Open Math. 2019; 17:1756–1763 DE GRUYTER

**Open Mathematics**

**Research Article**

Shuzhen Luo and Xiaoquan Xu*

# Split Hausdorff internal topologies on posets

https://doi.org/10.1515/math-2019-0136
Received January 23, 2019; accepted November 10, 2019

**Abstract:** In this paper, the concepts of weak quasi-hypercontinuous posets and weak generalized finitely regular relations are introduced. The main results are: (1) when a binary relation $\rho : X \rightharpoonup Y$ satisfies a certain condition, $\rho$ is weak generalized finitely regular if and only if $(\varphi_\rho(X, Y), \subseteq)$ is a weak quasi-hypercontinuous poset if and only if the interval topology on $(\varphi_\rho(X, Y), \subseteq)$ is split $T_2$; (2) the relation $\nleq$ on a poset $P$ is weak generalized finitely regular if and only if $P$ is a weak quasi-hypercontinuous poset if and only if the interval topology on $P$ is split $T_2$.

**Keywords:** split $T_2$; $T_2$ property; weak generalized finitely regular; weak quasi-hypercontinuous poset

**MSC:** 06B35, 54H10, 06A11

## 1 Introduction

In domain theory, the interval topology and the Lawson topology are two important "two-sided" topologies on posets. A basic problem (see [1-5]) is: When do the interval topology and the Lawson topology have $T_2$ properties? In [5] (see also [3, 4]), Gierz and Lawson have discussed this problem for the Lawson topology, and proved that a complete lattice is a quasicontinuous lattice if and only if the Lawson topology is $T_2$. However, $T_2$ properties for the interval topology on posets have attracted a considerable deal of attention (see [6-18]). Especially, Erné [1] obtained several equivalent characterizations about $T_2$ properties of the interval topology on posets. For a complete lattice $L$, Gierz and Lawson [5] proved that the interval topology on $L$ is $T_2$ if and only if $L$ is a generalized bicontinuous lattice.

The regularity of binary relations was first characterized by Zareckiǐ [18]. In [18] he proved the following remarkable result: a binary relation $\rho$ on a set $X$ is regular if and only if the complete lattice $(\Phi_\rho(X), \subseteq)$ is completely distributive, where $\Phi_\rho(X) = \{\rho(A) : A \subseteq X\}, \rho(A) = \{y \in X : \exists a \in A \text{ with } (a, y) \in \rho\}$. Further criteria for regularity were given by Markowsky [19] and Schein [20] (see also [21] and [22]). Motivated by the fundamental works relative Zareckiǐ on regular relations, Xu and Liu [23] introduced the concepts of finitely regular relations and generalized finitely regular relations, respectively. It is proved that a relation $\rho$ is generalized finitely regular if and only if the interval topology on $(\Phi_\rho(X), \subseteq)$ is $T_2$. Especially, in complete lattices, this condition turns out to be equivalent both to the $T_2$ interval topology and to the quasi-hypercontinuous lattices.

In this paper, we mainly concentrate on the $T_2$ interval topology of posets by using the regularity of binary relations. Therefore, we introduce the concepts of the split $T_2$ interval topology on posets and weak generalized finitely regular relations. Meanwhile, in order to characterize split $T_2$ interval topology of posets by a order structure, like the equivalence of the $T_2$ interval topology and quasi-hypercontinuous lattices in

**Shuzhen Luo:** Faculty of Science, Jiangxi University of Science and Technology, Ganzhou, Jiangxi 341000, China, E-mail: shuzhenluo@163.com
***Corresponding Author: Xiaoquan Xu:** School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian 363000, China, E-mail: xiqxu2002@163.com

[23], we give the notion of a weak quasi-hypercontinuous poset. It is proved that when a binary relation $\rho : X \rightharpoonup Y$ satisfies property $M$, $\rho$ is weak generalized finitely regular if and only if $(\varphi_\rho(X, Y), \subseteq)$ is a weak quasi-hypercontinuous poset if and only if the interval topology on $(\varphi_\rho(X, Y), \subseteq)$ is split $T_2$, where $\varphi_\rho(X, Y) = \{\rho(x) : x \in X\}$. For a poset $P$, the relation $\not\leq$ on $P$ is weak generalized finitely regular if and only if $P$ is a weak quasi-hypercontinuous poset if and only if the interval topology on $P$ is split $T_2$, which generalizes the corresponding works in [12, 16, 17].

## 2 Preliminaries

In this section, we recall some basic concepts needed in this paper; other non-explicitly stated elementary notions please refer to [4, 23, 24].

Let $P$ be a poset. For all $x \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. $A^{\uparrow}$ and $A^{\downarrow}$ denote the sets of all upper and lower bounds of $A$, respectively. Let $A^{\delta} = (A^{\uparrow})^{\downarrow}$ and $\delta(P) = \{A^{\delta} : A \subseteq P\}$. To avoid ambiguities, we also denote $A^{\uparrow}$, $A^{\downarrow}$ and $A^{\delta}$ on $P$ by $A_P^{\uparrow}$, $A_P^{\downarrow}$ and $A_P^{\delta}$, respectively. $(\delta(P), \subseteq)$ is called the *normal completion*, or the *Dedekind-MacNeille completion* of $P$ (see [25]). The topology generated by the collection of sets $P \backslash \downarrow x$ (as a subbase) is called the *upper topology* and denoted by $\upsilon(P)$; the *lower topology* $\omega(P)$ on $P$ is defined dually. The topology $\theta(P) = \upsilon(P) \vee \omega(P)$ is called the *interval topology* on $P$. For any set $X$, let $X^{(<\omega)} = \{F \subseteq X : F$ is nonempty and finite$\}$.

For two sets $X$ and $Y$, we call $\rho : X \rightharpoonup Y$ a binary relation if $\rho \subseteq X \times Y$. When $X = Y$, $\rho$ is usually called a binary relation on X.

**Definition 2.1.** *Let $\rho : X \rightharpoonup Y$, $\tau : Y \rightharpoonup Z$ be two binary relations. Define*

(1) $\tau \circ \rho = \{(x, z) : \exists y \in Y, (x, y) \in \rho, (y, z) \in \tau\}$. *The relation $\tau \circ \rho : X \rightharpoonup Z$ is called the* composition *of $\rho$ and $\tau$.*
(2) $\rho^{-1} = \{(y, x) \in Y \times X : (x, y) \in \rho\}$.
(3) $\rho^c = X \times Y \backslash \rho$.
(4) $\rho(A) = \{y \in Y : \exists x \in A$ *with* $(x, y) \in \rho\}$, *we call it the image of $A$ under a binary relation $\rho$. Instead of $\rho(\{x\})$, we write $\rho(x)$ for short.*
(5) $\Phi_\rho(X, Y) = \{\rho(A) : A \subseteq X\}$.
(6) $\varphi_\rho(X, Y) = \{\rho(x) : x \in X\}$.
(7) $\varphi_y = \{\rho(u) \in \varphi_\rho(X, Y) : y \notin \rho(u)\}$.

