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Hankel determinant of order three for familiar subsets of analytic functions related with sine function

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Abstract: In this paper we define and consider some familiar subsets of analytic functions associated with sine functions in the region of unit disk on the complex plane. For these classes our aim is to find the Hankel determinant of order three.

Keywords: subordinations, trigonometric function, Hankel determinant

MSC 2010: 30C45, 30C50

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}), \tag{1.1}$$

which are analytic in the region $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote a subclass of $\mathcal{A}$ which contains univalent functions in $\mathbb{D}$. An analytic function f is subordinate to an analytic function g (written as $f \prec g$) if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{D}$ such that $f(z) = g(w(z))$. In particular if g is univalent in $\mathbb{D}$, then $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. The familiar classes of starlike, convex and bounded turning functions are defined respectively in terms of subordination as

$$\begin{aligned} \mathcal{S}^{\star} &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \psi(z), \quad z \in \mathbb{D} \right\}, \\ \mathcal{C} &= \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z), \quad z \in \mathbb{D} \right\}, \\ \mathcal{R} &= \left\{ f \in \mathcal{S} : f'(z) \prec \psi(z), \quad z \in \mathbb{D} \right\}, \end{aligned} \tag{1.2}$$

where $\psi(z) = \frac{1+z}{1-z}$. By choosing suitable function ψ in (1.2), we obtain several subfamilies of $\mathcal{A}$ which have interesting geometric interpretation, see [1–11]. From these subfamilies, we recall here which are connected

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with trigonometric functions and are defined as follows:

$$\mathcal{C}_{\sin} = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \sin(z) \right\}, \tag{1.3}$$

$$\mathcal{S}^{\star}_{\sin} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \sin(z) \right\}, \tag{1.4}$$

$$\mathcal{R}_{\sin} = \left\{ f \in \mathcal{S} : f'(z) \prec 1 + \sin(z) \right\}. \tag{1.5}$$

The class $\mathcal{S}^{\star}_{\sin}$ of analytic function defined in (1.4) was introduced by Cho et al. [6] and they also study the radii problems for this class of functions.

The familiar coefficient conjecture for the functions $f \in \mathcal{S}$ having the series form (1.1), was given by Bieberbach [12] in 1916 and it was later proved by de-Branges [13] in 1985. It was one of the most celebrated conjecture in classical analysis, one that has stood as a challenge to mathematician for a very long time. During this period, many mathematicians worked hard to prove this conjecture and as result they established coefficient bounds for some subfamilies of the class $\mathcal{S}$ of univalent functions. They also develop some new inequalities related with coefficient bounds of some subclasses of univalent functions. Fekete-Szegő inequality is one of the inequality for the coefficients of univalent analytic functions found by Fekete and Szegő (1933), related to the Bieberbach conjecture. An other coefficient problem which is closely related with Fekete and Szegő is Hankel determinant. Hankel determinants are very useful in the investigations of the singularities and power series with integral coefficients.

For given parameters $q,\ n \in \mathbb{N} = \{1, 2, \ldots\}$, the Hankel determinant $H_{q,n}(f)$ of a function $f \in \mathcal{S}$ of the form (1.1) was introduced by Pommerenke [14, 15] as:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.6}$$

The growth of $H_{q,n}(f)$ has been investigated for different subfamilies of univalent functions. In particular, the absolute sharp bound of the functional $H_{2,2}(f) = a_2 a_4 - a_3^2$ for each of the sets $\mathcal{C}$, $\mathcal{S}^{\star}$ and $\mathcal{R}$ were found by Janteng et al. [16, 17] while the exact estimate of this determinant for the family of close-to-convex functions is still unknown (see [18]). On the other hand, the best estimate of $|H_{2,2}(f)|$ for the set of Bazilevič functions was proved by Krishna et al. [19]. For more work on $H_{2,2}(f)$, see [20–24] and reference therein.
The $H_{3,1}(f)$ determinant for a function f of the from (1.1) is defined as follows:

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

The sharp estimation of $|H_{3,1}(f)|$ is tedious as compared to $|H_{2,2}(f)|$. Babalola [25] obtained the upper bound of $|H_{3,1}(f)|$ for the subfamilies of $\mathcal{S}^{\star}$, $\mathcal{C}$ and $\mathcal{R}$ in 2010. Later on, the upper bound of $|H_{3,1}(f)|$ for various subfamilies of analytic and univalent functions were studied, see [26–33]. In 2017, Zaprawa [34] improved the results of Babalola and showed that

$$|H_{3,1}(f)| \le \begin{cases} 1, & \text{for } f \in \mathcal{S}^{\star}, \\ \frac{49}{540}, & \text{for } f \in \mathcal{C}, \\ \frac{41}{60}, & \text{for } f \in \mathcal{R}. \end{cases}$$

