Z-soft approximation space. Shao et al.

Yongwei Yang: School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China
***Corresponding Author: Kuanyun Zhu:** School of Information and Mathematics, Yangtze University, Jingzhou 434023, China; E-mail: kyzhu@whu.edu.cn
Xiaolong Xin: School of Mathematics, Northwest University, Xi'an 710127, China

introduced the notions of rough filters, multi-granulation rough filters, and rough fuzzy filters in pseudo-BCI algebras [9]. The lower and upper approximations in various hyperstructures were also discussed by many authors in many literatures [10–12]. Furthermore, some authors considered rough sets in a fuzzy algebraic system, such as [13, 14] studied some types of fuzzy covering rough set models and their generalizations over fuzzy lattices. The generalization of Pawlak rough set was introduced for two universes on general binary relations. Thus, equivalence relations should be extended to two universes for algebraic sets. It follows from this point of view that Davvaz [15] and Yamak et al. [16] put forward the notion of set-valued homomorphism for groups and rings, respectively.

In particular, Davvaz applied the notion of fuzzy ideal of a ring for definitions of the lower and upper approximations in a ring and studied the characterizations of the approximations [17]. In 2014, Xiao et al. [18] studied rough set model on ideals in lattices. In [18], let I be an ideal in a lattice L. Then θ_I is a joint-congruence on L. θ_I is a congruence on L if and only if L is distributive. Based on these congruences, they discussed the algebraic properties of rough sets induced by ideals in lattices. Since fuzzy set is an extension of classical set, it is meaningful to use fuzzy set instead of classical set. Be inspired of [17, 18], we focus on discussing the algebraic properties of rough sets induced by fuzzy ideals in distributive lattices. A novel congruence relation $U(\mu, t)$ induced by a fuzzy ideal μ of a distributive lattice is introduced. Some properties of this congruence relation are also investigated. Further, we discuss the lower and upper approximations of a subset of a distributive lattice with respect to a fuzzy ideal. Some characterizations of the above approximations are made and some examples are discussed.

This paper is organized as follows. In Section 2, we recall some concepts and results on lattices, fuzzy sets and rough sets. In Section 3, we study the rough sets which are constructed by a novel congruence relation $U(\mu, t)$. In particular, in Section 4, we introduce a special class of set-valued homomorphism with respect to a fuzzy ideal and discuss the properties of the generalized rough set.

2 Preliminaries

In this section, we recall some basic notions and results about lattices, fuzzy sets and rough sets. Throughout this paper, L is always a distributive lattice with the minimum element 0.

Definition 2.1. [19] *Let L be a lattice and $\emptyset \neq I \subseteq L$. Then I is called an ideal of L if it satisfies the following conditions: for any $x, y \in L$,*

(1) $x \in I$ and $y \in I$ imply $x \vee y \in I$;
(2) $x \in L$ and $x \leq y$ imply $x \in I$.

Let A, B be subsets of L, we define the join and meet as follows:

$$A \vee B = \{a \vee b | a \in A, b \in B\} \text{ and } A \wedge B = \{a \wedge b | a \in A, b \in B\}.$$

Let I, J be ideals of L, then $I \vee J$ is an ideal of L [18].

Definition 2.2. [19] *Let L be a lattice. A relation R is called an equivalence relation on L if for all $a, b, c \in L$,*

(1) Reflexive: $(a, a) \in R$;
(2) Symmetry: $(a, b) \in R$ implies $(b, a) \in R$;
(3) Transitivity: $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$.

An equivalence relation R is called a congruence relation on L, if for all $a, b, c, d \in L$, $(a, b) \in R$, $(c, d) \in R$, then $(a \vee c, b \vee d) \in R$ and $(a \wedge c, b \wedge d) \in R$.

Definition 2.3. [20] *Let μ be a fuzzy set of a lattice L. Then μ is called a fuzzy sublattice of L if $\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in L$.*

Let μ be a fuzzy sublattice of L. Then μ is a fuzzy ideal of L, if $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for all $x, y \in L$.

Proposition 2.4. [20] *Let μ be a fuzzy sublattice of a lattice L. Then μ is a fuzzy ideal of L if and only if $x \leq y$ implies that $\mu(x) \geq \mu(y)$, for all $x, y \in L$.*

Proposition 2.5. [21] *Let μ be a fuzzy set of a lattice L. Then μ is a fuzzy ideal of L if and only if any one of the following sets of conditions is satisfied: for all $x, y \in L$,*

(1) $\mu(0) = 1$ and $\mu(x \vee y) = \mu(x) \wedge \mu(y)$;
(2) $\mu(0) = 1$, $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ and $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

Let μ be a fuzzy subset of a lattice L and $t \in [0, 1]$. Then the set $\mu_t = \{x \in L | \mu(x) \geq t\}$ is called a t-level subset of μ.

Remark 2.6. *A fuzzy set μ is a fuzzy ideal of a lattice L if and only if every subset μ_t is an ideal of L for all $t \in [0, 1]$.*

Definition 2.7. [1] *Let R be an equivalence relation on the universe U and (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is called definable if $R_*X = R^*X$; otherwise, X is said to be a rough set, where two operators are defined as:*

$$R_*X = \left\{x \in U \middle| [x]_R \subseteq X\right\}, R^*X = \left\{x \in U \middle| [x]_R \cap X \neq \emptyset\right\}.$$

Definition 2.8. [1] Let X and Y be two non-empty sets and $B \subseteq Y$. Let $T : X \to \mathscr{P}^*(Y)$ be a set-valued mapping, where $\mathscr{P}^*(Y)$ denotes the family of all non-empty subsets of Y. The lower and upper approximations $\underline{T}(B)$ and $\overline{T}(B)$ are defined by

$$\underline{T}(B) = \{x \in U | T(x) \subseteq B\},$$
$$\overline{T}(B) = \{x \in U | T(x) \cap B \neq \emptyset\},$$

respectively. If $\underline{T}(B) \neq \overline{T}(B)$, then the pair $(\underline{T}(B), \overline{T}(B))$ is said to be a generalized rough set.

3 A novel congruence relation induced by a fuzzy ideal in a distributive lattice

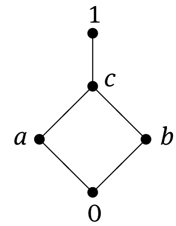
In this section, we introduce a novel congruence relation $U(\mu, t)$ induced by a fuzzy ideal μ in a distributive lattice. We define the join and meet of two non-empty subsets in a lattice as follows: $A \vee B = \{a \vee b | a \in A, b \in B\}$, $A \wedge B = \{a \wedge b | a \in A, b \in B\}$.

Definition 3.1. *Let μ be a fuzzy ideal of L. For each $t \in [0, 1]$, the set*

$$U(\mu, t) = \left\{(x, y) \in L \times L \middle| \bigvee\{\mu(a) | a \vee x = a \vee y, \exists a \in L\} \geq t\right\}$$

is called a t-level relation of μ.

Example 3.2. *Let $L = \{0, a, b, c, 1\}$. We define the binary relation $\leq$ in the following Hasse diagram. It is easy to check that L is a distributive lattice. Let $\mu = \frac{1}{0} + \frac{0.8}{a} + \frac{0.6}{b} + \frac{0.4}{c} + \frac{0}{1}$. Then it is clear that μ is a fuzzy ideal of L. Choose $t = 0.9$, then we have $U(\mu, 0.9) = \{(0, 0), (a, a), (b, b), (c, c), (1, 1)\}$. Thus $U(\mu, 0.9)$ is called a 0.9-level relation of μ.*



Now we prove that $U(\mu, t)$ is a congruence relation on L.

Lemma 3.3. *Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then $U(\mu, t)$ is a congruence relation on L.*

Proof. It is easy to see that $\mu(0) = 1$ and for any $x \in L$, $\bigvee_{a\vee x=a\vee x} \mu(a) = \bigvee \mu(a) \geq \mu(0) = 1 \geq t$. From Definition 3.1, we get that $(x, x) \in U(\mu, t)$, i.e., $U(\mu, t)$ is reflexive. Obviously, $U(\mu, t)$ is symmetric. Let $(x, y) \in U(\mu, t)$ and $(y, z) \in U(\mu, t)$. Then we have

$$\bigvee_{a\vee x=a\vee y} \mu(a) \geq t, \quad \bigvee_{b\vee y=b\vee z} \mu(b) \geq t,$$

and so $\left(\bigvee_{a\vee x=a\vee y} \mu(a)\right) \wedge \left(\bigvee_{b\vee y=b\vee z} \mu(b)\right) \geq t$. Since μ is a fuzzy ideal of L, we obtain that

$$\left(\bigvee_{a\vee x=a\vee y} \mu(a)\right) \wedge \left(\bigvee_{b\vee y=b\vee z} \mu(b)\right) = \bigvee_{a\vee x=a\vee y, b\vee y=b\vee z} \left(\mu(a) \wedge \mu(b)\right) = \bigvee_{a\vee x=a\vee y, b\vee y=b\vee z} \mu(a \vee b).$$

