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**Research Article**

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# Regular Banach space net and abstract-valued Orlicz space of range-varying type

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**Abstract:** This paper investigates the abstract-valued Orlicz space of range-varying type. We firstly give the notions and examples of partially continuous modular net and regular Banach space net of type (II), then deal with the definitions, constructions, and geometrical properties of the range-varying Orlicz spaces, including representation of the dual $L^{\varphi}_{+}(I, X_{\theta(\cdot)})^{*}$, and reflexivity of $L^{\varphi}(I, X_{\theta(\cdot)})$, under some reasonable conditions. As an application, we finally make another approach to the real interpolation spaces constructed by a generalized $\Phi$–function.

**Keywords:** Banach space net; partially continuous modular net; range-varying Orlicz space; real interpolation space

**MSC 2010:** 46B10, 46E30, 46E40

## 1 Introduction and preliminaries

This paper is devoted to studying the abstract-valued Orlicz space of range-varying type. Orlicz space was firstly introduced in [1]. Due to the power in dealing with the nonstandard growing phenomena, it has wide applications in many fields of applied mathematics, such as the model porus medium problem (see [2]), compressible Navier-Stokes equation (see [3]) and nonlinear obstacle problem (see [4]) etc. Roughly speaking, Orlicz space is a special type of semimodular space, where the semimodular $\varrho_{\varphi}$ is commonly constructed by a generalized $\Phi$–function $\varphi$ (refer to [5, §2.3]), namely

$$\varrho_{\varphi}(f) = \int_{I} \varphi(t, |f(t)|)d\mu,\ f \in L^0(I, \mu),$$

where $(I, \mu)$ is a complete measure space. Given a Banach space $X$, if we replace $L^0(I, \mu)$ with $L^0(I, X)$ the collection of all strongly measurable $X$–valued functions, and replace $|f(t)|$ with $\|f(t)\|_X$ for $f \in L^0(I, X)$, then we obtain the abstract-valued Orlicz space, which was receiving a growing interesting in recent decades (cf. [6–9] etc).

Here we focus on a special type of the abstract-valued Orlicz space, whose members have a varying range. This new type of function space was firstly introduced in [10], and later studied in [11–13]. As the value space varies as $t$ changes, it is crucial to give a suitable description of measurability for the functions of this type. In [10], the authors introduced the notions of totally bounded topological lattice $\mathcal{A}$, regular Banach space

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net $\{X_\alpha : \alpha \in \mathcal{A}\}$, and order-continuous map $\theta : I \to \mathcal{A}$. Based on these notions, they introduced a suitably measurable $X_{\theta(\cdot)}$–valued function space $L^0(I, X_{\theta(\cdot)})$ on an interval $I$. Functions in this space have a common property, that is the norm function $t \mapsto \|f(t)\|_{X_{\theta(\cdot)}}$ is measurable. There are three types of subspaces of $L^0(I, X_{\theta(\cdot)})$ according to their constructions: continuous type $C^-(I, X_{\theta(\cdot)})$ (cf. [10]), norm-modular type $L^{p(\cdot)}(I, X_{\theta(\cdot)})$, $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$ (cf. [11]), and pure modular type $L^{\varrho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ (cf. [12]). Each of which owns useful examples, such as $C^-(I, (X, D(A))_{1-1/p(\cdot),p(\cdot)})$, trace of the maximal regular space $W_0^{1,p(\cdot)}(I, X) \cap L^{1,p(\cdot)}(I, D(A))$ associated with a sectorial operator $A$, raised in [10] and treated in [13], Lebesgue space $L^{p(x,t)}(I \times \Omega)$ with the double variable exponent, and the anisotropic space $\{u \in L^2(I, L^2(\Omega)) : \partial_{x_i} u(t) \in L^{p_i(\cdot,t)}(\Omega)\}$, used in [14–17] to deal with the anisotropic parabolic equations.

We should admit that the notion Banach space net introduced in [11] has a limitation in application: It does not incorporates the following space family

$$\{L^{p(\cdot)}(\Omega) : 1 < \underline{p} \le \operatorname{essinf}_{x\in\Omega} p(x) \le \operatorname{esssup}_{x\in\Omega} p(x) \le \bar{p} < \infty\} \tag{1.1}$$

when $\Omega$ is an unbounded domain in $\mathbb{R}^N$, because of the restriction

– $X_\beta \hookrightarrow X_\alpha$ provided $\alpha \prec \beta$

in its definition. In order to incorporate (1.1) into our framework, in this paper, the above hypothesis is replaced by

– $X_\alpha \cap X_\gamma \hookrightarrow X_\beta \hookrightarrow X_\alpha + X_\gamma$ whenever $\alpha \prec \beta \prec \gamma$.

Adapted to this change, continuity and successive assumption are revised slightly. After these modifications, except for (1.1), a lot of space families including the complex interpolation series $\{[X_0, X_1]_s : s \in [0, 1]\}$ and the real interpolation series $\{(X_0, X_1)_{p,s} : 0 < a \le s \le b < 1\}$ become regular Banach space nets. This makes the application of the range-varying function spaces much wider. In order to distinguish the notions of Banach space net defined in [10, 11] and this paper, herein after we name them type (I) and type (II) respectively.

Analogous to [11], here we also pay attention to the partially continuous semimodular net $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$. We will prove that, if the norms of $X_\alpha \cap X_\gamma$, $X_\alpha + X_\gamma$ and $X_\beta$ are produced by $\varrho_\alpha \wedge \varrho_\gamma$, $\varrho_\alpha \vee \varrho_\gamma$ and $2\varrho(\cdot/2)$ respectively, then under some reasonable hypotheses, every partially continuous semimodular net generates a regular Banach space net (II). This gives a beneficial supplement of that in [11].

This paper is organised as follows. In section 1, we make some reviews on pre-semimodular and semimodular, including $\varrho_\alpha \wedge \varrho_\gamma$, a semimodular, producing an equivalent norm of $X_{\varrho_\alpha} \cap X_{\varrho_\gamma}$, and $\varrho_\alpha \vee \varrho_\gamma$, a pre-semimodular, producing an equivalent norm of $X_{\varrho_\alpha} + X_{\varrho_\gamma}$. In section 2, we give notions of regular Banach space net (II) and partially continuous semimodular net, together with three useful examples, namely complex interpolation space scale $\{[X_0, X_1]_s : s \in [0, 1]\}$, real interpolation space scale $\{(X_0, X_1)_{s,q} : s \in [a, b]\}$ $(0 < a < b < 1, 1 < q < \infty)$, and $\{L^{p(\cdot)}(\Omega) : p \in \mathcal{P}_0(\Omega)\}$ on a unbounded open set $\Omega \subseteq \mathbb{R}^N$.