Clearly, $\varphi_\rho(X, Y) \subseteq \Phi_\rho(X, Y)$, and $(\Phi_\rho(X, Y), \subseteq)$ is a complete lattice in which the join operation $\vee$ is the set union operator $\cup$. But in general $(\varphi_\rho(X, Y), \subseteq)$ is not a complete lattice. For example, let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Define a relation $\rho = \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_3), (x_3, y_1), (x_3, y_3)\}$. Then $\rho(x_1) = \{y_1, y_2\}, \rho(x_2) = \{y_2, y_3\}$ and $\rho(x_3) = \{y_1, y_3\}$. It is easy to see that there is no lease upper bound of $\rho(x_1), \rho(x_2)$ in $(\varphi_\rho(X, Y), \subseteq)$.

**Definition 2.2.** *[12] Let $P$ be a poset and $x \in P$, $A \subseteq P$.*

(1) *Define a relation $\prec$ on $P$ by $A \prec_P x$ iff $x \in int_{\upsilon(P)} \uparrow A$. Without causing confusion, we write $A \prec x$ for short.*
(2) *$P$ is called* quasi-hypercontinuous *if for all $x \in P$, $\uparrow x = \bigcap\{\uparrow F : F$ is finite and $F \prec x\}$ and $\{F \in P^{(<\omega)} : F \prec x\}$ is directed.*

A complete lattice which is quasi-hypercontinuous as a poset is called a quasi-hypercontinuous lattice (see [12]). In [12], it has been proved that $L$ is a quasi-hypercontinuous lattice if for all $x \in L$, and $U \in \upsilon(L)$ with $x \in U$, there exists $F \in L^{(<\omega)}$ such that $x \in \mathrm{int}_{\upsilon(L)} \uparrow F \subseteq \uparrow F \subseteq U$ .

**Definition 2.3.** *[12]A binary relation $\rho : X \rightharpoonup Y$ is called* generalized finitely regular, $\forall(x, y) \in \rho$, $\exists\{u_1, u_2, \dots, u_n\} \in X^{(<\omega)}$ *and* $\{v_1, v_2, \dots, v_m\} \in Y^{(<\omega)}$ *such that*

(a) $(u_i, y) \in \rho, (x, v_j) \in \rho (i = 1, 2, \dots, n; j = 1, 2, \dots, m)$, *and*
(b) $\forall\{s_1, s_2, \dots, s_m\} \in X^{(<\omega)}, \{t_1, t_2, \dots, t_n\} \in Y^{(<\omega)}$, *if* $(u_i, t_i) \in \rho(i = 1, 2, \dots, n), (s_j, v_j) \in \rho(j = 1, 2, \dots, m)$, *then* $\exists(l, k) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ *such that* $(s_l, t_k) \in \rho$.

**Theorem 2.4.** *[12] Let $\rho : X \rightharpoonup Y$ be a binary relation. Then the following conditions are equialent:*

(1) *$\rho$ is generalized finitely regular;*
(2) *$(\Phi_\rho(X, Y), \subseteq)$ is a quasi-hypercontinuous lattice.*

**Definition 2.5.** *[12] Let $\tau$ and $\delta$ be two topologies on a poset $P$. $\alpha = \tau \vee \delta$ is called* split $T_2$ *or* split Hausdorff *about $\tau$ and $\delta$, if for any $x, y$ with $x \not\leq y$, there exists $(U, V) \in \tau \times \delta$ such that $x \in U, y \in V$ with $U \cap V = \emptyset$. We call it split $T_2$ internal topology on a poset $P$, if the internal topology $\theta(P)$ is split $T_2$ about $\upsilon(P)$ and $\omega(P)$.*

In [12, 24], it is pointed that split $T_2$ is strictly stronger than $T_2$ property.

# 3 Weak generalized finitely regular relations

In this section, we consider the split $T_2$ interval topology of posets by using the regularity of binary relations, and obtain that the relation $\not\leq$ on a poset $P$ is weak generalized finitely regular if and only if $P$ is a weak quasi-hypercontinuous poset if and only if the interval topology on $P$ is split $T_2$.

**Definition 3.1.** *A poset $P$ is called* weak quasi-hypercontinuous, *if* $\uparrow x = \bigcap\{\uparrow F : F \in P^{(<\omega)}, F \prec x\}$ *for all* $x \in P$.

In contrast to quasi-hypercontinuous posets, a weak quasi-hypercontinuous poset need not be the case that the set $\{F \in P^{(<\omega)} : F \prec x\}$ is directed. Clearly, $P$ is a quasi-hypercontinuous poset $\Rightarrow$ $P$ is weak quasi-hypercontinuous, and If $P$ is a sup-semilattice, then they are equivalent.

**Definition 3.2.** *A binary relation $\rho : X \rightharpoonup Y$ is called* weak generalized finitely regular, *w-generalized finitely regular for short, if for any $(x, y) \in \rho$, there are $\{u_1, u_2, \dots, u_n\} \in X^{(<\omega)}$ and $\{v_1, v_2, \dots, v_m\} \in Y^{(<\omega)}$ such that*

*(a)* $(u_i, y) \in \rho, (x, v_j) \in \rho(i = 1, 2, \dots, n; j = 1, 2, \dots, m)$, *and*
*(b)* $\forall s \in X, \{t_1, t_2, \dots, t_n\} \subseteq Y$, *if* $(u_i, t_i) \in \rho(i = 1, 2, \dots, n), (s, v_j) \in \rho(j = 1, 2, \dots, m)$, *then there is a* $k \in \{1, 2, \dots, m\}$ *such that* $(s, t_k) \in \rho$.

Obviously, if $\rho$ is generalized finitely regular, then $\rho$ is w-generalized finitely regular.

**Proposition 3.3.** *For a binary relation $\rho : X \rightharpoonup Y$, the following conditions are equivalent:*

(1) *$\rho$ is w-generalized finitely regular;*
(2) $\forall(x, y) \in \rho, \exists(U, V) \in X^{(<\omega)} \times Y^{(<\omega)}$ *such that*
   (i) $U \subseteq \rho^{-1}(y), V \subseteq \rho(x)$;
   (ii) $\forall(s, T) \in X \times Y^{(<\omega)}$, *if* $U \subseteq \rho^{-1}(T)$ *and* $V \subseteq \rho(s)$, *then* $T \cap \rho(s) \neq \emptyset$.

