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L-topological-convex spaces generated by L-convex bases

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Abstract: In this paper, axiomatic definitions of both L-convex bases and L-convex subbases are introduced and their relations with L-convex spaces are studied. Based on this, the notion of L-topological-convex space is introduced as a triple $(X, \mathcal{T}, \mathcal{C})$, where X is a nonempty set, $\mathcal{C}$ is an L-convex structure on X and $\mathcal{T}$ is an L-cotopology on X compatible with $\mathcal{C}$. It can be characterized by many means.

Keywords: L-convex space, L-convex enclose relation space, L-topological-convex space, L-topological-convex enclosed relation space

MSC 2010: 54A40, 52A01

1 Introduction

By convex sets, we traditionally refer to convex sets in Euclidean spaces, where the property 'convexity' was originally inspired by some elementary geometric problems such as shapes of circles and characterizations of polytopes [1]. However, with increasing fields that convex sets involved and expanding scopes that convex sets were applied, many complex problems compelled people to engage in an axiomatic research of convex sets. This leads to abstract convex structure which is a set-theoretic structure satisfying several axioms [2]. Now, its theory involves many mathematical structures including graphs [3], posets [4], median algebras [5], metric spaces [6], lattices [7] and vector spaces [2].

Convex structure has been extended into fuzzy settings in several ways. Fuzzy convex structures defined by Rosa [8] was further extended into M-convex structures by Maruyama [9]. Latter, some characterizations of L-convex spaces were discussed [10–15]. Actually, an M-convex structure is a crisp family of M-fuzzy sets satisfying certain set of axioms similar to these that an abstract convex structure has. However, from a totally different point of view, Shi and Xiu introduced M-fuzzifying convex structures [16]. Now, many subsequent properties in M-fuzzifying convex spaces have been studied [17–25]. Shi and Xiu also introduced (L, M)-fuzzy convex spaces unifying both L-convex spaces and M-fuzzifying convex spaces [26]. Based on this, many characterizations of L-convex spaces and (L, M)-fuzzy convex spaces have been discussed [27, 28].

In this paper, we introduce axiomatic notions of both L-convex bases and L-convex subbase and discuss their relations with L-convex spaces. Further, based on L-concave bases, we introduce the notion of L-topological-convex spaces and obtain several of its characterizations in a view point of category aspect.

This paper is arranged as follows. In Section 2, we recall some notions and results related to L-convex spaces and L-cotopological spaces. In Section 3, we introduce L-convex bases and study its relations with L-

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convex spaces. In Section 4, we introduce L-convex subbases and study its relations with L-convex spaces and L-convex bases. In Section 5, we introduce the notion of L-topological-convex spaces and obtain several of its characterizations. In Section 6, we introduce the notion of L-topological-convex enclosed relation spaces and further obtain some other characterizations of L-topological-convex spaces.

2 Preliminaries

Throughout this paper, X and Y are nonempty sets. The power set of X is denoted by 2^X. The set of all finite subsets of X is denoted by 2^X_{fin}.

$(L, \vee, \wedge)$ is a completely distributive lattice. The least (resp. largest) element in L is denoted by $\perp$ (resp. $\top$). An element $a \in L$ is called a co-prime, if for all $b, c \in L$, $a \leq b \wedge c$ implies $a \leq b$ or $a \leq c$. The set of all co-primes in $L\backslash\{\perp\}$ is denoted by $J(L)$. For any $a \in L$, there is $L_1 \subseteq J(L)$ such that $a = \bigvee_{b \in L_1} b$ [29]. A binary relation $\prec$ on L is defined by $a \prec b$ iff for each $L_1 \subseteq L$, $b \leq \bigvee L_1$ implies the existence of $d \in L_1$ such that $a \leq d$. The mapping $\beta : L \to 2^L$, defined by $\beta(a) = \{b : b \prec a\}$, satisfies $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$ for $\{a_i\}_{i \in I} \subseteq L$. For any $a \in L$, $\beta(a)$ and $\beta^\star(a) = \beta(a) \cap J(L)$ satisfies $a = \bigvee \beta(a) = \bigvee \beta^\star(a)$ [29].

L^X is the set of all L-fuzzy sets on X. The lease (resp, largest) element in L^X is denoted by $\underline{\perp}$ (resp. $\underline{\top}$). A subset $\{A_i\}_{i \in I} \subseteq L^X$ is called an up-directed set, simply denoted by $\{A_i\}_{i \in I} \overset{dir}{\subseteq} L^X$, if for all $i, j \in I$, there is $k \in I$ such that $A_i, A_j \leq A_k$. For convenience, if $\psi \subseteq L^X$, we adopt $\bigvee \psi = \bigvee_{A \in \psi} A$ and $\bigwedge \psi = \bigwedge_{A \in \psi} A$. Further, if ψ is up-directed, we also adopt $\bigvee \psi = \bigvee^{dir}_{A \in \psi} A$. For $A \in L^X$, we denote

$$\mathfrak{F}(A) = \{F \in L^X : \exists \varphi \in 2^{\beta^\star(A)}_{fin}, F = \bigvee \varphi\},$$

where $\beta^\star(A) = \bigcup\{x_\lambda : \lambda \in \beta^\star(A(x))\}$. In particular, we write $\mathfrak{F}(L^X)$ for $\mathfrak{F}(\underline{\top})$ consisting of all L-fuzzy finite sets on X [28]. Clearly, for $A, B \in L^X$, $B \leq A$ iff $\mathfrak{F}(B) \subseteq \mathfrak{F}(A)$. In addition, it has been proved that $\beta^\star(A) \subseteq \mathfrak{F}(A) \overset{dir}{\subseteq} L^X$, $\bigvee \mathfrak{F}(A) = A$ and $\mathfrak{F}(\bigvee_{i \in I} A_i) = \bigcup_{i \in I} \mathfrak{F}(A_i)$ for all $A \in L^X$ and $\{A_i\}_{i \in I} \overset{dir}{\subseteq} L^X$ [28].

For a mapping $f : X \to Y$, the L-fuzzy mapping $f_L^\to : L^X \to L^Y$ is defined by $f_L^\to(A)(y) = \bigvee\{A(x) : f(x) = y\}$ for $A \in L^X$ and $y \in Y$, and the mapping $f_L^\leftarrow : L^Y \to L^X$ is defined by $f_L^\leftarrow(B)(x) = B(f(x))$ for $B \in L^Y$ and $x \in X$ [30].

Terminologies of Category Theory (resp. Convex Theory) used this paper can be seen in [31] (resp. [2]). Next, we recall some basic definitions and results related to L-convex spaces and L-cotopological spaces.

Definition 2.1. [9] A subset $\mathcal{C} \subseteq L^X$ is called an L-convex structure and the pair $(X, \mathcal{C})$ is called an L-convex space, if $\mathcal{C}$ satisfies

(LC1) $\underline{\perp}, \underline{\top} \in \mathcal{C}$;

(LC2) $\bigwedge \mathcal{C}_1$ for any $\mathcal{C}_1 \subseteq \mathcal{C}$;

(LC3) $\bigvee \mathcal{C}_2$ for any $\mathcal{C}_2 \overset{dir}{\subseteq} \mathcal{C}$.

Theorem 2.2. [11] *Let $(X, \mathcal{C})$ be an L-convex space. The L-hull operator $co_\mathcal{C} : L^X \to L^X$ (briefly, co) of $(X, \mathcal{C})$, defined by $co(A) = \bigwedge\{B \in \mathcal{C} : A \leq B\}$, satisfies*

(LCO1) $co(\underline{\perp}) = \underline{\perp}$;

(LCO2) $A \leq co(A)$ for all $A \in L^X$;

(LCO3) if $A \leq B$, then $co(A) \leq co(B)$;

(LCO4) $co(co(A)) = co(A)$ for all $A \in L^X$;

(LCO5) $co(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} co(A_i)$ for any $\{A_i\}_{i \in I} \overset{dir}{\subseteq} L^X$.

Conversely, if $co : L^X \to L^X$ satisfies (LCO1)–(LCO5), then the set $\mathcal{C}_{co} = \{A \in L^X : co(A) = A\}$ is an L-convex structure satisfying $co_{\mathcal{C}_{co}} = co$.

In [28], it showed that an operator $co : L^X \to L^X$ satisfying (LCO1)–(LCO4) is the L-hull operator of some L-convex space iff it satisfies (LDF).

(LDF) $co(A) = \bigvee_{F\in\mathfrak{F}(A)} co(F)$ for $A \in L^X$.

Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L-convex spaces. A mapping $f : X \to Y$ is called an L-convex structure preserving mapping, if $f_L^{\leftarrow}(A) \in \mathcal{C}_X$ for any $A \in \mathcal{C}_Y$. The category of L-convex spaces and L-convex structure preserving mappings is denoted by L-**CS** [28].

Definition 2.3. [32] A subset $\mathcal{T} \subseteq L^X$ is called an L-cotopology and the pair $(X, \mathcal{T})$ is called an L-cotopological space, if $\mathcal{T}$ satisfies

(LT1) $\underline{\perp}, \underline{\top} \in \mathcal{T}$;

(LT2) $\bigwedge \mathcal{T}_1 \in \mathcal{T}$ for any $\mathcal{T}_1 \subseteq \mathcal{T}$;

(LT3) $A \vee B \in \mathcal{T}$ for all $A, B \in \mathcal{T}$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-cotopological spaces. A mapping $f : X \to Y$ is called an L-continuous mapping if $f_L^{\leftarrow}(A) \in \mathcal{T}_X$ for any $A \in \mathcal{T}_Y$. The category of L-cotopological spaces and L-continuous mappings is denoted by L-**CTS**.

Definition 2.4. [12] A subset $\varphi \subseteq L^X$ is called an L-closure structure and the pair (X, φ) is called an L-closure space, if it satisfies

(LCS1) $\underline{\perp}, \underline{\top} \in \varphi$;

(LCS2) $\bigwedge \varphi_1 \in \varphi$ for any $\varphi_1 \subseteq \varphi$.

An L-closure structure is an L-cotopology (resp. L-convex structure) iff it satisfies (LT3) (resp. (LC3)). The L-closure operator of an L-closure space (X, φ) is defined by $cl_\varphi(A) = \bigwedge\{B \in\in \varphi : A \leq B\}$ for all $A \in L^X$. Then cl_φ satisfies

(LCL1) $cl(\underline{\perp}) = \underline{\perp}$;

(LCL2) $A \leq cl(A)$ for all $A \in L^X$;

(LCL3) if $A \leq B$, then $cl(A) \leq cl(B)$;

(LCL4) $cl(cl(A)) = cl(A)$ for all $A \in L^X$.

Conversely, if an operator $cl : L^X \to L^X$ satisfies (LCL1)–(LCL4), then the set $\varphi_{cl} = \{A \in L^X : cl(A) = A\}$ is an L-closure structure satisfying $cl_{\varphi_{cl}} = cl$.

Let (X, φ_X) and (Y, φ_Y) be L-closure spaces. A mapping $f : X \to Y$ is called an L-closure structure preserving mapping if $f_L^{\leftarrow}(A) \in \varphi_X$ for any $A \in \varphi_Y$. The category of L-closure spaces and L-closure structure preserving mappings is denoted by L-**CSS**.

Definition 2.5. [33] A binary relation $\preccurlyeq$ on L^X is called an L-topological enclosed relation and the pair $(X, \preccurlyeq)$ is called an L-topological enclosed relation space, if $\preccurlyeq$ satisfies

(LTER1) $\underline{\perp} \preccurlyeq \underline{\perp}$;

(LTER2) $A \preccurlyeq B$ implies $A \leq B$;

(LTER3) $A \preccurlyeq \bigwedge_{i\in I} B_i$ iff $A \preccurlyeq B_i$ for all $i \in I$;

(LTER4) $A \preccurlyeq B$ implies $C \in L^X$ with $A \preccurlyeq C \preccurlyeq B$;

(LTER5) $A \vee B \preccurlyeq C$ iff $A \preccurlyeq C$ and $B \preccurlyeq C$.

In an L-topological enclosed relation space $(X, \preccurlyeq)$, $A \leq B \preccurlyeq C \leq D$ implies that $A \preccurlyeq D$. Also, $A \preccurlyeq D$ implies some $C \in L^X$ such that $A \leq C \preccurlyeq C \leq D$ [33].

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-topological enclosed relation spaces. A mapping $f : X \to Y$ is called an L-topological enclosed relation dual-preserving mapping, if $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B)$ for all $A \preccurlyeq_Y B$ [33]. The category of L-topological enclosed relation spaces and L-topological enclosed relation dual-preserving mappings is denoted by L-**TERS**.

Theorem 2.6. [33] *(1) For an L-topological enclosed relation space $(X, \preccurlyeq)$, the operator $cl_\preccurlyeq : L^X \to L^X$, defined by*

$$cl_\preccurlyeq(A) = \bigwedge\{B \in L^X : A \preccurlyeq B\},$$

is a closure operator of some L-cotopology, denoted by $\mathcal{T}_{\preccurlyeq}$.

(2) For an L-cotopological space $(X, \mathcal{T})$, the binary operator $\preccurlyeq_{\mathcal{T}}$, defined by

$$A \preccurlyeq_{\mathcal{T}} B \text{ iff } cl_{\mathcal{T}}(A) \le B.$$

is an L-topological enclosed relation.