Section 3 is devoted to investigating the construction and geometrical properties of abstract-valued Orlicz spaces. With the aid of the associate space $L^\varphi(I, X)'$, we show the equivalence between the dual space $L^\varphi(I, X)^\star$ and the $X^\star$–valued function space $L^{\varphi'}(I, X^\star)$, i.e.

$$L^\varphi(I, X)^\star \cong L^\varphi(I, X)' \cong L^{\varphi'}(I, X^\star) \tag{1.2}$$

under the assumptions that $\varphi$ is a locally integrable generalized $\Phi$–function, and the dual space $X^\star$ satisfies the Radon-Nikodym's property. Equivalence (1.2) is a natural but not trivial extension of the corresponding result from the scalar case to the vector-valued case. Based on this extension, representation of the dual space of the range-varying Orlicz space constructed by the regular Banach space net (II) is derived, that is

$$L_+^\varphi(I, X_{\theta(\cdot)})^\star \cong L^{\varphi'}(I, X_{\theta(\cdot)}^\star). \tag{1.3}$$

It is worth remarking that, representation (1.3) also holds in case that $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a regular Banach space net (I). Taking into account that $\varphi$ is only a locally integral generalized $\Phi$–function, and the extra assumption that $X_\alpha^\star$ is norm-attainable is dropped here, (1.2) can be viewed as an improvement of that in [11].

To illustrate the application of the range-varying Orlicz spaces, in the last section, we make another approach to the real interpolation space, where the usual $p$–power $\tau^p$ is replaced by a generalized $\Phi$–function $\varphi$, from which, four different intermediate spaces $(X_0, X_1)_{s,\phi,\theta,K}$, $(X_0, X_1)_{s,\phi,\theta,J}$, $(X_0, X_1)_{s,\phi,\vartheta,K}$ and $(X_0, X_1)_{s,\phi,\vartheta,J}$ are constructed. All of them are produced naturally from the range-varying Orlicz spaces, two ones are the quotient spaces, and the other two are closed subspaces. We will show that if the lower index $\underline{p}_\varphi$ and the upper index $\bar{p}_\varphi$ satisfy $1 < \underline{p}_\varphi \le \bar{p}_\varphi < \infty$, then the four intermediate spaces are mutually equivalent, i.e.

$$(X_0, X_1)_{s,\phi,\theta,K} \cong (X_0, X_1)_{s,\phi,\theta,J}, \text{ and } (X_0, X_1)_{s,\phi,\vartheta,K} \cong (X_0, X_1)_{s,\phi,\vartheta,J}.$$

Moreover, for the dual space, we have

$$(X_0, X_1)^\star_{s,\phi,\theta,J} \cong (X_0^\star, X_1^\star)_{s,\phi,\vartheta,K}, \text{ and } (X_0, X_1)^\star_{s,\phi,\theta,K} \cong (X_0^\star, X_1^\star)_{s,\phi,\vartheta,J}.$$

In spite that the general interpolated property of the four intermediate spaces linear operators does not remain any more, we have a weak version of the interpolation, that is

$$\max\{\|u\|_{s,\phi,\theta,K}, \|u\|_{s,\phi,\theta,J}, \|u\|_{s,\phi,\vartheta,K}, \|u\|_{s,\phi,\vartheta,J}\} \le C_{s,\varphi}\|u\|_0^{1-s}\|u\|_1^s$$

for all $u \in X_0 \cap X_0$. In this sense, the four intermediate spaces can also be viewed as the interpolation spaces between $X_0$ and $X_1$. Finally, in concrete applications, the $\Phi$–function $\varphi$ can take the form $\tau^{p(t)}$, $\tau^p w(t)$ or $\tau(\log(1+\tau))^{p(t)}$ etc.

Before the main parts of this paper, as preliminaries, let us firstly make some reviews and arguments on the semimodular and semimodular space. Let $X$ be a complex or real linear space and $\varrho : X \to [0, \infty]$ be a convex functional with $\varrho(0) = 0$. If $\varrho(\lambda u) = \varrho(u)$ whenever $|\lambda| = 1$, and $\varrho(\lambda u) = 0$ for all $\lambda > 0$ leads to $u = 0$, then $\varrho$ is called a pre-semimodular. In addition, if $\varrho$ is left-continuous, i.e.

$$\lim_{\lambda \to 1^-} \varrho(\lambda u) = \varrho(u),$$

then $\varrho$ is called a semimodular. Furthermore, if additionally $\varrho(u) = 0$ implies $u = 0$, then $\varrho$ is said to be a modular.

Similar to the semimodular, for a pre-semimodular $\varrho$, the induced space

$$X_\varrho = \{u \in X : \varrho(\lambda u) < \infty \text{ for some } \lambda > 0\}$$

is a normed linear space with the Luxemmburg norm

$$\|u\|_\varrho = \inf\left\{\lambda > 0 : \varrho\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

If $\varrho$ is a semimodular, then $X_\varrho$ is called a semimodular space on which $\varrho$ is lower semicontinuous, and the unit ball property

$$\{u \in X_\varrho : \|u\|_\varrho \le 1\} = \{u \in X_\varrho : \varrho(u) \le 1\}$$

holds (refer to [5, §2.1]).

**Theorem 1.1.** *Suppose that $\varrho_\alpha$ and $\varrho_\gamma$ are two semimodulars on $X$, generating two normed spaces $X_{\varrho_\alpha}$ and $X_{\varrho_\gamma}$ respectively. Then the functional*

$$\varrho_{\alpha\vee\gamma}(u) := \inf\left\{\varrho_\alpha(u_1) + \varrho_\gamma(u_2) : u = u_1 + u_2\right\}$$

*is a pre-semimodular on $X$, and the induced space $X_{\varrho_{\alpha\vee\gamma}}$ is exactly the sum $X_{\varrho_\alpha} + X_{\varrho_\gamma}$ with the equivalent norms.*

**Proof** : Taking two points $u, v \in X$ and arbitrary $\lambda \in [0, 1]$, notice that for any decompositions $u = u_1 + u_2$ and $v = v_1 + v_2$, $((1-\lambda)u_1 + \lambda v_1) + ((1-\lambda)u_2 + \lambda v_2)$ is a decomposition of the convex combination $(1-\lambda)u + \lambda v$. Thus using the convexity of $\varrho_\alpha$ and $\varrho_\gamma$, we get the convexity of $\varrho_{\alpha\vee\gamma}$.

Obviously $\varrho_{\alpha\vee\gamma}(0) = 0$. Conversely, if $\varrho_{\alpha\vee\gamma}(\lambda u) = 0$ for all $\lambda > 0$, then by the definition of $\varrho_{\alpha\vee\gamma}$, there exist two sequences $\{u_{1,k}\}$ and $\{u_{2,k}\}$ such that $u = u_{1,k} + u_{2,k}$, and

$$\varrho_\alpha(ku_{1,k}) + \varrho_\gamma(ku_{2,k}) \le 1.$$

The above inequality shows that both $\|u_{1,k}\|_{\varrho_\alpha}$ and $\|u_{2,k}\|_{\varrho_\gamma}$ are no more than $1/k$. Let $k \to \infty$, we have

$$u_{1,k} \to 0 \text{ in } X_{\varrho_\alpha},\ u_{2,k} \to 0 \text{ in } X_{\varrho_\gamma},$$

consequently

$$u_{1,k} + u_{2,k} \to 0 \text{ in } X_{\varrho_\alpha} + X_{\varrho_\gamma},$$

i.e. $u = 0$.

The property $\varrho_{\alpha\vee\gamma}(\lambda u) = \varrho_{\alpha\vee\gamma}(u)$ for $|\lambda| = 1$ comes from that of $\varrho_\alpha$ and $\varrho_\gamma$. Hence $\varrho_{\alpha\vee\gamma}$ is a pre-semimodular.