*Proof.* (1) $\Rightarrow$ (2) For any $(x, y) \in \rho$, since $\rho$ is w-generalized finitely regular, $\exists\{u_1, u_2, \dots, u_n\} \in X^{(<\omega)}$ and $\{v_1, v_2, \dots, v_m\} \in Y^{(<\omega)}$ such that

(a) $(u_i, y) \in \rho, (x, v_j) \in \rho(i = 1, 2, \dots, n; j = 1, 2, \dots, m)$, and

(b) $\forall s \in X, \{t_1, t_2, \dots, t_n\} \subseteq Y$, if $(u_i, t_i) \in \rho(i = 1, 2, \dots, n)$, $(s, v_j) \in \rho(j = 1, 2, \dots, m)$, then $\exists k \in \{1, 2, \dots, m\}$ such that $(s, t_k) \in \rho$.

Let $U = \{u_1, u_2, \dots, u_n\}$, $V = \{v_1, v_2, \dots, v_m\}$. Then $(U, V) \in X^{(<\omega)} \times Y^{(<\omega)}$. By the condition (a), we have that $U \subseteq \rho^{-1}(y)$, $V \subseteq \rho(x)$, i.e., the condition (i) in (2) is satisfied. Now we check the condition (ii) in (2). $\forall (s, T) \in X \times Y^{(<\omega)}$, if $U \subseteq \rho^{-1}(T)$ and $V \subseteq \rho(s)$, then $\forall i \in \{1, 2, \dots, n\}$, $\exists t_i \in T$ such that $(u_i, t_i) \in \rho$, and $\forall j \in \{1, 2, \dots, m\}$, $(s, v_j) \in \rho$, by the condition (b), $\exists k \in \{1, 2, \dots, m\}$ such that $(s, t_k) \in \rho$. Thus $T \cap \rho(s) \neq \emptyset$.

(2) $\Rightarrow$ (1) Let $(x, y) \in \rho$. By (2), $\exists (U, V) \in X^{(<\omega)} \times Y^{(<\omega)}$ such that

(i) $U \subseteq \rho^{-1}(y)$, $V \subseteq \rho(x)$, and
(ii) $\forall (s, T) \in X \times Y^{(<\omega)}$, if $U \subseteq \rho^{-1}(T)$ and $V \subseteq \rho(s)$, then $T \cap \rho(s) \neq \emptyset$.

Let $U = \{u_1, u_2, \dots, u_n\}$, $V = \{v_1, v_2, \dots, v_m\}$. Then by condition (i), we have that $(u_i, y) \in \rho$, $(x, v_j) \in \rho(i = 1, 2, \dots, n; j = 1, 2, \dots, m)$. For any $s \in X$, $\{t_1, t_2, \dots, t_n\} \subseteq Y$, if $(u_i, t_i) \in \rho(i = 1, 2, \dots, n)$, $(s, v_j) \in \rho(j = 1, 2, \dots, m)$, let $T = \{t_1, t_2, \dots, t_n\}$. Then $U \subseteq \rho^{-1}(T)$ and $V \subseteq \rho(s)$. By the condition (ii), $T \cap \rho(s) \neq \emptyset$, i.e., $\exists k \in \{1, 2, \dots, m\}$ such that $(s, t_k) \in \rho$. Thus (1) holds. □

**Definition 3.4.** *Let $\rho : X \rightharpoonup Y$ be a binary relation. We call $\rho$ satisfies property* M *if for any $y \in Y$, $\varphi_y = \emptyset$ or $\varphi_y$ has the greatest element, where $\varphi_y = \{\rho(u) \in \varphi_\rho(X, Y) : y \notin \rho(u)\}$.*

**Example 3.5.** *(1) Let $E$ be a binary relation on a set $X$ with reflexive and transitive. Then the relation $E^c = X^2 \backslash E$ satisfies property* M.

*In fact, for any $y \in X$, since $E$ is reflexive, $y \notin E^c(y)$. Thus $\varphi_y \neq \emptyset$. Let $u \in X$ with $y \notin E^c(u)$, i.e., $(u, y) \in E$. Suppose that $E^c(u) \nsubseteq E^c(y)$, then there is a $t \in E^c(u)$ such that $t \notin E^c(y)$, i.e., $(u, t) \notin E$ and $(y, t) \in E$, we have $(u, t) \in E$ since $E$ is transitive, which contradicts $(u, t) \notin E$. Thus $E^c(y)$ is the greatest element of $\varphi_y$. Hence, the relation $E^c$ satisfies property* M.

*(2) Let $X$ be a set and $Y = \{y\}$. Define a function $f : X \to Y$ by $f(x) = y$ for any $x \in X$. Then $f$ satisfies property* M, *since $\varphi_y = \emptyset$ for any $y \in Y$.*

*(3) Let $X, Y$ be two sets and $g : X \to Y$ a injective function. If $|X| > 2$, then $g$ is not satisfy property* M, *since for any $x_1, x_2 \in X$, $g(x_1) \nsubseteq g(x_2)$.*

For any poset $P$, the relation $\leq$ on $P$ is reflexive and transitive, by Example 3.5(1), we have the following corollary.

**Corollary 3.6.** *For any poset $P$, the relation $\nleq$ on $P$ satisfies property* M .

**Lemma 3.7.** *Let $\rho : X \rightharpoonup Y$ be a binary relation. If $\rho$ satisfies property* M, *then $\delta((\varphi_\rho(X, Y), \subseteq))$ is order isomorphism to $(\Phi_\rho(X, Y), \subseteq)$.*

*Proof.* For any $A \subseteq X$, define $\eta : \delta((\varphi_\rho(X, Y), \subseteq)) \to (\Phi_\rho(X, Y), \subseteq)$ by $\eta(\{\rho(x) : x \in A\}^\delta_{\varphi_\rho(X,Y)}) = \rho(A)$ and $\psi : (\Phi_\rho(X, Y), \subseteq) \to \delta((\varphi_\rho(X, Y), \subseteq))$ by $\psi(\rho(A)) = \{\rho(x) : x \in A\}^\delta_{\varphi_\rho(X,Y)}$.

