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Research Article

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Some properties of pre-quasi operator ideal of type generalized Cesáro sequence space defined by weighted means

Abstract: Let E be a generalized Cesáro sequence space defined by weighted means and by using $s-$numbers of operators from a Banach space X into a Banach space Y. We give the sufficient (not necessary) conditions on E such that the components

$$S_E(X, Y) := \left\{ T \in L(X, Y) : ((s_n(T))_{n=0}^{\infty} \in E \right\},$$

of the class S_E form pre-quasi operator ideal, the class of all finite rank operators are dense in the Banach pre-quasi ideal S_E, the pre-quasi operator ideal formed by the sequence of approximation numbers is strictly contained for different weights and powers, the pre-quasi Banach Operator ideal formed by the sequence of approximation numbers is small and the pre-quasi Banach operator ideal constructed by $s-$numbers is simple Banach space. Finally the pre-quasi operator ideal formed by the sequence of $s-$numbers and this sequence space is strictly contained in the class of all bounded linear operators, whose sequence of eigenvalues belongs to this sequence space.

Keywords: pre-quasi operator ideal, $s-$numbers, generalized Cesáro sequence space, weighted means, simple Banach space

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1 Introduction

Through the paper

$$L(X, Y) = \left\{ T : X \to Y; T \text{ is a bounded linear operator } ; X \text{and } Y \text{ are Banach Spaces} \right\},$$

and if $X = Y$, we write $L(X)$, by w, we denote the space of all real sequences and θ is the zero vector of E. As an aftereffect of the enormous applications in geometry of Banach spaces, spectral theory, theory of eigenvalue distributions and fixed point theorems etc., the theory of operator ideals goals possesses an uncommon essentialness in useful examination. Some of operator ideals in the class of Banach spaces or Hilbert spaces are defined by different scalar sequence spaces. For example the ideal of compact operators is defined by the

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space c_0 of null sequence and Kolmogorov numbers. Pietsch [1], examined the quasi-ideals formed by the approximation numbers and classical sequence space $\ell^p (0 < p < \infty)$. He proved that the ideals of nuclear operators and of Hilbert Schmidt operators between Hilbert spaces are defined by ℓ^1 and ℓ^2 respectively. He proved that the class of all finite rank operators are dense in the Banach quasi-ideal and the algebra $L(\ell^p)$, where $(1 \leq p < \infty)$ contains one and only one non-trivial closed ideal. Pietsch [2], showed that the quasi Banach Operator ideal formed by the sequence of approximation numbers is small. Makarov and Faried [3], proved that the quasi-operator ideal formed by the sequence of approximation numbers is strictly contained for different powers i.e., for any infinite dimensional Banach spaces X, Y and for any $q > p > 0$, it is true that $S_{\ell^p}^{app}(X, Y) \subsetneq S_{\ell^q}^{app}(X, Y) \subsetneq L(X, Y)$. In [4], Faried and Bakery studied the operator ideals constructed by approximation numbers, generalized Cesáro and Orlicz sequence spaces ℓ_M. In [5], Faried and Bakery introduced the concept of pre-quasi operator ideal which is more general than the usual classes of operator ideal, they studied the operator ideals constructed by $s-$ numbers, generalized Cesáro and Orlicz sequence spaces ℓ_M, and proved that the operator ideal formed by the previous sequence spaces and approximation numbers is small under certain conditions. There are articles on bounded linear operators transforming between sequence spaces those have been studied by Tripathy and Paul [6], Tripathy and Saikia [7], and Tripathy and Das [8], from different aspects such as spectra, resolvent spectra etc. The aim of this paper is to study a generalized class S_E by using the sequence of s-numbers and E (generalized Cesáro sequence space defined by weighted means), we give sufficient (not necessary) conditions on E such that S_E constructs an pre-quasi operator ideal, which gives a negative answer of Rhoades [9] open problem about the linearity of $E-$type spaces S_E. The components of S_E as a pre-quasi Banach operator ideal containing finite dimensional operators as a dense subset and its completeness are proved. The pre-quasi operator ideal formed by the sequence of approximation numbers is strictly contained for different weights and powers are determined. Finally, we show that the pre-quasi Banach operator ideal formed by E and approximation numbers is small under certain conditions. Furthermore, the sufficient conditions for which the pre-quasi Banach operator ideal constructed by $s-$numbers is simple Banach space. Also the pre-quasi operator ideal formed by the sequence of $s-$numbers and this sequence space is strictly contained in the class of all bounded linear operators whose its sequence of eigenvalues belongs to this sequence space.

2 Definitions and preliminaries

Definition 2.1. *[10] An s-number function is a map defined on $L(X, Y)$ which associate to each operator $T \in L(X, Y)$ a non-negative scaler sequence $(s_n(T))_{n=0}^{\infty}$ assuming that the taking after states are verified:*
(a) $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \ldots \geq 0$, *for $T \in L(X, Y)$,*
(b) $s_{m+n-1}(T_1 + T_2) \leq s_m(T_1) + s_n(T_2)$ *for all $T_1, T_2 \in L(X, Y)$, $m, n \in \mathbb{N}$,*
(c) *ideal property: $s_n(RVT) \leq \|R\| s_n(V) \|T\|$ for all $T \in L(X_0, X)$, $V \in L(X, Y)$ and $R \in L(Y, Y_0)$, where X_0 and Y_0 are arbitrary Banach spaces,*
(d) *if $G \in L(X, Y)$ and $\lambda \in \mathbb{R}$, we obtain $s_n(\lambda G) = |\lambda| s_n(G)$,*
(e) *rank property: If $rank(T) \leq n$, then $s_n(T) = 0$ for each $T \in L(X, Y)$,*
(f) *norming property: $s_{r \geq n}(I_n) = 0$ or $s_{r < n}(I_n) = 1$, where I_n represents the unit operator on the n-dimensional Hilbert space ℓ_2^n.*

There are several examples of s-numbers, we mention the following:

(1) The n-th approximation number, denoted by $\alpha_n(T)$, is defined by
$\alpha_n(T) = \inf\{ \|T - B\| : B \in L(X, Y) \text{ and } rank(B) \leq n \}$.
(2) The n-th Gel'fand number, denoted by $c_n(T)$, is defined by $c_n(T) = \alpha_n(J_Y T)$, where J_Y is a metric injection from the normed space Y to a higher space $l_\infty(\Lambda)$ for an adequate index set Λ. This number is independent of the choice of the higher space $l_\infty(\Lambda)$.