For $a \vee x = a \vee y$, $b \vee y = b \vee z$, we have $a \vee b \vee x = a \vee b \vee y$, $a \vee b \vee y = a \vee b \vee z$. Thus $a \vee b \vee x = a \vee b \vee z$, i.e., $c \vee x = c \vee z$, where $c = a \vee b \in L$. It follows that

$$t \leq \bigvee_{a\vee x=a\vee y, b\vee y=b\vee z} \mu(a \vee b) \leq \bigvee_{c\vee x=c\vee z} \mu(c),$$

and so $\bigvee_{c\vee x=c\vee z} \mu(c) \geq t$. According to Definition 3.1, we get that $(x, z) \in U(\mu, t)$. Therefore, $U(\mu, t)$ is an equivalence relation on L. Now we show that $U(\mu, t)$ is a congruence relation on L. Let $(x, y) \in U(\mu, t)$ and $(u, v) \in U(\mu, t)$. Then

$$\bigvee_{a\vee x=a\vee y} \mu(a) \geq t, \quad \bigvee_{b\vee u=b\vee v} \mu(b) \geq t,$$

and so

$$\left(\bigvee_{a\vee x=a\vee y} \mu(a)\right) \wedge \left(\bigvee_{b\vee y=b\vee z} \mu(b)\right) \geq t.$$

Further, we have

$$\left(\bigvee_{a\vee x=a\vee y} \mu(a)\right) \wedge \left(\bigvee_{b\vee u=b\vee v} \mu(b)\right) = \bigvee_{a\vee x=a\vee y, b\vee u=b\vee v} \left(\mu(a) \wedge \mu(b)\right) = \bigvee_{a\vee x=a\vee y, u\vee y=b\vee v} \mu(a \vee b).$$

For $a \vee x = a \vee y$, $b \vee u = b \vee v$, we have $a \vee b \vee (x \vee u) = a \vee b \vee (y \vee v)$, i.e., $c \vee (x \vee u) = c \vee (y \vee v)$, where $c = a \vee b \in L$. Hence,

$$t \leq \bigvee_{a\vee x=a\vee y, u\vee y=b\vee v} \mu(a \vee b) \leq \bigvee_{c\vee(x\vee u)=c\vee(y\vee v)} \mu(c).$$

Consequently, $\bigvee_{c\vee(x\vee u)=c\vee(y\vee v)} \mu(c) \geq t$, which implies that $(x \vee u, y \vee v) \in U(\mu, t)$.

Further, let $(x_1, y_1) \in U(\mu, t)$ and $(x_2, y_2) \in U(\mu, t)$. Then

$$\bigvee_{b\vee x_1=b\vee y_1} \mu(b) \geq t, \quad \bigvee_{c\vee x_2=c\vee y_2} \mu(c) \geq t.$$

So

$$\left(\bigvee_{b\vee x_1=b\vee y_1} \mu(b)\right) \wedge \left(\bigvee_{c\vee x_2=c\vee y_2} \mu(c)\right) \geq t.$$

For $b \vee x_1 = b \vee y_1$ and $c \vee x_2 = c \vee y_2$, we have

$$(b \vee x_1) \wedge (c \vee x_2) = (b \vee y_1) \wedge (c \vee y_2).$$

On the other hand, since L is a distributive lattice, we have

$$\big[(b \wedge c) \vee (x_1 \wedge c) \vee (x_2 \wedge b)\big] \vee (x_1 \wedge x_2) = \big[(b \wedge c) \vee (y_1 \wedge c) \vee (y_2 \wedge b)\big] \vee (y_1 \wedge y_2).$$

Since $(b \vee x_1) \wedge c = (c \vee y_1) \wedge c$ and $(c \vee x_2) \wedge b = (c \vee y_2) \wedge b$, we have

$$(b \wedge c) \vee (x_1 \wedge c) \vee (x_2 \wedge b) = (b \wedge c) \vee (y_1 \wedge c) \vee (y_2 \wedge b).$$

Notice that μ is a fuzzy ideal of L, we get that

$$\mu[(b \wedge c) \vee (x_1 \wedge c) \vee (x_2 \wedge b)] = \mu(b \wedge c) \wedge \mu(x_1 \wedge c) \wedge \mu(x_2 \wedge b).$$

It follows from $b \wedge c \le b$, $x_1 \wedge c \le c$, $x_2 \wedge b \le b$ that

$$\mu(b \wedge c) \wedge \mu(x_1 \wedge c) \wedge \mu(x_2 \wedge b) \ge \mu(b) \wedge \mu(c).$$

Thus

$$\begin{aligned} t &\le \Big(\bigvee_{b \vee x_1 = b \vee y_1} \mu(b)\Big) \wedge \Big(\bigvee_{c \vee x_2 = c \vee y_2} \mu(c)\Big) \\ &= \bigvee_{b \vee x_1 = b \vee y_1, c \vee x_2 = c \vee y_2} \Big(\mu(b) \wedge \mu(c)\Big) \\ &\le \bigvee_{b \vee x_1 = b \vee y_1, c \vee x_2 = c \vee y_2} \Big(\mu(b \wedge c) \wedge \mu(x_1 \wedge c) \wedge \mu(x_2 \wedge b)\Big) \\ &\le \bigvee_{[(b \wedge c) \vee (x_1 \wedge c) \vee (x_2 \wedge b)] \vee (x_1 \wedge x_2) = [(b \wedge c) \vee (y_1 \wedge c) \vee (y_2 \wedge b)] \vee (y_1 \wedge y_2)} \Big(\mu(b \wedge c) \wedge \mu(x_1 \wedge c) \wedge \mu(x_2 \wedge b)\Big) \\ &\le \bigvee_{a \vee (x_1 \wedge x_2) = a \vee (y_1 \wedge y_2)} \mu(a), \end{aligned}$$

and therefore $(x_1 \wedge x_2, y_1 \wedge y_2) \in U(\mu, t)$. According to the above discussing, we get that $U(\mu, t)$ is a congruence relation on L. □

Remark 3.4. *In Lemma 3.3, we say x is congruent to y mod μ, written $x \equiv_t y$ (mod μ) if*

$$\bigvee_{a \vee x = a \vee y} \mu(a) \ge t.$$

It follows from Definition 3.1 and Lemma 3.3 that we can get many useful properties of these congruence relations. We denote by $[x]_{(\mu,t)}$ the equivalence class of $U(\mu, t)$ containing x of L.

Lemma 3.5. *Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then for all $x, y \in L$,*

(1) $[x]_{(\mu,t)} \vee [y]_{(\mu,t)} \subseteq [x \vee y]_{(\mu,t)}$;
(2) $[x]_{(\mu,t)} \wedge [y]_{(\mu,t)} \subseteq [x \wedge y]_{(\mu,t)}$.

Proof. The proof is easy, and we omit the details. □

Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then $U(\mu, t)$ is a congruence relation on L. Thus, when $U = L$ and R is the above equivalence relation (congruence relation), then we use (L, μ, t) instead of approximation space (U, R).

Definition 3.6. *Let μ be a fuzzy ideal of L, $t \in [0, 1]$ and $\emptyset \subsetneq X \subseteq L$. Then*

$$\underline{U(\mu, t)}(X) = \{x \in L | [x]_{(\mu,t)} \subseteq X\}$$

and

$$\overline{U(\mu, t)}(X) = \{x \in L | [x]_{(\mu,t)} \cap X \neq \emptyset\}$$

are called the lower approximation and the upper approximation of the set X with respect to μ and t, respectively. It is easy to know that $\underline{U(\mu, t)}(X) \subseteq X \subseteq \overline{U(\mu, t)}(X)$.

Lemma 3.7. *Let μ and ν be two fuzzy ideals of L such that* $\mu \subseteq \nu$ *and* $t \in [0, 1]$. *Then* $[x]_{(\mu,t)} \subseteq [x]_{(\nu,t)}$ *for all* $x \in L$.

Proof. Let $a \in [x]_{(\mu,t)}$. Then we have $(a, x) \in U(\mu, t)$, i.e., $\bigvee_{b\vee a=b\vee x} \mu(b) \geq t$. Since $\mu \subseteq \nu$, we have $\mu(b) \leq \nu(b)$. Thus $\bigvee_{b\vee a=b\vee x} \nu(b) \geq \bigvee_{b\vee a=b\vee x} \mu(b) \geq t$, which implies that $(a, x) \in U(\nu, t)$, i.e., $a \in [x]_{(\nu,t)}$. Therefore, $[x]_{(\mu,t)} \subseteq [x]_{(\nu,t)}$. □

From Lemma 3.7, we get the the following conclusion easily.

Lemma 3.8. *Let μ and ν be two fuzzy ideals of L such that* $\mu \subseteq \nu$, $t \in [0, 1]$ *and* $\emptyset \subsetneq X \subseteq L$. *Then*

(1) $\underline{U(\nu, t)}(X) \subseteq \underline{U(\mu, t)}(X)$;
(2) $\overline{U(\mu, t)}(X) \subseteq \overline{U(\nu, t)}(X)$;
(3) $\underline{U(\mu, t)}(X) \cup \underline{U(\nu, t)}(X) \subseteq \underline{U(\mu \cap \nu, t)}(X)$;
(4) $\overline{U(\mu \cap \nu, t)}(X) \subseteq \overline{U(\mu, t)}(X) \cap \overline{U(\nu, t)}(X)$.

The following example shows that the containedness in (3) and (4) of Lemma 3.8 need not be an equality.