The results in [34] are not sharp for the families $\mathcal{S}^{\star}$, $\mathcal{C}$ and $\mathcal{R}$. Moreover, Zaprawa found the sharp bounds of $|H_{3,1}(f)|$ for subfamilies of $\mathcal{S}^{\star}$, $\mathcal{C}$ and $\mathcal{R}$ comprising of m-fold symmetric functions. Recently, Kowalczyk et al. [35] and Lecko et al. [36] obtained the sharp inequalities

$$|H_{3,1}(f)| \le 4/135, \text{ and } |H_{3,1}(f)| \le 1/9,$$

for the sets $\mathcal{C}$ and $\mathcal{S}^{\star}(1/2)$ respectively, where $\mathcal{S}^{\star}(1/2)$ is the family of starlike functions of order $1/2$. More recently in 2019, Kwon et al. [37] got an improved bound $|H_{3,1}(f)| \le 8/9$ for $f \in \mathcal{S}^{\star}$. In this paper, our aim is to study $|H_{3,1}(f)|$ for the families of functions defined in the relations (1.3), (1.4) and (1.5).

2 A set of lemmas

Let $\mathcal{P}$ denote the family of all functions p which are analytic in $\mathbb{D}$ with $\mathfrak{Re}\left(p(z)\right) > 0$ and has the following series representation

$$p\left(z\right) = 1 + \sum_{n=1}^{\infty} c_n\, z^n \; \left(z \in \mathbb{D}\right). \tag{2.1}$$

Lemma 2.1. *If $p \in \mathcal{P}$ and has the form* (2.1), *then*

$$\left|c_n\right| \le 2 \;\; for \;\; n \ge 1, \tag{2.2}$$
$$\left|c_{n+k} - \mu c_n c_k\right| < 2, \; for\; 0 \le \mu \le 1, \tag{2.3}$$
$$\left|c_m c_n - c_k c_l\right| \le 4 \;\; for \;\; m + n = k + l, \tag{2.4}$$
$$\left|c_{n+2k} - \mu c_n c_k^2\right| \le 2(1 + 2\mu); \;\; for \;\; \mu \in \mathbb{R}, \tag{2.5}$$
$$\left|c_2 - \frac{c_1^2}{2}\right| \le 2 - \frac{\left|c_1\right|^2}{2}, \tag{2.6}$$

and for complex number λ, we have

$$\left|c_2 - \lambda c_1^2\right| \le \max\left\{2, 2\left|\lambda - 1\right|\right\}. \tag{2.7}$$

For the results in (2.2), (2.6), (2.3), (2.5), (2.4) *see [38]. Also, see [39] for the inequality* (2.7).

Lemma 2.2. *Let $p \in \mathcal{P}$ and has the form* (2.1), *then*

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \le 2\left|J\right| + 2\left|K - 2J\right| + 2\left|J - K + L\right|. \tag{2.8}$$

Proof. Consider the left hand side of (2.8) and rearranging the terms, we have

$$\begin{aligned}
\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| &= \left|J\left(c_1^3 - 2c_1c_2 + c_3\right) - \left(K - 2J\right)\left(c_1c_2 - c_3\right) + \left(J - K + L\right)c_3\right| \\
&\le \left|J\right|\left|c_1^3 - 2c_1c_2 + c_3\right| + \left|K - 2J\right|\left|c_1c_2 - c_3\right| + \left|J - K + L\right|\left|c_3\right| \\
&\le 2\left|J\right| + 2\left|K - 2J\right| + 2\left|J - K + L\right|,
\end{aligned}$$

where we have used (2.2), (2.3) and the result $\left|c_1^3 - 2c_1c_2 + c_3\right| \le 2$ due to [40]. □

3 Bound of $|H_{3,1}(f)|$ for the set $\mathcal{C}_{\sin}$

Theorem 3.1. *If $f \in \mathcal{C}_{\sin}$ and of the form* (1.1), *then*

$$\left|a_2\right| \le \frac{1}{2}, \tag{3.1}$$
$$\left|a_3\right| \le \frac{1}{6}, \tag{3.2}$$
$$\left|a_4\right| \le \frac{13}{144}, \tag{3.3}$$
$$\left|a_5\right| \le \frac{409}{7200}. \tag{3.4}$$

The first two bounds are best possible.

Proof. If $f \in \mathcal{C}_{\sin}$, then the relation (1.1) leads us to

$$1 + \frac{zf''\left(z\right)}{f'\left(z\right)} = 1 + \sin\left(w\left(z\right)\right), \tag{3.5}$$

where w is a Schwarz function. Consider a function p such that

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots, \tag{3.6}$$

then $p \in \mathcal{P}$. This implies that

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

From (1.1) , we can write

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= 1 + 2a_2 z + \left(6a_3 - 4a_2^2\right) z^2 + \left(12a_4 - 18a_2 a_3 + 8a_2^3\right) z^3 \\ &\quad + \left(20a_5 - 18a_3^2 - 32a_2 a_4 + 48a_2^2 a_3 - 16a_2^4\right) z^4 + \cdots \end{aligned} \tag{3.7}$$