(3) The n-th Kolmogorov number, denoted by $d_n(T)$, is defined by

$$d_n(T) = \inf_{dimY \leq n} \sup_{\|x\| \leq 1} \inf_{y \in Y} \|Tx - y\|.$$

(4) The n-th Weyl number, denoted by $x_n(T)$, is defined by

$$x_n(T) = \inf\{\alpha_n(TB) : \|B : \ell_2 \to X\| \leq 1\}.$$

(5) The n-th Chang number, denoted by $y_n(T)$, is defined by

$$y_n(T) = \inf\{\alpha_n(BT) : \|B : Y \to \ell_2\| \leq 1\}.$$

(6) The n-th Hilbert number, denoted by $h_n(T)$, is defined by

$$h_n(T) = \sup\{\alpha_n(BTA) : \|B : Y \to \ell_2\| \leq 1 \text{ and } \|A : \ell_2 \to X\| \leq 1\}.$$

Remark 2.2. *[10] Among all the s-number sequences defined above, it is easy to verify that the approximation number, $\alpha_n(T)$ is the largest and the Hilbert number, $h_n(T)$ is the smallest s-number sequence, i.e., $h_n(T) \leq s_n(T) \leq \alpha_n(T)$ for any bounded linear operator T. If T is compact and defined on a Hilbert space, then all the s-numbers coincide with the eigenvalues of $|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$.*

Theorem 2.3. *[10, p.115] If $T \in L(X, Y)$, then*

$$h_n(T) \leq x_n(T) \leq c_n(T) \leq \alpha_n(T) \text{ and } h_n(T) \leq y_n(T) \leq d_n(T) \leq \alpha_n(T).$$

Definition 2.4. *[1] A finite rank operator is a bounded linear operator whose dimension of the range space is finite. The space of all finite rank operators on E is denoted by F(E).*

Definition 2.5. *[1] A bounded linear operator $A : E \to E$ (where E is a Banach space) is called approximable if there are $S_n \in F(E)$, for all $n \in \mathbb{N}$ such that $\lim_{n \to \infty} \|A - S_n\| = 0$. The space of all approximable operators on E is denoted by $\Psi(E)$, and the space of all approximable operators from E to F is denoted by $\Psi(E, F)$.*

Lemma 2.6. *[1] Let $T \in L(X, Y)$. If T is not approximable, then there are operators $G \in L(X, X)$ and $B \in L(Y, Y)$ and $B \in L(X, Y)$ such that $BTGe_k = e_k$ for all $k \in \mathbb{N}$.*

Definition 2.7. *[1] A Banach space X is called simple if the algebra L(X) contains one and only one non-trivial closed ideal.*

Definition 2.8. *[1] A bounded linear operator $A : E \to E$ (where E is a Banach space) is called compact if $A(B_1)$ has compact closure, where B_1 denotes the closed unit ball of E. The space of all compact operators on E is denoted by $L_c(E)$.*

Theorem 2.9. *[1] If E is infinite dimensional Banach space, we have*

$$F(E) \subsetneqq \Psi(E) \subsetneqq L_c(E) \subsetneqq L(E).$$

Definition 2.10. *[1] Let L be the class of all bounded linear operators between any arbitrary Banach spaces. A sub class U of L is called an operator ideal if each element $U(X, Y) = U \cap L(X, Y)$ fulfill the following conditions:*
(i) *$I_F \in U$ wherever F represents Banach space of one dimension.*
(ii) *The space U(X, Y) is linear over $\mathbb{R}$.*
(iii) *If $T \in L(X_0, X)$, $V \in U(X, Y)$ and $R \in L(Y, Y_0)$ then, $RVT \in U(X_0, Y_0)$. See [11, 12].*

The concept of pre-quasi operator ideal which is more general than the usual classes of operator ideal.

Definition 2.11. *[5] A function $g : \Omega \to [0, \infty)$ is said to be a pre-quasi norm on the ideal Ω if the following conditions holds:*

(1) *For all $T \in \Omega(X, Y)$, $g(T) \geq 0$ and $g(T) = 0$ if and only if $T = 0$,*

(2) *there exists a constant $M \geq 1$ such that $g(\lambda T) \leq M|\lambda|g(T)$, for all $T \in \Omega(X, Y)$ and $\lambda \in \mathbb{R}$,*

(3) *there exists a constant $K \geq 1$ such that $g(T_1 + T_2) \leq K[g(T_1) + g(T_2)]$, for all $T_1, T_2 \in \Omega(X, Y)$,*

(4) *there exists a constant $C \geq 1$ such that if $T \in L(X_0, X)$, $P \in \Omega(X, Y)$ and $R \in L(Y, Y_0)$ then $g(RPT) \leq C \|R\|g(P) \|T\|$, where X_0 and Y_0 are normed spaces.*

Theorem 2.12. *[5] Every quasi norm on the ideal Ω is a pre-quasi norm on the ideal Ω.*

Let (a_n), (p_n) and (q_n) be sequences of positive reals with $p_n \geq 1$ for all $n \in \mathbb{N}$, Altay and Başar [13] defined the generalized Cesáro sequence space defined by weighted means as:

$$ces((a_n), (p_n), (q_n)) = \left\{ x = (x(k)) \in \omega : \rho(\lambda x) < \infty, \text{ for some } \lambda > 0 \right\}, \text{ where } \rho(x) = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k|x(k)| \right)^{p_n},$$

which is a Banach space with the Luxemburg norm defined by: $\|x\| = \inf\left\{ \lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1 \right\}$. If (p_n) is bounded, we can simply write

$$ces((a_n), (p_n), (q_n)) = \left\{ x = (x(k)) \in \omega : \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k|x(k)| \right)^{p_n} < \infty \right\}.$$

Remark 2.13. **(1)** *Taking $q_n = 1$ for all $n \in \mathbb{N}$, $ces((a_n), (p_n), (q_n))$ reduced to $ces(((a_n), (p_n))$, the sequence space defined and studied by Şengönül [14].*

(2) *Taking $a_n = \frac{1}{\sum_{k=0}^{n} q_k}$, then $ces((a_n), (p_n), (q_n))$ is reduced to $ces(((p_n), (q_n))$, the Ñorlund sequence spaces studied by Wang [15].*

(3) *Taking $a_n = \frac{1}{n+1}$ and $q_n = 1$, for all $n \in \mathbb{N}$, then $ces((a_n), (p_n), (q_n))$ is reduced to $ces((p_n))$ studied by Sanhan and Suantai [16].*

(4) *Taking $a_n = \frac{1}{n+1}$, $q_n = 1$ and $p_n = p$, for all $n \in \mathbb{N}$, then $ces((a_n), (p_n), (q_n))$ is reduced to ces_p. Different types of Cesáro summable sequence has been studied by many authors see [17-19].*

Definition 2.14. *[5] Let E be a linear space of sequences, then E is called a (sss) if:*

(1) *For $n \in \mathbb{N}$, $e_n \in E$,*

(2) *E is solid i.e., assuming $x = (x_n) \in w$, $y = (y_n) \in E$ and $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then $x \in E$,*

(3) *$(x_{[\frac{n}{2}]})_{n=0}^{\infty} \in E$, where $[\frac{n}{2}]$ indicates the integral part of $\frac{n}{2}$, whenever $(x_n)_{n=0}^{\infty} \in E$.*