*(3) L-**CTS** is isomorphic to L-**TERS**.*

Definition 2.7. A binary relation $\leqslant$ on L^X is called an L-convex enclosed relation and the pair $(X, \leqslant)$ is called an L-convex enclosed relation space, if $\leqslant$ satisfies

(LCER1) $\perp \leqslant \perp$;

(LCER2) $A \leqslant B$ implies $A \le B$;

(LCER3) $A \leqslant \bigwedge_{i \in I} B_i$ iff $A \leqslant B_i$ for all $i \in I$;

(LCER4) $\bigvee_{i \in I}^{dir} A_i \leqslant C$ iff $A_i \leqslant C$ for all $i \in I$;

(LCER5) $A \leqslant B$ implies the existence of $C \in L^X$ such that $A \leqslant C \leqslant B$.

Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be L-convex enclosed relation spaces. A mapping $f : X \to Y$ is called an L-convex enclosed relation dual-preserving mapping, if $f_L^{\leftarrow}(A) \leqslant_X f_L^{\leftarrow}(B)$ for all $A \leqslant_Y B$. The category of L-convex enclosed relation spaces and L-convex enclosed relation dual-preserving mappings is denoted by L-**CERS**.

Similar to Theorem 2.6, the following result is easy to check.

Theorem 2.8. *(1) For an L-convex enclosed relation space $(X, \leqslant)$, the operator $co_{\leqslant} : L^X \to L^X$, defined by*

$$co_{\leqslant}(A) = \bigwedge\{B \in L^X : A \leqslant B\},$$

is an L-hull operator of some L-convex structure, denoted by $\mathcal{C}_{\leqslant}$.

(2) For an L-convex space $(X, \mathcal{C})$, the binary operator $\leqslant_{\mathcal{C}}$, defined by

$$A \leqslant_{\mathcal{C}} B \text{ iff } co_{\mathcal{C}}(A) \le B.$$

is an L-convex enclosed relation.

*(3) The category L-**CS** is isomorphic to the category L-**CERS**.*

Remark 2.9. (1) L-cotopologies and L-convex structures have a uniform origination, i.e., L-closure structure. L-topological enclosed relations and L-convex enclosed relations also have a uniform origination defined by:

A binary operation $\sqsubset$ on X is called an L-enclosed relation and the pair $(X, \sqsubset)$ is called an L-enclosed relation space, if $\sqsubset$ satisfies the following conditions

(LER1) $\perp \sqsubset \perp$;

(LER2) $A \sqsubset B$ implies $A \le B$;

(LER3) $A \sqsubset \bigwedge_{i \in I} B_i$ iff $A \sqsubset B_i$ for all $i \in I$;

(LER4) $A \sqsubset B$ implies the existence of $C \in L^X$ such that $A \sqsubset C \sqsubset B$;

(LER5) $C \le A \sqsubset B$ implies $C \sqsubset B$.

Clearly, a binary relation satisfying (LER1)–(LER4) is an L-topological (resp. L-convex) enclosed relation if it satisfies (LTER5) (resp. (LCER5)).

For two L-enclosed relations $\sqsubset_1, \sqsubset_2$ on X, we say $\sqsubset_1$ is coarse than $\sqsubset_2$, denoted by $\sqsubset_1 \le \sqsubset_2$, if $A \sqsubset_1 B$ implies $A \sqsubset_2 B$ for all $A, B \in L^X$.

3 *L*-convex bases

In this section, we introduce L-convex bases and discuss its relations with L-convex spaces.

Definition 3.1. Let $(X, \mathcal{C})$ be an L-convex space. A subset $\mathcal{B} \subseteq \mathcal{C}$ is called an L-convex base of $\mathcal{C}$, if for any $A \in \mathcal{C}$, there is $\mathcal{B}_1 \overset{dir}{\subseteq} \mathcal{B}$ such that $A = \bigvee \mathcal{B}_1$.

Proposition 3.2. *Let $(X, \mathcal{C})$ be an L-convex space and let $\mathcal{B} \subseteq \mathcal{C}$. If $co(F) \in \mathcal{B}$ for any $F \in \mathfrak{F}(L^X)$, then $\mathcal{B}$ is an L-convex base of $\mathcal{C}$.*

Proof. Let $A \in \mathcal{C}$. Since $\{co(F) : F \in \mathfrak{F}(A)\} \overset{dir}{\subseteq} \mathcal{B}$, we have $A = co(A) = co(\bigvee^{dir}_{F\in\mathfrak{F}(A)} F) = \bigvee^{dir}_{F\in\mathfrak{F}(A)} co(F)$ by (LCO5). Thus $\mathcal{B}$ is an L-convex base of $\mathcal{C}$. □

Remark 3.3. A subset $\mathcal{B} \subseteq \mathcal{C}$ of a convex space $(X, \mathcal{C})$ is a base iff it contains all polytopes (i.e., $co(F) \in \mathcal{B}$ for all $F \in 2^X_{fin}$) [2]. However, if $L \neq \{\bot, \top\}$, the inverse result in Proposition 3.2 fails. For example, let $X = \{x\}$ and $L = [0, 1]$. Define $\mathcal{B} = [0, \frac{1}{3})^X \cup [\frac{1}{2}, 1]^X$ and $\mathcal{C} = \mathcal{B} \cup \{0, \frac{1}{3}\}^X$, where $\varphi^X = \{z_r \in L^X : z \in X, r \in \varphi\}$ for any $\varphi \subseteq L$. Then $\mathcal{B}$ is an L-convex base of $\mathcal{C}$. But $x_{\frac{1}{3}} \in \mathfrak{F}(L^X)$ and $co(x_{\frac{1}{3}}) = x_{\frac{1}{3}} \notin \mathcal{B}$. Thus Proposition 3.2 just gives a sufficient condition of L-convex bases. To obtain a necessary and sufficient condition for L-convex bases, we present the following results.

Theorem 3.4. *If $(X, \mathcal{C})$ is an L-convex space and $\mathcal{B} \subseteq \mathcal{C}$ is an L-convex base, then*

(LCB1) $\top = \bigvee \mathcal{B}_1$ for some $\mathcal{B}_1 \overset{dir}{\subseteq} \mathcal{B}$;

(LCB2) for $\{B_i\}_{i\in I} \subseteq \mathcal{B}$, there is $\mathcal{B}_1 \overset{dir}{\subseteq} \mathcal{B}$ such that $\bigwedge_{i\in I} B_i = \bigvee \mathcal{B}_1$;

(LCB3) if $\{A_i\}_{i\in I} \overset{dir}{\subseteq} L^X$, and $\{B_{ij}\}_{j\in J_i} \overset{dir}{\subseteq} \mathcal{B}$ such that $A_i = \bigvee_{j\in J_i} B_{ij}$ for each $i \in I$, then there is $\{B_k\}_{k\in K} \overset{dir}{\subseteq} \mathcal{B}$ such that $\bigvee_{i\in I} A_i = \bigvee_{k\in K} B_k$.

Proof. (LCB1): Since $\top \in \mathcal{C}$, there is $\mathcal{B}_1 \overset{dir}{\subseteq} \mathcal{B}$ with $\top = \bigvee \mathcal{B}_1$.

(LCB2): Since $\{B_i\}_{i\in I} \subseteq \mathcal{C}$, we have $\bigwedge_{i\in I} B_i \in \mathcal{C}$ by (LC2). Thus, by Definition 3.1, there is $\mathcal{B}_1 \overset{dir}{\subseteq} \mathcal{B}$ such that $\bigwedge_{i\in I} B_i = \bigvee \mathcal{B}_1$.

(LCB3): Let $\{A_i\}_{i\in I} \overset{dir}{\subseteq} L^X$, and let $\{B_{ij}\}_{j\in J_i} \overset{dir}{\subseteq} \mathcal{B}$ with $A_i = \bigvee_{j\in J_i} B_{ij}$ for each $i \in I$. For each $i \in I$, $A_i = \bigvee_{j\in J_i} B_{ij} \in \mathcal{C}$ by (LC3) and $\mathcal{B} \subseteq \mathcal{C}$. Thus $\bigvee_{i\in I} A_i \in \mathcal{C}$ since $\{A_i\}_{i\in I} \overset{dir}{\subseteq} \mathcal{C}$. Hence there is $\mathcal{B}_1 \overset{dir}{\subseteq} \mathcal{B}$ such that $\bigvee_{i\in I} A_i = \bigvee \mathcal{B}_1$. □

Theorem 3.5. *Let $\mathcal{B} \subseteq L^X$ be a set satisfying (LCA1)–(LCA3) in Theorem 3.4. Then there is a unique L-concave structure with $\mathcal{B}$ as an L-convex base.*

Proof. We prove that $\mathcal{C}_{\mathcal{B}}$ is an L-convex structure, where

$$\mathcal{C}_{\mathcal{B}} = \{A \in L^X : \exists \mathcal{B}_1 \overset{dir}{\subseteq} \mathcal{B}, A = \bigvee \mathcal{B}_1\}.$$

(LC1): By (LCB1), we have $\top \in \mathcal{C}_{\mathcal{B}}$. In addition, let $\emptyset \subseteq \mathcal{B}$, then $\emptyset$ is up-directed and $\bot = \bigvee \emptyset \in \mathcal{C}_{\mathcal{B}}$.

(LC2): If $\{A_i\}_{i\in I} \in \mathcal{C}_{\mathcal{B}}$ and $\mathcal{B}_i = \{B_{ij}\}_{j\in J_i} \overset{dir}{\subseteq} \mathcal{B}$ with $A_i = \bigvee_{j\in J_i} B_{ij}$, then

$$\bigwedge_{i\in I} A_i = \bigwedge_{i\in I} \bigvee_{j\in J_i} B_{ij} = \bigvee_{f\in \Pi_{i\in I} J_i} \bigwedge_{i\in I} B_{if(i)}.$$

By (LCB2), for each $f \in \Pi_{i\in I} J_i$, there is $\{B_{ik}\}_{k\in K_i} \overset{dir}{\subseteq} \mathcal{B}$ such that $\bigwedge_{i\in I} B_{if(i)} = \bigvee_{k\in K_i} B_{ik}$. Next, we prove that $\{\bigwedge_{i\in I} B_{if(i)} : f \in \Pi_{i\in I} J_i\} \overset{dir}{\subseteq} L^X$.

Let $f, g \in \Pi_{i\in I} J_i$. Then $f(i), g(i) \in J_i$ and $B_{if(i)}, B_{ig(i)} \in \mathcal{B}_i$ for each $i \in I$. Since $\mathcal{B}_i$ is up-directed, there is $B_{ij_i} \in \mathcal{B}_i$ such that $B_{if(i)}, B_{ig(i)} \leq B_{ij_i}$. Define $h : I \to \Pi_{i\in I} J_i$ by: $h(i) = B_{ij_i}$ for all $i \in I$. Thus $h \in \Pi_{i\in I} J_i$ and

$$\bigvee_{i\in I} B_{if(i)}, \bigvee_{i,g(i)} B_{ig(i)} \leq \bigvee_{i,h(i)} B_{ih(i)}.$$

Hence the aimed set is up-directed. By (LCB3), there is $\mathcal{D} \overset{dir}{\subseteq} \mathcal{B}$ such that

$$\bigwedge_{i\in I} A_i = \bigvee_{f\in \Pi_{i\in I} J_i} \bigwedge_{i\in I} B_{if(i)} = \bigvee \mathcal{D} \in \mathcal{C}_{\mathcal{B}}.$$

(LC3): Let $\{A_i\}_{i\in I} \overset{dir}{\subseteq} \mathcal{C}_{\mathcal{B}}$. For each $i \in I$, there is $\{B_{ij}\}_{j\in J_i} \overset{dir}{\subseteq} \mathcal{B}$ such that $A_i = \bigvee_{j\in J_i} B_{ij}$. Thus there is $\{B_k\}_{k\in K} \overset{dir}{\subseteq} \mathcal{B}$ such that $\bigvee_{i\in I} A_i = \bigvee_{k\in K} B_k$ by (LCB3). Therefore $\bigvee_{i\in I} A_i \in \mathcal{C}_{\mathcal{B}}$.

Finally, since $\mathcal{B} \subseteq \mathcal{C}_{\mathcal{B}}$ is an L-convex base, we know that $\mathcal{C}_{\mathcal{B}}$ is unique. □

By Theorems 3.4 and 3.5, we see that(LCB1)–(LCB3) is a necessity and sufficient condition for L-convex bases. Thus we present the axiomatic definition of L-convex bases as follows.

Definition 3.6. A subset $\mathcal{B} \subseteq L^X$ is called an L-convex base and the pair $(X, \mathcal{B})$ is called an L-convex base space, if $\mathcal{B}$ satisfies (LCB1)–(LCB3).

Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be L-convex bases. A mapping $f : X \to Y$ is called an L-convex base preserving mapping if $f_L^{\leftarrow}(B) \in \mathcal{B}_X$ for all $B \in \mathcal{B}_Y$. The category of L-convex base spaces and L-convex base preserving mappings is denoted by L-**CBS**. Next, we discuss relations between L-**CS** and L-**CBS**.