Example 3.9. *Consider the lattice L in Example 3.2, let* $\mu = \frac{1}{0} + \frac{0.6}{a} + \frac{0.8}{b} + \frac{0.4}{c} + \frac{0}{1}$, $\nu = \frac{1}{0} + \frac{0.8}{a} + \frac{0.5}{b} + \frac{0.3}{c} + \frac{0}{1}$. *Then it is clear that μ and ν are fuzzy ideals of L. Choose* $t = 0.8$, *then we have*

$$U(\mu, 0.8) = \{(0, 0), (a, a), (b, b), (c, c), (1, 1), (0, b), (a, c)\},$$
$$U(\nu, 0.8) = \{(0, 0), (a, a), (b, b), (c, c), (1, 1), (0, a), (b, c)\}.$$

Thus

$$U(\mu \cap \nu, 0.8) = \{(0, 0), (a, a), (b, b), (c, c), (1, 1)\}.$$

If $X = \{0, c\}$, *then*

$$\overline{U(\mu \cap \nu), t}(X) = \{0, c\}, \overline{U(\mu, t)}(X) \cap \overline{U(\nu, t)}(X) = \{0, a, b, c\}.$$

Therefore $\overline{U(\mu \cap \nu, t)}(X) \subsetneq \overline{U(\mu, t)}(X) \cap \overline{U(\nu, t)}(X)$. *Further, if* $X = \{c, 1\}$, *then*

$$\underline{U(\mu, t)}(X) \cup \underline{U(\nu, t)}(X) = \{1\}, \underline{U(\mu \cap \nu, t)}(X) = \{1, c\}.$$

Hence $\underline{U(\mu, t)}(X) \cup \underline{U(\nu, t)}(X) \subsetneq \underline{U(\mu \cap \nu, t)}(X)$.

The following definition is from Zadeh's expansion principle.

Definition 3.10. *Let μ and ν be two fuzzy sets over L. Define* $\mu \vee \nu$ *over L as follows:*

$$(\mu \vee \nu)(x) = \bigvee_{x=a\vee b} \left(\mu(a) \wedge \nu(b)\right)$$

for all $x \in L$.

Now we investigate the operations of lower approximations and upper approximations of the set X with respect to μ and t, respectively.

Proposition 3.11. *Let μ and ν be two fuzzy ideals of L, $t \in [0, 1]$ and $\emptyset \subsetneq X \subseteq L$. Then*

(1) $\underline{U(\mu \vee \nu, t)}(X) \subseteq \underline{U(\mu, t)}(X) \cap \underline{U(\nu, t)}(X)$;

(2) $\overline{U(\mu \vee \nu, t)}(X) \supseteq \overline{U(\mu, t)}(X) \cup \overline{U(\nu, t)}(X)$.

Proof. Since L is a distributive lattice, we have that $\mu \vee \nu$ is a fuzzy ideal of L. Let $x \in L$. Then $(\mu \vee \nu)(x) = \bigvee_{x=a\vee b} (\mu(a) \wedge \nu(b)) \geq \mu(x) \wedge \nu(0)$. Notice that ν is a fuzzy ideal of L, we obtain that $\nu(0) = 1$. It follows that

$$(\mu \vee \nu)(x) = \bigvee_{x=a\vee b} \Big(\mu(a) \wedge \nu(b)\Big) \geq \mu(x) \wedge \nu(0) \geq \mu(x) \wedge \nu(0) = \mu(x)$$

and so $\mu \subseteq \mu \vee \nu$. In a similar way, we have $\nu \subseteq \mu \vee \nu$. According to Lemma 3.8, we get that $\underline{U(\mu \vee \nu, t)}(X) \subseteq \underline{U(\mu, t)}(X) \cap \underline{U(\nu, t)}(X)$ and $\overline{U(\mu \vee \nu, t)}(X) \supseteq \overline{U(\mu, t)}(X) \cup \overline{U(\nu, t)}(X)$. □

Proposition 3.12. *Let μ and ν be two fuzzy ideals of L, $t \in [0, 1]$ and $\emptyset \subsetneq X \subseteq L$. Then*

(1) $U(\mu, t) \cap U(\nu, t)$ *is a congruence relation on L;*

(2) $\underline{U(\mu, t) \cap U(\nu, t)}(X) \supseteq \underline{U(\mu, t)}(X) \cup \underline{U(\nu, t)}(X)$;

(3) $\overline{U(\mu, t) \cap U(\nu, t)}(X) \subseteq \overline{U(\mu, t)}(X) \cap \overline{U(\nu, t)}(X)$.

Proof. It is straightforward. □

Theorem 3.13. *Let μ and ν be two fuzzy ideals of L, $t \in [0, 1]$ and $\emptyset \subsetneq X \subseteq L$. Then*

(1) $\underline{U(\mu \cap \nu, t)}(X) = \underline{U(\mu, t) \cap U(\nu, t)}(X)$;

(2) $\overline{U(\mu \cap \nu, t)}(X) = \overline{U(\mu, t) \cap U(\nu, t)}(X)$.

Proof. (1) We first show that $\underline{U(\mu, t) \cap U(\nu, t)}(X) \subseteq \underline{U(\mu \cap \nu, t)}(X)$. Let $x \in \underline{U(\mu, t) \cap U(\nu, t)}$ and $y \in [x]_{(\mu\cap\nu,t)}$. Then $(x, y) \in U(\mu \cap \nu, t)$,

$$\bigvee_{a\vee y=a\vee x} (\mu \cap \nu)(a) \geq t, \text{ i.e., } \bigvee_{a\vee y=a\vee x} \Big(\mu(a) \wedge \nu(a)\Big) \geq t.$$

Thus,

$$\bigvee_{a\vee y=a\vee x} \mu(a) \geq t \text{ and } \bigvee_{a\vee y=a\vee x} \nu(a) \geq t.$$

Hence, $y \in [x]_{(\mu,t)}$ and $y \in [x]_{(\nu,t)}$. So $y \in [x]_{(\mu,t)\cap(\nu,t)}$, and therefore $y \in X$, which implies that $x \in \underline{U(\mu \cap \nu, t)}(X)$. Therefore, $\underline{U(\mu, t) \cap U(\nu, t)}(X) \subseteq \underline{U(\mu \cap \nu, t)}(X)$.

Next we show that $\underline{U(\mu \cap \nu, t)}(X) \subseteq \underline{U(\mu, t) \cap U(\nu, t)}(X)$. Let $x \in \underline{U(\mu \cap \nu, t)}(X)$ and $x' \in [x]_{(\mu,t)\cap(\nu,t)}$. Then $x' \in [x]_{(\mu,t)}$ and $x' \in [x]_{(\nu,t)}$, i.e.,

$$\bigvee_{a\vee x'=a\vee x} \mu(a) \geq t \text{ and } \bigvee_{b\vee x'=b\vee x} \nu(b) \geq t.$$

For $a \vee x' = a \vee x$ and $b \vee x' = b \vee x$, we have

$$(a \vee x') \wedge (b \vee x') = (a \vee x) \wedge (b \vee x).$$

Since L is a distributive lattice and μ and ν are fuzzy ideals of L, we have

$$x' \vee (a \wedge b) = x \vee (a \wedge b) \text{ and } \mu(a \wedge b) \geq \mu(a), \nu(a \wedge b) \geq \nu(b),$$

i.e.,

$$\begin{aligned}
t &\le \Big(\bigvee_{a\vee x'=a\vee x}\mu(a)\Big)\wedge\Big(\bigvee_{b\vee x'=b\vee x}\nu(b)\Big)\\
&\le \bigvee_{x'\vee(a\wedge b)=x\vee(a\wedge b)}\Big(\mu(a)\wedge\nu(b)\Big)\\
&\le \bigvee_{x'\vee(a\wedge b)=x\vee(a\wedge b)}\Big(\mu(a\wedge b)\wedge\nu(a\wedge b)\Big)\\
&= \bigvee_{x'\vee(a\wedge b)=x\vee(a\wedge b)}(\mu\cap\nu)(a\wedge b).
\end{aligned}$$

Thus $x'\in[x]_{(\mu\cap\nu,t)}$, then $x'\in X$, which implies that $x\in\underline{U(\mu,t)\cap U(\nu,t)}(X)$. Thus

$$\underline{U(\mu,t)\cap U(\nu,t)}(X)\subseteq\underline{U(\mu\cap\nu,t)}(X).$$

Therefore, $\underline{U(\mu\cap\nu,t)}(X)=\underline{U(\mu,t)\cap U(\nu,t)}(X)$.

(2) Let $x\in\overline{U(\mu\cap\nu,t)}(X)$. Then there exists $x'\in[x]_{(\mu,t)\cap(\nu,t)}\cap X$, i.e., $x'\in X$ and $(x,x')\in U(\mu\cap\nu,t)$, so

$$\bigvee_{a\vee y=a\vee x}(\mu\cap\nu)(a)\ge t,\ \text{i.e.,}\ \bigvee_{a\vee y=a\vee x}\Big(\mu(a)\wedge\nu(a)\Big)\ge t.$$

Thus,

$$\bigvee_{a\vee x=a\vee x'}\mu(a)\ge t\text{ and }\bigvee_{a\vee x=a\vee x'}\nu(a)\ge t.$$

Hence, $x'\in[x]_{(\mu,t)}$ and $x'\in[x]_{(\nu,t)}$, which implies that $x\in\overline{U(\mu,t)\cap U(\nu,t)}(X)$. So

$$\overline{U(\mu\cap\nu,t)}(X)\subseteq\overline{U(\mu,t)\cap U(\nu,t)}(X).$$

In a similar way, we have $\overline{U(\mu\cap\nu,t)}(X)\supseteq\overline{U(\mu,t)\cap U(\nu,t)}(X)$. Therefore, $\overline{U(\mu\cap\nu,t)}(X)=\overline{U(\mu,t)\cap U(\nu,t)}(X)$. □