Definition 2.15. *[5] A subclass of the (sss) called a pre-modular (sss) assuming that we have a map $\rho : E \to [0, \infty[$ with the followings:*

(i) *For $x \in E$, $x = \theta \Leftrightarrow \rho(x) = 0$ with $\rho(x) \geq 0$,*

(ii) *for each $x \in E$ and scalar λ, we get a real number $L \geq 1$ for which $\rho(\lambda x) \leq |\lambda|L\rho(x)$,*

(iii) *$\rho(x + y) \leq K(\rho(x) + \rho(y))$ for each $x, y \in E$, holds for a few numbers $K \geq 1$,*

(iv) *for $n \in \mathbb{N}$, $|x_n| \leq |y_n|$, we obtain $\rho((x_n)) \leq \rho((y_n))$,*

(v) *the inequality,*

$\rho((x_n)) \leq \rho((x_{[\frac{n}{2}]})) \leq K_0\rho((x_n))$ holds, for some numbers $K_0 \geq 1$,

(vi) *$\overline{F} = E_\rho$, where F is the space of finite sequences,*

(vii) *there is a steady $\xi > 0$ such that $\rho(\lambda, 0, 0, 0, ...) \geq \xi|\lambda|\rho(1, 0, 0, 0, ...)$ for any $\lambda \in \mathbb{R}$.*

Condition (ii) gives the continuity of $\rho(x)$ at θ. The linear space E enriched with the metric topology formed by the premodular ρ will be indicated by E_ρ. Moreover condition (1) in definition (2.14) and condition (vi) in definition (2.15) explain that $(e_n)_{n \in \mathbb{N}}$ is a Schauder basis of E_ρ.

Notations 2.16. *[5]*

$$S_E := \left\{ S_E(X, Y); X \text{ and } Y \text{ are Banach Spaces} \right\}, \text{ where } S_E(X, Y) := \left\{ T \in L(X, Y) : ((s_i(T))_{i=0}^{\infty} \in E \right\}.$$

$$S_E^{app} := \left\{ S_E^{app}(X, Y); X \text{ and } Y \text{ are Banach Spaces} \right\}, \text{ where } S_E^{app}(X, Y) := \left\{ T \in L(X, Y) : ((\alpha_i(T))_{i=0}^{\infty} \in E \right\}.$$

Also

$$S_E^{Kol} := \left\{ S_E^{Kol}(X, Y); \ X \text{and} \ Y \text{ are Banach Spaces} \right\}, \ \text{where} \ S_E^{Kol}(X, Y) := \left\{ T \in L(X, Y) : ((d_i(T))_{i=0}^{\infty} \in E \right\}.$$

Theorem 2.17. *[5] If E is a (sss), then S_E is an operator ideal.*

Throughout, we denote $e_n = \{0, 0, \ldots, 1, 0, 0, \ldots\}$ where 1 appears at the nth place for all $n \in \mathbb{N}$ and the given inequality will be used in the sequel:

$$|a_n + b_n|^{p_n} \le H(|a_n|^{p_n} + |b_n|^{p_n}), \tag{1}$$

where $H = \max\{1, 2^{h-1}\}$, $h = \sup_n p_n$ and $p_n \ge 1$ for all $n \in \mathbb{N}$. See [13].

3 Linear problem

We examine here the operator ideals created by s−numbers also generalized Cesáro sequence space defined by weighted means such that those classes of all bounded linear operators T between arbitrary Banach spaces with $(s_n(T))_{n \in \mathbb{N}}$ in this sequence space type an ideal operator.

Theorem 3.1. *$ces((a_n), (p_n), (q_n))$ is a (sss), if the following conditions are satisfied:*
(a1) *The sequence (p_n) is increasing and bounded from above with $p_0 > 1$ for all $n \in \mathbb{N}$,*
(a2) *the sequence (a_n) of positive reals with $\sum_{n=0}^{\infty}(a_n)^{p_n} < \infty$,*
(a3) *either (q_n) is a monotonic decreasing sequence of positive reals or monotonic increasing bounded sequence and there exists a constant $C \ge 1$ such that $q_{2n+1} \le Cq_n$.*

Proof: (1-i) let $x, y \in ces((a_n), (p_n), (q_n))$. Since (p_n) is bounded, we have

$$\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |x_k + y_k| \right)^{p_n} \le 2^{h-1} \left(\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |x_k| \right)^{p_n} + \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |y_k| \right)^{p_n} \right) < \infty,$$

then $x + y \in ces((a_n), (p_n), (q_n))$.
(1-ii) let $\lambda \in \mathbb{R}$, $x \in ces((a_n), (p_n), (q_n))$ and since (p_n) is bounded, we have

$$\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |\lambda x_k| \right)^{p_n} \le \sup_n |\lambda|^{p_n} \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |x_k| \right)^{p_n} < \infty.$$

Then $\lambda x \in ces((a_n), (p_n), (q_n))$, from (1-i) and (1-ii) $ces((a_n), (p_n), (q_n))$ is a linear space.
Also to show that $e_m \in ces((a_n), (p_n), (q_n))$, for all $m \in \mathbb{N}$. Since (p_n) with $p_0 > 1$ and $\sum_{n=0}^{\infty}(a_n)^{p_n} < \infty$, we have

$$\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |e_m(k)| \right)^{p_n} = \sum_{n=m}^{\infty} (a_n q_m)^{p_n} \le \sup_n (q_m)^{p_n} \sum_{n=m}^{\infty} (a_n)^{p_n} \le \sup_n (q_m)^{p_n} \sum_{n=0}^{\infty} (a_n)^{p_n} < \infty.$$

Hence $e_m \in ces((a_n), (p_n), (q_n))$.

(2) Let $|x_n| \le |y_n|$ for all $n \in \mathbb{N}$ and $y \in ces((a_n), (p_n), (q_n))$. Since $a_n > 0$ and $q_n > 0$ for all $n \in \mathbb{N}$, then

$$\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |x_k| \right)^{p_n} \le \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |y_k| \right)^{p_n} < \infty,$$

we get $x \in ces((a_n), (p_n), (q_n))$.