Theorem 3.7. *An L-convex structure is an L-convex base of itself.*

Theorem 3.8. *Let $(X, \mathcal{B}_X)$ and $(X, \mathcal{B}_Y)$ be L-convex base spaces. If $f : X \to Y$ is an L-convex base preserving mapping, then $f : (X, \mathcal{C}_{\mathcal{B}_X}) \to (Y, \mathcal{C}_{\mathcal{B}_Y})$ is an L-convex structure preserving mapping.*

Proof. If $A_Y \in \mathcal{C}_{\mathcal{B}_Y}$, then there is $\{B_i\}_{i\in I} \overset{dir}{\subseteq} \mathcal{B}_Y$ such that $A_Y = \bigvee_{i\in I} B_i$. Thus $\{f_L^{\leftarrow}(B_i)\}_{i\in I} \overset{dir}{\subseteq} \mathcal{B}_X$ and $f_L^{\leftarrow}(A_Y) = \bigvee_{i\in I} f_L^{\leftarrow}(B_i) \in \mathcal{C}_{\mathcal{B}_X}$. □

By Theorems 3.7, the category L-**CS** is a subcategory of L-**CBS**. Thus we can define a factor $\mathbb{E}_b$:L-**CS**→ L-**CBS** by:

$$\mathbb{E}_b(X, \mathcal{C}) = (X, \mathcal{C}), \quad \mathbb{E}_b(f) = f.$$

By Theorems 3.5 and 3.8, we can define a factor $\mathbb{F}$:L-**CBS**→L-**CS** by:

$$\mathbb{F}(X, \mathcal{B}) = (X, \mathcal{C}_{\mathcal{B}}), \quad \mathbb{F}(f) = f.$$

Theorem 3.9. *$(\mathbb{E}_b, \mathbb{F})$ is a Galois's connection and $\mathbb{G}$ is a left inverse of $\mathbb{E}_b$.*

Proof. By Theorems 3.5, 3.7 and 3.8, $\mathbb{I}_{L-\mathbf{CS}} = \mathbb{F} \circ \mathbb{E}_b$ and $\mathbb{E}_b \circ \mathbb{F} \leq \mathbb{I}_{L-\mathbf{CBS}}$, where $\mathbb{I}_{L-\mathbf{CS}}$ and $\mathbb{I}_{L-\mathbf{CBS}}$ are identity factors of L-**CS** and L-**CBS**, respectively. □

Corollary 3.10. *L-**CS** can be embedded as a coreflective subcategory of L-**CBS**.*

An L-closure structure can generate an L-convex structure [28]. Actually, an L-closure structure is an L-convex base showed as follows.

Theorem 3.11. *An L-closure structure is an L-convex base.*

Proof. Let (X, φ) be an L-closure space. We verify that φ satisfies (LCB1)–(LCB3).

(LCB1): Since $\top \in \varphi$, the result is clear.

(LCB2): If $\{B_i\}_{i\in I} \in \varphi$ and $\mathcal{B}_i = \{B_{ij}\}_{j\in J_i} \overset{dir}{\subseteq} \varphi$ with $B_i = \bigvee_{j\in J_i} B_{ij}$, then

$$\bigwedge_{i\in I} B_i = \bigwedge_{i\in I} \bigvee_{j\in J_i} B_{ij} = \bigvee_{f\in \Pi_{i\in I} J_i} \bigwedge_{i\in I} B_{if(i)}.$$

In addition, it is direct to show that the set $\mathcal{D} = \{\bigwedge_{i\in I} B_{if(i)} : f \in \Pi_{i\in I} J_i\} \subseteq \varphi$ is up-directed. Hence $\bigwedge_{i\in I} B_i = \bigvee \mathcal{D}$ showing that (LCB2) hold for φ.

(LCB3): Let $\{A_i\}_{i\in I} \overset{dir}{\subseteq} L^X$ and let $\{B_{ij}\}_{j\in J_i} \overset{dir}{\subseteq} \varphi$ such that $A_i = \bigvee_{j\in J_i} B_{ij}$ for each $i \in I$. Further, take $A = \bigvee_{i\in I} A_i$, $\varphi_\star = \{B_{ij} : i \in I, j \in J_i\}$ and $\varphi_F = \{B \in \varphi_\star : F \leq B\}$ for each $F \in \mathfrak{F}(A)$.

Now, we prove that φ_F is nonempty for each $F \in \mathfrak{F}(A)$. Since

$$F \in \mathfrak{F}(A) = \mathfrak{F}(\bigvee_{i\in I}\bigvee_{j\in J_i} B_{ij}) = \bigcup_{i\in I}\bigcup_{j\in J_i} \mathfrak{F}(B_{ij}),$$

there are $i_F \in I$ and $j_F \in J_{i_F}$ such that $F \in \mathfrak{F}(B_{i_F j_F})$. Thus $B_{i_F j_F} \in \varphi_F$.

Further, we prove that the set $\{\bigwedge \varphi_F : F \in \mathfrak{F}(A)\}$ is up-directed.

If $F, G \in \mathfrak{F}(A)$, then $F \vee G \in \mathfrak{F}(A)$ and $\varphi_{F\vee G} \subseteq \varphi_F \cap \varphi_G$. Thus $\bigwedge \varphi_F, \bigwedge \varphi_G \leq \bigwedge \varphi_{F\vee G}$. Hence $\{\bigwedge \varphi_F : F \in \mathfrak{F}(A)\}$ is up-directed.

Finally, since $F \leq B_{i_F j_F} \leq A_{i_F} \leq A$, we have $F \leq \bigwedge \varphi_F \leq A$ and

$$A = \bigvee_{F\in\mathfrak{F}(A)} F \leq \bigvee_{F\in\mathfrak{F}(A)} \bigwedge \varphi_F \leq A.$$

Therefore $A = \bigvee_{F\in\mathfrak{F}(A)} \bigwedge \varphi_F$ showing that (LCB3) holds for φ. □

Remark 3.12. The inverse result of Theorem 3.11 fails. An L-convex base may not be an L-closure structure. To shows this, let $X = \{x\}$ and let $L = [0, 1]$. Define $\mathcal{B} = [0, \frac{1}{3})^X \cup (\frac{1}{3}, 1)^X$. Then $\mathcal{B}$ is an L-convex base. Let $\mathcal{B}_1 = (\frac{1}{3}, 1)^X \subseteq \mathcal{B}$. But $\underline{1} \notin \mathcal{B}$ and $\bigwedge \mathcal{B}_1 \notin \mathcal{B}$. Thus both (LC1) and (LC2) fail for $\mathcal{B}$.

Theorem 3.13. *L-**CS** is a bicoreflective subcategory of L-**CSS**, where L-**CSS** is the category of L-closure spaces and L-closure structure preserving mappings.*

Proof. To verify this, let (X, φ) be an L-closure space. By Theorems 3.5 and 3.11, $\mathcal{C}_\varphi$ is an L-convex structure. Thus we only need to prove that $id_X : (X, \mathcal{C}_\varphi) \to (X, \varphi)$ is a bicoreflector. It sufficient to show that the following statements hold.

(1) $id_X : (X, \mathcal{C}_\varphi) \to (X, \varphi)$ is an L-closure structure preserving mapping.

(2) for any L-convex space $(Y, \mathcal{C}_Y)$, if $f : (Y, \mathcal{C}_Y) \to (X, \varphi)$ is an L-closure structure preserving mapping, then $f : (Y, \mathcal{C}_Y) \to (X, \mathcal{C}_\varphi)$ is an L-convex structure preserving mapping.

Since $\varphi \subseteq \mathcal{C}_\varphi$, (1) is clear. For (2), if $A \in \mathcal{C}_\varphi$, then there is an up-directed set $\{B_i\}_{i\in I} \subseteq \varphi$ such that $A = \bigvee_{i\in I} B_i$. Since $f : (Y, \mathcal{C}_Y) \to (X, \varphi)$ is an L-closure structure preserving mapping, $f_L^{\leftarrow}(A) = \bigvee_{i\in I} f_L^{\leftarrow}(B_i) \in \mathcal{C}_Y$. Thus (2) holds. □

4 *L*-convex subbases

Definition 4.1. Let $(X, \mathcal{C})$ be an L-convex space. A subset $\mathcal{D} \subseteq \mathcal{C}$ is called an L-convex subbase of $\mathcal{C}$ if $\mathcal{B}_\mathcal{D} = \{\bigwedge \mathcal{F} : \mathcal{F} \subseteq \mathcal{D}\}$ is an L-convex base of $\mathcal{C}$.

Proposition 4.2. *Let $(X, \mathcal{C})$ be an L-convex space and let $\mathcal{D} \subseteq \mathcal{C}$.*

(1) If there is $\mathcal{D}_F \subseteq \mathcal{D}$ such that $co(F) = \bigwedge \mathcal{D}_F$ for any $F \in \mathfrak{F}(L^X)$, then $\mathcal{D}$ is an L-convex subbase of $\mathcal{C}$.

(2) An L-convex base is an L-convex subbase.

(3) $\mathcal{C}$ is the coarsest L-convex structure containing $\mathcal{D}$ if $\mathcal{D}$ is an L-convex subbase.

Proof. (1) and (2) directly follow from Proposition 3.2 and Definition 4.1.

(3): Let $\mathcal{C}_1$ be an L-convex structure with $\mathcal{D} \subseteq \mathcal{C}_1$. If $A \in \mathcal{C}$, then there is $\{B_i\}_{i\in I} \overset{dir}{\subseteq} \mathcal{B}_\mathcal{D}$ such that $A = \bigvee_{i\in I} B_i$. Since there is $\mathcal{D}_i \subseteq \mathcal{D}$ such that $B_i = \bigwedge \mathcal{D}_i$ for each $i \in I$, we have $B_i \in \mathcal{C}_1$ by (LC2), and $A \in \mathcal{C}_1$ by (LC3). Thus $\mathcal{C} \subseteq \mathcal{C}_1$. □

Remark 4.3. (1) of Proposition 4.2 just gives a sufficient condition of an L-convex subbase. In the example of Remark 3.3, $\mathcal{B}$ is an L-concave subbase of $\mathcal{C}$. However, $x_{\frac{2}{3}} \in \mathfrak{F}(L^X)$ and $co(x_{\frac{2}{3}}) = x_{\frac{2}{3}} \neq \bigwedge \mathcal{D}$ for any $\mathcal{D} \subseteq \mathcal{B}$. To obtain a necessary and sufficient condition, we present the following results.

Theorem 4.4. *If $(X, \mathcal{C})$ is an L-convex space with an L-convex subbase $\mathcal{D}$, then*

(LCSB) there is an up-directed set $\{B_i\}_{i\in I} \subseteq L^X$ such that $\top = \bigvee_{i\in I} B_i$, and there is subset $\{D_{ij}\}_{j\in J_i} \subseteq \mathcal{D}$ such that $B_i = \bigwedge_{j\in J_i} D_{ij}$ for each $i \in I$.

Proof. The result directly follows from (LCB1) of the L-convex base $\mathcal{B}_{\mathcal{D}}$. □

Theorem 4.5. *Let $\mathcal{D} \subseteq L^X$ be a set satisfying (LCSB), then there is a unique L-convex structure $\mathcal{C}_{\mathcal{D}}$ with $\mathcal{D}$ as an L-convex subbase.*

Proof. Let $\mathcal{B}_{\mathcal{D}} = \{\bigwedge \mathcal{F} : \mathcal{F} \subseteq \mathcal{D}\}$. We check that $\mathcal{B}_{\mathcal{D}}$ is an L-convex base.

(LCB1): It directly follows from (LCSB).

(LCB2): Let $\{B_i\}_{i\in I} \subseteq \mathcal{B}_{\mathcal{D}}$ and let $\{B_{ij}\}_{j\in J_i} \subseteq \mathcal{D}$ with $B_i = \bigwedge_{j\in J_i} B_{ij}$. Then $\bigwedge_{i\in I} B_i = \bigwedge_{i\in I}\bigwedge_{j\in J_i} B_{ij} \in \mathcal{B}_{\mathcal{D}}$. Since $\{\bigwedge_{i\in I} B_i\} \overset{dir}{\subseteq} \mathcal{B}_{\mathcal{D}}$, (LCB2) holds trivially.

(LCB3): Let $\{A_i\}_{i\in I} \subseteq L^X$ be up-directed, and let $\{B_{ij}\}_{j\in J_i} \overset{dir}{\subseteq} \mathcal{B}_{\mathcal{D}}$ such that $A_i = \bigvee_{j\in J_i} B_{ij}$ for each $i \in I$. Thus, for each $i \in I$ and each $j \in J_i$, there is a subfamily $\{D_{ijk}\}_{k\in K_{J_i}} \subseteq \mathcal{D}$ with $B_{ij} = \bigwedge_{k\in K_{J_i}} D_{ijk}$. Take $A = \bigvee_{i\in I} A_i$. We have $A = \bigvee_{i\in I} A_i = \bigvee_{i\in I}\bigvee_{j\in J_i} B_{ij}$. In addition, for each $F \in \mathfrak{F}(A)$, we have

$$F \in \mathfrak{F}(A) = \mathfrak{F}\Big(\bigvee_{i\in I}\bigvee_{j\in J_i} B_{ij}\Big) = \bigcup_{i\in I}\bigcup_{j\in J_j} \mathfrak{F}(B_{ij}).$$

Thus $F \in \mathfrak{F}(B_{ij})$ for some $i \in I$ and $j \in J_i$. Let $B_F = \bigwedge\{B_{ij} : F \in \mathfrak{F}(B_{ij})\}$. Then

$$B_F = \bigwedge_{F\in\mathfrak{F}(B_{ij})} B_{ij} = \bigwedge_{F\in\mathfrak{F}(B_{ij})}\bigwedge_{k\in K_{J_i}} D_{ijk} \in \mathcal{B}_{\mathcal{D}}.$$

Further, since $\{F : F \in \mathfrak{F}(A)\}$ is up-directed, $\{B_F : F \in \mathfrak{F}(A)\}$ is up-directed. So

$$F \le B_F = \bigwedge\{B_{ij} : F \in \mathfrak{F}(B_{ij})\} \le A.$$

This implies that

$$A = \bigvee_{F\in\mathfrak{F}(A)} F \le \bigvee_{F\in\mathfrak{F}(A)} B_F \le A.$$

Hence $A = \bigvee_{F\in\mathfrak{F}(A)} B_F$. Therefore (LCB3) holds for $\mathcal{B}_{\mathcal{D}}$.

