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Rough quotient in topological rough sets

Abstract: In this paper, we introduce a rough quotient. Also, we present conditions ensuring that $\overline{G}/H$ are partitions of $\overline{G}$. The rough projection map is also presented. We discuss first, second and third rough isomorphism theorems and other related results. At the end, an orbit and a stabilizer in topological rough groups are considered.

Keywords: rough quotient groups; the first rough isomorphism theorem; the second rough isomorphism theorem; the third rough isomorphism theorem

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1 Introduction

The rough set theory was introduced by Pawlak in [1]. Since then, many authors worked on rough set theory. For more details, see [2–6]. The classical rough set theory is based on the equivalence relations.

In 2016, Bagirmaz et al. introduced the notion of topological rough groups. They extended the notion of a topological group to include algebraic structures of rough groups in [7]. For more detailed definitions about rough groups, rough subgroups, rough normal subgroups and rough homomorphisms and kernel, see the recent paper [8].

The main purpose of this paper is to initiate rough quotient groups. For instance, we present conditions that we need to ensure that $\overline{G}/H$ are partitions of $\overline{G}$. We also define the rough projection maps. Moreover, the first, the second and the third rough isomorphism theorems are given with other important results. Moreover, the concepts of an orbit and stabilizer in topological rough groups are defined. For the details of topological group theory, we follow [9].

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2 Preliminaries

First, we give the definition of rough groups introduced by Biswas and Nanda in 1994.

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Let (U, R) be an approximation space such that U is any set and R is an equivalence relation on U. For a subset $X \subset U$,

$$\overline{X} = \{[x]_R : [x]_R \cap X \neq \emptyset\}$$

and

$$\underline{X} = \{[x]_R : [x]_R \subset X\},$$

then the set $X = (\underline{X}, \overline{X})$ is called a rough set of U.

Suppose that $(\star)$ is a binary operation defined on U. We will use xy instead of $x \star y$ for all composition of elements $x, y \in U$, as well as, for composition of subsets XY, where $X, Y \subseteq U$.

Definition 2.1. *[7] Let $G = (\underline{G}, \overline{G})$ be a rough set in the approximation space (U, R). Then $G = (\underline{G}, \overline{G})$ is called a rough group if the following conditions are satisfied:*

1. *$\forall x, y \in G, xy \in \overline{G}$ (closed);*
2. *$(xy)z = x(yz), \forall x, y, z \in \overline{G}$ (associative law);*
3. *$\forall x \in G, \exists e \in \overline{G}$ such that $xe = ex = x$ (e is the rough identity element);*
4. *$\forall x \in G, \exists y \in G$ such that $xy = yx = e$ (y is the rough inverse element of x. It is denoted as x^{-1}).*

Definition 2.2. *[7] A non-empty rough subset $H = (\underline{H}, \overline{H})$ of a rough group $G = (\underline{G}, \overline{G})$ is called a rough subgroup if it is a rough group itself.*

The rough set $G = (\underline{G}, \overline{G})$ is a trivial rough subgroup of itself. Also the rough set $e = (e, \overline{e})$ is a trivial rough subgroup of the rough group G if $e \in G$.

Theorem 2.1. *[7] A rough subset H is a rough subgroup of the rough group G if the two conditions are satisfied:*

1. *for all $x, y \in H, xy \in \overline{H}$;*
2. *for all $y \in H, y^{-1} \in H$.*

Also, a rough normal subgroup can be defined. Let N be a rough subgroup of the rough group G, then N is called a rough normal subgroup of G if for all $x \in G, xN = Nx$

Definition 2.3. *[5] Let (U_1, R_1) and (U_2, R_2) be two approximation spaces and $\star, \star'$ be two binary operations on U_1 and U_2, respectively. Suppose that $G_1 \subseteq U_1, G_2 \subseteq U_2$ are rough groups. If the mapping $\varphi : \overline{G_1} \to \overline{G_2}$ satisfies $\varphi(x \star y) = \varphi(x) \star' \varphi(y)$ for all $x, y \in \overline{G_1}$, then φ is called a rough homomorphism.*

Here, we present a topological rough group, which is an ordinary topology on a rough group, i.e., a topology τ on $\overline{G}$ induced a subspace topology τ_G on G. Suppose that (U, R) is an approximation space with a binary operation $\star$ on U. Let G be a rough group in U.