(3) Let $(x_n) \in ces((a_n),(p_n),(q_n))$, where (p_n) and (q_n) be increasing sequences of positive reals with a constant $C \geq 1$ such that $q_{2n+1} \leq Cq_n$, then we have

$$\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k|x_{[\frac{k}{2}]}|\right)^{p_n} = \sum_{n=0}^{\infty}\left(a_{2n}\sum_{k=0}^{2n}q_k|x_{[\frac{k}{2}]}|\right)^{p_{2n}} + \sum_{n=0}^{\infty}\left(a_{2n+1}\sum_{k=0}^{2n+1}q_k|x_{[\frac{k}{2}]}|\right)^{p_{2n+1}}$$

$$\leq \sum_{n=0}^{\infty}\left(a_{2n}\left[q_{2n}|x_n| + \sum_{k=0}^{2n}(q_{2k}+q_{2k+1})|x_k|\right]\right)^{p_n} + \sum_{n=0}^{\infty}\left(a_{2n+1}\left(\sum_{k=0}^{2n+1}(q_{2k}+q_{2k+1})|x_{[\frac{k}{2}]}|\right)\right)^{p_n}$$

$$\leq 2^{h-1}\left(\sum_{n=0}^{\infty}(Ca_n\sum_{k=0}^{n}q_k|x_k|)^{p_n} + \sum_{n=0}^{\infty}(2a_nC\sum_{k=0}^{n}q_k|x_k|)^{p_n}\right) + \sum_{n=0}^{\infty}\left(2Ca_n\sum_{k=0}^{n}q_k|x_k|\right)^{p_n}$$

$$\leq 2^{h-1}(2^h+1)C^h\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k|x_k|\right)^{p_n} + 2^hC^h\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k|x_k|\right)^{p_n}$$

$$\leq (2^{2h-1}+2^{h-1}+2^h)C^h\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k|x_k|\right)^{p_n} < \infty,$$

then $(x_{[\frac{n}{2}]}) \in ces((a_n),(p_n),(q_n))$.

By using Theorem (2.17), we can get the following corollary:

Corollary 3.2. *Let conditions (a1), (a2) and (a3) be satisfied, then $S_{ces((a_n),(p_n),(q_n))}$ is an operator ideal.*

4 Topological properties

The following question arises naturally; for which sufficient conditions (not necessary) on the pre-modular (sss) E_ρ, the ideal of the finite rank operators on the class of Banach spaces is dense in S_{E_ρ}? This gives a negative answer of Rhoades [9] open problem about the linearity of E_ρ type spaces (S_{E_ρ}).

Theorem 4.1. *$ces((a_n),(p_n),(q_n))$ is a pre-modular (sss), if conditions (a1), (a2) and (a3) are satisfied.*

Proof: Define a functional ρ on $ces((a_n),(p_n),(q_n))$ by $\rho(x) = \sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k|x_k|\right)^{p_n}$.
(i) Clearly, $\rho(x) \geq 0$ and $\rho(x) = 0 \Leftrightarrow x = \theta$.
(ii) There is a number $L = \max\left\{1, \sup_n|\lambda|^{p_n}\right\} \geq 1$ with $\rho(\lambda x) \leq L|\lambda|\rho(x)$ for all $x \in ces((a_n),(p_n))$ and $\lambda \in \mathbb{R}$.
(iii) We have the inequality $\rho(x+y) \leq 2^{h-1}(\rho(x)+\rho(y))$ for all $x, y \in ces((a_n),(p_n))$.
(iv) It is clear from (2) theorem 3.1.
(v) It is clear from (3) theorem 3.1, that $K_0 \geq (2^{2h-1}+2^{h-1}+2^h)C^h \geq 1$.
(vi) It is clear that $\overline{F} = ces((a_n),(p_n))$.
(vii) There exists a constant $0 < \xi \leq \sup_n|\lambda|^{p_n-1}$ such that $\rho(\lambda,0,0,0,...) \geq \xi|\lambda|\rho(1,0,0,0,...)$ for any $\lambda \neq 0$ and $\xi > 0$, when $\lambda = 0$.

We state the following theorem without proof, this can be established using standard technique.

Theorem 4.2. *The function $g(P) = \varrho(s_i(P))_{i=0}^{\infty}$ is a pre-quasi norm on S_{E_ϱ}, where E_ϱ is a pre-modular (sss).*

Lemma 4.3. *If E_ρ is a pre-modular (sss) and $(x_n) \in E_\rho$ is a monotonic decreasing sequence of positive reals, then*

$$\rho(\overbrace{0,0,0,...,0}^{2n},x_n,x_{n+1},x_{n+2},...) \leq K_0\rho(\overbrace{0,0,0,...,0}^{n},x_n,x_{n+1},x_{n+2},...).$$

Proof: By using the Definition (2.15-iv, v) and since $(x_n) \in E_\rho$ is monotonic decreasing we get

$$\rho(\overbrace{0,0,0,...,0}^{2n},x_n,x_{n+1},x_{n+2},...) \leq \rho(\overbrace{0,0,0,...,0}^{2n},x_n,x_n,x_{n+1},x_{n+1},...)$$

$$\le K_0\rho(\overbrace{0, 0, 0, ..., 0}^{n}, x_n, x_{n+1}, x_{n+2}, ...).$$

Theorem 4.4. *Let E_ρ be a pre-modular (sss). Then the linear space $F(X, Y)$ is dense in $S_{E_\rho}(X, Y)$, where $g(T) = \rho(s_n(T)_{n=0}^\infty)$.*

Proof: It is easy to prove that every finite mapping $T \in F(X, Y)$ belongs to $S_{E_\rho}(X, Y)$, since $e_m \in E_\rho$ for each $m \in \mathbb{N}$ and E_ρ is a linear space then every finite mapping $T \in F(X, Y)$ the sequence $(s_n(T))_{n=0}^\infty$ contains only finitely many numbers different from zero. To prove that $S_{E_\rho}(X, Y) \subseteq \overline{F(X, Y)}$, let $T \in S_E(X, Y)$, then $\rho((s_n(T))_{n=0}^\infty) < \infty$ and let $\varepsilon \in (0, 1)$, at that point there exists a $N \in \mathbb{N} - \{0\}$ such that $\rho((s_n(T))_{n=N}^\infty) < \varepsilon$. While $(s_n(T))_{n \in \mathbb{N}}$ is decreasing, we get

$$\rho((s_{2N}(T))_{n=N+1}^{2N}) \le \rho((s_n(T))_{n=N+1}^{2N}) \le \rho((s_n(T))_{n=N}^\infty) < \varepsilon. \tag{2}$$

Hence, there exists $A_N \in F(X, Y)$, rank$A \le N$ and

$$\rho((\|T - A\|)_{n=2N+1}^{3N}) \le \rho((\|T - A\|)_{n=N+1}^{2N}) < \varepsilon. \tag{3}$$

On considering

$$\rho((\|T - A\|)_{n=0}^{N}) < \varepsilon. \tag{4}$$

We have to prove that $\rho((s_n(T - A_N))_{n=0}^\infty) \to 0$ as $N \to \infty$. By taking $N = 8\eta$, where η is a natural number. From Definition (2.15-iii) we have

$$d(T, A_N) = \rho((s_n(T - A_N))_{n=0}^\infty) = \rho[(s_0(T - A_N), s_1(T - A_N), ..., s_{8\eta-1}(T - A_N), 0, 0, 0, ...)$$

$$+ (\overbrace{0, 0, 0, ..., 0}^{8\eta}, s_{8\eta}(T - A_N), s_{8\eta+1}(T - A_N), ..., s_{12\eta-1}(T - A_N), 0, 0, 0, ...)$$

$$+ (\overbrace{0, 0, 0, ..., 0}^{12\eta}, s_{12\eta}(T - A_N), s_{12\eta+1}(T - A_N), ...)]$$

$$\le K^2[\rho(s_0(T - A_N), s_1(T - A_N), ..., s_{8\eta-1}(T - A_N), 0, 0, 0, ...)$$

$$+ \rho(\overbrace{0, 0, 0, ..., 0}^{8\eta}, s_{8\eta}(T - A_N), s_{8\eta+1}(T - A_N), ..., s_{12\eta-1}(T - A_N), 0, 0, 0, ...)$$

$$+ \rho(\overbrace{0, 0, 0, ..., 0}^{12\eta}, s_{12\eta}(T - A_N), s_{12\eta+1}(T - A_N), ...)] = K^2[I_1(N) + I_2(\eta) + I_3(\eta)].$$