By Theorem 3.5, there is a unique L-convex structure with $\mathcal{B}_{\mathcal{D}}$ as an L-convex base. Thus it is the unique L-convex structure with $\mathcal{D}$ as an L-convex subbase. □

By Theorems 4.4 and 4.5, (LCSB) is a necessity and sufficient condition for L-convex subbases. Thus we present the following axiomatic definition.

Definition 4.6. A set $\mathcal{D} \subseteq L^X$ is called an L-convex subbase and the pair $(X, \mathcal{D})$ is called an L-convex subbase space provided that $\mathcal{D}$ satisfies (LCSB).

Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be L-convex subbase spaces. A mapping $f : X \to Y$ is called an L-convex subbase preserving mapping if $f_L^{\leftarrow}(D) \in \mathcal{D}_X$ for all $D \in \mathcal{D}_Y$.

The category of L-convex subbase spaces and L-convex subbase preserving mappings is denoted by L-**CSBS**. Next, we study relations between L-**CS** and L-**CSBS**.

Theorem 4.7. *An L-convex structure is an L-convex subbase of itself.*

Theorem 4.8. *Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be L-convex subbase spaces. If $f : X \to Y$ is an L-convex subbase preserving mapping, then $f : (X, \mathcal{C}_{\mathcal{D}_X}) \to (Y, \mathcal{C}_{\mathcal{D}_Y})$ is an L-convex structure preserving mapping.*

Proof. If $A \in \mathcal{C}_{\mathcal{D}_Y}$, then there is $\{B_i\}_{i\in I} \overset{dir}{\subseteq} L^Y$ with $A = \bigvee_{i\in I} B_i$ where there is $\{D_{ij}\}_{j\in J_i} \subseteq \mathcal{D}_Y$ such that $B_i = \bigwedge_{j\in J_i} D_{ij}$ for each $i \in I$. Thus

$$f_L^{\leftarrow}(B_i) = f_L^{\leftarrow}(\bigwedge_{j\in J_i} D_{ij}) = \bigwedge_{j\in J_i} f_L^{\leftarrow}(D_{ij})$$

Since $\{B_i\}_{i\in I}$ is up-directed and $f : X \to Y$ is an L-concave subbase preserving mapping, $\{f_L^{\leftarrow}(B_i)\}_{i\in I} \subseteq \mathcal{D}_X$ is up-directed. Hence $\{f_L^{\leftarrow}(B_i)\}_{i\in I} \in \mathcal{B}_{\mathcal{D}_X}$ and

$$f_L^{\leftarrow}(A) = f_L^{\leftarrow}(\bigvee_{i\in I}\bigwedge_{j\in J_i} D_{ij}) = \bigvee_{i\in I} f_L^{\leftarrow}(B_i) \in \mathcal{C}_{\mathcal{D}_X}.$$

Therefore $f : (X, \mathcal{C}_{\mathcal{D}_X}) \to (Y, \mathcal{C}_{\mathcal{D}_Y})$ is an L-convex preserving mapping. □

By Theorem 4.7, the category L-**CS** is a subcategory of the category L-**CSBS**. Thus we can define a factor $\mathbb{E}_s$:L-**CS**$\to$ L-**CSBS** by:

$$\mathbb{E}_s(X, \mathcal{C}) = (X, \mathcal{C}), \quad \mathbb{E}_s(f) = f.$$

By Theorems 4.5 and 4.8, we can define a factor $\mathbb{G}$:L-**CSBS**$\to$$L$-**CS** by:

$$\mathbb{G}(X, \mathcal{D}) = (X, \mathcal{C}_{\mathcal{D}}), \quad \mathbb{G}(f) = f.$$

Theorem 4.9. *$(\mathbb{E}_s, \mathbb{G})$ is a Galois's connection and $\mathbb{G}$ is a left inverse of $\mathbb{E}_s$.*

Proof. By Theorems 4.7, 4.5 and 4.8, $\mathbb{I}_{L-\mathbf{CS}} = \mathbb{G}\circ\mathbb{E}_s$ and $\mathbb{E}_s \circ \mathbb{G} \leq \mathbb{I}_{L-\mathbf{CSBS}}$, where $\mathbb{I}_{L-\mathbf{CS}}$ and $\mathbb{I}_{L-\mathbf{CSBS}}$ are identities factors in L-**CS** and L-**CSBS**, respectively. □

Corollary 4.10. *L-**CS** can be embedded as a coreflective subcategory of L-**CSBS**.*

From the proofs of Theorems 4.5 and 4.8, we have the following conclusion.

Corollary 4.11. *If $(X, \mathcal{D})$ is an L-convex subbase space, then the set $\mathcal{B}_{\mathcal{D}} = \{\bigwedge \mathcal{F} : \mathcal{F} \subseteq \mathcal{D}\}$ is an L-convex base. In addition, if $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ are L-convex subbase spaces and $f : X \to Y$ is an L-convex subbase preserving mapping, then $f : (X, \mathcal{B}_{\mathcal{D}_X}) \to (Y, \mathcal{B}_{\mathcal{D}_Y})$ is an L-convex base preserving mapping.*

Now, we consider relations between L-**CBS** and L-**CSBS**.

By (2) of Proposition 4.2, the category L-**CBS** is a subcategory of the category L-**CSBS**. Thus we can define a factor $\mathbb{E}$:L-**CBS** $\to$$L$-**CSBS** by:

$$\mathbb{E}((X, \mathcal{B})) = (X, \mathcal{B}), \quad \mathbb{E}(f) = f.$$

Conversely, by Corollary 4.11, we can define a factor $\mathbb{H}$: L-**CSBS**$\to$$L$-**CBS** by:

$$\mathbb{H}((X, \mathcal{B}_{\mathcal{D}})) = (X, \mathcal{D}), \quad \mathbb{H}(f) = f.$$

Theorem 4.12. *$(\mathbb{E}, \mathbb{H})$ is a Galois's connection and $\mathbb{H}$ is a left inverse of $\mathbb{E}$.*

Proof. By Proposition 4.2 and Corollary 4.11, $\mathbb{I}_{L-\mathbf{CBS}} = \mathbb{H} \circ \mathbb{E}$ and $\mathbb{E} \circ \mathbb{H} \leq \mathbb{I}_{L-\mathbf{CSBS}}$. □

Corollary 4.13. *L-**CBS** can be embedded as a coreflective subcategory of L-**CSBS**.*

5 L-topological-convex spaces

In [2], if X is a set equipped with a convex structure $\mathcal{C}$ and a cotopology $\mathcal{T}$, then the triple $(X, \mathcal{T}, \mathcal{C})$ is called a topological-convex space provided that $\mathcal{T}$ is compatible with $\mathcal{C}$, i.e., all polytopes are closed ($co_{\mathcal{C}}(F) \in \mathcal{T}$

for any $F \in 2^X_{fin}$). In this section, we extend this concept into L-fuzzy settings and obtain some of its characterizations. Before this, we give a brief observation of this concept.

Remark 5.1. Let X be a set equipped with a cotopology $\mathcal{T}$ and a convex structure $\mathcal{C}$. Then $\mathcal{T} \cap \mathcal{C}$ is a closure structure whose closure operator is denoted by $cl_{\mathcal{T}\cap\mathcal{C}}$.

(1) From definition of a topological-convex described as above, we can conclude that a set X, equipped with a cotopology $\mathcal{T}$ and a convex structure $\mathcal{C}$, is a topological-convex space iff $co_{\mathcal{C}}(F) = cl_{\mathcal{T}\cap\mathcal{C}}(F)$ for all $F \in 2^X_{fin}$. Indeed, if $(X, \mathcal{T}, \mathcal{C})$ is a topological-convex space, then $co_{\mathcal{C}}(F) = cl_{\mathcal{T}}(co_{\mathcal{C}}(F))$ for any $F \in 2^X_{fin}$. Thus $co_{\mathcal{C}}(F) = cl_{\mathcal{T}}(co_{\mathcal{C}}(F)) \in \mathcal{T} \cap \mathcal{C}$ which shows

$$co_{\mathcal{C}}(F) \leq cl_{\mathcal{T}\cap\mathcal{C}}(F) \leq cl_{\mathcal{T}}(co_{\mathcal{C}}(F)) = co_{\mathcal{C}}(F).$$

Conversely, if $co_{\mathcal{C}}(F) = cl_{\mathcal{T}\cap\mathcal{C}}(F)$ for all $F \in 2^X_{fin}$, then it is clear that $co_{\mathcal{C}}(F) \in \mathcal{T}$ for all $F \in 2^X_{fin}$. That is, $\mathcal{T}$ is compatible with $\mathcal{C}$.

(2) In a convex space $(X, \mathcal{C})$, a subset $\mathcal{B} \subseteq \mathcal{C}$ is called a base if for each $A \in \mathcal{C}$, there is an up-directed subset $\mathcal{B}_1 \subseteq \mathcal{B}$ such that $A = \bigcup \mathcal{B}$. As described in [2], a subset of a convex structure is a base iff it contains all polytopes. Thus, $\mathcal{T}$ is compatible with $\mathcal{C}$ iff $\mathcal{C}$ has a closed base (i.e., $\mathcal{C}$ has a base $\mathcal{B}$ contained in $\mathcal{T}$).

From (2) of Remark 5.1 and Definition 3.1, we extend the notion of topological-convex spaces into L-fuzzy settings as follows.

Definition 5.2. Let X be a set equipped with an L-cotopology $\mathcal{T}$ and an L-convex structure $\mathcal{C}$. The triple $(X, \mathcal{T}, \mathcal{C})$ is called an L-topological-convex space, if $\mathcal{T}$ is compatible with $\mathcal{C}$, that is, $\mathcal{C}$ has a closed L-convex base (i.e., there is $\mathcal{B} \subseteq \mathcal{T}$ such that $\mathcal{B}$ is a base of $\mathcal{C}$).

Next, we give some characterizations on L-topological-convex spaces as follows.

Theorem 5.3. *Let X be a set equipped with an L-cotopology $\mathcal{T}$ and an L-convex structure $\mathcal{C}$. Let $cl_{\mathcal{T}\cap\mathcal{C}}$ is the closure operator of the L-closure structure $\mathcal{T} \cap \mathcal{C}$. Then the following conditions are equivalent.*

(1) $(X, \mathcal{T}, \mathcal{C})$ is an L-topological-convex space.

(2) $co_{\mathcal{C}}(A) = \bigvee_{F\in\mathfrak{F}(A)} cl_{\mathcal{T}\cap\mathcal{C}}(F)$ for any $A \in L^X$.

(3) $co_{\mathcal{C}}(F) = \bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}\cap\mathcal{C}}(G)$ for any $F \in \mathfrak{F}(L^X)$.

Proof. (1) ⇒ (2): Let $\mathcal{B} \subseteq \mathcal{T} \cap \mathcal{C}$ be an L-convex base of $\mathcal{C}$. Let $A \in L^X$. By (LDF),

$$co_{\mathcal{C}}(A) = \bigvee_{F\in\mathfrak{F}(A)} co_{\mathcal{C}}(F) \leq \bigvee_{F\in\mathfrak{F}(A)} cl_{\mathcal{T}\cap\mathcal{C}}(F).$$

Conversely, since $\mathcal{B}$ is an L-convex base of $\mathcal{C}$, there is an up-directed set $\mathcal{B}_1 \subseteq \mathcal{B}$ such that $co_{\mathcal{C}}(A) = \bigvee \mathcal{B}_1$. For each $F \in \mathfrak{F}(A)$, we have

$$F \in \mathfrak{F}(A) \subseteq \mathfrak{F}(co_{\mathcal{C}}(A)) = \mathfrak{F}(\bigvee \mathcal{B}_1) = \bigcup_{B\in\mathcal{B}_1} \mathfrak{F}(B).$$

Thus there is $B \in \mathcal{B}$ such that $F \in \mathfrak{F}(B)$. Since $B \in \mathcal{B}_1 \subseteq \mathcal{B} \subseteq \mathcal{T} \cap \mathcal{C}$, we have

$$cl_{\mathcal{T}\cap\mathcal{C}}(F) \leq cl_{\mathcal{T}\cap\mathcal{C}}(B) = B \leq co_{\mathcal{C}}(A).$$

Hence $\bigvee_{F\in\mathfrak{F}(A)} cl_{\mathcal{T}\cap\mathcal{C}}(F) \leq co_{\mathcal{C}}(A)$.

(2) ⇒ (3): Clear.