Definition 2.4. *[7] A topological rough group is a rough group G with a topology τ_G on $\overline{G}$ satisfying the following conditions:*

1. *the product mapping $f : G \times G \to \overline{G}$ defined by $f(x, y) = xy$ is continuous with respect to a product topology on $G \times G$ and the topology τ on G induced by τ_G;*
2. *the inverse mapping $\iota : G \to G$ defined by $\iota(x) = x^{-1}$ is continuous with respect to the topology τ on G induced by τ_G.*

Elements in the topological rough group G are elements in the original rough set G with ignoring elements in approximations.

Example 2.1. *Let $U = \mathbb{R}$ and $U/R = \{\{x : x \geq 0\}, \{x : x < 0\}\}$ be a partition of $\mathbb{R}$. Consider $G = \mathbb{R}^\star = \mathbb{R} - 0$. Then G is a rough group with addition. It is also a topological rough group with the usual topology on $\mathbb{R}$.*

Example 2.2. *[7] Consider $U = S_4$ the set of all permutations of four objects. Let $(\star)$ be the multiplication operation of permutations. Let*

$$U/\mathcal{R} = \{E_1, E_2, E_3, E_4\}$$

be a classification of U, where

$$E_1 = \{1, (12), (13), (14), (23), (24), (34)\}$$

$$E_2 = \{(123), (132), (142), (124), (134), (143), (234), (243)\}$$

$$E_3 = \{(1234), (1243), (1342), (1324), (1423), (1432)\}$$

$$E_4 = \{(12)(34), (13)(24), (14)(23)\}.$$

Let $G = \{(12), (123), (132)\}$, then $\overline{G} = E_1 \cup E_2$. Clearly, G is a rough group. Consider a topology on $\overline{G}$ as $\tau_G = \{\emptyset, \overline{G}, \{(12)\}, \{1, (123), (132)\}, \{1, (12), (123), (132)\}\}$, then the relative topology on G is $\tau = \{\emptyset, G, \{(12)\}, \{(123), (132)\}\}$. The two conditions in Definition 2.4 are satisfied, hence G is a topological rough group.

3 Rough quotient

Let G be a rough group such that $\overline{G}$ is a group. Let H be a rough subgroup of G where both H and $\overline{H}$ are not subgroups in $\overline{G}$. Then $\overline{G}/H$ and $\overline{G}/\overline{H}$ do not divide $\overline{G}$ to cosets (partitions of $\overline{G}$). The following example confirms our argument.

Example 3.1. *Consider $Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$. Let R be an equivalence relation on Z_6 such that*

$$Z_6/R = \{\{\overline{2}, \overline{3}\}, \{\overline{0}, \overline{1}, \overline{4}\}, \{\overline{5}\}\}.$$

For the rough group $G = \{\overline{0}, \overline{1}, \overline{5}, \overline{3}\}$, we have its rough subgroup $H = \{\overline{0}, \overline{1}, \overline{5}\}$. Obviously, we have $\overline{G} = Z_6$ and $\overline{H} = \{\overline{0}, \overline{1}, \overline{4}, \overline{5}\}$. Note that H and $\overline{H}$ are not subgroups of $\overline{G}$. Then

$$\overline{G}/H = \{\{\overline{0}, \overline{1}, \overline{5}\}, \{\overline{1}, \overline{2}, \overline{0}\}, \{\overline{2}, \overline{3}, \overline{1}\}, \{\overline{3}, \overline{4}, \overline{2}\}, \{\overline{4}, \overline{5}, \overline{3}\}, \{\overline{5}, \overline{0}, \overline{4}\}\}$$

and

$$\overline{G}/\overline{H} = \{\{\overline{0}, \overline{1}, \overline{4}, \overline{5}\}, \{\overline{1}, \overline{2}, \overline{5}, \overline{0}\}, \{\overline{2}, \overline{3}, \overline{0}, \overline{1}\}, \{\overline{3}, \overline{4}, \overline{1}, \overline{2}\}, \{\overline{4}, \overline{5}, \overline{2}, \overline{3}\}, \{\overline{5}, \overline{0}, \overline{3}, \overline{4}\}\}$$

do not form a partition of $\overline{G}$.