1° $\eta$ is order preserving. Let $\{\rho(x) : x \in A\}^\delta_{\varphi_\rho(X,Y)} \subseteq \{\rho(y) : y \in B\}^\delta_{\varphi_\rho(X,Y)}$. Then $\{\rho(y) : y \in B\}^\uparrow_{\varphi_\rho(X,Y)} \subseteq \{\rho(x) : x \in A\}^\uparrow_{\varphi_\rho(X,Y)}$. Now we have to show that $\rho(A) \subseteq \rho(B)$. For any $w \in \rho(A)$, there is a $x_w \in A$ such that $w \in \rho(x_w)$. Since $\rho$ satisfies property $M$, let $N_w$ be the greatest element of $\varphi_w$ (if $\varphi_w = \emptyset$, let $N_w = \emptyset$). Then $\rho(x_w) \nsubseteq N_w$. Thus $N_w \notin \{\rho(x) : x \in A\}^\uparrow_{\varphi_\rho(X,Y)}$. Note that $\{\rho(y) : y \in B\}^\uparrow_{\varphi_\rho(X,Y)} \subseteq \{\rho(x) : x \in A\}^\uparrow_{\varphi_\rho(X,Y)}$, we have $N_w \notin \{\rho(y) : x \in B\}^\uparrow_{\varphi_\rho(X,Y)}$, it follows from that there is a $b \in B$ such that $\rho(b) \nsubseteq N_w$. By the definition of $N_w$, $w \in \rho(b) \subseteq \rho(B)$. Hence $\rho(A) \subseteq \rho(B)$.

2° $\psi$ is order preserving. Let $\rho(A) \subseteq \rho(B)$. We only have to show $\{\rho(y) : y \in B\}^\uparrow_{\varphi_\rho(X,Y)} \subseteq \{\rho(x) : x \in A\}^\uparrow_{\varphi_\rho(X,Y)}$. Suppose not, there is a $\rho(w) \in \{\rho(y) : y \in B\}^\uparrow_{\varphi_\rho(X,Y)}$ such that $\rho(w) \notin \{\rho(x) : x \in A\}^\uparrow_{\varphi_\rho(X,Y)}$. Thus for any $y \in B$, $\rho(y) \subseteq \rho(w)$ and $\rho(x_0) \nsubseteq \rho(w)$ for some $x_0 \in A$, it follows that there is a $z_0 \in \rho(x_0)$ with $z_0 \notin \rho(w)$. Since $\rho(x_0) \subseteq \rho(A) \subseteq \rho(B)$, there exists $y_0 \in B$ such that $z_0 \in \rho(y_0)$. Note that $\rho(y) \subseteq \rho(w)$ for any $y \in B$. Thus $z_0 \in \rho(w)$, which contradicts $z_0 \notin \rho(w)$. Thus $\{\rho(y) : y \in B\}^{\uparrow}_{\varphi_\rho(X,Y)} \subseteq \{\rho(x) : x \in A\}^{\uparrow}_{\varphi_\rho(X,Y)}$. Therefore, $\psi(\rho(A)) \subseteq \psi(\rho(B))$.

Obviously, $\eta \circ \psi = id_{(\Phi_\rho(X,Y),\subseteq)}$ and $\psi \circ \eta = id_{\delta((\varphi_\rho(X,Y),\subseteq))}$. All there show that $\delta((\varphi_\rho(X, Y), \subseteq)) \cong (\Phi_\rho(X, Y), \subseteq)$. □

From the Lemma 3.7, we can see that if $\rho$ satisfies property $M$, then $(\Phi_\rho(X, Y), \subseteq)$ is the normal completion of $(\varphi_\rho(X, Y), \subseteq)$.

**Definition 3.8.** *[24] A poset P is called* S-poset, *if for any $F, G \in P^{(<\omega)}\backslash\{\emptyset\}$, $F \subseteq G^{\downarrow}$, there exists $u \in P$ such that $F \subseteq \downarrow u \subseteq G^{\downarrow}$.*

**Lemma 3.9.** *[24] Let P be a sup-semilattice (inf-semilattice). Then P is an S-poset.*

**Lemma 3.10.** *Let $\rho : X \rightharpoonup Y$ be a relation with property* M. *$\mathcal{F} \in \varphi_\rho(X, Y)^{(<\omega)}$ and $\rho(x) \in \varphi_\rho(X, Y)$. Consider the following conditions.*

(1) $\mathcal{F} \prec_{\Phi_\rho(X,Y)} \rho(x)$;
(2) $\mathcal{F} \prec_{\varphi_\rho(X,Y)} \rho(x)$.

*Then* (1) ⇒ (2). *If $\varphi_\rho(X, Y)$ is an S-poset, then they are equivalent.*

*Proof.* (1) ⇒ (2) Let $\mathcal{F} \prec_{\Phi_\rho(X,Y)} \rho(x)$. Then there exist $\rho(A_1), \rho(A_2), \dots, \rho(A_m) \subseteq \Phi_\rho(X, Y)$ such that $\rho(x) \in \Phi_\rho(X, Y)\backslash \downarrow_{\Phi_\rho(X,Y)} \{\rho(A_1), \rho(A_2), \dots, \rho(A_m)\} \subseteq \uparrow_{\Phi_\rho(X,Y)} \mathcal{F}$. Thus for any $j \in \{1, 2, \dots, m\}$, $\rho(x) \nsubseteq \rho(A_j)$, and thus there is a $z_j \in \rho(x)$ with $z_j \notin \rho(A_j)$. Obviously, $\varphi_{z_j} \neq \emptyset$. Let $N_j$ be the greatest of $\varphi_{z_j}$. Then $\rho(x) \nsubseteq N_j$ for any $j \in \{1, 2, \dots, m\}$, and thus $\rho(x) \in \varphi_\rho(X)\backslash \downarrow_{\varphi_\rho(X)} \{N_1, N_2, \dots, N_j\}$. Now we show that $\varphi_\rho(X, Y)\backslash \downarrow_{\varphi_\rho(X,Y)} \{N_1, N_2, \dots, N_j\} \subseteq \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$. Let $\rho(w) \in \varphi_\rho(X, Y)\backslash \downarrow_{\varphi_\rho(X,Y)} \{N_1, N_2, \dots, N_j\}$. Then for any $j \in \{1, 2, \dots, m\}$, $\rho(w) \nsubseteq N_j$. By the definition of $N_j$, we have $z_j \in \rho(w)$, and thus $\rho(w) \nsubseteq \rho(A_j)$ (since $z_j \notin \rho(A_j)$). Hence $\rho(w) \in \Phi_\rho(X, Y)\backslash \downarrow_{\Phi_\rho(X,Y)} \{\rho(A_1), \rho(A_2), \dots, \rho(A_m)\}$, it follows that $\rho(w) \in \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$. Hence $\mathcal{F} \prec_{\varphi_\rho(X,Y)} \rho(x)$.