Theorem 3.14. *Let μ be a fuzzy ideal of L and $t\in[0,1]$. Then*

$$\underline{U(\mu,t)}(\mu_t)=\mu_t=\overline{U(\mu,t)}(\mu_t).$$

Proof. It is easy to know that $\underline{U(\mu,t)}(\mu_t)\subseteq\mu_t\subseteq\overline{U(\mu,t)}(\mu_t)$. Now we show that $\overline{U(\mu,t)}(\mu_t)\subseteq\mu_t\subseteq\underline{U(\mu,t)}(\mu_t)$. Let $x\in\overline{U(\mu,t)}(\mu_t)$. Then $[x]_{(\mu,t)}\cap\mu_t\neq\emptyset$, which means that there exists $y\in\mu_t$ and $y\in[x]_{(\mu,t)}$, i.e., $\mu(y)\ge t$ and $\bigvee_{a\vee y=a\vee x}\mu(a)\ge t$. So there exists $a\in L$ such that $\mu(a)\ge t$ satisfying $a\vee y=a\vee x$. Then we have $a\in\mu_t$. Since μ is a fuzzy ideal of L, we have μ_t is an ideal of L and $a\vee y\in\mu_t$. Thus $a\vee x\in\mu_t$. Since $x\le a\vee x$, we have $x\in\mu_t$, which implies that $\overline{U(\mu,t)}(\mu_t)\subseteq\mu_t$. Therefore, $\overline{U(\mu,t)}(\mu_t)=\mu_t$. Further, let $x\in\mu_t$ and $y\in[x]_{(\mu,t)}$. Then $(x,y)\in U(\mu,t)$, i.e., $\bigvee_{b\vee x=b\vee y}\mu(b)\ge t$. So there exists $b\in L$ such that $\mu(b)\ge t$ satisfying $b\vee y=b\vee x$. Then we have $b\in\mu_t$ and $b\vee y\in\mu_t$. Since $y\le a\vee y$, we have $y\in\mu_t$. So $[x]_{(\mu,t)}\subseteq\mu_t$, which implies that $x\in\underline{U(\mu,t)}(\mu_t)$. Hence $\mu_t\subseteq\underline{U(\mu,t)}(\mu_t)$. From the above, $\underline{U(\mu,t)}(\mu_t)=\mu_t=\overline{U(\mu,t)}(\mu_t)$. □

Theorem 3.15. *Let μ and ν be two fuzzy ideals of L and $t\in[0,1]$. Then $\mu_t=\overline{U(\mu,t)}(\mu\cap\nu)_t$.*

Proof. It is easy to know that $(\mu\cap\nu)_t=\mu_t\cap\nu_t$. Now we show that $\mu_t=\overline{U(\mu,t)}(\mu_t\cap\nu_t)$. Let $x\in\mu_t$, $y\in\nu_t$. Then $\mu(x)\ge t$. Since μ is a fuzzy ideal of L, we have μ_t is an ideal of L. Further, $x\wedge y\le x$ and $x\wedge y\le y$, we have $x\wedge y\in\mu_t$ and $x\wedge y\in\nu_t$, i.e., $x\wedge y\in\mu_t\cap\nu_t$. Since $x\vee x=x\vee(x\wedge y)$, we have $\bigvee_{a\vee x=a\vee(x\wedge y)}\mu(a)\ge\mu(x)\ge t$, which implies that $x\wedge y\in[x]_{(\mu,t)}$. Thus $[x]_{(\mu,t)}\cap(\mu_t\cap\nu_t)\neq\emptyset$. So $x\in\overline{U(\mu,t)}(\mu_t\cap\nu_t)$, that is, $\mu_t\subseteq\overline{U(\mu,t)}(\mu_t\cap\nu_t)$.

On the other hand, it is easy to see that $\overline{U(\mu, t)}(\mu_t \cap \nu_t) \subseteq \overline{U(\mu, t)}(\mu_t)$. Moreover, it follows from Theorem 3.14 that $\overline{U(\mu, t)}(\mu_t) = \mu_t$. So $\overline{U(\mu, t)}(\mu_t \cap \nu_t) \subseteq \mu_t$. Therefore, $\mu_t = \overline{U(\mu, t)}(\mu_t \cap \nu_t)$, i.e., $\mu_t = \overline{U(\mu, t)}(\mu \cap \nu)_t$. □

Corollary 3.16. *Let μ and ν be two fuzzy ideals of L and $t \in [0, 1]$. Then $\nu_t \subseteq \overline{U(\mu, t)}(\mu_t \vee \nu_t)$.*

Proof. Since μ and ν are two fuzzy ideals of L, we have μ_t and ν_t are ideals of L. Further, since L is a distributive lattice, we have $\mu_t \vee \nu_t$ is an ideal of L. Let $x \in \mu_t$ and $y \in \nu_t$. Then $x \vee y \in \mu_t \vee \nu_t$. On the other hand, $\bigvee_{b\vee x=b\vee(x\vee y)} \mu(b) \geq \mu(x) \geq t$, which implies that $x \vee y \in [x]_{(\mu,t)}$. Thus $[x]_{(\mu,t)} \cap (\mu_t \vee \nu_t) \neq \emptyset$. So $y \in \overline{U(\mu, t)}(\mu_t \vee \nu_t)$. Therefore, $\nu_t \subseteq \overline{U(\mu, t)}(\mu_t \vee \nu_t)$. □

In the following discussion, we denote by $\downarrow a = \{x \in L | x \leq a\}$ for $a \in L$.

Theorem 3.17. *Let μ be a fuzzy ideal of L, $t \in [0, 1]$. Then*

(1) $\overline{U(\mu, t)}(\downarrow a) = \mu_t$ for each $a \in \mu_t$;

(2) $\bigcup_{a\in\mu_t} \underline{U(\mu, t)}(\downarrow a) \subseteq \mu_t$.

Proof. (1) Since μ is a fuzzy ideal of L, we have μ_t is an ideal of L. It follows from the definition of $\downarrow a$ that $\downarrow a$ is an ideal and $\downarrow a \subseteq \mu_t$ for each $a \in \mu_t$. It follows from the Theorem 3.15 that $\overline{U(\mu, t)}(\downarrow a) = \mu_t$.

(2) Let $a \in \mu_t$. Then $\downarrow a \subseteq \mu_t$. It is easy to see that $\underline{U(\mu, t)}(\downarrow a) \subseteq \underline{U(\mu, t)}(\mu_t)$. Follows from Theorem 3.14, we obtain that $\underline{U(\mu, t)}(\mu_t) = \mu_t$. Thus $\underline{U(\mu, t)}(\downarrow a) \subseteq \mu_t$. Therefore, $\bigcup_{a\in\mu_t} \underline{U(\mu, t)}(\downarrow a) \subseteq \mu_t$. □

Theorem 3.18. *Let μ and ν be two fuzzy ideals of L and $t \in [0, 1]$. Then the followings are equivalent:*

(1) $\mu \subseteq \nu$;

(2) $\nu_t = \overline{U(\mu, t)}(\nu_t)$;

(3) $\nu_t = \underline{U(\mu, t)}(\nu_t)$.

Proof. (1) ⇒ (2) Let $\mu \subseteq \nu$ and $x \in \overline{U(\mu, t)}(\nu_t)$. Then $[x]_{(\mu,t)} \cap \nu_t \neq \emptyset$. This means that there exists $a \in \nu_t$ such that $a \in [x]_{(\mu,t)}$, i.e.,

$$\bigvee_{b\vee a=b\vee x} \mu(b) \geq t.$$

Since $\mu \subseteq \nu$, we have

$$\bigvee_{b\vee a=b\vee x} \nu(b) \geq \bigvee_{b\vee a=b\vee x} \mu(b) \geq t.$$

So there exists $b \in L$ such that $\nu(b) \geq t$ satisfying $b \vee a = b \vee x$, i.e., $b \in \nu_t$. So $b \vee a = b \vee x \in \nu_t$. Since $x \leq b \vee x$, we have $x \in \nu_t$. Hence, $\overline{U(\mu, t)}(\nu_t) \subseteq \nu_t$. On the other hand, it is easy to see that $\nu_t \subseteq \overline{U(\mu, t)}(\nu_t)$. Therefore, $\nu_t = \overline{U(\mu, t)}(\nu_t)$.

(2) ⇒ (1) If $\nu_t = \overline{U(\mu, t)}(\nu_t)$, it follows from Theorem 3.14 and Theorem 3.15 that $\mu_t = \overline{U(\mu, t)}(\mu_t \cap \nu_t) \subseteq \overline{U(\mu, t)}(\nu_t) = \nu_t$. Therefore, $\mu \subseteq \nu$.

(2) ⇒ (3) Let $\nu_t = \overline{U(\mu, t)}(\nu_t)$, $x \in \nu_t$ and $a \in [x]_{(\mu,t)}$. Assume that $a \notin \nu_t$, then $a \notin \overline{U(\mu, t)}(\nu_t)$. Thus, $[x]_{(\mu,t)} \cap \nu_t = \emptyset$, this implies that $a \notin \overline{U(\mu, t)}(\nu_t) = \nu_t$, which contradicts with $x \in \nu_t$. Thus $a \in \nu_t$. Hence, $[x]_{(\mu,t)} \subseteq \nu_t$, this means that $x \in \underline{U(\mu, t)}(\nu_t)$. Thus $\nu_t \in \underline{U(\mu, t)}(\nu_t)$. On the other hand, it is easy to see that $\underline{U(\mu, t)}(\nu_t) \subseteq \nu_t$. Therefore, $\nu_t = \underline{U(\mu, t)}(\nu_t)$.