After some simple calculations, we obtain

$$\begin{aligned} 1 + \sin(w(z)) &= 1 - w(z) + \frac{(w(z))^3}{3!} - \frac{(w(z))^5}{5!} + \frac{(w(z))^7}{7!} - \cdots \\ &= 1 + \frac{1}{2} c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{5}{48} c_1^3 + \frac{c_3}{2} - \frac{c_1 c_2}{2}\right) z^3 + \\ &\quad \left(\frac{c_4}{2} + \frac{5}{16} c_1^2 c_2 - \frac{c_1^4}{32} - \frac{c_1 c_3}{2} - \frac{c_2^2}{4}\right) z^4 + \cdots. \end{aligned} \tag{3.8}$$

From (3.5) , (3.7) and (3.8), it follows that

$$a_2 = \frac{c_1}{4}, \tag{3.9}$$

$$a_3 = \frac{c_2}{12}, \tag{3.10}$$

$$a_4 = \frac{1}{12}\left(\frac{c_3}{2} - \frac{c_1 c_2}{8} - \frac{c_1^3}{48}\right), \tag{3.11}$$

$$a_5 = \frac{1}{20}\left(\frac{5}{288} c_1^4 + \frac{c_4}{2} - \frac{c_1 c_3}{6} - \frac{c_1^2 c_2}{48} - \frac{c_2^2}{8}\right). \tag{3.12}$$

Applying relation (2.2) in (3.9) and (3.10), we obtain

$$|a_2| \le \frac{1}{2} \text{ and } |a_3| \le \frac{1}{6}.$$

By rearranging (3.11), it gives

$$|a_4| = \frac{1}{12}\left|\frac{1}{4}\left(c_3 - \frac{c_1 c_2}{2}\right) + \frac{1}{4}\left(c_3 - \frac{c_1^3}{12}\right)\right|.$$

Application of triangle inequality along with (2.3) and (2.5) leads us to

$$|a_4| \le \frac{1}{12}\left\{\frac{1}{2} + \frac{7}{12}\right\} = \frac{13}{144}.$$

From (3.12), it follows that

$$|a_5| = \frac{1}{20}\left|\frac{1}{4}\left(c_4 - \frac{7}{18} c_2^2\right) + \frac{1}{4}\left(c_4 - \frac{2}{3} c_1 c_3\right) - \frac{5}{144} c_1^2 \left(c_2 - \frac{c_1^2}{2}\right) - \frac{c_2}{36}\left(c_2 - \frac{c_1^2}{2}\right)\right|.$$

Using triangle inequality along with (2.2), (2.6) and (2.3), we get

$$|a_5| \le \frac{1}{20}\left\{\frac{1}{2} + \frac{1}{2} + \frac{5}{144} |c_1|^2 \left(2 - \frac{|c_1|^2}{2}\right) + \frac{1}{18}\left(2 - \frac{|c_1|^2}{2}\right)\right\}.$$

Suppose that $|c_1| = x$ and $x \in [0, 2]$, therefore

$$|a_5| \leq \frac{1}{20}\left\{1 + \frac{5}{144}x^2\left(2 - \frac{x^2}{2}\right) + \frac{1}{18}\left(2 - \frac{x^2}{2}\right)\right\}.$$

The above function attains its maximum value at $x = \sqrt{\frac{6}{5}} = 1.0954$, hence

$$|a_5| \leq \frac{409}{7200}.$$

Equality for the bounds given in (3.1) and (3.2) , is obtained by taking

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \sin(z)\,. \tag{3.13}$$

□

Theorem 3.2. *Let $f \in \mathcal{C}_{\sin}$. Then for a complex number γ*

$$\left|a_3 - \gamma a_2^2\right| \leq \max\left\{\frac{1}{6}, \frac{1}{12}|3\gamma - 2|\right\}. \tag{3.14}$$

This result is sharp.

Proof. Using (3.9) and (3.10), one may write

$$\left|a_3 - \gamma a_2^2\right| = \left|\frac{c_2}{12} - \frac{\gamma}{16}c_1^2\right| = \frac{1}{12}\left|\left(c_2 - \frac{1}{2}\left(\frac{3}{2}\gamma\right)c_1^2\right)\right|.$$

Application of relation (2.7), gives

$$\left|a_3 - \gamma a_2^2\right| \leq \max\left\{\frac{1}{6}, \frac{1}{12}|3\gamma - 2|\right\}.$$

Thus we have the required result. □

Taking $\gamma = 1$ in last Theorem, we obtain

Corollary 3.3. *If $f \in \mathcal{C}_{\sin}$ with the series form* (1.1), *then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{6}. \tag{3.15}$$

This inequality is sharp.