Here, we need that G is a rough group such that $\overline{G}$ is a group and H is a rough subgroup of G. We also need that H is a subgroup in $\overline{G}$ (or $\overline{H}$ is a subgroup in $\overline{G}$). Consequently, by group theory, $\overline{G}/H$ (or $\overline{G}/\overline{H}$) is a partition of $\overline{G}$. If also H (or $\overline{H}$) is a normal subgroup in $\overline{G}$, then $\overline{G}/H$ (or $\overline{G}/\overline{H}$) is a rough quotient group.

Example 3.2. *Consider the approximation space (Q_8, R) where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group. Let*

$$R = \{\{\pm i\}, \{\pm 1\}, \{\pm j, \pm k\}\}.$$

Let $G = \{\pm i, -1\}$, then $\overline{G} = \{\pm i, \pm 1\}$. It is easy to prove that G is a rough group and $\overline{G}$ is a group. Consider the rough subgroup $H = \{-1\}$ of G. We have its upper approximation $\overline{H} = \{\pm 1\}$, which is a subgroup of $\overline{G}$. Then

$$\overline{G}/\overline{H} = \{\{\pm i\}, \{\pm 1\}\},$$

is a partition of $\overline{G}$.

Let G be a topological rough group such that $\overline{G}$ is a group. Let H be a rough subgroup of G and H be a subgroup of $\overline{G}$. Then the left (or right) action (multiplication) of H on G induces a projection $\pi : \overline{G} \to \overline{G}/H$, that is, it is continuous and open. Then $\overline{G}/H$ is a rough homogeneous space of G with respect to the left (or right) action.

If H is a normal subgroup in $\overline{G}$, then $\overline{G}/H$ is a group and the projection map π is a homomorphism.

Theorem 3.1. *Let G be a topological rough group and let H be a rough subgroup of G such that $\overline{G}$ is a group and H is a subgroup of $\overline{G}$. If H is an open set in $\overline{G}$, then it is closed in $\overline{G}$.*

Proof. The left coset is $\overline{G}/H = \{gH : g \in \overline{G}\}$. By a left transformation, gH is open for every $g \in \overline{G}$ because that H is open in $\overline{G}$, . Each element of $\overline{G}/H$ can be written as a complement of union of all other elements. Hence $H = eH$ is closed in $\overline{G}$. □

Theorem 3.2. *Let H be a closed rough subgroup of a topological rough group G (where $\overline{G}$ is a group and H is a subgroup of $\overline{G}$). Then the family $\{\pi(xV) : V \in \overline{G}, e \in V\}$ is a local base of the space $\overline{G}/H$ at the point $xH \in \overline{G}/H$.*

Proof. Let W be an arbitrary open set in $\overline{G}/H$ such that $xH \in W$. Put $O = \pi^{-1}(W)$. The continuity of the projection map implies that the set O is open. It is clear that $x \in O$, because that $\pi^{-1}(xH) = x$. Let V be an open set in $\overline{G}$ such that $e \in V$ and $xV \subseteq O$. Then $\pi(xV) \subset W$ implies $\pi^{-1}(\pi(xV)) \subset O$. We can write the set $xVH = \pi(xV)$ as the union of left cosets yH where $y \in xV$. It follows that $\pi(xVH) \subset W$, where $\pi(xVH)$ is open by the projection map. □

Theorem 3.3. *Let G be a topological rough group such that $\overline{G}$ is a group. If H is a closed rough subgroup such that H is a subgroup of $\overline{G}$, then $\overline{G}/H$ is a T_1-space.*

Proof. We know that L_g is a homeomorphism, so $L_g(H) = gH = \pi(g)$. Thus for every $g \in \overline{G}$, $\pi(g)$ is closed in $\overline{G}/H$. It follows that $\overline{G}/H$ is a T_1-space. □

Theorem 3.4. *(The First Rough Isomorphism Theorem) Let $f : \overline{G} \to \overline{H}$ be a topological rough group homomorphism from $\overline{G}$ into $\overline{H}$ such that $\overline{G}$ is a group. Let $K = kerf$, then $\phi : \overline{G}/K \to \overline{H}$, defined by $\phi(xK) = f(x)$, is a continuous rough isomorphism. If f is open, then ϕ is a rough homeomorphism.*