(3) ⇒ (2) Assume that $\nu_t = \underline{U(\mu, t)}(\nu_t)$. Let $x \in \overline{U(\mu, t)}(\nu_t)$. Then $[x]_{(\mu,t)} \cap \nu_t \neq \emptyset$, which means that there exists $a \in \nu_t$ such that $a \in [x]_{(\mu,t)}$. Since $\nu_t = \underline{U(\mu, t)}(\nu_t)$, we have $[x]_{(\mu,t)} = [a]_{(\mu,t)} \subseteq \nu_t$, so $x \in \underline{U(\mu, t)}(\nu_t) = \nu_t$, i.e., $\overline{U(\mu, t)}(\nu_t) \subseteq \nu_t$. On the other hand, it is easy to see that $\nu_t \subseteq \overline{U(\mu, t)}(\nu_t)$. Therefore, $\nu_t = \overline{U(\mu, t)}(\nu_t)$. □

Theorem 3.19. *Let μ, ν and ω be fuzzy ideals of L such that $\mu \subseteq \omega$ and $t \in [0,1]$. Then*

$$\overline{U(\mu,t)}(\overline{U(\nu,t)}(\omega_t)) = \overline{U(\nu,t)}(\omega_t) = \overline{U(\nu,t)}(\overline{U(\mu,t)}(\omega_t)).$$

Proof. Since $\mu \subseteq \omega$, we have $\mu_t \subseteq \omega_t$. It follows from Theorem 3.14 that $\overline{U(\mu,t)}(\omega_t) = \omega_t$. So $\overline{U(\nu,t)}(\omega_t) = \overline{U(\nu,t)}(\overline{U(\mu,t)}(\omega_t))$. Next we show that $\overline{U(\mu,t)}(\overline{U(\nu,t)}(\omega_t)) = \overline{U(\nu,t)}(\omega_t)$. First of all, we prove that $\overline{U(\nu,t)}(\omega_t)$ is an ideal of L. Since ω is a fuzzy ideal of L, we have ω_t is an ideal of L. On the other hand, it is easy to see that $a \vee b \in \overline{U(\nu,t)}(\omega_t)$ for all $a, b \in \overline{U(\nu,t)}(\omega_t)$. Let $c \in L$, $d \in \overline{U(\nu,t)}(\omega_t)$ and $c \leq d$. Then there exists $e \in [d]_{(\nu,t)} \cap \omega_t$. Now let $f \in [c]_{(\nu,t)}$. Then $e \wedge f \in [d]_{(\nu,t)} \wedge [c]_{(\nu,t)} \subseteq [c \wedge d]_{(\nu,t)} = [c]_{(\nu,t)}$. Since $e \wedge f \leq e$, we have $e \wedge f \in \omega_t$. Thus $[c]_{(\nu,t)} \cap A \neq \emptyset$, this means that $c \in \overline{U(\nu,t)}(\omega_t)$. Thus $\overline{U(\nu,t)}(\omega_t)$ is an ideal of L. Further, $\mu_t \subseteq \omega_t \subseteq \overline{U(\nu,t)}(\omega_t)$. It follows from Theorem 3.14 that $\overline{U(\mu,t)}(\overline{U(\nu,t)}(\omega_t)) = \overline{U(\nu,t)}(\omega_t)$. □

Theorem 3.20. *Let μ, ν and ω be fuzzy ideals of L such that $\mu \subseteq \omega$ and $t \in [0,1]$. Then*

$$\overline{U(\mu,t) \cap U(\nu,t)}(\omega_t) = \overline{U(\mu,t)}(\omega_t) \cap \overline{U(\nu,t)}(\omega_t).$$

Proof. Let $x \in \overline{U(\mu,t)}(\omega_t) \cap \overline{U(\nu,t)}(\omega_t)$. Since μ and ω are two fuzzy ideals of L and $\mu \subseteq \omega$, we have $\mu_t \subseteq \omega_t$. It follows from Theorem 3.14 that $x \in \omega_t \cap \overline{U(\nu,t)}(\omega_t) = \omega_t \subseteq \overline{U(\nu,t) \cap U(\nu,t)}(\omega_t)$. So $\overline{U(\mu,t)}(\omega_t) \cap \overline{U(\nu,t)}(\omega_t) \subseteq \overline{U(\mu,t) \cap U(\nu,t)}(\omega_t)$. It follows from Proposition 3.12 that $\overline{U(\mu,t) \cap U(\nu,t)}(\omega_t) = \overline{U(\mu,t)}(\omega_t) \cap \overline{U(\nu,t)}(\omega_t)$. □

Theorem 3.21. *Let μ and ν be two fuzzy ideals of L such that $\mu \subseteq \nu$ and $t \in [0,1]$. If $\emptyset \subsetneq A \subseteq L$, then*

$$\overline{U(\mu,t)}(\nu_t \cap A) = \overline{U(\mu,t)}(\nu_t) \cap \overline{U(\mu,t)}(A).$$

Proof. It is easy to see that $\overline{U(\mu,t)}(\nu_t \cap A) \subseteq \overline{U(\mu,t)}(\nu_t) \cap \overline{U(\mu,t)}(A)$. Now we show that $\overline{U(\mu,t)}(\nu_t) \cap \overline{U(\mu,t)}(A) \subseteq \overline{U(\mu,t)}(\nu_t \cap A)$. Let $x \in \overline{U(\mu,t)}(\nu_t) \cap \overline{U(\mu,t)}(A)$. Since ν is a fuzzy ideal of L, we have ν_t is an ideal of L. It follows from Theorem 3.14 that $x \in \nu_t \cap \overline{U(\mu,t)}(A)$. Thus $x \in \nu_t$ and $x \in \overline{U(\mu,t)}(A)$, i.e., $[x]_{(\mu,t)} \cap A \neq \emptyset$. Thus there exists $a \in A$ such that $a \in [x]_{(\mu,t)}$, which implies that $\bigvee_{b \vee a = b \vee x} \mu(b) \geq t$. This means that there exists $b \in L$ such that $\mu(b) \geq t$ satisfying $b \vee a = b \vee x$, i.e., $b \in \mu_t$. Since $\mu \subseteq \nu$, we have $\mu_t \subseteq \nu_t$. Thus $b \in \nu_t$ and $b \vee a = b \vee x \in \nu_t$. Since $a \leq b \vee a$, we have $a \in \nu_t$. So $a \in A \cap \nu_t$, it follows that $x \in \overline{U(\mu,t)}(\nu_t \cap A)$. And therefore $\overline{U(\mu,t)}(\nu_t \cap A) = \overline{U(\mu,t)}(\nu_t) \cap \overline{U(\mu,t)}(A)$. □

Theorem 3.22. *Let μ be a fuzzy ideal of L and $t \in [0,1]$. If A, B are ideals of L and $\mu_t \subseteq A \cup B$, then*

(1) $\overline{U(\mu,t)}(A) \vee \overline{U(\mu,t)}(B) = \overline{U(\mu,t)}(A \vee B)$;

(2) $\underline{U(\mu,t)}(A) \vee \underline{U(\mu,t)}(B) \subseteq \underline{U(\mu,t)}(A \vee B)$.

Proof. (1) Let $x \in \overline{U(\mu,t)}(A) \vee \overline{U(\mu,t)}(B)$. Then there exist $y \in \overline{U(\mu,t)}(A)$ and $z \in \overline{U(\mu,t)}(B)$ such that $x = y \vee z$, i.e., $[y]_{(\mu,t)} \cap A \neq \emptyset$ and $[z]_{(\mu,t)} \cap B \neq \emptyset$, which means that there exist $a \in A$ and $b \in B$ such that $a \in [y]_{(\mu,t)}$ and $b \in [z]_{(\mu,t)}$, i.e.,

$$\bigvee_{y' \vee a = y' \vee y} \mu(y') \geq t, \quad \bigvee_{z' \vee b = z' \vee z} \mu(z') \geq t.$$

For $y' \vee a = y' \vee y$, $z' \vee b = z' \vee z$, we have $(y' \vee z') \vee (a \vee b) = (y' \vee z') \vee (y \vee z) = (y' \vee z') \vee x$. Thus

$$\begin{aligned} t &\leq \Big(\bigvee_{y' \vee a = y' \vee y} \mu(y')\Big) \wedge \Big(\bigvee_{z' \vee b = z' \vee z} \mu(z')\Big) \\ &\leq \bigvee_{(y' \vee z') \vee (a \vee b) = (y' \vee z') \vee (y \vee z)} \Big(\mu(y') \wedge \mu(z')\Big) \\ &\leq \bigvee_{(y' \vee z') \vee (a \vee b) = (y' \vee z') \vee x} \mu(y' \vee z'). \end{aligned}$$

So $a \vee b \in [x]_{(\mu,t)}$. Thus $[x]_{(\mu,t)} \wedge (A \vee B) \neq \emptyset$, i.e., $x \in \overline{U(\mu,t)}(A \vee B)$. Therefore, $\overline{U(\mu,t)}(A) \vee \overline{U(\mu,t)}(B) \subseteq \overline{U(\mu,t)}(A \vee B)$. Next we show that $\overline{U(\mu,t)}(A \vee B) \subseteq \overline{U(\mu,t)}(B)$. Since A and B are ideals of L and L is a distributive lattice, we have $A \vee B$ is also an ideal of L. Since $\mu_t \subseteq A \cup B$, we have $\mu_t \subseteq A \cup B \subseteq A \vee B$. According to Theorem 3.14, we get that $\overline{U(\mu,t)}(A \vee B) = A \vee B \subseteq \overline{U(\mu,t)}(A) \vee \overline{U(\mu,t)}(B)$. Therefore, $\overline{U(\mu,t)}(A) \vee \overline{U(\mu,t)}(B) = \overline{U(\mu,t)}(A \vee B)$.