Since $s_n(A_N) = 0$ for $n \ge N$ then

$$s_n(T - A_N) \le s_{n-N}(T). \tag{5}$$

By using Lemma 4.3, inequality (5) and Definition (2.15-iv), we get

$$I_3(\eta) = \rho(\overbrace{0, 0, 0, ..., 0}^{12\eta}, s_{12\eta}(T - A_N), s_{12\eta+1}(T - A_N), ...) \le \rho(\overbrace{0, 0, 0, ..., 0}^{12\eta}, s_{4\eta}(T), s_{4\eta+1}(T), ...)$$

$$\le K_0\rho(\overbrace{0, 0, 0, ..., 0}^{6\eta}, s_{4\eta}(T), s_{4\eta+1}(T), ...) \le K_0^2\rho(\overbrace{0, 0, 0, ..., 0}^{3\eta}, s_{4\eta}(T), s_{4\eta+1}(T), ...)$$

$$\le K_0^2\rho(\overbrace{0, 0, 0, ..., 0}^{3\eta}, s_{3\eta}(T), s_{3\eta+1}(T), ...) = K_0^2\rho((s_n(T))_{n=3\eta}^\infty) \to 0 \text{ as } \eta \to \infty.$$

Now using Lemma 4.3 and Definition (2.15-iv), we have

$$I_2(\eta) = \rho(\overbrace{0, 0, 0, ..., 0}^{8\eta}, s_{8\eta}(T - A_N), s_{8\eta+1}(T - A_N), ..., s_{12\eta-1}(T - A_N), 0, 0, 0, ...)$$

$$\le K_0\rho(\overbrace{0, 0, 0, ..., 0}^{4\eta}, s_{8\eta}(T - A_N), s_{8\eta+1}(T - A_N), ..., s_{12\eta-1}(T - A_N), 0, 0, 0, ...)$$

$$\le K_0\rho(s_0(T - A_N), s_1(T - A_N), ..., s_{8\eta-1}(T - A_N), 0, 0, 0, ...) = k_0 I_1(N).$$

Finally we have to show that $I_1(N) \to 0$ as $N \to \infty$. By taking $\varepsilon_1 = \frac{\varepsilon}{3LK}$ for each $n \geq N_0(\varepsilon_1)$, using the inequalities (2), (3), (4) and Definition (2.15-ii and iii) then we have

$$
\begin{aligned}
I_1(N) &= \rho(s_0(T - A_N), s_1(T - A_N), \ldots, s_{N-1}(T - A_N), 0, 0, 0, \ldots) \\
&\leq K[\rho(s_0(T - A_N), s_1(T - A_N), \ldots, s_{N_0-1}(T - A_N), 0, 0, 0, \ldots) \\
&\quad + \rho(\overbrace{0, 0, 0, \ldots, 0}^{N_0}, s_{N_0}(T - A_N), s_{N_0+1}(T - A_N), \ldots, s_{N-1}(T - A_N), 0, 0, 0, \ldots)] \\
&\leq K[\rho(\|T - A_N\|, \|T - A_N\|, \ldots, \|T - A_N\|, 0, 0, 0, \ldots) \\
&\quad + \rho(\overbrace{0, 0, 0, \ldots, 0}^{N_0}, 2s_n(T), 2s_n(T), \ldots, 2s_n(T), 0, 0, 0, \ldots)] \\
&\leq K[\varepsilon_1 + 2L\varepsilon_1] < \epsilon.
\end{aligned}
$$

This completes the proof.

Corollary 4.5. $\overline{F(X, Y)} = S_{ces((a_n),(p_n),(q_n))}(X, Y)$, *assuming that states (a1), (a2) and (a3) are fulfilled and the converse is not necessarily true.*

Proof: It follows from Theorem (4.4) and $ces((a_n), (p_n), (q_n))$ is pre-modular (sss). For the converse part, since $I_3 \in S_{ces((1),(1),(1))}$ but the condition (a2) is not satisfied which is a counter example. This establishes the proof. From Corollary (4.5), we can say that if (a1), (a2) and (a3) are satisfied, then every compact operators would be approximated by finite rank operators and the converse is not necessarily true.

5 Completeness of the pre-quasi closed ideal components

For which sequence space $ces((a_n), (p_n), (q_n))$, the components of pre-quasi closed operator ideal $S_{ces((a_n),(p_n),(q_n))}$ are complete?

Theorem 5.1. *If X and Y are Banach spaces, (a1), (a2) and (a3) are satisfied, then $\left(S_{ces((a_n),(p_n),(q_n))_\rho}, g\right)$, where $\rho(x) = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k|x(k)|\right)^{p_n}$ and $g(T) = \rho\left((s_n(T))_{n=0}^{\infty}\right)$ is a pre-quasi Banach operator ideal.*

Proof: Since $ces((a_n), (p_n), (q_n))_\rho$ is a pre-modular (sss), then the function $g(T) = \rho\left((s_n(T))_{n=0}^{\infty}\right)$ is a pre-quasi norm on $S_{ces((a_n),(p_n),(q_n))_\rho}$. Let (T_m) be a Cauchy sequence in $S_{ces((a_n),(p_n),(q_n))_\rho}(X, Y)$, then by definition (2.15-vii) there exists a constant $\xi > 0$ and since $L(X, Y) \supseteq S_{ces((a_n),(p_n),(q_n))_\rho}(X, Y)$, we get

$$
g(T_i - T_j) = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k s_k(T_i - T_j)\right)^{p_n} \geq \sum_{n=0}^{\infty} \left(a_n q_0 \|T_i - T_j\|\right)^{p_n} \geq \xi \|T_i - T_j\| \sum_{n=0}^{\infty} \left(a_n q_0\right)^{p_n},
$$

then $(T_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in $L(X, Y)$. While the space $L(X, Y)$ is a Banach space, so there exists $T \in L(X, Y)$ with $\lim_{m \to \infty} \|T_m - T\| = 0$ and while $(s_n(T_m))_{n=0}^{\infty} \in ces((a_n), (p_n), (q_n))_\rho$ for each $m \in \mathbb{N}$, hence using definition (2.15) conditions (iii), (iv), (v) and ρ is continuous at θ, we get

$$
\begin{aligned}
g(T) &= \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k(s_k(T))\right)^{p_n} = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k(s_k(T - T_m + T_m))\right)^{p_n} \\
&\leq K\left(\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_{[\frac{k}{2}]} s_{[\frac{k}{2}]}(T - T_m)\right)\right) + K\left(\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_{[\frac{k}{2}]} s_{[\frac{k}{2}]}(T_m)\right)^{p_n}\right) \\
&\leq K\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k \|T_m - T\|\right) + KK_0 \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k s_k(T_m)\right)^{p_n} < \varepsilon,
\end{aligned}
$$

We have $(s_n(T))_{n=0}^{\infty} \in ces((a_n), (p_n), (q_n))_\rho$, then $T \in S_{ces((a_n),(p_n),(q_n))_\rho}(X, Y)$.