(3) ⇒ (1): We have $\mathcal{B} = \{cl_{\mathcal{T}\cap\mathcal{C}}(F) : F \in \mathfrak{F}(L^X)\} \subseteq \mathcal{T} \cap \mathcal{C}$. Thus for each $C \in \mathcal{C}$,

$$C = co_{\mathcal{C}}(C) = \bigvee_{F\in\mathfrak{F}(C)} co_{\mathcal{C}}(F) = \bigvee_{F\in\mathfrak{F}(C)} \bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}\cap\mathcal{C}}(F) = \bigvee_{G\in\mathfrak{F}(C)} cl_{\mathcal{T}\cap\mathcal{C}}(G).$$

Since the set $\{G : G \in \mathfrak{F}(C)\}$ is up-directed, the set $\{cl_{\mathcal{T}\cap\mathcal{C}}(G) : G \in \mathfrak{F}(C)\} \subseteq \mathcal{B}$ is also up-directed. Therefore $\mathcal{B}$ is a closed L-convex base of $\mathcal{C}$. □

In [28], it has been proved that for an L-closure space (X, φ), the set

$$\mathcal{C}_\varphi = \{A \in L^X : \exists \mathcal{D} \overset{dir}{\subseteq} \varphi, A = \bigvee \mathcal{D}\}$$

is an L-convex structure generated by φ. Thus an L-cotopology $\mathcal{T}$ naturally induces an L-convex structure denoted by $\mathcal{C}_\mathcal{T}$. Moreover, we have the following result.

Theorem 5.4. *Let $(X, \mathcal{T})$ be an L-cotopological space. Then $(X, \mathcal{T}, \mathcal{C}_\mathcal{T})$ is an L-topological-convex space.*

Proof. Let $F \in \mathfrak{F}(L^X)$. Since $\mathcal{T} \subseteq \mathcal{C}_\mathcal{T}$, it directly follows from (LDF) that

$$co_{\mathcal{C}_\mathcal{T}}(F) = \bigvee_{G \in \mathfrak{F}(F)} co_{\mathcal{C}_\mathcal{T}}(G) \le \bigvee_{G \in \mathfrak{F}(F)} cl_{\mathcal{T} \cap \mathcal{C}_\mathcal{T}}(G).$$

Conversely, let $G \in \mathfrak{F}(F)$. For each $B \in \mathcal{C}_\mathcal{T}$ with $F \le B$, there is an up-directed set $\mathcal{D} \subseteq \mathcal{T}$ such that $B = \bigvee \mathcal{D}$. Thus $G \in \mathfrak{F}(B) = \bigcup_{D \in \mathcal{D}} \mathfrak{F}(D)$. Hence there is $D_G \in \mathcal{D}$ such that $G \in \mathfrak{F}(D_G)$. This shows $D_G \in \mathcal{T} \cap \mathcal{C}_\mathcal{T}$ and $G \le D_G \le B$. So

$$co_{\mathcal{C}_\mathcal{T}}(F) = \bigwedge\{B \in \mathcal{C}_\mathcal{T} : F \le B\} \ge D_G = cl_{\mathcal{T} \cap \mathcal{C}_\mathcal{T}}(D_G) \ge cl_{\mathcal{T} \cap \mathcal{C}_\mathcal{T}}(G).$$

Therefore $\bigvee_{G \in \mathfrak{F}(F)} cl_{\mathcal{T} \cap \mathcal{C}_\mathcal{T}}(G) \le co_{\mathcal{C}_\mathcal{T}}(F)$. □

Remark 5.5. (1) Unlike topological-convex spaces, an L-topological-convex space $(X, \mathcal{T}, \mathcal{C})$ fails to imply $co_\mathcal{C}(F) \in \mathcal{T}$ for all $F \in \mathfrak{F}(L^X)$. For example, let $X = \{x\}$ and $L = [0, 1]$. Define $\mathcal{T} = \{x_r : r \in [0, \frac{1}{2}) \cup \{1\}\}$ and $\mathcal{C} = \mathcal{T} \cup \{x_{\frac{1}{2}}\}$. Then $(X, \mathcal{T}, \mathcal{C})$ is an L-topological-convex space. However, $x_{\frac{1}{2}} \in \mathfrak{F}(L^X)$ and $co_\mathcal{C}(x_{\frac{1}{2}}) = x_{\frac{1}{2}} \notin \mathcal{T}$. Hence $cl_{\mathcal{T} \cap \mathcal{C}}(x_{\frac{1}{2}}) = cl_\mathcal{T}(x_{\frac{1}{2}}) = x_1$ which shows that $co_\mathcal{C}(x_{\frac{1}{2}}) \neq cl_{\mathcal{T} \cap \mathcal{C}}(x_{\frac{1}{2}})$.

(2) In the L-convex structure generated by an L-cotopology, each element in $\mathcal{C}$ is the supermum of an up-directed subset of $\mathcal{T}$. This property is also preserved by an L-topological-convex space. The L-convex structure generated by an L-cotopology containing this L-cotopology. But an L-topological-convex space $(X, \mathcal{T}, \mathcal{C})$, $\mathcal{T} \subseteq \mathcal{C}$ may fail. For example, let $X = \{x, y\}$ and let $L = [0, 1]$. Define $\mathcal{C} = \{\underline{0}, x_{\frac{1}{2}}, y_{\frac{1}{2}}, \underline{1}\}$ and $\mathcal{T} = \mathcal{C} \cup \{\underline{\frac{1}{2}}\}$. Then $(X, \mathcal{T}, \mathcal{C})$ is an L-topological-convex space. But $\mathcal{T} \nsubseteq \mathcal{C}$.

(3) For an L-cotopological space $(X, \mathcal{T})$, $\mathcal{C}_\mathcal{T}$ is an L-Alexander topology.

In fact, if $\mathcal{C}_\mathcal{T}$ is the L-convex space generated by an L-topology $\mathcal{T}$ and $\{A_i\}_{i \in I} \subseteq \mathcal{C}_\mathcal{T}$, then there is an up-directed set $\mathcal{D}_i \subseteq \mathcal{T}$ such that $A = \bigvee \mathcal{D}_i$ for any $i \in I$. Let $\mathcal{D} = \bigcup_{i \in I} \mathcal{D}_i$ and let $\mathcal{B} = \{\sqcup \mathcal{G} : \mathcal{G} \in 2^{\mathcal{D}}_{fin}\}$, where $\sqcup \mathcal{G}$ stands for $\bigvee \mathcal{G}$ ($\mathcal{G}$ is finite). Then $\mathcal{B} \subseteq \mathcal{T}$ is up-directed and $\bigvee_{i \in I} A_i = \bigvee \mathcal{B} \in \mathcal{C}_\mathcal{T}$. So $\mathcal{C}_\mathcal{T}$ is an L-Alexander topology. However, in an L-topological-convex space $(X, \mathcal{T}, \mathcal{C})$, $\mathcal{C}$ may not be an L-Alexander topology. The example in (1) is of this type.

Next, we get a weaker L-topology by an L-topology and an L-convex structure.

Lemma 5.6. *Let X be equipped with an L-cotopology $\mathcal{T}$ and an L-convex structure $\mathcal{C}$. Let $\varphi = \mathcal{T} \cap \mathcal{C}$. Define*

$$\mathcal{T}_w = \{A \in L^X : \exists \mathcal{B} \subseteq \varphi^\sqcup, A = \bigwedge \mathcal{B}\},$$

where $\varphi^\sqcup = \{\sqcup \mathcal{D} : \mathcal{D} \in 2^\varphi_{fin}\}$. Then $\mathcal{T}_w$ is an L-cotopology satisfying $\mathcal{T}_w = \bigwedge\{\mathcal{S} \in \mathfrak{T}(X) : \varphi \subseteq \mathcal{S}\}$ and $\mathcal{T}_w \subseteq \mathcal{T}$, where $\mathfrak{T}(X)$ is the set of all L-cotopologies on X.

Proof. (LT1): Since $\top, \bot \in \varphi \subseteq \varphi^\sqcup$, we have $\bot, \top \in \mathcal{T}_w$.

(LT2): Let $\{A_i\}_{i \in I} \subseteq \mathcal{T}_w$. For each $i \in I$, there is $\mathcal{B}_i \subseteq \varphi^\sqcup$ such that $A_i = \bigwedge \mathcal{B}_i$. Let $\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$. Then $\mathcal{B} \subseteq \varphi^\sqcup$ and $\bigwedge_{i \in I} A_i = \bigwedge \mathcal{B} \in \mathcal{T}_w$.

(LT3): We firstly prove that $A \vee B \in \varphi^\sqcup$ for all $A, B \in \varphi^\sqcup$.

Since $A, B \in \varphi^\sqcup$, there are $\mathcal{D}_1, \mathcal{D}_2 \in 2^\varphi_{fin}$ such that $\sqcup \mathcal{D}_1 = A$ and $\sqcup \mathcal{D}_2 = B$. Since $\mathcal{B} = \mathcal{D}_1 \cup \mathcal{D}_2$, $A \vee B = \sqcup \mathcal{B} \in \varphi^\sqcup$. Next, we prove that $\mathcal{T}_w$ satisfies (LT3).

If $A, B \in \mathcal{T}_w$, then there are $\{D_i\}_{i\in I}, \{D_j\}_{j\in J} \subseteq \varphi^{\sqcup}$ such that $A = \bigwedge_{i\in I} D_i$ and $B = \bigwedge_{j\in J} D_j$. Let $D_{ij} = D_i \vee D_j$ for any $i \in I$ and any $j \in J$. We have

$$A \vee B = \bigwedge_{i\in I, j\in J} D_i \vee D_j = \bigwedge_{i\in I, j\in J} D_{ij} \in \mathcal{T}_w.$$

Therefore $\mathcal{T}_w$ is an L-cotopology.

To prove that $\mathcal{T}_w \subseteq \mathcal{T}$, let $A \in \mathcal{T}_w$. Then there is $\mathcal{B} \subseteq \varphi^{\sqcup}$ such that $A = \bigwedge \mathcal{B}$. Since $\mathcal{B} \subseteq \varphi^{\sqcup} \subseteq \mathcal{T}$, we have $\mathcal{B} \subseteq \mathcal{T}$ and $A = \bigwedge \mathcal{B} \in \mathcal{T}$. Therefore $\mathcal{T}_w \subseteq \mathcal{T}$.

Finally, since $\mathcal{T}_w \supseteq \varphi^{\sqcup} \supseteq \varphi$, we have $\bigcap\{\mathcal{S} \in \mathfrak{T}(X) : \varphi \subseteq \mathcal{S}\} \subseteq \mathcal{T}_w$.

Conversely, let $\varphi \subseteq \mathcal{S} \in \mathfrak{T}(X)$. If $A \in \mathcal{T}_w$, then there is $\mathcal{B} \subseteq \varphi^{\sqcup}$ such that $A = \bigwedge \mathcal{B}$. Thus, for each $B \in \mathcal{B}$, there is $\mathcal{D} \in 2^{\varphi}_{fin}$ such that $B = \sqcup \mathcal{D}$. Hence $B \in \mathcal{S}$ and $A = \bigwedge \mathcal{B} \in \mathcal{S}$. So $\mathcal{T}_w \subseteq \bigcap\{\mathcal{S} \in \mathfrak{T}(X) : \varphi \subseteq \mathcal{S}\}$. □

With help of $\mathcal{T}_w$, we can characterize L-topological-convex space as following.

Theorem 5.7. *Let X be a set equipped with an L-cotopology $\mathcal{T}$ and an L-convex structure $\mathcal{C}$. Then $(X, \mathcal{T}, \mathcal{C})$ is an L-topological-convex space iff $(X, \mathcal{T}_w, \mathcal{C})$ is an L-topological-convex space.*

Proof. $\mathcal{T}_w \cap \mathcal{C} = \mathcal{T} \cap \mathcal{C}$ by Lemma 5.6. Thus the result follows from Theorem 5.3. □

6 *L*-topological-convex enclosed relation spaces

Except for Theorems 5.3 and 5.7, there are others ways to characterize L-topological-convex spaces. In this section, we introduce the notions of L-topological-convex enclosed relations, by which, we characterize L-topological-convex spaces.

For L-topological-convex spaces $(X, \mathcal{T}_X, \mathcal{C}_Y)$ and $(Y, \mathcal{T}_Y, \mathcal{C}_Y)$, a mapping $f : X \to Y$ is called an L-topological-convex structure preserving mapping, if $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is L-continuous and $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is L-convex structure preserving.

The category of L-topological-convex spaces and L-topological-convex structure preserving mappings is denoted by L-**TCS**.

Definition 6.1. Let X be a set equipped with an L-topological enclosed relation $\preccurlyeq$ and an L-convex enclosed relation $\leqslant$. The triple $(X, \preccurlyeq, \leqslant)$ is called an L-topological-convex enclosed relation spaces provided that $\preccurlyeq$ is compatible with $\leqslant$, that is, for any $F \in \mathfrak{F}(L^X)$ and any $B \in L^X$,

$$F \leqslant B \text{ implies } \forall G \in \mathfrak{F}(F), \exists D \in L^X \text{ such that } G \leq D \leqslant D \preccurlyeq D \leq B.$$

Let $(X, \preccurlyeq_X, \leqslant_X)$ and $(Y, \preccurlyeq_Y, \leqslant_Y)$ be L-topological-convex enclosed relation spaces. A mapping $f : X \to Y$ is called an L-topological-convex enclosed relation dual-preserving mapping, if $f : (X, \preccurlyeq_X) \to (Y, \preccurlyeq_Y)$ is an L-topological enclosed relation dual-preserving mapping, and $f : (X, \leqslant_X) \to (Y, \leqslant_Y)$ is an L-convex enclosed relation dual-preserving mapping. That is, for all $A, B \in L^Y$, $A \preccurlyeq_Y B$ (resp. $A \leqslant_Y B$) implies $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(B)$ (resp. $f_L^{\leftarrow}(A) \leqslant_X f_L^{\leftarrow}(B)$).