Proof. Consider the projection map $\pi : \overline{G} \to \overline{G}/K$. It is clear that $f = \phi \circ \pi$. We know that π is continuous and open. Thus, ϕ is continuous. Since K is a normal subgroup in $\overline{G}$, then π is a homomorphism. This implies that ϕ is a homomorphism. Let $x \in \overline{G}$ such that $\phi(xK) = e_H$. Then $f(x) = \phi(\pi(x)) = e_H$. If $x \in K$, then $xK = K$. Thus ϕ is a rough isomorphism. Now assume that f is open. Since π is onto, we have that ϕ is open (let W be an open set in $\overline{G}/K$, then the image $\phi(W) = f(\pi^{-1}(W))$ is open in H) which implies that ϕ^{-1} is continuous. Hence, ϕ is a rough homeomorphism. □

Before stating the second rough isomorphism theorem, we need the following proposition.

Proposition 3.1. *Let G and H be topological rough groups such that $\overline{G}$ and $\overline{H}$ are groups. Let $f : \overline{G} \to \overline{H}$ be a rough homeomorphism. Let G' be a rough subgroup of G and be a normal subgroup in $\overline{G}$. Take $H' = f(G')$, then $\phi : \overline{G}/G' \to \overline{H}/H'$ is a topological rough group homeomorphism.*

Proof. Consider the projection maps $\pi : \overline{G} \to \overline{G}/G'$ and $\pi' : \overline{H} \to \overline{H}/H'$. We have $\pi' \circ f = \phi \circ \pi$, then ϕ is a continuous open homomorphism due to the fact that f, π and π' are open continuous homomorphisms. Let $xG' \in \overline{G}/G'$ and set $y = f(x)$. If $\phi(xG') = H'$, then $\pi'(y) = H'$. Therefore, $y \in H'$ and $x \in G'$. Thus, the kernel of ϕ is trivial. By the first rough isomorphism theorem, ϕ is a topological rough group homeomorphism. □

Theorem 3.5. *(The Second Rough Isomorphism Theorem) Let G and H be topological rough groups such that $\overline{G}$ and $\overline{H}$ are groups. Let $f : \overline{G} \to \overline{H}$ be a rough homomorphism such that f is open. Let H' be a normal subgroup*

of $\overline{H}$. Let $G' = f^{-1}(H')$ and $K = f^{-1}(e_H)$. Then $\phi : ((\overline{G}/K)/(G'/K)) \to \overline{H}/H'$ is a topological rough group homeomorphism.

Proof. Let π' be the rough quotient homomorphism of $\overline{H}$ onto $\overline{H}/H'$. It is clear that π' is a continuous, open map and homomorphism. Then the composition $\pi' \circ f$ is continuous, open map and homomorphism from $\overline{G}$ onto $\overline{H}/H'$ with kernel $G' = f^{-1}(H')$. Hence the rough quotient group $\overline{G}/G'$ is a topological rough group homeomorphism to $\overline{H}/H'$ by the first rough isomorphism theorem. Also, by the same theorem, $\psi : \overline{G}/K \to \overline{H}$ defined by $\psi(xK) = H'$, is a topological rough group homeomorphism and $\psi(G'/K) = H'$. By applying Proposition 3.1, we conclude that ϕ is a topological rough group homeomorphism. $\square$

Theorem 3.6. *(The Third Rough Isomorphism Theorem) Let G be a topological rough group and H be a normal subgroup of $\overline{G}$. Let M be an arbitrary topological rough subgroup of G. If $\overline{G}, \overline{H}$ and $\overline{M}$ are groups, then the rough quotient $\overline{M}H/H$ is a topological rough group homeomorphism to the subgroup $\phi(\overline{M})$ of the rough quotient group $\overline{G}/H$, where $\pi : \overline{G} \to \overline{G}/H$.*

Proof. It is clear that $\overline{M}H = \pi^{-1}(\pi(\overline{M}))$. Consider the restriction map ϕ of π to $\overline{M}H$ onto $\pi(\overline{M})$. Let e be the rough identity of G. Then e is the identity in the group $\overline{G}$. Moreover, $\phi^{-1}(\phi(e)) = \pi^{-1}(\pi(e)) = H = ker(\phi)$. By the first rough isomorphism theorem, $\overline{M}H/H$ and $\phi(\overline{M})$ are topological rough group homeomorphisms. $\square$

Let (U, R) be an approximation space with a binary operation $\star$ defined on U. Let G be a topological rough group in U, and X be a topological space induced by topological space $\overline{X}$, where X is a rough set in U. Suppose that G acts on X from left (right). We can define an equivalence relation on $\overline{X}$ by setting $x \sim x'$ if there is an element $g \in \overline{G}$ such that $x' = gx$ ($x' = xg$).