Theorem 5.2. *If X and Y are normed spaces, (a1), (a2) and (a3) are satisfied, then $\left(S_{ces((a_n),(p_n),(q_n))_\rho}, g\right)$, where $\rho(x) = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k |x(k)|\right)^{p_n}$ and $g(T) = \rho\left((s_n(T))_{n=0}^{\infty}\right)$ is a pre-quasi closed operator ideal.*

Proof: Since $ces((a_n), (p_n), (q_n))_\rho$ is a pre-modular (sss), then the function $g(T) = \rho\left((s_n(T))_{n=0}^{\infty}\right)$ is a pre-quasi norm on $S_{ces((a_n),(p_n),(q_n))_\rho}$. Let $T_m \in S_{ces((a_n),(p_n),(q_n))_\rho}(X, Y)$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} g(T_m - T) = 0$, then by utilizing definition (2.15-vii) there exists a constant $\xi > 0$ and since $L(X, Y) \supseteq S_{ces((a_n),(p_n),(q_n))_\rho}(X, Y)$, we get

$$g(T - T_m) = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k s_k(T - T_m)\right)^{p_n} \geq \sum_{n=0}^{\infty} \left(a_n q_0 \|T - T_m\|\right)^{p_n} \geq \xi \|T - T_m\| \sum_{n=0}^{\infty} \left(a_n q_0\right)^{p_n},$$

then $(T_m)_{m\in\mathbb{N}}$ is a convergent sequence in $L(X, Y)$. While $(s_n(T_m))_{n=0}^{\infty} \in ces((a_n), (p_n), (q_n))_\rho$ for each $m \in \mathbb{N}$, hence using definition (2.15) conditions (iii), (iv), (v) and ρ is continuous at θ, we obtain

$$g(T) = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k (s_k(T))\right)^{p_n} = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k (s_k(T - T_m + T_m))\right)^{p_n}$$

$$\leq K\left(\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_{[\frac{k}{2}]} s_{[\frac{k}{2}]}(T - T_m)\right)\right) + K\left(\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_{[\frac{k}{2}]} s_{[\frac{k}{2}]}(T_m)\right)^{p_n}\right)$$

$$\leq K\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k \|T_m - T\|\right) + KK_0 \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k s_k(T_m)\right)^{p_n} < \varepsilon,$$

we have $(s_n(T))_{n=0}^{\infty} \in ces((a_n), (p_n), (q_n))_\rho$, then $T \in S_{ces((a_n),(p_n),(q_n))_\rho}(X, Y)$.

6 Minimum pre-quasi Banach operator ideal

We give here the sufficient conditions on the generalized Cesáro sequence space defined by weighted means such that the pre-quasi operator ideal formed by the sequence of approximation numbers and this sequence space is strictly contained for different weights and powers.

Theorem 6.1. *For any infinite dimensional Banach spaces X, Y and for any $1 < p_n^{(1)} < p_n^{(2)}$, $0 < a_n^{(2)} \leq a_n^{(1)}$ and $0 < q_n^{(2)} \leq q_n^{(1)}$ for all $n \in \mathbb{N}$, it is true that*

$$S^{app}_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}(X, Y) \subsetneqq S^{app}_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}(X, Y) \subsetneqq L(X, Y).$$

Proof: Let X and Y be infinite dimensional Banach spaces and for any $1 < p_n^{(1)} < p_n^{(2)}$, $0 < a_n^{(2)} \leq a_n^{(1)}$ and $0 < q_n^{(2)} \leq q_n^{(1)}$ for all $n \in \mathbb{N}$, if $T \in S^{app}_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}(X, Y)$, then $(\alpha_n(T)) \in ces((a_n^{(1)}), (p_n^{(1)}), (q_n^{(1)}))$. We have

$$\sum_{n=0}^{\infty} \left(a_n^{(2)} \sum_{k=0}^{n} q_k^{(2)} \alpha_k(T)\right)^{p_n^{(2)}} < \sum_{n=0}^{\infty} \left(a_n^{(1)} \sum_{k=0}^{n} q_k^{(1)} \alpha_k(T)\right)^{p_n^{(1)}} < \infty,$$

hence $T \in S^{app}_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}(X, Y)$. Next, if we take $(\alpha_n(T))_{n=0}^{\infty}$ such that $a_n^{(1)} \sum_{k=0}^{n} q_k^{(1)} \alpha_k(T) = \frac{1}{p_n^{(1)}\sqrt{n+1}}$, one can find $T \in L(X, Y)$ with

$$\sum_{n=0}^{\infty} \left(a_n^{(1)} \sum_{k=0}^{n} q_k^{(1)} \alpha_k(T)\right)^{p_n^{(1)}} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty,$$

and

$$\sum_{n=0}^{\infty} \left(a_n^{(2)} \sum_{k=0}^{n} q_k^{(2)} \alpha_k(T)\right)^{p_n^{(2)}} \leq \sum_{n=0}^{\infty} \left(a_n^{(1)} \sum_{k=0}^{n} q_k^{(1)} \alpha_k(T)\right)^{p_n^{(2)}} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^{\frac{p_n^{(2)}}{p_n^{(1)}}} < \infty.$$

Hence T does not belong to $S^{app}_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}(X, Y)$ and $T \in S^{app}_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}(X, Y)$.

It is easy to verify that $S^{app}_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}(X, Y) \subset L(X, Y)$. Next, if we take $(\alpha_n(T))_{n=0}^{\infty}$ such that $a_n^{(2)} \sum_{k=0}^{n} q_k^{(2)} \alpha_k(T) = \frac{1}{p_n^{(2)}\sqrt{n+1}}$. One can find $T \in L(X, Y)$ such that T does not belong to $S^{app}_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}(X, Y)$. This completes the proof.

Corollary 6.2. *For any infinite dimensional Banach spaces X, Y and $1 < p < q < \infty$, then*

$$S^{app}_{ces_p}(X, Y) \subsetneqq S^{app}_{ces_q}(X, Y) \subsetneqq L(X, Y).$$

In this part, we give the conditions for which the the pre-quasi Banach Operator ideal $S^{app}_{ces((a_n),(p_n),(q_n))}$ is minimum.