Suppose that $u \in X_{\varrho_{\alpha\vee\gamma}}$ and $\|u\|_{\varrho_{\alpha\vee\gamma}} \le 1$, then for arbitrary $\varepsilon > 0$, we have $\varrho_{\alpha\vee\gamma}(u/\sqrt{1+\varepsilon}) \le 1$, consequently there is a decomposition $u = u_{\varepsilon,1} + u_{\varepsilon,2}$ such that

$$\varrho_\alpha\Big(\frac{u_{\varepsilon,1}}{\sqrt{1+\varepsilon}}\Big) + \varrho_\gamma\Big(\frac{u_{\varepsilon,2}}{\sqrt{1+\varepsilon}}\Big) \le \sqrt{1+\varepsilon},$$

which yields $u_{\varepsilon,1} \in X_{\varrho_\alpha}$, $u_{\varepsilon,2} \in X_{\varrho_\gamma}$, and $\|u_{\varepsilon,1}\|_{\varrho_\alpha} \le 1+\varepsilon$, $\|u_{\varepsilon,2}\|_{\varrho_\gamma} \le 1+\varepsilon$. Therefore $u \in X_{\varrho_\alpha} + X_{\varrho_\gamma}$ and $\|u\|_{X_{\varrho_\alpha}+X_{\varrho_\gamma}} \le \|u_{\varepsilon,1}\|_{\varrho_\alpha} + \|u_{\varepsilon,2}\|_{\varrho_\gamma} \le 2(1+\varepsilon)$. By the arbitrariness of $\varepsilon$, we obtain $\|u\|_{X_{\varrho_\alpha}+X_{\varrho_\gamma}} \le 2$.

Conversely, if $u \in X_{\varrho_\alpha} + X_{\varrho_\gamma}$ and $\|u\|_{X_{\varrho_\alpha}+X_{\varrho_\gamma}} \le 1$. Then for every $\varepsilon > 0$, there is a decomposition $u = u_{\varepsilon,1} + u_{\varepsilon,2}$ verifying $\|u_{\varepsilon,1}\|_{\varrho_\alpha} + \|u_{\varepsilon,2}\|_{\varrho_\gamma} \le 1+\varepsilon$. Hence by the unit ball property of $\varrho_\alpha$ and $\varrho_\gamma$, we have

$$\varrho_\alpha\Big(\frac{u_{\varepsilon,1}}{1+\varepsilon}\Big) + \varrho_\gamma\Big(\frac{u_{\varepsilon,2}}{1+\varepsilon}\Big) \le \Big\|\frac{u_{\varepsilon,1}}{1+\varepsilon}\Big\|_{\varrho_\alpha} + \Big\|\frac{u_{\varepsilon,2}}{1+\varepsilon}\Big\|_{\varrho_\gamma} \le 1,$$

which means that $u \in X_{\varrho_{\alpha\vee\gamma}}$ and $\|u\|_{\varrho_{\alpha\vee\gamma}} \le 1+\varepsilon$. Similarly by the arbitrariness of $\varepsilon$, we get $\|u\|_{\varrho_{\alpha\vee\gamma}} \le 1$.

Finally, by the scaling arguments, we can derive that $X_{\varrho_{\alpha\vee\gamma}}$ is equivalent to $X_{\varrho_\alpha} + X_{\varrho_\gamma}$ with the estimate

$$\|u\|_{\varrho_{\alpha\vee\gamma}} \le \|u\|_{X_\alpha+X_\gamma} \le 2\|u\|_{\varrho_{\alpha\vee\gamma}},$$

and the proof has been completed. □

*Remark* 1.2. By laying the reflexive assumption on $X_{\varrho_\alpha}$ and $X_{\varrho_\gamma}$, we can obtain the left-continuity of $\varrho_{\alpha\vee\gamma}$. As a matter of fact, since the function $\lambda \mapsto \varrho_{\alpha\vee\gamma}(\lambda u)$ is increasing, we have $\lim_{\lambda\to 1^-} \varrho_{\alpha\vee\gamma}(\lambda u) \le \varrho_{\alpha\vee\gamma}(u)$. The reverse inequality can be proved by contradiction. Assume that $\varrho_{\alpha\vee\gamma}(u) < \infty$ and there is a number $\varepsilon > 0$ verifying

$$\varrho_{\alpha\vee\gamma}(u) - \varepsilon > \varrho_{\alpha\vee\gamma}(\lambda u)$$

for all $\lambda \in (0,1)$. Then for each $\lambda \in (0,1)$, there is a decomposition $u = u_{\lambda,1} + u_{\lambda,2}$ such that

$$\varrho_{\alpha\vee\gamma}(u) - \frac{\varepsilon}{2} > \varrho_{\alpha\vee\gamma}(\lambda u) + \frac{\varepsilon}{2} > \varrho_\alpha(\lambda u_{\lambda,1}) + \varrho_\gamma(\lambda u_{\lambda,2}). \tag{1.4}$$

This shows the boundedness of $\{\varrho_\alpha(\lambda u_{\lambda,1})\}$ and $\{\varrho_\gamma(\lambda u_{\lambda,2})\}$. Since for arbitrary semimodular $\varrho$,

$$\|u\|_\varrho \le \varrho(u) + 1,$$

we get the boundedness of $\{\lambda u_{\lambda,1}\}$ and $\{\lambda u_{\lambda,2}\}$ in $X_\alpha$ and $X_\gamma$ respectively. Thus by the reflexivity of $X_\alpha$ and $X_\gamma$, there is a sequence $\{\lambda_k\}$ convergent to 1 and two points $u_1 \in X_\alpha$, $u_2 \in X_\alpha$ such that $\lambda_k u_{\lambda_k,1} \rightharpoonup u_1$ in $X_\alpha$ and $\lambda_k u_{\lambda_k,2} \rightharpoonup u_2$ in $X_\gamma$ as $k \to \infty$. Notice that $\lambda_k u = \lambda_k u_{\lambda_k,1} + \lambda_k u_{\lambda_k,2}$ and $\lambda_k u \to u$ in $X_\alpha + X_\gamma$, let $k \to \infty$, we obtain the decomposition $u = u_1 + u_2$. Putting it into (1.4), we derive an impossible result

$$\varrho_\alpha(u_1) + \varrho_\gamma(u_2) - \frac{\varepsilon}{2} \ge \varrho_{\alpha\vee\gamma}(u) - \frac{\varepsilon}{2} \ge \varrho_\alpha(u_1) + \varrho_\gamma(u_2).$$

Hence the preceding assumption does not hold, and then inequality $\lim_{\lambda\to 1^-} \varrho_{\alpha\vee\gamma}(\lambda u) \ge \varrho_{\alpha\vee\gamma}(u)$ holds. The case $\varrho_{\alpha\vee\gamma}(u) = \infty$ can be dealt with in the similar way. □

*Remark* 1.3. It is easy to check that

$$\varrho_{\alpha\wedge\gamma}(u) := \varrho_\alpha(u) + \varrho_\gamma(u)$$

defines a semimodular on $X$, and the induced space $X_{\varrho_{\alpha\wedge\gamma}}$ is equivalent to the intersection $X_{\varrho_\alpha} \cap X_{\varrho_\gamma}$.