The category of L-topological-convex enclosed relation spaces and L-topological-convex enclosed relation dual-preserving mappings is denoted by L-**TCERS**.

Next, we prove that the L-convex enclosed relation generated by an L-topological enclosed relation is compatible with this L-topological enclosed relation.

Lemma 6.2. *Let $(X, \preccurlyeq)$ be an L-topological enclosed relation space. Define a binary operator $\leqslant_{\preccurlyeq}$ on X by:*

$$\forall A, B \in L^X, \ \ A \leqslant_{\preccurlyeq} B \text{ iff } \exists \mathcal{D} \overset{dir}{\subseteq} L^X, \ s.t. \ A = \bigvee \mathcal{D}; \forall D \in \mathcal{D}, D \preccurlyeq B.$$

Then $\leqslant_{\preccurlyeq}$ is an L-convex enclosed relation generated by $\preccurlyeq$. Moreover, for all $A, B \in L^X$, $A \leqslant_{\preccurlyeq} B$ iff $F \preccurlyeq B$ for all $F \in \mathfrak{F}(A)$.

Proof. (LCER1): $\bot \lessdot\!\preccurlyeq \bot$ since $\{\bot\} \overset{dir}{\subseteq} L^X$ and $\bot \preccurlyeq \bot$.

(LCER2): If $A \lessdot\!\preccurlyeq B$, then there is $\mathcal{D} \overset{dir}{\subseteq} L^X$ such that $A = \bigvee \mathcal{D}$ and $D \preccurlyeq B$ for all $D \in \mathcal{D}$. Thus $A \leq B$ by (LTER2).

(LCER3): Let $B = \bigwedge_{i \in I} B_i$. If $A \lessdot\!\preccurlyeq B$, then there is $\mathcal{D} \overset{dir}{\subseteq} L^X$ such that $A = \bigvee \mathcal{D}$ and $D \preccurlyeq B$ for each $D \in \mathcal{D}$. Since $B \leq B_i$ for each $i \in I$, we have $D \preccurlyeq B_i$ for each $D \in \mathcal{D}$. Thus $A \lessdot\!\preccurlyeq B_i$ for each $i \in I$.

Conversely, let $A \lessdot\!\preccurlyeq B_i$ for each $i \in I$ and let $\mathcal{D}_i \overset{dir}{\subseteq} L^X$ such that $A = \bigvee \mathcal{D}_i$ and $D_i \preccurlyeq B_i$ for all $D_i \in \mathcal{D}_i$. Let $\mathcal{S}$ be the set of all choice mappings $s : I \to \bigcup_{i \in I} \mathcal{D}_i$ with $s(i) \in \mathcal{D}_i$. Then the set $\mathcal{D} = \{\bigwedge_{i \in I} s(i) : s \in \mathcal{S}\}$ has the following properties.

(i) $\mathcal{D}$ is up-directed.

Let $B_1, B_2 \in \mathcal{D}$. Then there are $s_1, s_2 \in \mathcal{S}$ such that $B_1 = \bigwedge_{i \in I} s_1(i)$ and $B_2 = \bigwedge_{i \in I} s_2(i)$. Thus there is $D_i \in \mathcal{D}_i$ such that $s_1(i), s_2(i) \leq D_i$ for each $i \in I$. Define $s_3 : I \to \bigcup_{i \in I} \mathcal{D}_i$ by $s_3(i) = D_i$ for each $i \in I$. Then $s_3 \in \mathcal{S}$ and $B_1, B_2 \leq B_3 = \bigwedge_{i \in I} s_3(i) \in \mathcal{D}$. Therefore $\mathcal{D}$ is up-directed.

(ii) $A = \bigvee \mathcal{D}$.

Let $x_\lambda \prec A$. Since $A \lessdot\!\preccurlyeq B_i$ for each $i \in I$, there is $\mathcal{D}_i \overset{dir}{\subseteq} L^X$ such that $A = \bigvee \mathcal{D}_i$. Thus there is $D_i \in \mathcal{D}_i$ such that $x_\lambda \leq D_i$. Define $s : I \to \bigcup_{i \in I} \mathcal{D}_i$ by $s(i) = D_i$ for each $i \in I$. Then $s \in \mathcal{S}$ and $x_\lambda \leq \bigwedge_{i \in I} s(i) \in \mathcal{D}$. Hence $A \leq \bigvee \mathcal{D}$. Conversely, if $y_\mu \prec \bigvee \mathcal{D}$, then there is $s \in \mathcal{S}$ such that $y_\mu \leq \bigwedge_{i \in I} s(i)$. Since $s(i) \in \mathcal{D}_i$, we have $y_\mu \leq \bigvee \mathcal{D}_i = A$. Thus $\bigvee \mathcal{D} \leq A$. Therefore $A = \bigvee \mathcal{D}$.

(iii) $D \preccurlyeq B$ for any $D \in \mathcal{D}$.

For any $D \in \mathcal{D}$, there is $s \in \mathcal{S}$ such that $D = \bigwedge_{i \in I} s(i)$. Fix any $j \in I$. We have $D \leq s(j) \preccurlyeq B_j$. Thus $D \preccurlyeq B$. Therefore (iii) holds.

Combining (i)–(iii), we find that $A \lessdot\!\preccurlyeq \bigwedge_{i \in I} B_i$.

(LCER4): Let $\{A_i\}_{i \in I} \overset{dir}{\subseteq} L^X$ with $A = \bigvee^{dir}_{i \in I} A_i$. If $A \lessdot\!\preccurlyeq B$, then there is $\mathcal{D} \overset{dir}{\subseteq} L^X$ such that $A = \bigvee \mathcal{D}$ and $D \preccurlyeq B$ for each $D \in \mathcal{D}$. For each $i \in I$, it is clear that $\mathcal{D}_i = \{A_i \wedge D : D \in \mathcal{D}\} \overset{dir}{\subseteq} L^X$ and $A_i = \bigvee \mathcal{D}_i$. In addition, $A_i \wedge D \leq D \preccurlyeq B$ and so $A_i \wedge D \preccurlyeq B$ for each $D \in \mathcal{D}$. Thus $A_i \lessdot\!\preccurlyeq B$ for each $i \in I$.

Conversely, let $A_i \lessdot\!\preccurlyeq B$ for each $i \in I$. Then, for each $i \in I$, there is $\mathcal{D}_i \overset{dir}{\subseteq} L^X$ such that $A_i = \bigvee \mathcal{D}_i$ and $D_i \preccurlyeq B$ for all $D_i \in \mathcal{D}_i$. Let $\mathcal{D} = \bigcup_{i \in I} \mathcal{D}_i$, $G = \bigvee \mathcal{D}$ and $\varphi_F = \{D : F \leq D \in \mathcal{D}\}$ for all $F \in \mathfrak{F}(A)$. We have the following results.

(i) φ_F is nonempty for all $F \in \mathfrak{F}(A)$.

Since $\{A_i\}_{i \in I} \overset{dir}{\subseteq} L^X$, there is $i_F \in I$ such that $F \in \mathfrak{F}(A_{i_F}) = \mathfrak{F}(\bigvee \mathcal{D}_{i_F}) = \bigcup_{D_{i_F} \in \mathcal{D}_{i_F}} \mathfrak{F}(D_{i_F})$. So there is $D_{i_F} \in \mathcal{D}_{i_F} \subseteq \mathcal{D}$ with $F \in \mathfrak{F}(D_{i_F})$. Hence $D_{i_F} \in \varphi_F$.

(ii) $A = \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F$.

Since $F \leq D_{i_F} \leq A_{i_F} \leq A$ for all $F \in \mathfrak{F}(A)$, we have $F \leq \bigwedge \varphi_F \leq A$. Thus

$$A = \bigvee_{F \in \mathfrak{F}(A)} F \leq \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F \leq A.$$

Hence $A = \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F$.

(iii) $\{\bigwedge \varphi_F : F \in \mathfrak{F}(A)\} \subseteq \mathcal{D}$ is up-directed.

Let $F, G \in \mathfrak{F}(A)$. Then there is $H \in \mathfrak{F}(A)$ such that $F, G \leq H$. Since $D \in \varphi_F$ and $D \in \varphi_G$ for each $D \in \varphi_H$, we have $\bigwedge \varphi_F, \varphi_G \leq \varphi_H$. Thus (iii) holds.

(iv) $\bigwedge \varphi_F \preccurlyeq B$ for each $F \in \mathfrak{F}(A)$.

We have $\bigwedge \varphi(F) \leq D \preccurlyeq B$ for each $D \in \varphi(F)$. Thus $\bigwedge \varphi(F) \preccurlyeq B$.

Combining (i)–(iv), we conclude that $\bigvee^{dir}_{i \in I} A_i \lessdot\!\preccurlyeq B$.

(LCER5): If $A \lessdot\!\preccurlyeq B$, then there is $\mathcal{D} \overset{dir}{\subseteq} L^X$ such that $A = \bigvee \mathcal{D}$ and $D \preccurlyeq B$ for each $D \in \mathcal{D}$. Thus, for each $D \in \mathcal{D}$, there is $E_D \in L^X$ such that $D \leq E_D \preccurlyeq E_D \leq B$ by (LTER5). This shows $D \preccurlyeq E_D \preccurlyeq B$.

Let $\varphi_D = \{E_D \in L^X : D \preccurlyeq E_D \preccurlyeq B\}$ for each $D \in \mathcal{D}$. Then φ_D is nonempty. Further, the set $\{\bigwedge \varphi_D : D \in \mathcal{D}\}$ is up-directed and $D \preccurlyeq \bigwedge \varphi_D$ by (LTER3). Also, it is clear that $\bigwedge \varphi_D \preccurlyeq B$. Let $C = \bigvee_{D \in \mathcal{D}} \bigwedge \varphi_D$. We have $D \preccurlyeq C$ for each $D \in \mathcal{D}$. Hence $A \lessdot\!\preccurlyeq C \lessdot\!\preccurlyeq B$ as desired.

Therefore $\leqslant_{\preccurlyeq}$ is an L-convex enclosed relation.

Finally, let $A, B \in L^X$. We prove that $A \leqslant_{\preccurlyeq} B$ iff $F \prec B$ for each $F \in \mathfrak{F}(A)$.

If $A \leqslant_{\preccurlyeq} B$, then there is an up-directed set $\mathcal{D} \subseteq L^X$ such that $A = \bigvee \mathcal{D}$ and $D \prec B$ for each $D \in \mathcal{D}$. Thus, for each $F \in \mathfrak{F}(A)$, we have $F \in \mathfrak{F}(A) = \mathfrak{F}(\bigvee \mathcal{D}) = \bigcup_{D \in \mathcal{D}} \mathfrak{F}(D)$. Hence there is $D \in \mathcal{D}$ such that $F \leq D \prec B$. Therefore $F \prec B$.

Conversely, let $F \prec B$ for all $F \in \mathfrak{F}(A)$. Since $\{F : F \in \mathfrak{F}(A)\}$ is up-directed and $A = \bigvee_{F \in \mathfrak{F}(A)} G$, we have $A \leqslant_{\preccurlyeq} B$ by definition. □

Theorem 6.3. *For an L-topological enclosed relation space $(X, \prec)$, the triple $(X, \prec, \leqslant_{\preccurlyeq})$ is an L-topological-convex enclosed space.*

Proof. Let $F \in \mathfrak{F}(L^X)$ and let $B \in L^X$ with $F \leqslant_{\preccurlyeq} B$. If $G \in \mathfrak{F}(F)$, then $G \prec B$ by Lemma 6.2. Thus there is $D_G \in L^X$ such that $G \leq D_G \prec D_G \leq B$. Since $D_G \prec D_G$ and $\{D_G\} \subseteq L^X$ is up-directed, we have $D_G \leqslant_{\preccurlyeq} D_G$. Hence

$$G \leq D_G \leqslant_{\preccurlyeq} D_G \prec D_G \leq B.$$

Therefore $(X, \prec, \leqslant_{\preccurlyeq})$ is an L-topological-convex enclosed relation space. □

By Theorem 6.3, we list some L-topological-convex enclosed relations as follows.

Example 6.4. Let $(X, \mathscr{U})$ be a pointwise quasi-uniform space. Define

$$A \prec_{\mathscr{U}} B \text{ iff } \exists \mathcal{U} \in \mathscr{U} \text{ such that } A \leq \bigwedge_{y_\mu \not\leq B} \mathcal{U}(y_\mu);$$

$$A \leqslant_{\mathscr{U}} B \text{ iff } \exists \mathcal{D} \overset{dir}{\subseteq} L^X \text{ such that } A = \bigvee \mathcal{D};\ \forall D \in \mathcal{D}, D \prec_{\mathscr{U}} B.$$

Then $(X, \prec_{\mathscr{U}}, \leqslant_{\mathscr{U}})$ is an L-topological-convex enclosed relation space by Theorem 4.6 in [33], Lemma 6.2 and Theorem 6.3.

Example 6.5. Let (X, δ) be a pointwise S-quasi-proximity space. Define

$$A \prec_{\delta} B \text{ iff } \forall y_\mu \not\leq B,\ \ \delta(y_\mu, A) = 0;$$

$$A \leqslant_{\delta} B \text{ iff } \exists \mathcal{D} \overset{dir}{\subseteq} L^X \text{ such that } A = \bigvee \mathcal{D};\ \forall D \in \mathcal{D}, D \prec_{\delta} B.$$

Then $(X, \prec_{\delta}, \leqslant_{\delta})$ is an L-topological-convex enclosed relation space by Theorem 4.7 in [33], Lemma 6.2 and Theorem 6.3.