(2) ⇒ (1) Suppose that $\mathcal{F} \prec_{\varphi_\rho(X,Y)} \rho(x)$, then there exists $\{\rho(y_1), \rho(y_2), \dots, \rho(y_m)\} \subseteq \varphi_\rho(X, Y)$ such that $\rho(x) \in \varphi_\rho(X, Y)\backslash \downarrow_{\varphi_\rho(X,Y)} \{\rho(y_1), \rho(y_2), \dots, \rho(y_m)\} \subseteq \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$. Now we have to show that $\rho(x) \in \Phi_\rho(X, Y)\backslash \downarrow_{\Phi_\rho(X,Y)} \{\rho(y_1), \rho(y_2), \dots, \rho(y_m)\} \subseteq \uparrow_{\Phi_\rho(X,Y)} \mathcal{F}$. Obviously, $\rho(x) \in \Phi_\rho(X, Y)\backslash \downarrow_{\Phi_\rho(X,Y)} \{\rho(y_1), \rho(y_2), \dots, \rho(y_m)\}$. Assume that there is a $\rho(A) \in \Phi_\rho(X, Y)\backslash \downarrow_{\Phi_\rho(X,Y)} \{\rho(y_1), \rho(y_2), \dots, \rho(y_m)\}$ such that $\rho(A) \notin \uparrow_{\Phi_\rho(X,Y)} \mathcal{F}$. Let $\mathcal{F} = \{\rho(u_1), \rho(u_2), \dots, \rho(u_n)\}$. Then $\rho(A) \nsubseteq \rho(y_j)(j = 1, 2, \dots, m)$ and $\rho(u_i) \nsubseteq \rho(A)(i = 1, 2, \dots, n)$. Thus there exist $s_j \in A$ and $v_j \in \rho(s_j)$ such that $v_j \notin \rho(y_j)(j = 1, 2, \dots, m)$, and $t_i \in \rho(u_i)$ with $t_i \notin \rho(A)(i = 1, 2 \dots, n)$. We can conclude that there exist $k_0 \in \{1, 2, \dots, m\}$ and $l_0 \in \{1, 2, \dots, n\}$ such that $t_{l_0} \in \rho(s_{k_0})$. If not, then for any $k \in \{1, 2, \dots, m\}$ and $l \in \{1, 2, \dots, n\}$, $t_l \notin \rho(s_k)$. Let $N_l$ be the greatest element of $\varphi_{t_l}$. Then $\rho(s_k) \subseteq N_l$, so for any $k \in \{1, 2, \dots, m\}$, $\rho(s_k)$ is a lower bound of $\{N_1, N_2, \dots, N_n\}$. Since $\varphi_\rho(X, Y)$ is an S-poset, there exists $s \in X$ such that $\{\rho(s_1), \rho(s_2), \dots, \rho(s_m)\} \subseteq \downarrow \rho(s) \subseteq \{N_1, N_2, \dots, N_n\}^{\downarrow}$. Thus, $v_j \in \rho(s)$ and $\rho(s) \nsubseteq \rho(y_j)(j = 1, 2, \dots, m)$, that is $\rho(s) \in \varphi_\rho(X, Y)\backslash \downarrow_{\varphi_\rho(X,Y)} \{\rho(y_1), \rho(y_2), \dots, \rho(y_n)\}$, so $\rho(s) \in \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$. Thus there is a $l \in \{1, 2, \dots, n\}$ such that $\rho(u_l) \subseteq \rho(s)$. Notice that $t_l \in \rho(u_l)$, we have $t_l \in \rho(s)$. On the other side, since $\rho(s) \in \{N_1, N_2, \dots, N_n\}^{\downarrow}$, $\rho(s) \subseteq N_l$. By the definition of $N_l$, $t_l \notin \rho(s)$, a contradiction. Therefore, there exist $k_0 \in \{1, 2, \dots, m\}$ and $l_0 \in \{1, 2, \dots, n\}$ such that $t_{l_0} \in \rho(s_{k_0})$. Since $\rho(s_{k_o}) \subseteq \rho(A)$, $t_{l_0} \in \rho(A)$, which contradicts $t_l \notin \rho(A)$ for any $l \in \{1, 2 \dots, n\}$. Hence $\mathcal{F} \prec_{\Phi_\rho(X,Y)} \rho(x)$. □

**Theorem 3.11.** *For a binary relation $\rho : X \rightharpoonup Y$ with property* M, *consider the following conditions:*

(1) *$\rho$ is w-generalized finitely regular;*

(2) $(\varphi_\rho(X, Y), \subseteq)$ *is a weak quasi-hypercontinuous poset;*
(3) *the interval topology on* $(\varphi_\rho(X, Y), \subseteq)$ *is split* $T_2$;
(4) $(\Phi_\rho(X, Y), \subseteq)$ *is a quasi-hypercontinuous lattice.*

*Then* $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftarrow (4)$. *If* $\varphi_\rho(X, Y)$ *is an S-poset, then* $(1) - (4)$ *are equivalent.*

*Proof.* $(1) \Rightarrow (2)$ For any $\rho(x) \in \varphi_\rho(X, Y)$, if $\rho(x) \nsubseteq \rho(u)$, then there is a $y \in \rho(x)$ such that $y \notin \rho(u)$. Since $\rho$ is w-generalized finitely regular, there are $\{u_1, u_2, \ldots, u_n\} \in X^{(<\omega)}$ and $\{v_1, v_2, \ldots, v_m\} \in Y^{(<\omega)}$ such that

(a) $(u_i, y) \in \rho, (x, v_j) \in \rho (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, and
(b) $\forall s \in X, T = \{t_1, t_2, \ldots, t_n\} \subseteq Y$, if $(u_i, t_i) \in \rho (i = 1, 2, \ldots, n), (s, v_j) \in \rho (j = 1, 2, \ldots, m)$, then $\exists k \in \{1, 2, \ldots, m\}$ such that $(s, t_k) \in \rho$.