Theorem 3.4. *Let $f \in \mathcal{C}_{\sin}$. Then*

$$|a_2 a_3 - a_4| \leq \frac{13}{144}. \tag{3.16}$$

Proof. From (3.9), (3.10) and (3.11), we have

$$|a_2 a_3 - a_4| = \left|\frac{c_1 c_2}{32} + \frac{c_1^3}{576} - \frac{c_3}{24}\right|.$$

Application of Lemma 2.2, Leads us to

$$|a_2 a_3 - a_4| \leq \frac{13}{144}.$$

This completes the proof. □

Theorem 3.5. *Let $f \in \mathcal{C}_{\sin}$. Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{7}{144}. \tag{3.17}$$

Proof. From (3.9), (3.10) and (3.11), we may write

$$\left|a_2a_4 - a_3^2\right| = \left|\frac{c_1c_3}{96} - \frac{c_1^4}{2304} - \frac{c_1^2c_2}{384} - \frac{c_2^2}{144}\right| = \left|\frac{c_1}{288}\left(c_3 - \frac{3}{4}c_1c_2\right) + \frac{(c_1c_3 - c_2^2)}{144} - \frac{c_1^4}{2304}\right|.$$

Application of triangle inequality as well as (2.2), (2.3) and (2.4), we obtain

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{288} + \frac{4}{144} + \frac{16}{2304} = \frac{7}{144}.$$

This completes the result. □

Theorem 3.6. *Let $f \in \mathcal{C}_{\sin}$. Then*

$$|H_{3,1}(f)| \leq \frac{13333}{518400}.$$

Proof. From (1.6) , it is easy to see that

$$H_{3,1}(f) = a_3\left(a_2a_4 - a_3^2\right) - a_4(a_4 - a_2a_3) + a_5\left(a_3 - a_2^2\right),$$

where $a_1 = 1$. This implies that

$$|H_{3,1}(f)| \leq |a_3|\left|a_2a_4 - a_3^2\right| + |a_4||a_4 - a_2a_3| + |a_5|\left|a_3 - a_2^2\right|.$$

By using (3.1), (3.2), (3.3), (3.4), (3.15), (3.16) and (3.17), we obtain the required result. □

4 Bound of $|H_{3,1}(f)|$ for the set $\mathcal{S}^{\star}_{\sin}$

Theorem 4.1. *Let $f \in \mathcal{S}^{\star}_{\sin}$. Then*

$$|a_2| \leq 1, \tag{4.1}$$

$$|a_3| \leq \frac{1}{2}, \tag{4.2}$$

$$|a_4| \leq \frac{13}{36}, \tag{4.3}$$

$$|a_5| \leq \frac{409}{1440}. \tag{4.4}$$

The first two coefficients bounds are best possible.

Proof. Let $f \in \mathcal{S}^{\star}_{\sin}$. Then we can write (1.4), in terms of Schwarz function as

$$\frac{zf'(z)}{f(z)} = 1 + \sin(w(z)), \quad (z \in \mathbb{D}).$$

Since,

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + \left(2a_3 - a_2^2\right)z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right)z^3 + \\ + \left(4a_5 - 2a_3^2 - 4a_2a_4 + 4a_2^2a_3 - a_2^4\right)z^4 + \cdots. \tag{4.5}$$

By comparing (4.5) and (3.8), we may write

$$a_2 = \frac{c_1}{2}, \tag{4.6}$$

$$a_3 = \frac{c_2}{4}, \tag{4.7}$$

$$a_4 = \frac{1}{3}\left(\frac{c_3}{2} - \frac{c_1c_2}{8} - \frac{c_1^3}{48}\right), \tag{4.8}$$

$$a_5 = \frac{1}{4}\left(\frac{5}{288}c_1^4 + \frac{c_4}{2} - \frac{c_1c_3}{6} - \frac{c_1^2c_2}{48} - \frac{c_2^2}{8}\right). \tag{4.9}$$

Using coefficient bounds for class $\mathcal{P}$ given in (2.2) in (4.6) and (4.7), we get

$$|a_2| \le 1 \text{ and } |a_3| \le \frac{1}{2}.$$

From (4.7), we write

$$|a_4| = \frac{1}{12}\left|\frac{1}{4}\left(c_3 - \frac{c_1c_2}{2}\right) + \frac{1}{4}\left(c_3 - \frac{c_1^3}{12}\right)\right|.$$

Application of triangle inequality along with (2.3) and (2.5) lead us to

$$|a_4| \le \frac{1}{3}\left\{\frac{1}{2} + \frac{7}{12}\right\} = \frac{13}{36}.$$

Now

$$|a_5| = \frac{1}{4}\left|\frac{1}{4}\left(c_4 - \frac{2}{3}c_1c_3\right) + \frac{1}{4}\left(c_4 - \frac{7}{18}c_2^2\right) - \frac{5}{144}c_1^2\left(c_2 - \frac{c_1^2}{2}\right) - \frac{c_2}{36}\left(c_2 - \frac{c_1^2}{2}\right)\right|.$$

By using (2.2), (2.6) and (2.3), we have

$$|a_5| \le \frac{1}{4}\left\{\frac{1}{2} + \frac{1}{2} + \frac{5}{144}|c_1|^2\left(2 - \frac{|c_1|^2}{3}\right) + \frac{1}{18}\left(2 - \frac{|c_1|^2}{3}\right)\right\}.$$