Consider the quotient space $\overline{X}/\overline{G}$ and the projection map $\pi : \overline{X} \to \overline{X}/\overline{G}$. Each element in $\overline{X}/\overline{G}$ is

$$\overline{G}x = \{gx : g \in \overline{G}\}$$

or

$$(x\overline{G} = \{xg : g \in \overline{G}\}).$$

It is called an orbit of x by G. Each orbit is an equivalence class of $\sim$.

Now, let G be a topological rough group such that $\overline{G}$ is a group. Given a rough action of G on X. For an element x of $\overline{X}$, consider the set $G_x = \{g \in \overline{G} : gx = x\}$ (or $G_x = \{g \in \overline{G} : xg = x\}$). Then G_x is called the stabilizer of x.

Theorem 3.7. *The stabilizer of x is a subgroup of $\overline{G}$.*

Proof.   1. For $g_1, g_2 \in G_x$, we have $g_1 x = x$ and $g_2 x = x$. Then $g_1 g_2 x = g_1 x = x$. Thus, $g_1 g_2 \in G_x$.
2. The identity element $e \in G_x$ (since $ex = x$).
3. Let $g \in G_x$. Then $g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = ex = x$. Hence, for every $g \in G_x$, we have $g^{-1} \in G_x$.
From (1), (2) and (3), we conclude that G_x is a subgroup of $\overline{G}$.

$\square$

Remark 3.1. *In definition of the stabilizer, we always need $\overline{G}$ to be a group. Without this condition, we cannot confirm that G_x is closed under multiplication.*

Now, for $x \in \overline{X}$, we define the map $\mu_x : \overline{G} \to \overline{X}$ by $\mu_x(g) = gx$ (or $\mu_x(g) = xg$), where μ_x is continuous.

Remark 3.2. *Let G be a topological rough group such that $\overline{G}$ is a group acting on X. Then*
(i) G acts transitively on X $\iff$ μ_x is surjective;
(ii) G acts effectively on X $\iff$ $\cap_{x \in X} G_x = \{e\}$.

Theorem 3.8. *Let X be a topologically rough homogeneous space of G.*

1. *μ_x induces a bijection $h_x : \overline{G}/G_x \to \overline{X}$ such that $\mu_x = h_x \circ \pi_x$, where $\pi_x : \overline{G} \to \overline{G}/G_x$ is the projection.*
2. *If μ_x is an open map, then h_x is a homeomorphism.*

Proof.   1. We have that X is a rough homogeneous space of G. That is, G acts transitively, so we have μ_x is surjective. Observe that $\pi_x(g_1) = \pi_x(g_2) \iff g_1^{-1}g_2 \in G_x \iff g_1^{-1}g_2 x = x \iff \mu_x(g_1) = \mu_x(g_2)$. Since $h_x(\pi_x(g)) = \mu_x(g)$, h_x is injective. We also have μ_x and π_x are both surjective, so h_x is surjective.
2. From above, we have h_x is a bijection. Now, let V be an open set of $\overline{X}$, then $\mu_x^{-1}(V) = \pi_x^{-1}(h_x^{-1}(V))$ is open and so $h_x^{-1}(V)$ is open. Thus, h_x is a continuous bijection.

   Let O be an open set in $\overline{G}/G_x$. If μ_x is open map, then $\mu_x(\pi^{-1}(O)) = h_x(O)$ is open. Therefore, h_x is an open map, and hence h_x is a homeomorphism.

□

4 Conclusion

In this paper, we investigated cosets and quotients in topological rough groups giving conditions ensuring that $\overline{G}/H$ or $\overline{G}/\overline{H}$ are partitions of $\overline{G}$ and $\overline{G}/H$ or $\overline{G}/\overline{H}$ is a group. We also have discussed rough isomorphism theorems and other related results in this theory. So far, we have many applications of rough sets into making decision theory, but we could not see more applications of rough algebraic structures. In future, we want to find applications of these rough algebraic topological structures with into making decision theory.

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