(2) It follows from Theorem 3.15 that $\underline{U(\mu,t)}(A \vee B) = A \vee B$. Since $\underline{U(\mu,t)}(A) \vee \underline{U(\mu,t)}(B) \subseteq A \vee B$, we have $\underline{U(\mu,t)}(A) \vee \underline{U(\mu,t)}(B) \subseteq \underline{U(\mu,t)}(A \vee B)$. □

Let μ and ν be two fuzzy ideals of L and $t \in [0,1]$. The composition of $U(\mu,t)$ and $U(\nu,t)$ is defined as follows:

$$U(\mu,t) * U(\nu,t) = \left\{(x,y) \in L \times L \,\middle|\, \exists z \in L \text{ such that } (x,z) \in U(\mu,t) \text{ and } (z,y) \in U(\nu,t)\right\}$$

It is not difficult to check that $U(\mu,t) * U(\nu,t)$ is a congruence relation on L if and only if $U(\mu,t) * U(\nu,t) = U(\nu,t) * U(\mu,t)$.

Theorem 3.23. *Let μ and ν be two fuzzy ideals of L, $t \in [0,1]$ and $U(\mu,t) * U(\nu,t) = U(\nu,t) * U(\mu,t)$.*

(1) *If A is a non-empty subset of L, then $\underline{U(\mu,t) * U(\nu,t)}(A) \subseteq \underline{U(\mu,t)}(A) \cap \underline{U(\mu,t)}(A)$.*

(2) *If A is a sublattice of L, then $\overline{U(\mu,t)}(A) \cap \overline{U(\mu,t)}(A) \subseteq \overline{U(\mu,t) * U(\nu,t)}(A)$.*

Proof. (1) Let $x \in \underline{U(\mu,t) * U(\nu,t)}(A)$ and $a \in [x]_{(\mu,t)}$. Since $x \in [x]_{(\nu,t)}$, we have $a \in [x]_{(\mu,t)*(\nu,t)}$. Thus $a \in A$. So $x \in \underline{U(\mu,t)}(A)$. In a similar way, we have $x \in \underline{U(\nu,t)}(A)$. Therefore,

$$\underline{U(\mu,t) * U(\nu,t)}(A) \subseteq \underline{U(\mu,t)}(A) \cap \underline{U(\mu,t)}(A).$$

(2) Let $x \in \overline{U(\mu,t)}(A) \cap \overline{U(\mu,t)}(A)$. Then there exist $y, z \in A$ such that $y \in [x]_{(\mu,t)}$ and $z \in [x]_{(\nu,t)}$, i.e.,

$$\bigvee_{a \vee y = a \vee x} \mu(a) \geq t, \quad \bigvee_{b \vee z = b \vee x} \nu(a) \geq t.$$

For $a \vee y = a \vee x$, $b \vee z = b \vee x$, we have $(z \vee y) \vee a = (z \vee x) \vee a$, $(z \vee x) \vee b = x \vee b$. Hence

$$\bigvee_{(z \vee y) \vee a = (z \vee x) \vee a} \mu(a) \geq \bigvee_{a \vee y = a \vee x} \mu(a) \geq t,$$

and

$$\bigvee_{(z \vee x) \vee b = x \vee b} \nu(a) \geq \bigvee_{z \vee b = x \vee b} \nu(b) \geq t.$$

Thus $(z \vee y) \in [z \vee x]_{(\mu,t)}$, $(z \vee x) \in [x]_{(\nu,t)}$, i.e., $(z \vee y) \in [x]_{(\mu,t)*(\nu,t)}$. Since A is a sublattice of L, we have $z \vee y \in A$. Thus $z \vee y \in [x]_{(\mu,t)*(\nu,t)} \cap A$, i.e., $x \in \overline{U(\mu,t) * U(\nu,t)}(A)$. Therefore, $\overline{U(\mu,t)}(A) \cap \overline{U(\mu,t)}(A) \subseteq \overline{U(\mu,t) * U(\nu,t)}(A)$. □

The following example shows that the containedness in Theorem 3.22 (2) and Theorem 3.23 need not be an equality.

Example 3.24. *Consider the lattice in Example 3.2. Let $\mu = \frac{1}{0} + \frac{0.8}{a} + \frac{0.6}{b} + \frac{0.4}{c} + \frac{0}{1}$ and $\nu = \frac{1}{0} + \frac{0.7}{a} + \frac{0.8}{b} + \frac{0.3}{c} + \frac{0}{1}$. Then it is clear that μ and ν are fuzzy ideals of L. Choose $t = 0.8$, then $\mu_t = \{0, a\}$ and $\nu_t = \{0, b\}$. Now let $A = \{a, b\}$, $B = \{0, b\}$. Then we have $\mu_t \subseteq A \cup B$ and $A \vee B = \{a, b, c\}$. Thus*

$$\underline{U(\mu,t)}(A) \vee \underline{U(\mu,t)}(B) = \emptyset \text{ and } \underline{U(\mu,t)}(A \vee B) = \{b, c\}.$$

Therefore,

$$\underline{U(\mu,t)}(A) \vee \underline{U(\mu,t)}(B) \subsetneq \underline{U(\mu,t)}(A \vee B).$$

Let $A = \{a, b, c\}$. *Then* $\underline{U(\mu, t)}(A) = \{b, c\}$, $\underline{U(\nu, t)}(A) = \{a, c\}$, *and*

$$\underline{U(\mu, t) \star U(\nu, t)}(A) = \emptyset, \ \underline{U(\mu, t)}(A) \cap \underline{U(\nu, t)}(A) = \{c\}.$$

Therefore, $\underline{U(\mu, t) \star U(\nu, t)}(A) \subseteq \underline{U(\mu, t)}(A) \cap \underline{U(\mu, t)}(A)$.

Let $A = \{a, c\}$ *be a sublattice of* L. *Then* $\overline{U(\mu, t)}(A) = \{0, a, b, c\}$, $\overline{U(\nu, t)}(A) = \{a, c\}$, *and*

$$\overline{U(\mu, t)}(A) \cap \overline{U(\mu, t)}(A) = \{b, c\} \text{ and } \overline{U(\mu, t) \star U(\nu, t)}(A) = \{0, a, b, c\}.$$

Therefore, $\overline{U(\mu, t)}(A) \cap \overline{U(\mu, t)}(A) \subseteq \overline{U(\mu, t) \star U(\nu, t)}(A)$.

Theorem 3.25. *Let* μ *and* ν *be two fuzzy ideals of* L, $t \in [0, 1]$, $U(\mu, t) \star U(\nu, t) = U(\nu, t) \star U(\mu, t)$ *and* A *be an ideal of* L.

(1) *If* $\mu_t \subseteq A$, *then* $\underline{U(\mu, t) \star U(\nu, t)}(A) = \underline{U(\mu, t)}(A) \cap \underline{U(\nu, t)}(A)$.

(2) *If* $\mu_t, \nu_t \subseteq A$, *then* $\overline{U(\mu, t) \star U(\nu, t)}(A) = \overline{U(\mu, t)}(A) \cap \overline{U(\nu, t)}(A)$.

Proof. (1) Let $x \in \underline{U(\mu, t)}(A) \cap \underline{U(\nu, t)}(A)$ and $x' \in [x]_{(\mu,t)\star(\nu,t)}$. Then there exists $y \in L$ such that $x' \in [y]_{(\mu,t)}$ and $y \in [x]_{(\nu,t)}$. So $\bigvee_{x' \vee d = y \vee d} \mu(d) \geq t$ and $y \in A$, which means that there exists $d \in L$ such that $\mu(d) \geq t$ satisfying $x' \vee d = y \vee d$. Thus $d \in \mu_t$. Since A is an ideal of L and $\mu_t \subseteq A$, we get that $y \vee d \in A$. Further, since $x' \vee d = y \vee d \geq x'$, we have $x' \in A$. So $x \in \underline{U(\mu, t) \star U(\nu, t)}(A)$. Therefore, $\underline{U(\mu, t)}(A) \cap \underline{U(\nu, t)}(A) \subseteq \underline{U(\mu, t) \star U(\nu, t)}(A)$. On the other hand, it follows from Theorem 3.23 that $\underline{U(\mu, t) \star U(\nu, t)}(A) = \underline{U(\mu, t)}(A) \cap \underline{U(\nu, t)}(A)$.

(2) Let $x \in \overline{U(\mu, t) \star U(\nu, t)}(A)$. Then there exist $x' \in A$ and $y \in L$ such that $x' \in [y]_{(\mu,t)}$ and $y \in [x]_{(\nu,t)}$. So $y \in \overline{U(\mu, t)}(A)$. Since A is an ideal of L and $\mu_t \subseteq A$, it follows from Theorem 3.15 that $\overline{U(\mu, t)}(A) = A$. So $y \in A$. Thus $x \in \overline{U(\nu, t)}(A)$. Since $U(\mu, t) \star U(\nu, t) = U(\nu, t) \star U(\mu, t)$, we have $x \in \overline{U(\mu, t)}(A)$. Therefore, $\overline{U(\mu, t) \star U(\nu, t)}(A) \subseteq \overline{U(\mu, t)}(A) \cap \overline{U(\nu, t)}(A)$. From Theorem 3.23, we get that $\overline{U(\mu, t) \star U(\nu, t)}(A) = \overline{U(\mu, t)}(A) \cap \overline{U(\nu, t)}(A)$. □

Proposition 3.26. *Let* μ *be a fuzzy ideal of* L *and* $t \in [0, 1]$. *Then*

(1) $[0]_{(\mu,t)}$ *is an ideal of* L;

(2) $[0]_{(\mu,t)} = \mu_t$.