Let $\mathcal{F} = \{\rho(u_1), \rho(u_2), \ldots \rho(u_n)\}$. Then $\mathcal{F} \in \varphi_\rho(X, Y)^{(<\omega)}$ and $\rho(u_i) \nsubseteq \rho(u) (i = 1, 2, \ldots, n)$, thus $\rho(u) \notin \mathcal{F}$. Let $N_j$ be the greatest element of $\varphi_{v_j}$ (if $\varphi_{v_j} = \emptyset$, let $N_j = \emptyset$ ). Then $\rho(x) \in \varphi_\rho(X, Y) \setminus \downarrow_{\varphi_\rho(X,Y)} \{N_1, N_2, \ldots, N_m\}$ since $v_j \in \rho(x) (j = 1, 2, \ldots m)$. For any $\rho(s) \in \varphi_\rho(X, Y) \setminus \downarrow_{\varphi_\rho(X,Y)} \{N_1, N_2, \ldots, N_m\}, \rho(s) \nsubseteq N_j (j = 1, 2, \ldots, m)$. By the definition of $N_j$, $v_j \in \rho(s)$. If $\rho(s) \notin \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$, then for any $i \in \{1, 2, \ldots, n\}, \rho(u_i) \nsubseteq \rho(s)$, so there is a $t_i \in \rho(u_i)$ with $t_i \notin \rho(s)$. By the condition (b), there is a $k \in \{1, 2, \ldots, m\}$ such that $(s, t_k) \in \rho$, i.e.,$t_k \in \rho(s)$, which contradicts $t_i \notin \rho(s)$ for any $i \in \{1, 2, \ldots, n\}$. Thus $\rho(s) \in \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$.

All above show that $\rho(x) \in \varphi_\rho(X, Y) \setminus \downarrow_{\varphi_\rho(X,Y)} \{N_1, N_2, \ldots, N_m\} \subseteq \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$, i.e.,$\mathcal{F} \prec_{\varphi_\rho(X,Y)} \rho(x)$. Note that $\rho(u) \notin \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$. Hence, for any $\rho(x) \in \varphi_\rho(X, Y)$, $\uparrow_{\varphi_\rho(X,Y)} \rho(x) = \bigcap\{\uparrow_{\varphi_\rho(X,Y)} \mathcal{F} : \mathcal{F} \in \varphi_\rho(X, Y)^{(<\omega)}$ and $\mathcal{F} \prec_{\varphi_\rho(X,Y)} \rho(x)\}$. Therefore, $(\varphi_\rho(X, Y), \subseteq)$ is a weak quasi-hypercontinuous poset.

$(2) \Rightarrow (3)$ For any $\rho(x), \rho(y) \in \varphi_\rho(X, Y)$ with $\rho(x) \nsubseteq \rho(y)$. By (2), there exists $\mathcal{F} \in \varphi_\rho(X, Y)^{(<\omega)}$ such that $\mathcal{F} \prec_{\varphi_\rho(X,Y)} \rho(x)$ and $\rho(y) \notin \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$. By the definition of $\prec$, we have that $\rho(x) \in \mathrm{int}_{\upsilon((\varphi_\rho(X,Y),\subseteq))} \uparrow_{\varphi_\rho(X,Y)} \mathcal{F} \subseteq \uparrow_{\varphi_\rho(X,Y)} \mathcal{F} \subseteq \varphi_\rho(X, Y) \setminus \downarrow_{\varphi_\rho(X,Y)} \rho(y)$. Let $\mathcal{U} = \mathrm{int}_{\upsilon((\varphi_\rho(X,Y),\subseteq))} \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$ and $\mathcal{V} = \varphi_\rho(X, Y) \setminus \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$. Then $\rho(x) \in \mathcal{U} \in \upsilon((\varphi_\rho(X, Y), \subseteq)), \rho(y) \in \mathcal{V} \in \omega((\varphi_\rho(X, Y), \subseteq))$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Hence, the interval topology on $(\varphi_\rho(X, Y), \subseteq)$ is split $T_2$;

$(3) \Rightarrow (1)$ For any $(x, y) \in \rho$, let $N_y$ be the greatest element of $\varphi_y$ (if $\varphi_y = \emptyset$, let $N_y = \emptyset$). Then $\rho(x) \nsubseteq N_y$. Since the interval topology on $(\varphi_\rho(X, Y), \subseteq)$ is split $T_2$, there exist $\{\rho(x_1), \rho(x_2), \ldots, \rho(x_m)\} \in \varphi_\rho(X, Y)^{(<\omega)}$ and $\{\rho(u_1), \rho(u_2), \ldots, \rho(u_n)\}^{(<\omega)}$ such that $\rho(x) \in \varphi_\rho(X, Y) \setminus \downarrow_{\varphi_\rho(X,Y)} \{\rho(x_1), \rho(x_2), \ldots, \rho(x_m)\}, N_y \in \varphi_\rho(X, Y) \setminus \uparrow_{\varphi_\rho(X,Y)} \{\rho(u_1), \rho(u_2), \ldots, \rho(u_n)\}$ and $\downarrow_{\varphi_\rho(X,Y)} \{\rho(x_1), \rho(x_2) \ldots, \rho(x_m)\} \bigcup \uparrow_{\varphi_\rho(X,Y)} \{\rho(u_1), \rho(u_2), \ldots, \rho(u_n)\} = \varphi_\rho(X, Y)$.

Since $\rho(x) \nsubseteq \rho(x_j) (j = 1, 2, \ldots, m)$, choose $v_j \in \rho(x)$ and $v_j \notin \rho(x_j)$. On the other side, $\rho(u_i) \nsubseteq N_y (i = 1, 2, \ldots, n)$. By the definition of $N_y$, $y \in \rho(u_i) (i = 1, 2, \ldots, n)$. Thus $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$ satisfy the condition (a) of Definition 3.2. $\forall s \in X, \{t_1, t_2, \ldots, t_n\} \subseteq Y$, if $(u_i, t_i) \in \rho (i = 1, 2, \ldots, n), (s, v_j) \in \rho (j = 1, 2, \ldots, m)$, we have $\rho(s) \nsubseteq \rho(x_j)$ for any $j \in \{1, 2, \ldots, m\}$. Thus $\rho(s) \in \uparrow_{\varphi_\rho(X,Y)} \{\rho(u_1), \rho(u_2), \ldots, \rho(u_n)\}$, i.e., there is a $k \in \{1, 2, \ldots, n\}$ such that $\rho(u_k) \subseteq \rho(s)$. Notice that $t_k \in \rho(u_k)$, thus $t_k \in \rho(s)$, i.e., $(s, t_k) \in \rho$. Hence $\rho$ is w-generalized finitely regular.

$(4) \Rightarrow (1)$ By Theorem 2.4, $\rho$ is generalized finitely regular. Hence (1) holds.