Let $|c_1| = x \in [0, 2]$. Then

$$|a_5| \le \frac{1}{4}\left\{1 + \frac{5}{144}x^2\left(2 - \frac{x^2}{3}\right) + \frac{1}{18}\left(2 - \frac{x^2}{3}\right)\right\}.$$

Suppose that

$$\Phi(x) = \frac{1}{4}\left\{1 + \frac{5}{144}x^2\left(2 - \frac{x^2}{3}\right) + \frac{1}{18}\left(2 - \frac{x^2}{3}\right)\right\}.$$

The function Φ has maximum value at $x = \sqrt{\frac{6}{5}}$, therefore

$$|a_5| \le \frac{409}{1440}.$$

Equality is attained for the first two bounds for the function

$$f(z) = z\exp\left(\int_0^z \frac{\sin t - 1}{t}dt\right) = z + z^2 + \frac{z^3}{2} + \frac{z^4}{9} - \frac{z^5}{72} + \cdots. \tag{4.10}$$

□

Theorem 4.2. *Let $f \in \mathcal{S}^*_{\sin}$. Then for a complex number γ*

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{\frac{1}{2}, \frac{1}{2}|\gamma - 1|\right\}. \tag{4.11}$$

Equality is acheived for the function defined in (4.10) .

Proof. Using (4.6) and (4.7), we may write

$$\left|a_3 - \gamma a_2^2\right| = \left|\frac{c_2}{4} - \frac{\gamma}{4}c_1^2\right| = \frac{1}{4}\left|\left(c_2 - \gamma c_1^2\right)\right|.$$

Application of (2.7), leads us to

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{\frac{1}{2}, |\gamma - 1|\right\}.$$

Hence the result is completed. □

If we put $\gamma = 1$ in the last result, we get the following result.

Corollary 4.3. *If $f \in \mathcal{S}^{\star}_{\sin}$, then*

$$\left|a_3 - a_2^2\right| \le \frac{1}{2}. \tag{4.12}$$

This result is sharp.

Theorem 4.4. *If $f \in \mathcal{S}^{\star}_{\sin}$, then*

$$|a_2 a_3 - a_4| \le \frac{7}{18}. \tag{4.13}$$

Proof. From (4.6), (4.7) and (4.8), we have

$$|a_2 a_3 - a_4| = \left|\frac{c_1 c_2}{6} + \frac{c_1^3}{144} - \frac{c_3}{6}\right|.$$

Application of triangle inequality and Lemma 2.2, leads us to

$$|a_2 a_3 - a_4| \le \frac{7}{18}.$$

Thus the proof is completed. □

Theorem 4.5. *If $f \in \mathcal{S}^{\star}_{\sin}$, then*

$$\left|a_2 a_4 - a_3^2\right| \le \frac{17}{36}. \tag{4.14}$$

Proof. From (4.6), (4.7) and (4.8), we have

$$\left|a_2 a_4 - a_3^2\right| = \left|\frac{c_1 c_3}{12} - \frac{c_1^4}{288} - \frac{c_1^2 c_2}{48} - \frac{c_2^2}{16}\right| = \left|\frac{1}{16}\left(c_1 c_3 - c_2^2\right) + \frac{c_1}{48}\left(c_3 - c_1 c_2\right) - \frac{c_1^4}{288}\right|.$$

Using triangle inequality as well as (2.2), (2.3) and (2.4), we get

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{4} + \frac{1}{12} + \frac{1}{18} = \frac{17}{36}.$$

□

Theorem 4.6. *If $f \in \mathcal{S}^{\star}_{\sin}$, then*

$$|H_{3,1}(f)| \le \frac{13441}{25920}.$$

Proof. From (1.6), we may write

$$|H_{3,1}(f)| \le |a_3| \left|a_2 a_4 - a_3^2\right| + |a_4|\,|a_4 - a_2 a_3| + |a_5| \left|a_3 - a_2^2\right|.$$

By using(4.1), (4.2), (4.3), (4.4), (4.12), (4.13) and (4.14), we obtain the required result. □

5 Bound of $|H_{3,1}(f)|$ for the set $\mathcal{R}_{\sin}$

Theorem 5.1. *If $f \in \mathcal{R}_{\sin}$, then*

$$|a_2| \le \frac{1}{2}, \tag{5.1}$$

$$|a_3| \le \frac{1}{3}, \tag{5.2}$$

$$|a_4| \le \frac{1}{4}, \tag{5.3}$$

$$|a_5| \le \frac{17}{40}. \tag{5.4}$$

Proof. Let $f \in \mathcal{R}_{\sin}$. Then we can write

$$f'(z) = 1 + \sin(w(z)), \quad (z \in \mathbb{D}).$$

Since,

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \tag{5.5}$$

By comparing (5.5) and (3.8), we may get

$$a_2 = \frac{c_1}{4}, \tag{5.6}$$

$$a_3 = \frac{1}{3}\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right), \tag{5.7}$$

$$a_4 = \frac{1}{4}\left(\frac{5}{48}c_1^3 + \frac{c_3}{2} - \frac{c_1 c_2}{2}\right), \tag{5.8}$$

$$a_5 = \frac{1}{5}\left(\frac{c_4}{2} + \frac{5}{16}c_1^2 c_2 - \frac{c_1^4}{32} - \frac{c_1 c_3}{2} - \frac{c_2^2}{4}\right). \tag{5.9}$$