Theorem 6.3. *If conditions (a1), (a2), (a3) and $(a_n \sum_{j=0}^{n} q_j) \notin \ell^{(p_n)}$ are satisfied, then the pre-quasi Banach Operator ideal $S^{app}_{ces((a_n),(p_n),(q_n))}$ is minimum.*

Proof: Let conditions (a1), (a2), (a3) and $(a_n \sum_{j=0}^{n} q_j) \notin \ell^{(p_n)}$ be satisfied. Then $(S^{app}_{ces((a_n),(p_n),(q_n))}, g)$, where $g(T) = \left[\sum_{i=0}^{\infty} (a_i \sum_{j=0}^{i} q_j \alpha_j(T))^{p_i} \right]^{\frac{1}{h}}$ is a pre-quasi Banach Operator ideal. Let X and Y be any two Banach spaces. Suppose that $S^{app}_{ces((a_n),(p_n),(q_n))} = L(X, Y)$, then there exists a constant $C > 0$ such that $g(T) \le C\|T\|$ for all $T \in L(X, Y)$. Assume that X and Y be infinite dimensional Banach spaces. Hence by Dvoretzky's theorem [20] for $m \in \mathbb{N}$, we have quotient spaces X/N_m and subspaces M_m of Y which can be mapped onto ℓ_2^m by isomorphisms H_m and A_m such that $\|H_m\|\|H_m^{-1}\| \le 2$ and $\|A_m\|\|A_m^{-1}\| \le 2$. Let I_m be the identity map on ℓ_2^m, Q_m be the quotient map from X onto X/N_m and J_m be the natural embedding map from M_m into Y. Let u_n be the Bernstein numbers [21] then

$$\begin{aligned}
1 = u_n(I_m) &= u_n(A_m A_m^{-1} I_m H_m H_m^{-1}) \\
&\le \|A_m\| u_n(A_m^{-1} I_m H_m)\|H_m^{-1}\| \\
&= \|A_m\| u_n(J_m A_m^{-1} I_m H_m)\|H_m^{-1}\| \\
&\le \|A_m\| d_n(J_m A_m^{-1} I_m H_m)\|H_m^{-1}\| \\
&= \|A_m\| d_n(J_m A_m^{-1} I_m H_m Q_m)\|H_m^{-1}\| \\
&\le \|A_m\| \alpha_n(J_m A_m^{-1} I_m H_m Q_m)\|H_m^{-1}\|,
\end{aligned}$$

for $0 \le i \le m$. Now

$$\sum_{j=0}^{i} q_j \le \sum_{j=0}^{i} \|A_m\| q_j \alpha_j(J_m A_m^{-1} I_m H_m Q_m)\|H_m^{-1}\| \Rightarrow$$

$$a_i \sum_{j=0}^{i} q_j \le \|A_m\|(a_i \sum_{j=0}^{i} q_j \alpha_j(J_m A_m^{-1} I_m H_m Q_m))\|H_m^{-1}\| \Rightarrow$$

$$\left(a_i \sum_{j=0}^{i} q_j \right)^{p_i} \le (\|A_m\|\|H_m^{-1}\|)^{p_i}(a_i \sum_{j=0}^{i} q_j \alpha_j(J_m A_m^{-1} I_m H_m Q_m))^{p_i}.$$

Therefore,

$$\left[\sum_{i=0}^{m}\left(a_i\sum_{j=0}^{i}q_j\right)^{p_i}\right]^{\frac{1}{h}} \le L\|A_m\|\|H_m^{-1}\|\left[\sum_{i=0}^{m}(a_i\sum_{j=0}^{i}q_j\alpha_j(J_mA_m^{-1}I_mH_mQ_m))^{p_i}\right]^{\frac{1}{h}} \Rightarrow$$

$$\left[\sum_{i=0}^{m}\left(a_i\sum_{j=0}^{i}q_j\right)^{p_i}\right]^{\frac{1}{h}} \le L\|A_m\|\|H_m^{-1}\|\left[\sum_{i=0}^{m}(a_i\sum_{j=0}^{i}q_j\alpha_j(J_mA_m^{-1}I_mH_mQ_m))^{p_i}\right]^{\frac{1}{h}} \Rightarrow$$

$$\left[\sum_{i=0}^{m}\left(a_i\sum_{j=0}^{i}q_j\right)^{p_i}\right]^{\frac{1}{h}} \le L\|A_m\|\|H_m^{-1}\|g(J_mA_m^{-1}I_mH_mQ_m) \Rightarrow$$

$$\left[\sum_{i=0}^{m}\left(a_i\sum_{j=0}^{i}q_j\right)^{p_i}\right]^{\frac{1}{h}} \le LC\|A_m\|\|H_m^{-1}\|\|J_mA_m^{-1}I_mH_mQ_m\| \Rightarrow$$

$$\left[\sum_{i=0}^{m}\left(a_i\sum_{j=0}^{i}q_j\right)^{p_i}\right]^{\frac{1}{h}} \le LC\|A_m\|\|H_m^{-1}\|\|J_mA_m^{-1}\|\|I_m\|\|H_mQ_m\| = LC\|A_m\|\|H_m^{-1}\|\|A_m^{-1}\|\|I_m\|\|H_m\| \Rightarrow$$

$$\left[\sum_{i=0}^{m}\left(a_i\sum_{j=0}^{i}q_j\right)^{p_i}\right]^{\frac{1}{h}} \le 4LC,$$

for some $L \ge 1$. We arrive at a contradiction, since m is an arbitrary and $(a_n\sum_{j=0}^{n}q_j) \notin \ell^{(p_n)}$. Thus X and Y both cannot be infinite dimensional when $S^{app}_{ces((a_n),(p_n),(q_n))} = L(X, Y)$. Hence, the result.

Theorem 6.4. *If conditions (a1), (a2), (a3) and $(a_n\sum_{j=0}^{n}q_j) \notin \ell^{(p_n)}$ are satisfied, then the pre-quasi Banach Operator ideal $S^{Kol}_{ces((a_n),(p_n),(q_n))}$ is minimum.*

Corollary 6.5. *If $1 < p < \infty$, then the quasi Banach Operator ideal $S^{app}_{ces_p}$ is minimum.*

Corollary 6.6. *If $1 < p < \infty$, then the quasi Banach Operator ideal $S^{Kol}_{ces_p}$ is minimum.*

7 Pre-quasi simple Banach operator ideal

The following question arises naturally; for which generalized Cesáro sequence space defined by weighted means, the pre-quasi Banach ideal is simple?