Recall that (refer to [5, §2.2]), given a semimodular $\varrho$ on $X$ with the semimodular space $X_\varrho$, the dual functional

$$\varrho^\star(\xi) = \sup_{u \in X_\varrho} \{\langle \xi, u\rangle - \varrho(u)\}$$

is also a semimodular on the dual space $X_\varrho^\star$, and the induced space $(X_\varrho^\star)_{\varrho^\star}$ is equivalent to $X_\varrho^\star$. Furthermore, for the double dual $\varrho^{\star\star}$, we have

$$\varrho^{\star\star}(u) = \varrho(u), \ \forall\, u \in X_\varrho. \tag{1.5}$$

## 2 Regular Banach space net of type (II)

**Definition 2.1.** Suppose that $\mathcal{A}$ is a topological space on which there is also defined an order $\prec$. We say that the order $\prec$ is compatible with the topology, if for any net $\{\alpha_i : i \in \mathcal{I}\}$ convergent to $\alpha$ in $\mathcal{A}$ according to the topology, and $\alpha_i \prec \beta$ for all $i \in \mathcal{I}$, one has $\alpha \prec \beta$ definitely. In a word, the order can be preserved through the process of convergence. Under this situation, $\mathcal{A}$ is called an ordered topological space. Furthermore, if for every order-bounded subset of $\mathcal{A}$, its order-supremum and order-infimum are both existing, then $\mathcal{A}$ is called an topological lattice. For the convenience of use, throughout this paper, we always assume that $\mathcal{A}$ is a totally order-bounded topological lattice, or $BTL$ in symbol. Its total order-supremum and total order-infimum are denoted by $\alpha^+$ and $\alpha^-$ respectively. Given a sequence $\{\alpha_k\} \subset \mathcal{A}$ and a point $\beta \in \mathcal{A}$, we say $\{\alpha_k\}$ is upper (or lower)-approaching $\beta$, or $\alpha_k \uparrow \beta$ (or $\alpha_k \downarrow \beta$ ) in symbol, if $\alpha_k \prec \beta$ ( or $\beta \prec \alpha_k$) and $\lim_{k\to\infty} \alpha_k = \beta$ are fulfilled at the same time.

**Definition 2.2.** Attached to the $BTL$ $\mathcal{A}$, let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be a family of Banach spaces.

- We say $\{X_\alpha\}$ is a Banach space net of type (II), or $BSN$ (II) for short, provided for all $\alpha, \beta, \gamma \in \mathcal{A}$, $\alpha \prec \beta \prec \gamma$, there are
$$X_\alpha \cap X_\gamma \hookrightarrow X_\beta \hookrightarrow X_\alpha + X_\gamma. \tag{2.1}$$
- $\{X_\alpha\}$ is called uniformly bounded if the imbedding constants of (2.1) are independent of $\alpha, \beta$ and $\gamma$.

Suppose that $\{\alpha_k\}$ and $\{\gamma_k\}$ are two sequence upper- and lower-approaching $\beta \in \mathcal{A}$ respectively.

- If for every $u \in X_\beta$, the limit
$$\lim_{k\to\infty} \|u\|_{X_{\alpha_k}+X_{\gamma_k}} = \|u\|_\beta \tag{2.2}$$
holds, then $\{X_\alpha\}$ is called norm-continuous, where $\|\cdot\|_\beta$ denotes the norm in $X_\beta$.
- If $u \in \bigcap_{k=1}^\infty (X_{\alpha_k} + X_{\gamma_k})$, and
$$\sup_{k\geq 1} \|u\|_{X_{\alpha_k}+X_{\gamma_k}} \leq K \tag{2.3}$$
means that $u \in X_\beta$, and $\|u\|_\beta \leq K$, then $\{X_\alpha\}$ is called successive.

Finally, we say $\{X_\alpha\}$ is a regular $BSN$ (II), if it is uniformly bounded, norm-continuous and successive simultaneously.

*Remark* 2.3. Previous notion of Banach space net defined in [10, 11] is called $BSN$ (I) here.

*Remark* 2.4. For the sake of convenience in applications, in the coming arguments, we always assume that

$$\langle \xi, u\rangle_{X_\alpha^\star \times X_\alpha} = \langle \xi, u\rangle_{X_\gamma^\star \times X_\gamma}$$

whenever $\xi \in X_\alpha^\star \cap X_\gamma^\star$ and $u \in X_\alpha \cap X_\gamma$. Under this assumption, it is easy to check that if $u_1 + u_2 = v_1 + v_2$, $u_1, v_1 \in X_\alpha$ $u_2, v_2 \in X_\gamma$, and $\xi \in X_\alpha^\star \cap X_\gamma^\star$, then

$$\langle \xi, u_1 \rangle_{X_\alpha^\star \times X_\alpha} + \langle \xi, u_2 \rangle_{X_\gamma^\star \times X_\gamma} = \langle \xi, v_1 \rangle_{X_\alpha^\star \times X_\alpha} + \langle \xi, v_2 \rangle_{X_\gamma^\star \times X_\gamma}.$$

Thus we can define the dual product in $(X_\alpha^\star \cap X_\gamma^\star) \times (X_\alpha + X_\gamma)$ as follows

$$\langle \xi, u \rangle_{(X_\alpha^\star \cap X_\gamma^\star) \times (X_\alpha + X_\gamma)} := \langle \xi, u_1 \rangle_{X_\alpha^\star \times X_\alpha} + \langle \xi, u_2 \rangle_{X_\gamma^\star \times X_\gamma},$$

where $\xi \in X_\alpha^\star \cap X_\gamma^\star$ and $u = u_1 + u_2 \in X_\alpha + X_\gamma$.

For the dual space of $X_\alpha \cap X_\gamma$, we have $(X_\alpha + X_\gamma)^* \subset X_\alpha^\star + X_\gamma^\star$. Moreover, if $X_\alpha \cap X_\gamma$ is densely imbedded in both $X_\alpha$ and $X_\gamma$, then we have (refer to [18, p. 69])

$$(X_\alpha \cap X_\gamma)^* = X_\alpha^\star + X_\gamma^\star, \text{ and } (X_\alpha + X_\gamma)^* = X_\alpha^\star \cap X_\gamma^\star. \tag{2.4}$$

**Example 2.1.** Let $X_0$ and $X_1$ be two Banach spaces embedded continuously into a topological linear space $\mathcal{X}$. Assume also $X_0 \cap X_1$ is dense in both $X_0$ and $X_1$. For each $s \in [0, 1]$, consider the complex interpolation space $[X_0, X_1]_s$. Recall that if $u \in [X_0, X_1]_s$, then there exists $f \in \mathcal{F}$, s.t. $f(s) = u$. Here $\mathcal{F}$ is the collection of all $X_0 + X_1$–valued functions which are analytic in the open strip $S = \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$, continuous and bounded on the closed strip $\overline{S} = \{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}$ and satisfy

- $f(it)$ is continuous in $X_0$, and
$$\lim_{t\to\infty} \|f(it)\|_{X_0} = 0;$$
- $f(1 + it)$ is continuous in $X_1$, and
$$\lim_{t\to\infty} \|f(1 + it)\|_{X_1} = 0,$$

where $i$ denotes the unit imaginary number. We know that, endowed with the norm

$$|||f||| = \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{X_1}\},$$

$\mathcal{F}$ becomes a Banach space. For each $u \in [X_0, X_1]_s$, let

$$\|u\|_s = \inf\{|||f||| : f \in \mathcal{F}, f(s) = u\},$$

then we define a norm in $[X_0, X_1]_s$ making it be a Banach space.