Proof. (1) Let $x, y \in [0]_{(\mu,t)}$. Then $x \vee y \in [0]_{(\mu,t)} \vee [0]_{(\mu,t)} \subseteq [0 \vee 0]_{(\mu,t)} = [0]_{(\mu,t)}$. Thus, $x \vee y \in [0]_{(\mu,t)}$. Now let $x \in L$, $a \in [0]_{(\mu,t)}$ and $x \leq a$. Then $(a, 0) \in U(\mu, t)$, i.e., $\bigvee_{a \vee c = 0 \vee c} \mu(c) \geq t$. For $a \vee c = 0 \vee c$, we have $x \leq a \leq c$. Thus $\bigvee_{x \vee d = 0 \vee d} \mu(d) \geq \mu(c) \geq t$, i.e., $x \in [0]_{(\mu,t)}$. Therefore, $[0]_{(\mu,t)}$ is an ideal of L.

(2) We first show that $\mu_t \subseteq [0]_{(\mu,t)}$. Let $x \in \mu_t$. Then $\mu(x) \geq t$. Thus $\bigvee_{a \vee x = a \vee 0} \mu(a) \geq \mu(x) \geq t$. It follows from Definition 3.1 that $(0, x) \in U(\mu, t)$, i.e., $x \in [0]_{(\mu,t)}$. Therefore, $[0]_{(\mu,t)} \subseteq \mu_t$. Now we prove that $[0]_{(\mu,t)} \subseteq \mu_t$. Let $y \in [0]_{(\mu,t)}$. Then $(y, 0) \in U(\mu, t)$, i.e., $\bigvee_{a \vee y = a \vee 0} \mu(a) \geq t$. For $a \vee y = a \vee 0$, we know that $y \leq a$. Since μ is a fuzzy ideal of L, we have $\mu(y) \geq \mu(a)$. Thus $\mu(y) \geq \bigvee_{a \vee y = a \vee 0} \mu(a) \geq t$, i.e., $y \in \mu_t$. Therefore, $[0]_{(\mu,t)} \subseteq \mu_t$. □

4 Generalized roughness in distributive lattices with respect to fuzzy ideals

In this section, we investigate generalized roughness in a distributive lattice L with respect to a fuzzy ideal μ and t, where $t \in [0, 1]$. Let J be a distributive lattice and $\eta : L \to \mathscr{P}^\star(J)$ be a set-valued mapping, where

$\mathscr{P}^{\star}(J)$ denotes the family of all non-empty subsets of J. Let μ be a fuzzy ideal of J, $t \in [0,1]$ and X be a non-empty subset of J. We denote $\eta_{\mu}^{t}(x) = \{b \in [a]_{(\mu,t)} | a \in \eta(x)\}$ for all $x \in L$. Obviously, η_{μ}^{t} is a set-valued mapping from L to $\mathscr{P}^{\star}(J)$. Further, $\eta(x) \subseteq \eta_{\mu}^{t}(x)$ for all $x \in L$. Thus, $\underline{\eta_{\mu}^{t}}(X) = \{x \in L | \eta_{\mu}^{t}(x) \subseteq X\}$ and $\overline{\eta_{\mu}^{t}}(X) = \{x \in L | \eta_{\mu}^{t}(x) \cap X \neq \emptyset\}$ are called generalized lower and upper approximations of X with respect to μ and t, respectively. In this section, J is always a distributive lattice and $\mathscr{P}^{\star}(J)$ denotes the set of all non-empty subsets of J.

Definition 4.1. *Let $\eta : L \to \mathscr{P}^{\star}(J)$ be a mapping. Then*

(1) η is called a $\vee$-homomorphic set-valued mapping if $\eta(x) \vee \eta(y) \subseteq \eta(x \vee y)$ for all $x, y \in L$.

(2) η is called a $\wedge$-homomorphic set-valued mapping if $\eta(x) \wedge \eta(y) \subseteq \eta(x \wedge y)$ for all $x, y \in L$.

η is called a homomorphic set-valued mapping if it is both a $\vee$-homomorphic set-valued mapping and a $\wedge$-homomorphic set-valued mapping.

Theorem 4.2. *Let μ and ν be fuzzy ideals of J, $t \in [0,1]$ and $\eta : L \to \mathscr{P}^{\star}(J)$ be a homomorphic set-valued mapping. Then*

(1) η_{μ}^{t} is a homomorphic set-valued mapping.

(2) $\eta_{\mu}^{t} \cap \eta_{\nu}^{t}$ is a homomorphic set-valued mapping.

Proof. (1) Let $x, y \in L$ and $z \in \eta_{\mu}^{t}(x) \vee \eta_{\mu}^{t}(y)$. Then there exist $x' \in \eta_{\mu}^{t}(x)$ and $y' \in \eta_{\mu}^{t}(y)$ such that $z = x' \vee y'$. It follows from the definition of η_{μ}^{t} that there exist $a \in \eta(x)$, $b \in \eta(y)$ such that $x' \in [a]_{(\mu,t)}$ and $y' \in [b]_{(\mu,t)}$, i.e.,

$$\bigvee_{x' \vee c = a \vee c} \mu(c) \geq t, \quad \bigvee_{y' \vee d = b \vee d} \mu(d) \geq t.$$

For $x' \vee c = a \vee c$, $y' \vee d = b \vee d$, we have $(x' \vee y') \vee (c \vee d) = (a \vee b) \vee (c \vee d)$. Since μ is a fuzzy ideal of J, we get that $\mu(c \vee d) = \mu(c) \wedge \mu(d)$. Thus,

$$\begin{aligned} t &\leq \Big(\bigvee_{y' \vee d = b \vee d} \mu(c)\Big) \wedge \Big(\bigvee_{y' \vee d = b \vee d} \mu(d)\Big) \\ &\leq \bigvee_{(x' \vee y') \vee (c \vee d) = (a \vee b) \vee (c \vee d)} (\mu(c) \wedge \mu(d)) \\ &= \bigvee_{(x' \vee y') \vee (c \vee d) = (a \vee b) \vee (c \vee d)} \mu(c \vee d), \end{aligned}$$

and so $z = x' \vee y' \in [a \vee b]_{(\mu,t)}$. Since η is a homomorphic set-valued mapping, we have $a \vee b \in \eta(x) \vee \eta(y) \subseteq \eta(x \vee y)$. Thus $z = x' \vee y' \in \eta_{\mu}^{t}(x \vee y)$. Therefore, $\eta_{\mu}^{t}(x) \vee \eta_{\mu}^{t}(y) \subseteq \eta_{\mu}^{t}(x \vee y)$. In a similar way, we have $\eta_{\mu}^{t}(x) \wedge \eta_{\mu}^{t}(y) \subseteq \eta_{\mu}^{t}(x \wedge y)$. Hence, η_{μ}^{t} is a homomorphic set-valued mapping.

(2) Let $x, y \in L$ and $z \in (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(x) \vee (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(y)$. Then there exist $x' \in (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(x)$ and $y' \in (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(y)$ such that $z = x' \vee y'$, which means that there exist $a, b \in \eta(x)$ and $c, d \in \eta(y)$ such that $x' \in [a]_{(\mu,t)} \cap [b]_{(\mu,t)}$ and $y' \in [c]_{(\mu,t)} \cap [d]_{(\mu,t)}$. Thus

$$x' \vee y' \in \Big([a]_{(\mu,t)} \vee [c]_{(\mu,t)}\Big) \cap \Big([b]_{(\nu,t)} \vee [d]_{(\nu,t)}\Big) \subseteq [a \vee c]_{(\mu,t)} \cap [b \vee d]_{(\nu,t)}.$$

Since η is a homomorphic set-valued mapping, we have $a \vee c$, $b \vee d \in \eta(x) \vee \eta(y) \subseteq \eta(x \vee y)$. It follows that $z \in (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(x \vee y)$, and so $(\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(x) \vee (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(y) \subseteq (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(x \vee y)$. In a similar way, we have $(\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(x) \wedge (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(y) \subseteq (\eta_{\mu}^{t} \cap \eta_{\nu}^{t})(x \wedge y)$. Therefore, $\eta_{\mu}^{t} \cap \eta_{\nu}^{t}$ is a homomorphic set-valued mapping. □

Theorem 4.3. *Let μ be a fuzzy ideal of J, $t \in [0,1]$ and $\eta : L \to \mathscr{P}^{\star}(J)$ be a homomorphic set-valued mapping. If $\emptyset \subsetneq X, Y \subseteq J$, then*

(1) $\eta^t_\mu(X) \vee \eta^t_\mu(Y) \subseteq \eta^t_\mu(X \vee Y)$;

(2) $\eta^t_\mu(X) \wedge \eta^t_\mu(Y) \subseteq \eta^t_\mu(X \wedge Y)$.