By using (2.2) in (5.6), we obtain

$$|a_2| \le \frac{1}{2}.$$

Now using (2.6), in (5.7), we get

$$|a_3| \le \frac{1}{3}.$$

Application of triangle inequality and Lemma 2.2 in (5.8) lead us to

$$|a_4| \le \frac{1}{4}.$$

From (5.9), we may write

$$|a_5| = \frac{1}{5}\left|\frac{c_1^2 c_2}{8} + \frac{1}{2}(c_4 - c_1 c_3) + \frac{c_1^2}{16}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{c_2}{4}\left(c_2 - \frac{c_1^2}{2}\right)\right|.$$

By using (2.2), (2.6) and (2.3), we obtain

$$|a_5| \le \frac{1}{5}\left\{\frac{|c_1|^2}{4} + 1 + \frac{|c_1|^2}{16}\left(2 - \frac{|c_1|^2}{2}\right) + \frac{1}{2}\left(2 - \frac{|c_1|^2}{2}\right)\right\}.$$

Let $|c_1| = x \in [0, 2]$. Then

$$|a_5| \le \frac{1}{5}\left\{\frac{x^2}{4} + 1 + \frac{x^2}{16}\left(2 - \frac{x^2}{2}\right) + \frac{1}{2}\left(2 - \frac{x^2}{2}\right)\right\}.$$

Consider the function

$$\Phi_1(x) = \frac{1}{5}\left\{\frac{x^2}{4} + 1 + \frac{x^2}{16}\left(2 - \frac{x^2}{2}\right) + \frac{1}{2}\left(2 - \frac{x^2}{2}\right)\right\}.$$

The above function has its maximum at $x = \sqrt{2}$, therefore we obtain

$$|a_5| \le \frac{17}{40}.$$

□

Theorem 5.2. *If $f \in \mathcal{R}_{\sin}$, then*

$$\left|a_3 - \gamma a_2^2\right| \le \max\left\{\frac{2}{3}, \frac{1}{12}|3\gamma - 4|\right\}, \tag{5.10}$$

where γ is any complex number.

Proof. From (5.6) and (5.7), we may get

$$\left|a_3-\gamma a_2^2\right| = \left|\frac{c_2}{3}-\frac{c_1^2}{12}-\frac{\gamma}{16}c_1^2\right| = \frac{1}{3}\left|\left\{c_2-\frac{1}{2}\left(\frac{4+3\gamma}{8}\right)c_1^2\right\}\right|.$$

Application of (2.7), leads us to

$$\left|a_3-\gamma a_2^2\right| \le \max\left\{\frac{2}{3}, \frac{1}{12}\left|3\gamma-4\right|\right\}.$$

Thus the proof is completed. □

Making $\gamma = 1$ in the last result, we get the following result.

Corollary 5.3. *If $f \in \mathcal{R}_{\sin}$, then*

$$\left|a_3-a_2^2\right| \le \frac{2}{3}. \tag{5.11}$$

Theorem 5.4. *If $f \in \mathcal{R}_{\sin}$, then*

$$\left|a_2a_3-a_4\right| \le 0.39434. \tag{5.12}$$

Proof. From (5.6), (5.7) and (5.8), we have

$$\left|a_2a_3-a_4\right| = \left|\frac{c_1c_2}{6}-\frac{3}{64}c_1^3-\frac{c_3}{8}\right| = \left|\frac{3}{32}c_1\left(c_2-\frac{c_1^2}{2}\right)-\frac{1}{8}\left(c_3-\frac{7}{12}c_1c_2\right)\right|.$$

Application of triangle inequality along with (2.2), (2.6) and (2.3) give us

$$\left|a_2a_3-a_4\right| \le \left\{\frac{3}{32}\left|c_1\right|\left(2-\frac{\left|c_1\right|^2}{2}\right)+\frac{1}{4}\right\}.$$

If we replace $|c_1| = x \in [0, 2]$, then

$$\left|a_2a_3-a_4\right| \le \frac{3}{32}x\left(2-\frac{x^2}{2}\right)+\frac{1}{4}.$$