Taking any $f \in \mathcal{F}$, define

$$M(t) = \sup_{y\in\mathbb{R}} \|f(t + iy)\|_{X_0+X_1}.$$

Since $\log M(\cdot)$ is convex (cf. [19, §VI. 10]), for every $s \in (0, 1)$ and arbitrary $\delta > 0$, we have

$$\begin{aligned} \|f(s)\|_{X_0+X_1} &\le (\sup_{y\in\mathbb{R}} e^{\delta(iy-s)^2} \|f(iy)\|_{X_0+X_1})^{1-s} (\sup_{y\in\mathbb{R}} e^{\delta(1+iy-s)^2} \|f(1 + iy)\|_{X_0+X_1})^s \\ &\le e^{\delta s(1-s)} M(0)^{1-s} M(1)^s. \end{aligned}$$

Then by the arbitrariness of $\delta$, we can conclude that,

$$\|u\|_{X_0+X_1} \le \|u\|_s. \tag{2.5}$$

for all $u \in [X_0, X_1]_s$.

As for the dual space, we know that (cf. [18, §1.11.3] or [20, §4.5]) if one of $X_j$ $(j = 0, 1)$ is reflexive, then

$$[X_0, X_1]_s^\star = [X_0^\star, X_1^\star]_s.$$

Moreover, by the density of $X_0 \cap X_1$ in $X_0$ and $X_1$, we can deduce the density of $X_0^\star \cap X_1^\star$ in $X_0^\star$ and $X_1^\star$. Consequently, for all $\alpha \in [0, 1]$ , $X_0 \cap X_1$ and $X_0^\star \cap X_1^\star$ are dense in $X_\alpha$ and $X_\alpha^\star$ respectively ([20, §4.1, §4.5]).

Let $\mathcal{A} = [0, 1]$ with the natural topology and the general order $\leq$, and let $X_s = [X_0, X_1]_s$ for $s \in (0, 1)$, then we obtain a family of Banach space $\{X_s : s \in [0, 1]\}$. Here spaces $X_j$, $j = 0, 1$ can be replaced by the interpolation spaces $[X_0, X_1]_j$, $j = 0, 1$, since the latter ones are the closed subspaces of the former ones respectively, and (refer to [21])

$$[[X_0, X_1]_0, [[X_0, X_1]_1]_s = [X_0, X_1]_s$$

for all $s \in [0, 1]$. Suppose that both of $X_0$ and $X_1$ are reflexive, then $\{X_s : s \in [0, 1]\}$ is a regular $BSN$ of type (II).

Firstly, for all $\alpha, s, \gamma \in [0, 1]$, $\alpha \leq s \leq \gamma$, we have

$$[X_\alpha, X_\gamma]_\lambda = [X_0, X_1]_s$$

with the equal norms (cf. [20, §4.6]). This fact, together with (2.5), leads to the inlusion $X_s \hookrightarrow X_\alpha + X_\gamma$, and

$$\|u\|_{X_\alpha + X_\gamma} \leq \|u\|_s, \ \forall\, u \in X_s.$$

In addition, since $[\cdot, \cdot]_s$ is an exact interpolation functor (cf. [21]), we have $X_\alpha \cap X_\gamma \hookrightarrow X_s$, and

$$\|u\|_s = \|u\|_{[X_\alpha, X_\gamma]_\lambda} \leq \|u\|_\alpha^{1-\lambda} \|u\|_\gamma^\lambda \leq \|u\|_{X_\alpha \cap X_\gamma}, \ \forall\, u \in X_\alpha \cap X_\gamma,$$

where $\lambda \in [0, 1]$ making $s = (1 - \lambda)\alpha + \lambda\gamma$. Therefore $\{X_s\}$ is a uniformly bounded $BSN$, and

$$\limsup_{k\to\infty} \|u\|_{X_{\alpha_k} + X_{\gamma_k}} \leq \|u\|_s, \ \forall u \in X_s, \tag{2.6}$$

$$\liminf_{k\to\infty} \|u\|_{X_{\alpha_k} \cap X_{\gamma_k}} \geq \|u\|_s, \ \forall u \in X_0 \cap X_1, \tag{2.7}$$

if $\alpha_k \uparrow \beta$ and $\gamma_k \downarrow \beta$.

In order to show the norm-continuous and successive properties of $\{X_s\}$, we need to prove the inverse inequalities of (2.6), (2.7). To this end, let $s \subseteq [0, 1]$, and $u \in X_0 \cap X_1$. If $0 < \alpha < s$, then

$$\|u\|_\alpha \leq \|u\|_0^{1-\alpha/s} \|u\|_s^{\alpha/s}.$$

Letting $\alpha \uparrow s$, we obtain

$$\limsup_{\alpha\uparrow s} \|u\|_\alpha \leq \|u\|_s.$$

Similarly, if $s < \gamma < 1$, then

$$\|u\|_\gamma \leq \|u\|_s^{(1-\gamma)/(1-s)} \|u\|_1^{(\gamma-s)/(1-s)}.$$

letting $\gamma \downarrow s$, we obtain

$$\limsup_{\gamma\downarrow s} \|u\|_\gamma \leq \|u\|_s.$$

Therefore, for the sequences $\alpha_k \uparrow \beta$ and $\gamma_k \downarrow \beta$, we have

$$\limsup_{k\to\infty} \|u\|_{X_{\alpha_k} \cap X_{\gamma_k}} \leq \|u\|_s,$$

which, combined with (2.7), leads to the equality

$$\lim_{k\to\infty} \|u\|_{X_{\alpha_k} \cap X_{\gamma_k}} = \|u\|_s. \tag{2.8}$$

Now we can prove the norm-continuity of $\{X_\alpha\}$. If $u \in X_0 \cap X_1$, then using (2.4) and (2.8) for the dual spaces, we have

$$\begin{aligned}\liminf_{k\to\infty} \|u\|_{X_{\alpha_k} + X_{\gamma_k}} &= \liminf_{k\to\infty} \sup \Big\{ \frac{\langle \xi, u \rangle}{\|\xi\|_{X^\star_{\alpha_k} \cap X^\star_{\gamma_k}}} : \xi \in X_0^\star \cap X_1^\star, \xi \neq 0 \Big\} \\ &\geq \sup \Big\{ \frac{\langle \xi, u \rangle}{\|\xi\|_{X_s^\star}} : \xi \in X_0^\star \cap X_1^\star, \xi \neq 0 \Big\} = \|u\|_{X_s}.\end{aligned}$$

This inequality, together with (2.6), produces

$$\lim_{k\to\infty}\|u\|_{X_{\alpha_k}+X_{\gamma_k}} = \|u\|_s. \tag{2.9}$$

If $u \in X_s$, then for arbitrary $\varepsilon > 0$, there is a $u_\varepsilon \in X_0 \cap X_1$ such that $\|u_\varepsilon - u\|_s < \varepsilon$. Thus

$$\begin{aligned}\big|\|u\|_{X_{\alpha_k}+X_{\gamma_k}} - \|u\|_s\big| &\le \|u_\varepsilon - u\|_{X_{\alpha_k}+X_{\gamma_k}} + \big|\|u_\varepsilon\|_{X_{\alpha_k}+X_{\gamma_k}} - \|u_\varepsilon\|_s\big| + \|u_\varepsilon - u\|_s\\ &\le 2\|u_\varepsilon - u\|_s + \big|\|u_\varepsilon\|_{X_{\alpha_k}+X_{\gamma_k}} - \|u_\varepsilon\|_s\big|\\ &\le 2\varepsilon + \big|\|u_\varepsilon\|_{X_{\alpha_k}+X_{\gamma_k}} - \|u_\varepsilon\|_s\big|,\end{aligned}$$

which yields (2.9) for $u \in X_s$.