Theorem 7.1. *For any infinite dimensional Banach spaces X, Y. If $(p_n^{(1)})$, $(p_n^{(2)})$ are bounded sequences with $1 < p_n^{(1)} < p_n^{(2)}$, $0 < a_n^{(2)} \le a_n^{(1)}$ and $0 < q_n^{(2)} \le q_n^{(1)}$ for all $n \in \mathbb{N}$, then*

$$L\left(S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}, S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}\right) = \Psi\left(S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}, S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}\right).$$

Proof: Suppose that there exists $T \in L\left(S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}, S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}\right)$ which is not approximable. According to Lemma (2.6), we can find $X \in L(S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}, S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))})$ and $B \in L(S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}, S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))})$ with $BTXI_k = I_k$. Then it follows for all $k \in \mathbb{N}$ that

$$\|I_k\|_{S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}} = \left(\sum_{n=0}^{\infty}\left(a_n^{(1)}\sum_{i=0}^{n}q_i^{(1)}s_i(I_k)\right)^{p_n^{(1)}}\right)^{\frac{1}{\sup p_n^{(1)}}}$$

$$\le \|BTX\|\|I_k\|_{S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}} \le \left(\sum_{n=0}^{\infty}\left(a_n^{(2)}\sum_{i=0}^{n}q_i^{(2)}s_i(I_k)\right)^{p_n^{(2)}}\right)^{\frac{1}{\sup p_n^{(2)}}}.$$

This contradicts Theorem(6.1). Hence $T \in \Psi\left(S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}, S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}\right)$, which finishes the proof.

Corollary 7.2. *For any infinite dimensional Banach spaces X, Y. If $(p_n^{(1)})$, $(p_n^{(2)})$ are bounded sequences with $1 < p_n^{(1)} < p_n^{(2)}$, $0 < a_n^{(2)} \le a_n^{(1)}$ and $0 < q_n^{(2)} \le q_n^{(1)}$ for all $n \in \mathbb{N}$, then*

$$L\left(S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}, S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}\right) = L_C\left(S_{ces((a_n^{(2)}),(p_n^{(2)}),(q_n^{(2)}))}, S_{ces((a_n^{(1)}),(p_n^{(1)}),(q_n^{(1)}))}\right).$$

Proof: Since every approximable operator is compact.

Theorem 7.3. *For a bounded sequence (p_n) with $1 < p_n < \infty$, $q_n > 0$ and $a_n > 0$ for all $n \in \mathbb{N}$, the pre-quasi Banach space $S_{ces((a_n),(p_n),(q_n))}$ is simple.*

Proof: Suppose that the closed ideal $L_C(S_{ces((a_n),(p_n),(q_n))})$ contains an operator T which is not approximable. According to Lemma (2.6), we can find $X, B \in L(S_{ces((a_n),(p_n),(q_n))})$ with $BTXI_k = I_k$. This means that $I_{S_{ces((a_n),(p_n),(q_n))}} \in L_C(S_{ces((a_n),(p_n),(q_n))})$. Consequently $L(S_{ces((a_n),(p_n),(q_n))}) = L_C(S_{ces((a_n),(p_n),(q_n))})$. Therefore $\Psi(S_{ces((a_n),(p_n),(q_n))})$ is the only non-trivial closed ideal in $L(S_{ces((a_n),(p_n),(q_n))})$.

8 Eigenvalues of s-type operators

We give here the sufficient conditions on the generalized Cesáro sequence space defined by weighted means such that the pre-quasi operator ideal formed by the sequence of $s-$numbers and this sequence space is strictly contained in the class of all bounded linear operators whose sequence of eigenvalues belongs to this sequence space.

Notation 8.1.

$$S_E^\lambda := \left\{ S_E^\lambda(X, Y); \ X \text{and } Y \text{ are Banach Spaces} \right\}, \ \text{where}$$

$$S_E^\lambda(X, Y) := \left\{ T \in L(X, Y) : ((\lambda_i(T))_{n=0}^\infty \in E \text{ and } \|T - \lambda_n(T)\| \text{ is not invertible for all } n \in \mathbb{N} \right\}.$$

Theorem 8.2. *For any infinite dimensional Banach spaces X and Y. If conditions (a1), (a2), and (a3) are satisfied with $\inf_n q_n > 0$ and $\inf_n (n + 1)a_n > 0$, then*

$$S_{ces((a_n),(p_n),(q_n))}(X, Y) \subsetneqq S_{ces((a_n),(p_n),(q_n))}^\lambda(X, Y).$$

Proof: Let $T \in S_{ces((a_n),(p_n),(q_n))}(X, Y)$, then $(s_n(T))_{n=0}^\infty \in ces((a_n), (p_n), (q_n))$. Since $\inf_n q_n > 0$, $\inf_n (n+1)a_n > 0$ and (p_n) is increasing bounded sequence with $p_n > 1$ for all $n \in \mathbb{N}$, we have

$$\sum_{n=0}^\infty \left(a_n \sum_{k=0}^n q_k s_k(T) \right)^{p_n} \ge \inf_n \left(\inf_n [(n + 1)a_n q_n] \right)^{p_n} \sum_{n=0}^\infty [s_n(T)]^{p_n}.$$

Hence $(s_n(T))_{n=0}^\infty \in \ell^{(p_n)}$, so $\lim_{n\to\infty} s_n(T) = 0$. Suppose $\|T - s_n(T)\|$ is invertible for all $n \in \mathbb{N}$, then $\|T - s_n(T)\|^{-1}$ exists and bounded for all $n \in \mathbb{N}$. This gives $\lim_{n\to\infty} \|T - s_n(T)\|^{-1} = \|T\|^{-1}$ exists and bounded. Since $(S_{ces((a_n),(p_n),(q_n))}, g)$ is a pre-quasi operator ideal, we have

$$I = TT^{-1} \in S_{ces((a_n),(p_n),(q_n))}(X, Y) \Rightarrow (s_n(I))_{n=0}^\infty \in ces((a_n), (p_n), (q_n)) \Rightarrow \lim_{n\to\infty} s_n(I) = 0.$$

But $\lim_{n\to\infty} s_n(I) = 1$. This is a contradiction, then $\|T - s_n(T)\|$ is not invertible for all $n \in \mathbb{N}$. Therefore the sequence $(s_n(T))_{n=0}^\infty$ is the eigenvalues of T. Next, on considering $(s_n(T))_{n=0}^\infty$ such that $a_n \sum_{k=0}^n q_k s_k(T) = \frac{1}{p_n\sqrt{n+1}}$, we find $T \in L(X, Y)$ with

$$\sum_{n=0}^\infty \left(a_n \sum_{k=0}^n q_k s_k(T) \right)^{p_n} = \sum_{n=0}^\infty \frac{1}{n + 1} = \infty,$$

and if we take $(\lambda_n(T))_{n=0}^{\infty}$ such that $a_n \sum_{k=0}^{n} q_k \lambda_k(T) = \frac{1}{n+1}$. Hence T does not belong to $S_{ces((a_n),(p_n),(q_n))}(X, Y)$ and $T \in S^{\lambda}_{ces((a_n),(p_n),(q_n))}(X, Y)$. This finishes the proof.

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