Example 6.6. Let (X, d) be a pointwise pseudo-metric space. Define

$$A \prec_{d} B \text{ iff } \forall y_\mu \not\leq B,\ \bigwedge_{x_\eta \leq A} d(x_\eta, y_\mu) > 0;$$

$$A \leqslant_{d} B \text{ iff } \exists \mathcal{D} \overset{dir}{\subseteq} L^X \text{ such that } A = \bigvee \mathcal{D};\ \forall D \in \mathcal{D}, D \prec_{d} B.$$

Then $(X, \prec_{d}, \leqslant_{d})$ is an L-topological-convex enclosed relation space by Theorem 4.2 in [33], Lemma 6.2 and Theorem 6.3.

Next, we discuss relationships between L-**TCERS** and L-**TCS**.

Theorem 6.7. *If $(X, \prec, \leqslant)$ is an L-topological-convex enclosed relation space, then $(X, \mathcal{T}_{\preccurlyeq}, \mathcal{C}_{\leqslant})$ is an L-topological-convex space.*

Proof. We firstly verify that $co_{\leqslant} : L^X \to L^X$, defined by $co_{\leqslant}(A) = \bigwedge\{B \in L^X : A \leqslant B\}$ in Theorem 2.8, satisfies $co_{\leqslant}(A) = \bigvee_{F \in \mathfrak{F}(A)} co_{\leqslant}(F)$ for all $A \in L^X$.

Clearly, $\bigvee_{F\in\mathfrak{F}(A)} co_{\leqslant}(F) \le co_{\leqslant}(A)$. Conversely, for each $G \in \mathfrak{F}(A)$, we have

$$G \leqslant co_{\leqslant}(G) \le \bigvee_{F\in\mathfrak{F}(A)} co_{\leqslant}(F).$$

Thus

$$A = \bigvee_{G\in\mathfrak{F}(A)} G \leqslant \bigvee_{F\in\mathfrak{F}(A)} co_{\leqslant}(F).$$

Hence $co_{\leqslant}(A) \le \bigvee_{F\in\mathfrak{F}(A)} co_{\leqslant}(F)$.

To prove the desired result, let $F \in \mathfrak{F}(L^X)$. We need to prove that

$$co_{\mathcal{C}_{\leqslant}}(F) = \bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}_{\preccurlyeq}\cap\mathcal{C}_{\leqslant}}(G).$$

Since $\mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant} \subseteq \mathcal{C}_{\leqslant}$, we have $cl_{\mathcal{C}_{\leqslant}} \le cl_{\mathcal{T}_{\preccurlyeq}\cap\mathcal{C}_{\leqslant}}$. Thus

$$co_{\mathcal{C}_{\leqslant}}(F) = \bigvee_{G\in\mathfrak{F}(F)} co_{\mathcal{C}_{\leqslant}}(G) \le \bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}_{\preccurlyeq}\cap\mathcal{C}_{\leqslant}}(G).$$

Conversely, let $G \in \mathfrak{F}(F)$. To prove that $cl_{\mathcal{T}_{\preccurlyeq}\cap\mathcal{C}_{\leqslant}}(G) \le co_{\mathcal{C}_{\leqslant}}(F)$, let $B \in L^X$ with $co_{\mathcal{C}_{\leqslant}}(F) \le B$. We next prove that $cl_{\mathcal{T}_{\preccurlyeq}\cap\mathcal{C}_{\leqslant}}(G) \le B$.

Since $co_{\mathcal{C}_{\leqslant}}(F) \le B$, we have $F \leqslant B$. Thus there is $D \in L^X$ such that $G \le D \leqslant D \preccurlyeq D \le B$ and so

$$G \le D = cl_{\mathcal{T}_{\preccurlyeq}}(D) = co_{\mathcal{C}_{\leqslant}}(D).$$

Hence $G \le D \in \mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}$ which implies $cl_{\mathcal{T}_{\preccurlyeq}\cap\mathcal{C}_{\leqslant}}(G) \le D \le B$. In conclusion, $\bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}_{\preccurlyeq}\cap\mathcal{C}_{\leqslant}}(G) \le B$. Therefore $\bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}_{\preccurlyeq}\cap\mathcal{C}_{\leqslant}}(G) \le co_{\mathcal{C}_{\leqslant}}(F)$.

Thus, by Theorem 5.3, $(X, \mathcal{T}_{\preccurlyeq}, \mathcal{C}_{\leqslant})$ is an L-topological-convex space. □

Theorem 6.8. *If $(X, \mathcal{T}, \mathcal{C})$ is an L-topological-convex space, then $(X, \preccurlyeq_{\mathcal{T}}, \leqslant_{\mathcal{C}})$ is an L-topological-convex enclosed space.*

Proof. Let $F \in \mathfrak{F}(L^X)$ and $B \in L^X$ such that $F \leqslant_{\mathcal{C}} B$. Then

$$\bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}\cap\mathcal{C}}(G) = co_{\mathcal{C}}(F) \le B.$$

Let $D_G = cl_{\mathcal{T}\cap\mathcal{C}}(G)$ for each $G \in \mathfrak{F}(F)$. Then $G \le D_G \in \mathcal{T} \cap \mathcal{C}$. By $D_G \in \mathcal{C}$, we have $co_{\mathcal{C}}(D_G) = D_G$ and so $D_G \leqslant_{\mathcal{C}} D_G$. Similarly, by $D_G \in \mathcal{T}$, we have $cl_{\mathcal{T}}(D_G) = D_G$ and so $D_G \preccurlyeq_{\mathcal{T}} D_G$. Hence

$$G \le D_G \leqslant_{\mathcal{C}} D_G \preccurlyeq_{\mathcal{T}} D_G \le B.$$

Therefore $(X, \preccurlyeq_{\mathcal{T}}, \leqslant_{\mathcal{C}})$ is an L-topological-convex enclosed space. □

The following three results directly follow from Theorems 2.6 and 2.8.

Theorem 6.9. *Let $(X, \preccurlyeq_X, \leqslant_X)$ and $(Y, \preccurlyeq_Y, \leqslant_Y)$ be L-topological-convex enclosed relation spaces. If $f : X \to Y$ is an L-topological-convex enclosed relation dual-preserving mapping, then $f : (X, \mathcal{T}_{\preccurlyeq_X}, \mathcal{C}_{\leqslant_X}) \to (Y, \mathcal{T}_{\preccurlyeq_Y}, \mathcal{C}_{\leqslant_Y})$ is an L-topology-convexity preserving mapping.*

Theorem 6.10. *Let $(X, \mathcal{T}_X, \mathcal{C}_X)$ and $(Y, \mathcal{T}_Y, \mathcal{C}_Y)$ be L-topological-convex spaces. If $f : X \to Y$ is L-topological-convex structure preserving, then $f : (X, \preccurlyeq_{\mathcal{T}_X}, \leqslant_{\mathcal{C}_X}) \to (Y, \preccurlyeq_{\mathcal{T}_Y}, \leqslant_{\mathcal{C}_Y})$ is L-topological-convex enclosed relation dual-preserving.*

Theorem 6.11. *If $(X, \preccurlyeq, \leqslant)$ is an L-topological-convex enclosed relation space, then $\preccurlyeq_{\mathcal{T}_{\preccurlyeq}} = \preccurlyeq$ and $\leqslant_{\mathcal{C}_{\leqslant}} = \leqslant$. Conversely, if $(X, \mathcal{T}, \mathcal{C})$ is an L-topological-convex space, then $\mathcal{T}_{\preccurlyeq_{\mathcal{T}}} = \mathcal{T}$ and $\mathcal{C}_{\leqslant_{\mathcal{C}}} = \mathcal{C}$.*

From Theorems 6.7 and 6.9, we can define a factor $\mathbb{T} : L$-**TCERS** $\to L$-**TCS** by

$$\mathbb{T}(X, \preccurlyeq, \leqslant) = (X, \mathfrak{T}_{\preccurlyeq}, \mathfrak{C}_{\leqslant}) \text{ and } \mathbb{T}(f) = f.$$

From Theorems 6.7 to 6.11, $\mathbb{T}$ is an isomorphic factor.

Corollary 6.12. *L-**TCERS** is isomorphic to L-**TCS**.*

Next, we obtain a new L-topological enclosed relation via an L-topological enclosed relation and an L-concave enclosed relation.

Lemma 6.13. *Let X be a set equipped with an L-topological enclosed relation $\preccurlyeq$ and an L-convex enclosed relation $\leqslant$. Define a binary operator $\sqsubset_w$ on X by*

$$A \sqsubset_w B \text{ iff } \overline{cl}(A) \le B,$$

where the operator $\overline{cl} : L^X \to L^X$ is defined by:

$$\overline{cl}(A) = \bigwedge\{\widetilde{cl}(A) : cl_{\preccurlyeq} \vee co_{\leqslant} \le \widetilde{cl} \in \mathfrak{C}(X)\},$$

where $\mathfrak{C}(X)$ is the set of all L-closure operator on X and $cl_{\preccurlyeq} \vee co_{\leqslant} \le \widetilde{cl}$ means $cl_{\preccurlyeq}(A) \vee co_{\leqslant}(A) \le \widetilde{cl}(A)$ for any $A \in L^X$. Then $\sqsubset_w$ is an L-enclosed relation generated by $\preccurlyeq$ and $\leqslant$. In addition, $\sqsubset_w$ is the biggest L-enclosed relation with respect to $\sqsubset_w \le \preccurlyeq$ and $\sqsubset_w \le \leqslant$.

Proof. Note that the set $\{\widetilde{cl} \in \mathfrak{C}(X) : cl_{\preccurlyeq} \vee co_{\leqslant} \le \widetilde{cl}\}$ is not empty since it contains the closure operator of the indiscrete L-topology on X. To prove that $\sqsubset_w$ is an L-enclosed operator, we only need to verify that $\overline{cl}$ is an L-closure operator.

(LCL1)–(LCL3) are easy.

(LCL4): Let $A \in L^X$. Since $\overline{cl} \le \widetilde{cl}$ for any $\widetilde{cl} \in \mathfrak{C}(X)$ with $cl_{\preccurlyeq} \vee co_{\leqslant} \le \widetilde{cl}$,

$$\begin{aligned}
\overline{cl}(\overline{cl}(A)) &= \bigwedge\{\widetilde{cl}(\overline{cl}(A)) : cl_{\preccurlyeq} \vee co_{\leqslant} \le \widetilde{cl} \in \mathfrak{C}(X)\} \\
&\le \bigwedge\{\widetilde{cl}(\widetilde{cl}(A)) : cl_{\preccurlyeq} \vee co_{\leqslant} \le \widetilde{cl} \in \mathfrak{C}(X)\} \\
&\le \bigwedge\{\widetilde{cl}(A) : cl_{\preccurlyeq} \vee co_{\leqslant} \le \widetilde{cl} \in \mathfrak{C}(X)\} \\
&= \overline{cl}(A).
\end{aligned}$$

Thus $\overline{cl}(\overline{cl}(A)) = \overline{cl}(A)$.

Hence $\overline{cl}$ is an L-closure operator. Therefore $\sqsubset_w$ is an L-enclosed relation.

Finally, $\sqsubset_w \le \preccurlyeq$ and $\sqsubset_w \le \leqslant$ by $cl_{\preccurlyeq} \vee co_{\leqslant} \le \overline{cl}$. Now, let $\sqsubset$ be any L-enclosed relation on X with $\sqsubset \le \preccurlyeq$ and $\sqsubset \le \leqslant$. Then $cl_{\preccurlyeq} \vee co_{\leqslant} \le cl_{\sqsubset}$ and so $\overline{cl} \le cl_{\sqsubset}$. Hence

$$\sqsubset = \preccurlyeq_{cl_{\sqsubset}} \le \preccurlyeq_{\overline{cl}} = \sqsubset_w.$$

So $\sqsubset_w$ is the biggest L-enclosed relation with respect to $\sqsubset_w \le \preccurlyeq$ and $\sqsubset_w \le \leqslant$. □

Theorem 6.14. *Let X be a set equipped with an L-topological enclosed relation $\preccurlyeq$ and an L-convex enclosed relation $\leqslant$. Let $\sqsubset_w$ be the L-enclosed relation generated by $\preccurlyeq$ and $\leqslant$. Then the following results are valid.*

(1) $\varphi_{\sqsubset_w} = \mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\leqslant}$, where $\varphi_{\sqsubset_w}$ is the L-closure structure induced by $\sqsubset_w$.

(2) $\mathfrak{T}_{\sqsubset_w} = \mathfrak{T}_w \subseteq \mathfrak{T}_{\preccurlyeq}$, where $\mathfrak{T}_{\sqsubset_w}$ is the L-cotopology generated by $\varphi_{\sqsubset_w}$, and $\mathfrak{T}_w$ is the L-cotopology generated by $\mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\leqslant}$.

(3) Define a binary operator $\preccurlyeq_{\sqsubset_w}$ on X by:

$$A \preccurlyeq_{\sqsubset_w} B \text{ iff } \forall \sqsubset_w \le \widetilde{\preccurlyeq} \in \mathfrak{E}(X),\ A \widetilde{\preccurlyeq} B,$$

where $\mathfrak{E}(X)$ is the set of all L-topological enclosed relation on X. Then $\preccurlyeq_{\sqsubset_w} = \preccurlyeq_{\mathfrak{T}_{\sqsubset_w}}$ which is called the L-topological enclosed relation generated by $\sqsubset_w$.