Finally, suppose that $u \in \bigcap_{k=1}^\infty (X_{\alpha_k} + X_{\gamma_k})$ satisfying (2.3) for some $K > 0$. Then for all $\xi \in X_0^\star \cap X_1^\star$, we have

$$\langle \xi, u\rangle \le \|u\|_{X_{\alpha_k}+X_{\gamma_k}}\|\xi\|_{X^\star_{\alpha_k}\cap X^\star_{\gamma_k}} \le K\|\xi\|_{X^\star_{\alpha_k}\cap X^\star_{\gamma_k}}.$$

Let $k \to \infty$, using (2.8) for the dual spaces, we get

$$\langle \xi, u\rangle \le K\|\xi\|_{X_s^\star}.$$

Since $X_0^\star \cap X_1^\star$ is dense in $X_s^\star$, we have $u \in X_s$, and $\|u\|_s \le K$. This proves the successive property of $\{X_\alpha\}$. Putting all the properties together, we conclude that $\{X_\alpha\}$ is a regular $BSN$ (II).

*Remark* 2.5. As a byproduct, we can find that, under the present hypotheses, the dual space family $\{X_s^\star : s \in [0, 1]\}$ is also a regular $BSN$ (II), called the dual space net of $\{X_s : s \in [0, 1]\}$.

**Example 2.2.** Let $(X_0, X_1)$ be an interpolation couple as above, and $(X_0, X_1)_{s,q}$ be the real interpolation space between $X_0$ and $X_1$ for $s \in (0, 1)$ and $1 < q < \infty$, i.e.

$$(X_0, X_1)_{s,q} = \Big\{u \in X_0 + X_1 : \int_0^\infty (t^{-s}J(t, f(t)))^q \frac{dt}{t} < \infty \text{ for some } f \in \mathcal{S}(u)\Big\},$$

with the norm

$$\|u\|_{s,q} = \inf_{f\in\mathcal{S}(u)}\Big\{\int_0^\infty (t^{-s}J(t, f(t)))^q \frac{dt}{t}\Big\}^{1/q}.$$

Here $\mathcal{S}(u)$ is the collection of all $X_0 \cap X_1$–valued functions strongly measurable in the sum space $X_0 + X_1$ and satisfying

$$u = \int_0^\infty f(t)\frac{dt}{t} \text{ in } X_0 + X_1,$$

and

$$J(t, w) = \max\{\|w\|_0, t\|w\|_1\}$$

is the equivalent norm of $w \in X_0 \cap X_1$.

By [18, §1.6.1, 1.11.2] or [20, §3.3, 3.4, 3.7], we know that $X_0 \cap X_1$ is dense in $(X_0, X_1)_{s,q}$. Moreover, if $X_0 \cap X_1$ is dense in both $X_0$ and $X_1$, then

$$(X_0, X_1)^\star_{s,q} \cong (X_0^\star, X_1^\star)_{s,q'},$$

where $1/q + 1/q' = 1$.

Given $0 < a < b < 1$ and $1 < q < \infty$, let $\mathcal{A} = [a, b]$ be the $BTL$ as above, $X_{s,q} = (X_0, X_1)_{s,q}$ for $a \le s \le b$, then under all the assumptions in the previous example, the real interpolation space family $\{X_{s,q} : s \in [a, b]\}$ is a regular $BSN$ (II). Firstly, for all $a \le \alpha < s < \gamma \le b$ and $1 < p, q, r < \infty$, we have (cf. [18, §1.10.2], [20, §3.5]) or [22, Theorem 7.21])

$$X_{\alpha,p} \cap X_{\gamma,r} \hookrightarrow X_{s,q} \cong (X_{\alpha,p}, X_{\gamma,r})_{\lambda,q} \hookrightarrow X_{\alpha,p} + X_{\gamma,r}, \tag{2.10}$$

where $0 < \lambda < 1$, and $(1 - \lambda)\alpha + \lambda\gamma = s$. This infers that $\{X_{s,q} : s \in [a, b]\}$ is a $BSN$ (II).

Notice that the equivalent constant in (2.10) is proportional to $(\gamma - \alpha)^{-1/q'}$ and consequently blows up as $\alpha \uparrow s$ and $\gamma \downarrow s$, hence we could not get the unform boundedness of $\{X_{s,q} : s \in [a, b], 1 < q < \infty\}$ from (2.10). By this reason, we fix the second exponent $q$ in this example, and use the splitting method to derive the unform boundedness of $\{X_{s,q} : s \in [a, b]\}$. More precisely, for all $a \le \alpha \le s \le \gamma \le b$, $u \in X_{s,q}$ and $f \in \mathcal{S}(u)$, let $f_1 = f\chi_{(0,1]}$, $f_2 = f\chi_{(1,\infty)}$, and

$$u_i = \int_0^\infty f_i(t)\frac{dt}{t}, \quad i = 1, 2.$$

Obviously, $f_i \in \mathcal{S}(u_i)$ and $u = u_1 + u_2$ in $X_0 + X_1$. Since

$$\int_0^\infty (t^{-\alpha}J(t, f_1(t)))^q \frac{dt}{t} \le \int_0^1 (t^{-s}J(t, f(t)))^q \frac{dt}{t},$$

and

$$\int_0^\infty (t^{-\gamma}J(t, f_2(t)))^q \frac{dt}{t} \le \int_1^\infty (t^{-s}J(t, f(t)))^q \frac{dt}{t},$$

we can deduce that $u_1 \in X_{\alpha,q}$, $u_2 \in X_{\gamma,q}$, and $\|u_1\|_{\alpha,q} + \|u_2\|_{\alpha,q} \le \|u\|_{s,q}$, which in turn yields

$$\|u\|_{X_{\alpha,q}+X_{\gamma,q}} \le \|u\|_{s,q}. \tag{2.11}$$

On the other hand, if $u \in X_{\alpha,q} \cap X_{\gamma,q}$, then

$$\int_0^\infty (t^{-s}J(t, f(t)))^q \frac{dt}{t} \le \int_0^1 (t^{-\gamma}J(t, f(t)))^q \frac{dt}{t} + \int_1^\infty (t^{-\alpha}J(t, f(t)))^q \frac{dt}{t},$$

which implies that

$$\|u\|_{s,q} \le 2\|u\|_{X_{\alpha,q}\cap X_{\gamma,q}}. \tag{2.12}$$

Inequalities (2.11) and (2.12) jointly show the uniform boundedness of $\{X_{s,q} : s \in [a, b]\}$ and

$$\limsup_{k\to\infty} \|u\|_{X_{\alpha_k,q}+X_{\gamma_k,q}} \le \|u\|_{s,q}$$

as $\alpha_k \uparrow s$ and $\gamma_k \downarrow s$ for $u \in X_{s,q}$. Moreover, using Lebesgue's convergence theorem, we have

$$\lim_{\beta\to s} \|u\|_{\beta,q} = \|u\|_{s,q}$$

provided $u \in X_0 \cap X_1$. Then similar to the previews example, using the reflexivity of the dual interpolation spaces, and the density of $X_0 \cap X_1$ in $X_0$ and $X_1$, we can derive the norm-continuity and the successive property of $\{X_{s,q} : s \in [a, b]\}$.