Proof. Let $c \in \eta^t_\mu(X) \vee \eta^t_\mu(Y)$. Then there exist $x \in \eta^t_\mu(X)$ and $y \in \eta^t_\mu(Y)$ such that $c = x \vee y$. Thus there exist $x' \in X, y' \in Y$ and $a \in \eta(x)$, $b \in \eta(y)$ such that $x' \in [a]_{(\mu,t)}$, $y' \in [b]_{(\mu,t)}$. So $x' \vee y' \in [a \vee b]_{(\mu,t)} \cap (A \cap B)$ and $a \vee b \in \eta(x) \vee \eta(y) \subseteq \eta(x \vee y)$. Hence, $\eta^t_\mu(x \vee y) \cap (A \cap B) \neq \emptyset$, i.e., $c \in \eta^t_\mu(X \vee Y)$. Therefore, $\eta^t_\mu(X) \vee \eta^t_\mu(Y) \subseteq \eta^t_\mu(X \vee Y)$.

(2) The proof is similar to that of (1). □

Proposition 4.4. *Let μ and ν be fuzzy ideals of J, $t \in [0,1]$ and $\eta : L \to \mathscr{P}^\star(J)$ be a homomorphic set-valued mapping. If $\emptyset \subsetneq X \subseteq J$ and $\mu \subseteq \nu$, then*

(1) $\underline{\eta^t_\nu}(X) \subseteq \underline{\eta^t_\mu}(X)$.

(2) $\overline{\eta^t_\mu}(X) \subseteq \overline{\eta^t_\nu}(X)$.

Proof. It is straightforward. □

According to Proposition 4.4, we can get the following result easily.

Corollary 4.5. *Let μ and ν be fuzzy ideals of J, $t \in [0,1]$ and $\eta : L \to \mathscr{P}^\star(J)$ be a homomorphic set-valued mapping. If $\emptyset \subsetneq X \subseteq J$, then*

(1) $\underline{\eta^t_\mu}(X) \cup \underline{\eta^t_\nu}(X) \subseteq \underline{\eta^t_{\mu\cap\nu}}(X)$.

(2) $\overline{\eta^t_{\mu\cap\nu}}(X) \subseteq \overline{\eta^t_\mu}(X) \cap \overline{\eta^t_\nu}(X)$.

Lemma 4.6. *Let μ and ν be fuzzy ideals of J, $t \in [0,1]$ and $\eta : L \to \mathscr{P}^\star(J)$ be a homomorphic set-valued mapping. Then*

$$\eta^t_{\mu\cap\nu}(x) \subseteq \eta^t_\mu(x) \cap \eta^t_\nu(x)$$

for all $x \in L$.

Proof. Let $x \in L$ and $a \in \eta^t_{\mu\cap\nu}(x)$. Then there exists $b \in \eta(x)$ such that $a \in [b]_{(\mu\cap\nu,t)}$, i.e., $\bigvee_{a\vee c=b\vee c} (\mu\cap\nu)(c) \geq t$. On the other hand,

$$t \leq \bigvee_{a\vee c=b\vee c} (\mu\cap\nu)(c) = \bigvee_{a\vee c=b\vee c} \Big(\mu(c) \wedge \nu(c)\Big) = \Big(\bigvee_{a\vee c=b\vee c} \mu(c)\Big) \wedge \Big(\bigvee_{a\vee c=b\vee c} \nu(c)\Big),$$

that is,

$$\bigvee_{a\vee c=b\vee c} \mu(c) \geq t \text{ and } \bigvee_{a\vee c=b\vee c} \nu(c) \geq t,$$

which means that $a \in [b]_{(\mu,t)}$ and $a \in [b]_{(\nu,t)}$. And so, $a \in \eta^t_\mu(x) \cap \eta^t_\nu(x)$. Therefore, $\eta^t_{\mu\cap\nu}(x) \subseteq \eta^t_\mu(x) \cap \eta^t_\nu(x)$. □

From Lemma 4.6, we get the following result.

Theorem 4.7. *Let μ and ν be fuzzy ideals of J, $t \in [0,1]$ and $\eta : L \to \mathscr{P}^\star(J)$ be a homomorphic set-valued mapping. If $\emptyset \subsetneq X \subseteq J$, then*

(1) $\underline{\eta^t_{\mu\cap\nu}}(X) \supseteq \underline{\eta^t_\mu \cap \eta^t_\mu}(X)$.

(2) $\overline{\eta^t_{\mu\cap\nu}}(X) \subseteq \overline{\eta^t_\mu \cap \eta^t_\mu}(X)$.

Lemma 4.8. *Let μ be a fuzzy ideal of J, $t \in [0,1]$ and $\eta : L \to \mathscr{P}^\star(J)$ be a homomorphic set-valued mapping. Let $x \in L$. Then the following statements are equivalent:*

(1) $\eta(x) \subseteq \mu_t$;
(2) $\eta^t_\mu(x) = \mu_t$.

Proof. (1) ⇒ (2) Let $a \in \eta^t_\mu(x)$. Then there exists $b \in \eta(x) \subseteq \mu_t$ such that $a \in [b]_{(\mu,t)}$, that is, $\bigvee_{a\vee c=b\vee c} \mu(c) \geq t$, which means that there exists $c \in J$ such that $\mu(c) \geq t$ satisfying $a\vee c = b\vee c$. Thus $c \in \mu_t$ and $a\vee c = b\vee c \in \mu_t$. Since $a \leq a\vee c$, we have $a \in \mu_t$. Therefore, $\eta^t_\mu(x) \subseteq \mu_t$. Next we show that $\mu_t \subseteq \eta^t_\mu(x)$. Let $f \in \mu_t$. Since $\eta(x) \neq \emptyset$, we have there exists $d \in \eta(x) \subseteq \mu_t$, i.e., $\mu(d) \geq t$. On the other hand, since μ be a fuzzy ideal of J, we have $\mu(f \vee d) = \mu(f) \wedge \mu(d) \geq t$. Thus $\bigvee_{f\vee e=d\vee e} \mu(e) \geq \mu(f \vee d) \geq t$. So $f \in [d]_{(\mu,t)}$. Hence, $f \in \eta^t_\mu(x)$, i.e., $\mu_t \subseteq \eta^t_\mu(x)$. Therefore, $\eta^t_\mu(x) = \mu_t$.

(2) ⇒ (1) Let $g \in \eta(x)$. Since $g \in [g]_{(\mu,t)}$, we have $g \in \eta^t_\mu(g) \subseteq \mu_t$. Therefore, $\eta(x) \subseteq \mu_t$. □

Theorem 4.9. *Let μ and ν be fuzzy ideals of J, $t \in [0,1]$ and $\eta : L \to \mathscr{P}^*(J)$ be a homomorphic set-valued mapping. If $\mu_t \subseteq X \subseteq J$ and $\eta(x) \subseteq \mu_t$ for all $x \in L$, then $\underline{\eta^t_\mu}(x) = \overline{\eta^t_\mu}(x) = L$.*

Proof. According to Lemma 4.8, we get the conclusion easily. □

Theorem 4.10. *Let μ and ν be fuzzy ideals of J, $t \in [0,1]$, $\mu \subseteq \nu$ and $\eta : L \to \mathscr{P}^*(J)$ be a homomorphic set-valued mapping. If $x \in \eta(x)$ for all $x \in L$, then the following are equivalent:*

(1) *$\eta(x) \subseteq \nu_t$ for all $x \in \nu_t$;*
(2) *$\underline{\eta^t_\mu}(\nu_t) = \nu_t$.*

Proof. (1) ⇒ (2) Let $x \in \underline{\eta^t_\mu}(\nu_t)$. Then $\eta^t_\mu(x) \subseteq \nu_t$. Since $x \in \eta(x) \subseteq \eta^t_\mu(x)$, we have $x \in \nu_t$. Now let $a' \in \nu_t$. Then for any $y \in \eta^t_\mu(x)$, there exists $a' \in \eta(x)$ such that $y \in [a']_{(\mu,t)}$, i.e., $\bigvee_{y\vee c=a'\vee c} (\mu)(c) \geq t$, which means that there exists $c \in J$ such that $\mu(c) \geq t$ satisfying $y \vee c = a' \vee c$. Thus $c \in \mu_t$. Since $\mu \subseteq \nu$, we have $\mu_t \subseteq \nu_t$. On the other hand, since $\eta(x) \subseteq \nu_t$, we have $a \vee c \in \nu_t$. So $y \in \nu_t$. Thus, $\underline{\eta^t_\mu}(\nu_t) \subseteq \nu_t$. Therefore, $\underline{\eta^t_\mu}(\nu_t) = \nu_t$.

(2) ⇒ (1) Let $x \in \nu_t$ and $y \in \eta(x)$. Since $\eta(x) \subseteq \eta^t_\mu(x)$, we have $y \in \eta^t_\mu(x)$. On the other hand, $\underline{\eta^t_\mu}(\nu_t) = \nu_t$, we have $\eta^t_\mu(x) \subseteq \nu_t$. Thus $y \in \nu_t$. Therefore, $\eta(x) \subseteq \nu_t$ for all $x \in \nu_t$. □

5 Conclusion

The study of rough sets in the distributive lattice theory is an interesting topic of rough set theory. In this paper, we introduce the special class of rough sets and generalized rough sets with respect to a fuzzy ideal in a distributive lattice, that is the universe of objects is endowed with a distributive lattice and a congruence relation is defined with respect to a fuzzy ideal. The main conclusions in this paper and the further work to do are listed as follows.

(1) A novel congruence relation $U(\mu, t)$ induced by a fuzzy ideal μ of a distributive lattice is introduced.
(2) Roughness in distributive lattices with respect to fuzzy ideals are investigated,
(3) Generalized roughness in distributive lattices with respect to fuzzy ideals are investigated.

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