$(2) \Rightarrow (4)$ For any $\rho(A) \in (\Phi_\rho(X, Y), \subseteq)$, let $\rho(B) \in \Phi_\rho(X, Y)$ with $\rho(A) \nsubseteq \rho(B)$. Then there is a $y \in \rho(A)$ such that $y \notin \rho(B)$. Choose $x \in A$ with $y \in \rho(x)$. Let $N_y$ be the greatest element of $\varphi_y$. Then $\rho(x) \nsubseteq N_y$ and $\rho(B) \subseteq N_y$. Since $(\varphi_\rho(X, Y), \subseteq)$ is weak quasi-hypercontinuous, there exists $\mathcal{F} \in \varphi_\rho(X, Y)^{(<\omega)}$ such that $\mathcal{F} \prec_{\varphi_\rho(X,Y)} \rho(x)$ with $N_y \notin \uparrow_{\varphi_\rho(X,Y)} \mathcal{F}$. By Lemma 3.10 and $\rho(B) \subseteq N_y, \mathcal{F} \prec_{\Phi_\rho(X,Y)} \rho(x) \subseteq \rho(A)$ and $\rho(B) \notin \uparrow_{\Phi_\rho(X,Y)} \mathcal{F}$. Hence $(\Phi_\rho(X, Y), \subseteq)$ is a quasi-hypercontinuous lattice. □

For any poset $P$, let $\rho = \nleq$ on $P$. Then $\delta(P)$ is order isomorphism to $(\Phi_{\nleq}(P), \subseteq)$ (define $\delta(P) \to (\Phi_{\nleq}(P), \subseteq)$ by $A^\delta \mapsto P \setminus A^\uparrow$) (see [24]). Furthermore, we can check that $P$ is order isomorphism to $(\varphi_{\nleq}(P), \subseteq)$. In deed, define $f : P \to (\varphi_{\nleq}(P), \subseteq)$ by $f(x) = P \setminus \uparrow x$ and $g : (\varphi_{\nleq}(P), \subseteq) \to P$ by $g(\rho(x)) = x$. One can easily check that $f, g$ are order preserving and $f \circ g = id_{(\varphi_{\nleq}(P),\subseteq)}$, $g \circ f = id_P$. Therefore, using Corollary 3.6 and Theorem 3.11, we have the following.

**Theorem 3.12.** *Let P be a poset. Consider the following conditions.*

(1) *P is a weak quasi-hypercontinuous poset;*

(2) *the relation* $\not\leq$ *is w-generalized finitely regular;*
(3) *for any* $x, y \in P$ *with* $x \not\leq y$, *there are* $\{u_1, u_2, \ldots, u_n\}, \{v_1, v_2, \ldots, v_m\} \in P^{(<\omega)}$ *such that*
   (a) $u_i \not\leq y, x \not\leq v_j (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, *and*
   (b) $\forall z \in P, \exists k \in \{1, 2, \ldots, n\}$ *such that* $u_k \leq z$ *or* $\exists l \in \{1, 2, \ldots, m\}$ *such that* $z \leq v_l$;
(4) *for any* $x, y \in P$ *with* $x \not\leq y$, *there exist* $F, G \in P^{(<\omega)}$ *such that*
   (a) $x \notin\downarrow G, y \notin\uparrow F$, *and*
   (b) $\downarrow G \bigcup \uparrow F = P$;
(5) $\theta(P)$ *is split* $T_2$;
(6) *there is a w-generalized finitely regular relation* $\rho : X \rightharpoonup Y$ *satisfying property* M *such that* $P \cong (\varphi_\rho(X, Y), \subseteq)$;
(7) *there is a w-generalized finitely regular relation* $\rho : X \rightharpoonup X$ *satisfying property* M *such that* $P \cong (\varphi_\rho(X), \subseteq)$;
(8) $(\delta(P), \subseteq)$ *is a quasi-hypercontinuous lattice.*

*Then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftarrow$ (8). *If P is an S-poset, then* (1) – (8) *are equivalent.*

*Proof.* (1) $\Rightarrow$ (2) Let $\rho =\not\leq$ on $P$. By Theorem 3.11 and $P \cong (\varphi_{\not\leq}(P), \subseteq)$.

(2) $\Rightarrow$ (3) Let $x, y \in P$ with $x \not\leq y$. By the definition of w-generalized finitely regular, there are $\{u_1, u_2, \ldots, u_n\}, \{v_1, v_2, \ldots, v_m\} \in P^{(<\omega)}$ such that

(i) $u_i \not\leq y, x \not\leq v_j (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, and
(ii) $\forall s \in X, \{t_1, t_2, \ldots, t_n\} \subseteq Y$, if $u_i \not\leq t_i (i = 1, 2, \ldots, n), s \not\leq v_j (j = 1, 2, \ldots, m)$, then $\exists k \in \{1, 2, \ldots, m\}$ such that $s \not\leq t_k$.

For any $z \in P$, let $s = z$ and $t_i = z (i = 1, 2, \ldots, n)$. Then by (ii), $\exists k \in \{1, 2, \ldots, n\}$ such that $u_k \leq z$ or $\exists l \in \{1, 2, \ldots, m\}$ such that $z \leq v_l$.

(3) $\Rightarrow$ (1) Let $x, y \in P$ with $x \not\leq y$. By (3), there are $\{u_1, u_2, \ldots, u_n\}, \{v_1, v_2, \ldots, v_m\} \in P^{(<\omega)}$ such that

(i) $u_i \not\leq y, x \not\leq v_j (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$, and
(ii) $\forall z \in P, \exists k \in \{1, 2, \ldots, n\}$ such that $u_k \leq z$ or $\exists l \in \{1, 2, \ldots, m\}$ such that $z \leq v_l$.

Let $F = \{u_1, u_2, \ldots, u_n\}$. Then we have $y \notin\uparrow F$ and $x \in P \backslash \downarrow \{v_1, v_2, \ldots, v_m\}$. Now we show that $P \backslash \downarrow \{v_1, v_2, \ldots, v_m\} \subseteq\uparrow F$. For any $z \in P \backslash \downarrow \{v_1, v_2, \ldots, v_m\}$, $z \not\leq v_l (l = 1, 2, \ldots, m)$, by (ii), there is a $k \in \{1, 2, \ldots, n\}$ such that $u_k \leq z$. Thus $z \in\uparrow F$, and thus $F \prec x$. Therefore $P$ is a weak quasi-hypercontinuous poset.

(3) $\Leftrightarrow$ (4) See [24].

(4) $\Rightarrow$ (5) Let $x, y \in P$ with $x \not\leq y$. By (4) there exist $F, G \in P^{(<\omega)}$ such that

(i) $x \notin\downarrow G, y \notin\uparrow F$;
(ii) $\downarrow G \bigcup \uparrow F = P$.

Let $U = P \backslash \downarrow G$, $V = P \backslash \uparrow F$. Then $x \in U \in \upsilon(P), y \in V \in \omega(P)$ and $U \cap V = \emptyset$, thus $\theta(P)$ is split $T_2$.