**Definition 2.6.** Let $X$ be a linear space, and $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$ be a family of semimodulars defined on $X$. Suppose that every $\varrho_\alpha$ generates a Banach space $X_{\varrho_\alpha}$ with the Luxemburg norm $\|\cdot\|_{\varrho_\alpha}$. We say $\{\varrho_\alpha\}$ is a partially continuous semimodular net, or *PCMN* in symbol, provided the following two hypotheses are satisfied:

1. There are constants $C_1 > 0$, $C_2 \ge 0$ such that for all $\alpha, \beta, \gamma \in \mathcal{A}$, $\alpha \prec \beta \prec \gamma$, inequalities
$$\frac{1}{C_1}\varrho_{\alpha\vee\gamma}(u) - C_2 \le \varrho_\beta(u) \le C_1\varrho_{\alpha\wedge\gamma}(u) + C_2 \tag{2.13}$$
hold for all $u \in \mathcal{X}$.
2. When the sequences $\{\varrho_{\alpha_k}\}$ and $\{\varrho_{\gamma_k}\}$ upper- and lower-approach $\beta \in \mathcal{A}$ respectively and $u \in X_{\rho_{\alpha^-}} \cap X_{\rho_{\alpha^+}}$, limits
$$\lim_{k\to\infty} \varrho_{\alpha_k}(u) = \lim_{k\to\infty} \varrho_{\gamma_k}(u) = \varrho_\beta(u) \tag{2.14}$$
hold simultaneously.

*Remark* 2.7. Unlike the continuous semimodular net, the dual semimodular family of a partially semimodular net is no longer a semimodular net in general.

In the sequel, we will use $\|u\|_{\varrho_{\alpha\vee\gamma}}$ as the norm of $X_{\varrho_\alpha}+X_{\varrho_\gamma}$, and use $2\varrho_\beta(u/2)$ to produce the norm of $X_{\varrho_\beta}$. We also assume that the space nets $\{X_{\varrho_\alpha}\}$ and $\{X^\star_{\varrho_\alpha}\}$ are compatible, i.e. for all $\alpha,\beta,\gamma\in\mathcal{A}$, $\alpha\prec\beta\prec\gamma$, and all $\xi\in X^\star_{\varrho_\alpha}\cap X^\star_{\varrho_\gamma}$, all $u\in X_\beta$, the dual products $\langle\xi,u\rangle_{(X^\star_{\varrho_\alpha}\cap X^\star_{\varrho_\gamma})\times(X_{\varrho_\alpha}+X_{\varrho_\gamma})}$ and $\langle\xi,u\rangle_{X^\star_{\varrho_\beta}\times X_{\varrho_\beta}}$ are equal. Due to these conventions, in the following arguments, we will omit the subscript and only use $\langle\xi,u\rangle$ to denote the dual product between $\xi$ and $u$.

**Lemma 2.8.** *Let $(\varrho^\star)^\star_{\alpha\wedge\gamma}$ be the dual functional of $(\varrho^\star)_{\alpha\wedge\gamma}=\varrho^\star_\alpha+\varrho^\star_\gamma$, then we have*

$$(\varrho^\star)^\star_{\alpha\wedge\gamma}(u)\le\varrho_{\alpha\vee\gamma}(u),\ \forall\, u\in X_{\varrho_\alpha}+X_{\varrho_\gamma}. \tag{2.15}$$

*and particularly,*

$$(\varrho^\star)^\star_{\beta\wedge\beta}(u)=\varrho_{\beta\vee\beta}(u),\ \forall\, u\in X_{\varrho_\beta}. \tag{2.16}$$

*Hence, $\varrho_{\beta\vee\beta}$ is also a semimodular.*

**Proof** : For each splitting $u=u_1+u_2$ of $u\in X_{\varrho_\beta}$, by the definition of dual semimodular, we have

$$\begin{aligned}(\varrho^\star)^\star_{\alpha\wedge\gamma}(u)&=\sup_{\xi\in X^\star_{\varrho_\alpha}\cap X^\star_{\varrho_\gamma}}\{\langle\xi,u_1\rangle+\langle\xi,u_2\rangle-\varrho^\star_\alpha(\xi)-\varrho^\star_\gamma(\xi)\}\\&\le\sup_{\xi\in X^\star_{\varrho_\alpha}}\{\langle\xi,u_1\rangle-\varrho^\star_\alpha(\xi)\}+\sup_{\xi\in X^\star_{\varrho_\gamma}}\{\langle\xi,u_2\rangle-\varrho^\star_\gamma(\xi)\}\\&=\varrho_\alpha(u_1)+\varrho_\gamma(u_2).\end{aligned}$$

Taking infimum over the set of all the splitting $u=u_1+u_2$, we obtain (2.15). Equality (2.16) is a straight consequence of (2.15). □

**Theorem 2.9.** *Suppose that $\{\varrho_\alpha:\alpha\in\mathcal{A}\}$ is a PCMN on X, each semimodular $\varrho_\alpha$ generates a Banach space $X_{\varrho_\alpha}$ with the equivalent norm $\|\cdot\|_{2\rho_\alpha(\cdot/2)}$. Suppose also $X_{\rho_{\alpha^-}}\cap X_{\rho_{\alpha^+}}$ is embedded densely into $X_{\rho_\alpha}$ for all $\alpha\in\mathcal{A}$, and*

$$\lim_{k\to\infty}\varrho^\star_{\alpha_k}(\xi)=\lim_{k\to\infty}\varrho^\star_{\gamma_k}(\xi)=\varrho^\star_\beta(\xi),\ \forall\,\xi\in X^\star_{\rho_{\alpha^-}}\cap X^\star_{\rho_{\alpha^+}}, \tag{2.17}$$

*whenever $\{\alpha_k\}$ and $\{\gamma_k\}$ lower- and upper-approach $\beta$ in $\mathcal{A}$ respectively. Then $\{X_{\varrho_\alpha}:\alpha\in\mathcal{A}\}$ is a regular BSN (II).*