Proof. (1): Let $A \in L^X$. We have

$$\begin{aligned} A \in \varphi_{\sqsubset_w} &\Leftrightarrow A \sqsubset_w A \\ &\Leftrightarrow A = \overline{cl}(A) \geq cl_{\preccurlyeq}(A) \vee co_{\leqslant}(A) \geq A \\ &\Rightarrow cl_{\preccurlyeq}(A) = co_{\leqslant}(A) = A \\ &\Leftrightarrow A \in \mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}. \end{aligned}$$

Thus $\varphi_{\sqsubset_w} \subseteq \mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}$. Conversely, since $\mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}$ is an L-closure structure, we have

$$cl_{\preccurlyeq} \vee co_{\leqslant} \leq cl_{\mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}} \in \mathfrak{C}(X).$$

Thus $\overline{cl} \leq cl_{\mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}}$. Hence, for any $A \in \mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}$, we have

$$A \leq \overline{cl}(A) \leq cl_{\mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}}(A) = A.$$

This implies $\overline{cl}(A) = A$ and $A \in \varphi_{\sqsubset_w}$. Therefore $\mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant} \subseteq \varphi_{\sqsubset_w}$.

(2): The result directly follows from (1).

(3): For any $\widetilde{\preccurlyeq} \in \mathfrak{E}(X)$ with $\sqsubset_w \leq \widetilde{\preccurlyeq}$, we have $\varphi_{\sqsubset_w} \subseteq \mathcal{T}_{\widetilde{\preccurlyeq}}$. Thus it follows that $\mathcal{T}_{\sqsubset_w} = \bigcap\{\mathcal{T} : \varphi_{\sqsubset_w} \subseteq \mathcal{T} \in \mathfrak{T}(X)\} \subseteq \mathcal{T}_{\widetilde{\preccurlyeq}}$. Hence

$$\begin{aligned} A \preccurlyeq_{\mathcal{T}_{\sqsubset_w}} B &\Leftrightarrow cl_{\mathcal{T}_{\sqsubset_w}}(A) \leq B \\ &\Rightarrow \forall \sqsubset_w \leq \widetilde{\preccurlyeq} \in \mathfrak{E}(X),\ cl_{\mathcal{T}_{\widetilde{\preccurlyeq}}}(A) \leq cl_{\mathcal{T}_{\sqsubset_w}}(A) \leq B \\ &\Rightarrow \forall \sqsubset_w \leq \widetilde{\preccurlyeq} \in \mathfrak{E}(X),\ A \preccurlyeq_{\mathcal{T}_{\widetilde{\preccurlyeq}}} B \\ &\Leftrightarrow \forall \sqsubset_w \leq \widetilde{\preccurlyeq} \in \mathfrak{E}(X),\ A \widetilde{\preccurlyeq} B \\ &\Leftrightarrow A \preccurlyeq_{\sqsubset_w} B. \end{aligned}$$

Therefore $\preccurlyeq_{\mathcal{T}_{\sqsubset_w}} \leq \preccurlyeq_{\sqsubset_w}$.

Conversely, if $A \preccurlyeq_{\sqsubset_w} B$, then $A \widetilde{\preccurlyeq} B$ (i.e., $cl_{\widetilde{\preccurlyeq}}(A) \leq B$) for each $\sqsubset_w \leq \widetilde{\preccurlyeq} \in \mathfrak{E}(X)$. Let $D = \bigwedge_{\sqsubset_w \leq \widetilde{\preccurlyeq} \in \mathfrak{E}(X)} cl_{\widetilde{\preccurlyeq}}(A)$. Then for each $\sqsubset_w \leq \widetilde{\preccurlyeq} \in \mathfrak{E}(X)$, we have

$$D \leq cl_{\mathcal{T}_{\sqsubset_w}}(D) \leq cl_{\mathcal{T}_{\widetilde{\preccurlyeq}}}(D) \leq cl_{\mathcal{T}_{\widetilde{\preccurlyeq}}}(cl_{\widetilde{\preccurlyeq}}(A)) = cl_{\widetilde{\preccurlyeq}}(A) \leq B.$$

By arbitrariness of $\sqsubset_w \leq \widetilde{\preccurlyeq} \in \mathfrak{E}(X)$, we have

$$D \leq cl_{\mathcal{T}_{\sqsubset_w}}(D) \leq \bigwedge_{\sqsubset_w \leq \widetilde{\preccurlyeq} \in \mathfrak{E}(X)} cl_{\widetilde{\preccurlyeq}}(A) = D.$$

Thus $cl_{\mathcal{T}_{\sqsubset_w}}(A) \leq D \in \mathcal{T}_{\sqsubset_w}$ which shows $A \preccurlyeq_{\mathcal{T}_{\sqsubset_w}} B$. Therefore $\preccurlyeq_{\sqsubset_w} \leq \preccurlyeq_{\mathcal{T}_{\sqsubset_w}}$. □

Theorem 6.15. *Let X be a set equipped with an L-topological enclosed relation $\preccurlyeq$ and an L-convex enclosed relation $\leqslant$. Let $\sqsubset_w$ be the L-enclosed relation generated by $\preccurlyeq$ and $\leqslant$, and $\preccurlyeq_{\sqsubset_w}$ be the L-topological enclosed relation generated by $\sqsubset_w$. For any $F \in \mathfrak{F}(L^X)$, we denote $\mathfrak{F}_F(L^X) = \{H \in \mathfrak{F}(L^X) : F \in \mathfrak{F}(H)\}$ and $C_F = \bigwedge_{H \in \mathfrak{F}_F(L^X)} co_{\leqslant}(H)$. Then the following conditions are equivalent.*

(1) $(X, \preccurlyeq, \leqslant)$ is an L-topological-convex enclosed relation space;

(2) $(X, \mathcal{T}_{\sqsubset_w}, \mathcal{C}_{\leqslant})$ is an L-topological-convex space;

(3) $(X, \preccurlyeq_{\sqsubset_w}, \leqslant)$ is an L-topological-convex enclosed relation space;

(4) $C_F \in \mathcal{T}_{\sqsubset_w}$ for all $F \in \mathfrak{F}(L^X)$;

(5) $C_F \in \mathcal{T}_{\preccurlyeq}$ for all $F \in \mathfrak{F}(L^X)$;

(6) $C_F \in \varphi_{\sqsubset_w}$ for all $F \in \mathfrak{F}(L^X)$.

Proof. (1) ⇒ (2): By (2) of Theorem 6.14, $\mathcal{T}_{\sqsubset_w} = \mathcal{T}_w$ which is an L-topology with $\mathcal{T}_{\preccurlyeq} \cap \mathcal{C}_{\leqslant}$ as its closed subbase. Let $F \in \mathfrak{F}(L^X)$. By (LDF),

$$co_{\leqslant}(F) = \bigvee_{G \in \mathfrak{F}(F)} co_{\leqslant}(G) \leq \bigvee_{G \in \mathfrak{F}(F)} cl_{\mathcal{T}_{\sqsubset_w} \cap \mathcal{C}_{\leqslant}}(G).$$

Conversely, since $(X, \preccurlyeq, \lessdot)$ is an L-topological-convex enclosed relation space,

$$co_{\lessdot}(F) = \bigvee_{G \in \mathfrak{F}(F)} cl_{\mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\lessdot}}(G).$$

Thus $cl_{\mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\lessdot}}(G) \leq co_{\lessdot}(F)$ for each $G \in \mathfrak{F}(F)$. Further, for any $G \in \mathfrak{F}(F)$,

$$G \in \mathfrak{F}(F) = \mathfrak{F}(\bigvee_{R \in \mathfrak{F}(F)} R) = \bigcup_{R \in \mathfrak{F}(F)} \mathfrak{F}(R).$$

Thus there is $R \in \mathfrak{F}(F)$ such that $G \in \mathfrak{F}(R)$. Let $B_G = \bigwedge_{H \in \mathfrak{F}_G(L^X)} cl_{\mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\lessdot}}(H)$. We have $G \leq B_G \in \mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\lessdot}$ which implies $cl_{\mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}}(G) \leq B_G$.

Since $R \in \mathfrak{F}(F)$ and $G \in \mathfrak{F}(R)$, we have $B_G \leq cl_{\mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\lessdot}}(R) \leq co_{\preccurlyeq}(F)$. Hence $cl_{\mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}}(G) \leq co_{\preccurlyeq}(F)$ for any $G \in \mathfrak{F}(F)$. So $\bigvee_{G \in \mathfrak{F}(F)} cl_{\mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}}(G) \leq co_{\lessdot}(F)$. Therefore $(X, \mathfrak{T}_{\sqsubseteq_w}, \mathfrak{C}_{\lessdot})$ is an L-topological-convex space.

(2) ⇔ (3): It directly follows from (3) of Theorem 6.14 and Theorem 6.8.

(2) ⇒ (4): If $x_\lambda \nleq C_F$, then there is $H \in \mathfrak{F}_F(L^X)$ such that $x_\lambda \nleq co_{\lessdot}(H)$. Since $F \in \mathfrak{F}(H)$, there is $R \in \mathfrak{F}(H)$ such that $F \in \mathfrak{F}(R)$.

Since $(X, \mathfrak{T}_{\sqsubseteq_w}, \mathfrak{C}_{\lessdot})$ is an L-topological-convex space, we have

$$cl_{\mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}}(R) \leq \bigvee_{G \in \mathfrak{F}(H)} cl_{\mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}}(G) = co_{\lessdot}(H).$$

Thus $x_\lambda \nleq cl_{\mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}}(R)$. Further, we have $cl_{\sqsubseteq_w}(C_F) \leq cl_{\mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}}(R)$ since

$$C_F = \bigwedge_{H \in \mathfrak{F}_F(L^X)} co_{\lessdot}(H) \leq co_{\lessdot}(R) \leq cl_{\mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}}(R) \in \mathfrak{T}_{\sqsubseteq_w} \cap \mathfrak{C}_{\lessdot}.$$

Hence $x_\lambda \nleq cl_{\sqsubseteq_w}(C_F)$. Therefore $cl_{\sqsubseteq_w}(C_F) \leq C_F$ which shows that $C_F \in \mathfrak{T}_{\sqsubseteq_w}$.

(4) ⇒ (5): Since $\mathfrak{T}_{\sqsubseteq_w} = \mathfrak{T}_w \subseteq \mathfrak{T}_{\preccurlyeq}$, the result is clear.

(5) ⇒ (6): Since $C_F \in \mathfrak{T}_{\preccurlyeq}$, we have

$$cl_{\preccurlyeq}(C_F) = C_F = \bigwedge_{H \in \mathfrak{F}_F(L^X)} co_{\lessdot}(H) \in \mathfrak{C}_{\lessdot}.$$

Thus $C_F \in \mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\lessdot} = \varphi_{\sqsubseteq_w}$ by (1) of Theorem 6.14.

(6) ⇒ (1): Let $B \in L^X$ with $F \lessdot B$. Then $co_{\lessdot}(F) \leq B$. Let $G \in \mathfrak{F}(F)$. Then $F \in \mathfrak{F}_G(L^X)$ and so $C_G \leq co_{\lessdot}(F)$. In addition, by (2) of Theorem 6.14 and (6), $G \leq C_G \in \varphi_{\sqsubseteq_w} = \mathfrak{T}_{\preccurlyeq} \cap \mathfrak{C}_{\lessdot}$. Thus

$$G \leq C_G \lessdot C_G \preccurlyeq C_G \leq co_{\lessdot}(F) \leq B.$$

Therefore $(X, \preccurlyeq, \lessdot)$ is an L-topological-convex enclosed relation space. □

7 Conclusions

In this paper, axiomatic definitions of both L-convex bases and L-convex subbases are introduced. It is proved that the category of L-convex spaces is a coreflective subcategory of both the category of L-convex base spaces and the category of L-convex subbase spaces. In particular, it is also proved that the category of L-convex spaces is a bireflective subcategory of the category of L-closure spaces. Further, by L-convex bases, the notion of L-topological-convex space is introduced which is a triple consisting of an L-cotopology and a compatible L-convex structure on the same set. L-topological-convex spaces can be characterized by many means including L-topological-convex enclosed relation spaces.

L-convex bases are quite different to convex bases when $L \neq \{\bot, \top\}$. Specifically, a subset of a convex structure is a base iff it contains all polytopes [2]. That is, if $(X, \mathcal{C})$ is a convex space, then $\mathcal{B} \subseteq \mathcal{C}$ is a base of $\mathcal{C}$ iff $co(F) \in \mathcal{B}$ for any $F \in 2^X_{fin}$. However, as we can see in Remark 3.3, an L-convex base $\mathcal{B}$ of an L-convex space

$(X, \mathcal{C})$ may fails to imply $co(F) \in \mathcal{B}$ for all $F \in \mathfrak{F}(L^X)$. Thus we didn't directly extend the original definition of topological-convex spaces to L-setting.

As we can see, L-convergence spaces are closely related to L-topological spaces [34–37]. Similar to the compatibility between an L-cotopology and an L-convex structure, it could be possible to discuss the compatibility between an L-convergence structure and an L-convex structure. Further, it could be possible to characterize L-topological-convex spaces by such compatibility.

Topological-convex spaces in Convex Theory is a basic notion in combining Topology Theory and Convex Theory. With such spaces, many combined properties can be investigated including continuities of hull operators, compactness and uniformity of convex spaces. Thus this paper could be helpful in discussing L-topological-concave spaces in the future.

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