(5) $\Rightarrow$ (4) Let $x, y \in P$ with $x \not\leq y$. By (5), there are $U \in \upsilon(P), V \in \omega(P)$ such that $x \in U, y \in V$ with $U \cap V = \emptyset$. Thus there exist $F, G \in P^{(<\omega)}$ such that $x \in P \backslash \downarrow G \subseteq U$ and $y \in P \backslash \uparrow F \subseteq V$. It is easy to see that $F, G$ satisfy the conditions (a) and (b) of (4).

(2) $\Rightarrow$ (6) Let $X = Y = P$ and $\rho =\not\leq$. By Corollary 3.6.

(6) $\Rightarrow$ (7) Obviously.

(7) $\Rightarrow$ (1) Follows from Theorem 3.11.

(8) $\Rightarrow$ (1) By $P \cong (\varphi_{\not\leq}(P), \subseteq)$, $\delta(P) \cong (\Phi_{\not\leq}(P), \subseteq)$ and Theorem 3.11.

(1) $\Rightarrow$ (8) Let $P$ be an S-poset. Since $P \cong (\varphi_{\not\leq}(P), \subseteq)$ and $\delta(P) \cong (\Phi_{\not\leq}(P), \subseteq)$, by Theorem 3.11, the condition (8) holds. □

**Corollary 3.13.** *Let P be a poset. Then P is weak quasi-hypercontinuous if and only if* $P^{op}$ *is weak quasi-hypercontinuous.*

**Corollary 3.14.** *Let P be a sup-semilattice. Then the following two conditions are equivalent:*

(1) *P is a quasi-hypercontinuous poset;*
(2) *$(\delta(P), \subseteq)$ is a quasi-hypercontinuous lattice.*

Follows from Corollary 3.14, we establish a necessary and sufficient condition for a quasi-hypercontinuous poset to have a normal completion which is a quasi-hypercontinuous lattice, that is, $P$ only need to be a sup-semilattice. This condition is weaker than that appears in Theorem 5.2 of [29].

**Acknowledgements:** The authors would like to thank the referees for their very careful reading of the manuscript and valuable comments. This research is supported by the National Natural Science Foundation of China (No. 11661057, 11661040), the Ganpo 555 project for leading talents of Jiangxi Province and the Natural Science Foundation of Jiangxi Province (No. 20192ACBL20045), the Foundation of Education Department of Jiangxi Province (No. GJJ160660).

# References

[1] Erne M., Separation axioms for interval topologies, Proc. Amer. Math. Soc., 1980, 79, 185–190.
[2] Frink O., Topology in lattices, Trans. Amer. Math. Soc., 1942, 51, 569–582.
[3] Gierz G., Hofmann K.H., Keimel K., Lawson J.D., Mislove M., Scott D., A Compendium of Continuous Lattices, Springer Verlag, 1980.
[4] Gierz G., Hofmann K.H., Keimel K., Lawson J.D., Mislove M., Scott D., Continuous Lattices and Domains, Cambridge University Press, Cambridge, 2003.
[5] Gierz G., Lawson J.D., Generalized continuous and hypercontinuous lattices, Rocky Mountain J. Math., 1981, 11, 271–296.
[6] Northam F.S., The interval topology of a lattice, Proc. Amer. Math. Soc., 1953, 4, 824–827.
[7] Kogan S.A., Solution of three problems in lattice theory, Uspehi Math. Nauk., 1956, 11, 185–190.
[8] Wolk E.S., Topologies on a partially ordered set, Proc. Amer. Math. Soc., 1958, 9, 524–529.
[9] Matsushima Y., Hausdorff interval topology on a partially ordered set, Proc. Amer. Math. Soc., 1960, 11, 233–235.
[10] Kolibiar M., Bemerkungen über Intervalltopologie in halbgeordneten Mengen, in: General topology and its Relations to Modern Analysis and Algebra (Proc. Sympos, Prague, 1961), Academic Press, New York; Publ. Hause Czech. Acad. Sci., Prague, 1962, 252–253.
[11] Birkhoff G., Lattice Theory, vol. 25 of AMS Colloquium Publications, revised edition, RhodeIsland, 1967.
[12] Xu X.Q., Relational representations of complete lattices and their applications, PhD thesis, Sichuan University, 2004.
[13] Xu X.Q., Liu Y.M., Regular relations and strictly completely regular ordered spaces, Topology Appl., 2004, 135, 1–12.
[14] Xu X.Q., Liu Y.M., Regular relations and monotone normal ordered spaces, Chinese Ann. Math., 2004, 25B, 157–164.
[15] Xu X.Q., Luo M.K., Regular relations and normality of topologies, Semigroup Forum, 2006, 72, 477–480.
[16] Xu X.Q., Liu Y.M., Regular relations and completely regular spaces, Chinese Ann. Math., 2008, 29A, 819–828.
[17] Xu X.Q., Luo M.K., Regular relations and normal spaces, Acta Math. Sinica (Chin. Ser.), 2009, 52, 393–402.
[18] Zareckiǐ A., The semigroup of binary relations, Mat.Sbornik, 1963, 61, 291–305.
[19] Markowsky G., Idempotents and product representations with applications to the semigroup of binary relations, Semigroup Forum, 1972, 5, 95–119.
[20] Schein B.M., Regular elements of the semigroup of all binary relations, Semigroup Forum, 1976, 13, 95–102.
[21] Bandelt H.J., Regularity and complete distributivity, Semigroup Forum, 1980, 19, 123–126.
[22] Bandelt H.J., On regularity classes of binary relations, In: Universal Algebra and Applications, Banach Center Publications, PWN, Warsaw, 1982, 9, 329–333.
[23] Xu X.Q., Liu Y.M., Relational representations of hypercontinuous lattices, In: Domain Theory, Logic, and Computation, Kluwer Academic, Dordrecht, 2003, 65–74.
[24] Xu X.Q., Order and topology (in chinese), Science Press, Beijin, 2016.
[25] Erné M., The Dedekind-MacNeille completion as a reflector, Order, 1991, 8, 159–173.
[26] Jiang G.H., Xu L.S., Conjugative relations and applications, Semigroup Forum, 2010, 1, 85–91.
[27] Jiang G.H., Xu L.S., Cai C.,, Han G.W., Normal relations on sets and applications, Int. J. Contemp. Math. Sci, 2011, 6, 721–726.
[28] Jiang G.H., Xu L.S., Dually normal relations on sets, Semigroup Forum, 2012, 85, 75–80.
[29] Xu X.Q., Zhang W.F., The upper topology and interval topology on quasi-hypercontinuous posets, Topology and its Applications, 2017, 230, 539–549.