Consider the function

$$\Phi_2(x) = \frac{3}{32}x\left(2-\frac{x^2}{2}\right)+\frac{1}{4}.$$

The function Φ_2 has its maximum value at $x = 1.15472$. Therefore

$$\left|a_2a_3-a_4\right| \le 0.39434$$

and this completes the proof. □

Theorem 5.5. *If $f \in \mathcal{R}_{\sin}$, then*

$$\left|a_2a_4-a_3^2\right| \le 0.45324. \tag{5.13}$$

Proof. From (5.6), (5.7), and (5.8), we have

$$\left|a_2a_4-a_3^2\right| = \left|\frac{7}{288}c_1^2c_2+\frac{c_1c_3}{32}-\frac{c_1^4}{2304}-\frac{c_2^2}{9}\right| = \left|\frac{c_1}{32}\left(c_3-c_1c_2\right)-\frac{c_1^4}{2304}-\frac{c_2}{9}\left(c_2-\frac{c_1^2}{2}\right)\right|.$$

Using triangle inequality along with (2.2), (2.6) and (2.3), we get

$$\left|a_2a_4-a_3^2\right| \le \left\{\frac{\left|c_1\right|}{16}+\frac{\left|c_1\right|^4}{2304}+\frac{2}{9}\left(2-\frac{\left|c_1\right|^2}{2}\right)\right\}.$$

If we put $|c_1| = x \in [0, 2]$, then we have

$$\left|a_2a_4-a_3^2\right| \le \frac{x}{16}+\frac{x^4}{2304}+\frac{2}{9}\left(2-\frac{x^2}{2}\right).$$

Now consider the function

$$\Phi_3(x) = \frac{x}{16} + \frac{x^4}{2304} + \frac{2}{9}\left(2 - \frac{x^2}{2}\right).$$

Then the above function takes its maximum at $x = 0.2814$. Thus

$$\left|a_2 a_4 - a_3^2\right| \le 0.45324.$$

This completes the proof. □

Theorem 5.6. *If $f \in \mathcal{R}_{\sin}$, then*

$$|H_{3,1}(f)| \le 0.53299.$$

Proof. From (1.6) , we may write

$$|H_{3,1}(f)| \le |a_3| \left|a_2 a_4 - a_3^2\right| + |a_4| \left|a_4 - a_2 a_3\right| + |a_5| \left|a_3 - a_2^2\right|.$$

By using (5.1), (5.2), (5.3), (5.4), (5.11), (5.12) and (5.13), we get the required result. □

6 Bounds of $|H_{3,1}(f)|$ for %2-fold symmetric and 3-fold symmetric functions

Let $m \in \mathbb{N} = \{1, 2, \ldots\}$. A domain Λ is said to be m-fold symmetric if a rotation of Λ about the origin through an angle $2\pi/m$ carries Λ on itself. It is easy to see that, an analytic function f is m-fold symmetric in $\mathbb{D}$, if

$$f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z), \ (z \in \mathbb{D}).$$

By $\mathcal{S}^{(m)}$, we mean the set of m-fold symmetric univalent functions having the following Taylor series form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \ \ (z \in \mathbb{D}). \tag{6.1}$$

The subfamilies $\mathcal{R}_{\sin}^{(m)}$, $\mathcal{S}_{\sin}^{*(m)}$ and $\mathcal{C}_{\sin}^{(m)}$ of $\mathcal{S}^{(m)}$ are the sets of m-fold symmetric bounded turning, starlike and convex functions respectively associated with sine functions. More intuitively, an analytic function f of the form (6.1) , belongs to the families $\mathcal{R}_{\sin}^{(m)}$, $\mathcal{S}_{\sin}^{*(m)}$ and $\mathcal{C}_{\sin}^{(m)}$, if and only if

$$f'(z) = 1 + \sin\left(\frac{p(z) - 1}{p(z) + 1}\right), \ p \in \mathcal{P}^{(m)}, \tag{6.2}$$

$$\frac{zf'(z)}{f(z)} = 1 + \sin\left(\frac{p(z) - 1}{p(z) + 1}\right), \ p \in \mathcal{P}^{(m)}, \tag{6.3}$$

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \sin\left(\frac{p(z) - 1}{p(z) + 1}\right), \ p \in \mathcal{P}^{(m)}, \tag{6.4}$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk}, \ \ (z \in \mathbb{D})\right\}. \tag{6.5}$$

Now we can prove the following theorem.

Theorem 6.1. *Let $f \in \mathcal{R}_{\sin}^{(2)}$, be of the form* (6.1). *Then*

$$|H_{3,1}(f)| \le \frac{1}{15}.$$

Proof. Since $f \in \mathcal{R}_{\sin}^{(2)}$, therefore there exists a function $p \in \mathcal{P}^{(2)}$, such that

$$f'(z) = 1 + \sin\left(\frac{p(z) - 1}{p(z) + 1}\right).$$

For $f \in \mathcal{R}_{\sin}^{(2)}$, using the series form (6.1) and (6.5), when $m = 2$ in the above relation, we can write

$$a_3 = \frac{1}{6}c_2, \; a_5 = -\frac{1}{20}c_2^2 + \frac{1}{10}c_4.$$

It is clear that for $f \in \mathcal{R}_{\sin}^{(2)}$,

$$H_{3,1}(f) := a_3 a_5 - a_3^3.$$

Therefore,

$$H_{3,1}(f) = \frac{1}{60}c_2(c_4 - \frac{2}{9}c_2^2).$$

Using (2.3) and triangle inequality, we get

$$|H_{3,1}(f)| \le \frac{1}{15}.$$

Hence the proof is completed. □

Theorem 6.2. *If $f \in \mathcal{R}_{\sin}^{(3)}$, then*

$$|H_{3,1}(f)| \le \frac{1}{16}.$$

This result is sharp for the function

$$f(z) = \int_0^z \sin(t^3)dt = z + \frac{1}{4}z^4 - \frac{1}{54}z^{10} + \cdots.$$

Proof. Since $f \in \mathcal{R}_{\sin}^{(3)}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$f'(z) = 1 + \sin\left(\frac{p(z) - 1}{p(z) + 1}\right).$$

For $f \in \mathcal{R}_{\sin}^{(3)}$, using the series form (6.1) and (6.5), when $m = 2$ in the above relation, we can write

$$a_4 = \frac{1}{8}c_3.$$

It is easy to see that

$$H_{3,1}(f) := -a_4^2.$$

Therefore,

$$H_{3,1}(f) = \frac{-1}{64}c_3^2.$$

Using coefficient estimates for class $\mathcal{P}$ and triangle inequality, we get

$$|H_{3,1}(f)| \le \frac{1}{16}.$$

Hence the proof is completed. □

Theorem 6.3. *Let $f \in \mathcal{S}_{\sin}^{*(2)}$, be of the form* (6.1). *Then*

$$|H_{3,1}(f)| \le \frac{1}{8}.$$

Proof. Since $f \in \mathcal{S}^{\star(2)}_{\sin}$, therefore there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right).$$

Using the series form (6.1) and (6.5) , when $m = 2$ in the above relation, we can write

$$a_3 = \frac{1}{4}c_2,\ a_5 = -\frac{1}{32}c_2^2 + \frac{1}{8}c_4.$$

Now

$$H_{3,1}(f) := a_3a_5 - a_3^3.$$

Therefore,

$$H_{3,1}(f) = \frac{1}{32}c_2(c_4 - \frac{3}{4}c_2^2).$$

Using (2.3) and triangle inequality, we get

$$|H_{3,1}(f)| \le \frac{1}{8}.$$

Thus the proof is completed. □

Theorem 6.4. *If $f \in \mathcal{S}^{\star(3)}_{\sin}$, then*

$$|H_{3,1}(f)| \le \frac{1}{9}.$$

This result is sharp for the function

$$f(z) = z\exp\left(\int_0^z (\frac{\sin(t^3)-1}{t})dt\right) = z + \frac{1}{3}z^4 + \frac{1}{18}z^7 + \cdots.$$

Proof. Since $f \in \mathcal{S}^{\star(3)}_{\sin}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right).$$

Using the series form (6.1) and (6.5) , when $m = 2$ in the above relation, we can write

$$a_4 = \frac{1}{6}c_3.$$

Then

$$H_{3,1}(f) := -a_4^2.$$

Therefore,

$$H_{3,1}(f) = \frac{-1}{36}c_3^2.$$

Using coefficient estimates for class $\mathcal{P}$ and triangle inequality, we get

$$|H_{3,1}(f)| \le \frac{1}{9}.$$

Hence the proof is completed. □

Theorem 6.5. *Let $f \in \mathcal{C}^{(2)}_{\sin}$, be of the form* (6.1). *Then*

$$|H_{3,1}(f)| \le \frac{1}{120}.$$

Proof. Since $f \in \mathcal{C}^{(2)}_{\sin}$, therefore there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right).$$

Using the series form (6.1) and (6.5) , when $m = 2$ in the above relation, we can write

$$a_3 = \frac{1}{12}c_2, \; a_5 = -\frac{1}{160}c_2^2 + \frac{1}{40}c_4.$$

Therefore,

$$H_{3,1}(f) = \frac{1}{480}c_2(c_4 - \frac{19}{36}c_2^2).$$

Using (2.3) and triangle inequality, we get

$$|H_{3,1}(f)| \leq \frac{1}{120}.$$

Hence the proof is completed. □

Theorem 6.6. *Let $f \in \mathcal{C}^{(3)}_{\sin}$, be of the form* (6.1) *. Then*

$$|H_{3,1}(f)| \leq \frac{1}{144}.$$

This result is sharp for the function

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \sin(z^3) = 1 + z^3 - \frac{1}{6}z^9 + \cdots.$$

Proof. Since $f \in \mathcal{C}^{(3)}_{\sin}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$f'(z) = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right).$$

Using the series form (6.1) and (6.5) , when $m = 2$ in the above relation, we can write

$$a_4 = \frac{1}{24}c_3.$$

Therefore,

$$H_{3,1}(f) = \frac{-1}{576}c_3^2.$$

Using coefficient estimates for class $\mathcal{P}$ and triangle inequality, we get

$$|H_{3,1}(f)| \leq \frac{1}{144}.$$

This completes the proof. □

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