**Proof** : Imbedding (2.1) with the uniform bounds comes from (2.13). The remaining task is to show the norm-continuity and successive property of $\{X_{\varrho_\alpha}\}$. Taking two sequences $\{\alpha_k\}$ and $\{\gamma_k\}$ in $\mathcal{A}$ with $\alpha_k\uparrow\beta$ and $\gamma_k\downarrow\beta$, since $X_{\rho_{\alpha^-}}\cap X_{\rho_{\alpha^+}}$ is a dense subspace of $X_{\rho_\beta}$, by virtue of [11], it suffices to prove

$$\lim_{k\to\infty}\varrho_{\alpha_k\vee\gamma_k}(u)=2\varrho_\beta\Big(\frac{u}{2}\Big),\ \forall\, u\in X_{\rho_{\alpha^-}}\cap X_{\rho_{\alpha^+}}. \tag{2.18}$$

Firstly, with the aid of (2.14), we have

$$\limsup_{k\to\infty}\varrho_{\alpha_k\vee\gamma_k}(u)\le\lim_{k\to\infty}\varrho_{\alpha_k}\Big(\frac{u}{2}\Big)+\lim_{k\to\infty}\varrho_{\gamma_k}\Big(\frac{u}{2}\Big)=2\varrho_\beta\Big(\frac{u}{2}\Big). \tag{2.19}$$

Secondly, since $X_{\rho_{\alpha^-}}\cap X_{\rho_{\alpha^+}}$ is dense in $X_{\varrho_\alpha}$, $X^\star_{\rho_{\alpha^-}}\cap X^\star_{\rho_{\alpha^+}}$ is also dense in $X^\star_{\varrho_\alpha}$. Consequently, by virtue of (2.15) and (2.17), we can derive that

$$\begin{aligned}\liminf_{k\to\infty}\varrho_{\alpha_k\vee\gamma_k}(u)&\ge\liminf_{k\to\infty}\sup_{\xi\in X^\star_{\rho_{\alpha^-}}\cap X^\star_{\rho_{\alpha^+}}}\{\langle\xi,u\rangle-\varrho^\star_{\alpha_k}(\xi)-\varrho^\star_{\gamma_k}(\xi)\}\\&\ge\sup_{\xi\in X^\star_{\rho_{\alpha^-}}\cap X^\star_{\rho_{\alpha^+}}}\{\langle\xi,u\rangle-2\varrho^\star_\beta(\xi)\}=2\varrho_\beta\Big(\frac{u}{2}\Big).\end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20), we obtain (2.18), and the desired result has been reached. □

**Example 2.3.** Let $\Omega$ be an unbounded open subset of $\mathbb{R}^N$, and $\mathcal{P}_0(\Omega)$ be the collection of all variable exponents $p(\cdot)$ measurable on $\Omega$ with $1 < p^- \leq p^+ < \infty$, where $p^- = \operatorname{essinf}_{x\in\Omega} p(x)$ and $p^+ = \operatorname{esssup}_{x\in\Omega} p(x)$. For every variable exponent $p(\cdot) \in \mathcal{P}_0(\Omega)$, define the $L^{p(\cdot)}$–space as follows:

$$L^{p(\cdot)}(\Omega) = \Big\{f \in L^0(\Omega) : \varrho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty\Big\}.$$

Recall that $\varrho_{p(\cdot)}$ is a modular on the linear space $L^0(\Omega)$, and $L^{p(\cdot)}(\Omega)$ is a Banach space endowed with the Luxemburg norm $\|\cdot\|_{p(\cdot)}$. Moreover the dual space $L^{p(\cdot)}(\Omega)^\star$ is equivalent to $L^{p'(\cdot)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$ for a.e. $x \in \Omega$.

Fixe two numbers $\underline{p}$ and $\bar{p}$ in $(1, \infty)$, and let $\mathcal{A} = \{p(\cdot) \in \mathcal{P}_0(\Omega) : \underline{p} \leq p^- \leq p^+ \leq \bar{p}\}$. Equip $\mathcal{A}$ with the natural topology: $p_k(\cdot)$ converges to $q(\cdot)$ in $\mathcal{A}$ if and only if $p_k(x) \to q(x)$ for a.e. $x \in \Omega$ as $k \to \infty$, and the order $\prec$: $p(\cdot) \prec q(\cdot)$ if and only if $p(x) \leq q(x)$ for a.e $x \in \Omega$, $\mathcal{A}$ becomes a BTL. We will show that the collection of semimodular $\{\varrho_{p(\cdot)} : p(\cdot) \in \mathcal{A}\}$ constitute a partially continuous semimodular net fulfilling all the hypotheses in Theorem 2.9. Consequently $\{L^{p(\cdot)}(\Omega) : p(\cdot) \in \mathcal{A}\}$ is a regular *BSN* (II). For this purpose, firstly take any $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{A}$ with $p(\cdot) \prec q(\cdot) \prec r(\cdot)$, and any $f \in L^0(\Omega)$, let $E = \{x \in \Omega : |f(x)| \leq 1\}$, and $f_1 = f\chi_{E_1}$, $f_2 = f - f_1$, where $\chi_{E_1}$ stands for the characteristic function of $E_1$. Employing this splitting, we can deduce that

$$\begin{aligned}\varrho_{p(\cdot)\vee r(\cdot)}(f) &\leq \varrho_{p(\cdot)}(f_2) + \varrho_{r(\cdot)}(f_1)\\ &\leq \varrho_{q(\cdot)}(f_2) + \varrho_{q(\cdot)}(f_1) = \varrho_{q(\cdot)}(f)\\ &\leq \varrho_{p(\cdot)}(f_1) + \varrho_{r(\cdot)}(f_2) \leq \varrho_{p(\cdot)\wedge r(\cdot)}(f).\end{aligned}$$

This proves (2.15) with $C_1 = 1$, $C_2 = 0$.

Next notice that $L^{\underline{p}} \cap L^{\bar{p}}$ is dense in $L^{p(\cdot)}(\Omega)$, and for all $f \in L^{\underline{p}}(\Omega) \cap L^{\bar{p}}(\Omega)$, $\varrho_{\underline{p}}(f) < \infty$ and $\varrho_{\bar{p}}(f) < \infty$. Let $p_k(\cdot) \prec q(\cdot)$, $r_k(\cdot) \prec q(\cdot)$, and $p_k(\cdot) \to q(\cdot)$, $r_k(\cdot) \to q(\cdot)$ in $\mathcal{A}$ as $k \to \infty$. Since

$$\max\{|f(x)|^{p_k(x)}, |f(x)|^{r_k(x)}\} \leq |f(x)|^{\underline{p}(x)} + |f(x)|^{\bar{p}(x)},$$

we can use Lebesgue's convergence theorem to derive that

$$\lim_{k\to\infty} \varrho_{p_k(\cdot)}(f) = \lim_{k\to\infty} \varrho_{r_k(\cdot)}(f) = \varrho_{q(\cdot)}(f).$$

This proves the partial continuity of $\{\varrho_{p(\cdot)} : p(\cdot) \in \mathcal{A}\}$. By the same procedure, we can derive that

$$\lim_{k\to\infty} \varrho'_{p_k(\cdot)}(f) = \lim_{k\to\infty} \varrho'_{r_k(\cdot)}(f) = \varrho'_{q(\cdot)}(f)$$

for all $f \in L^{\underline{p}'}(\Omega) \cap L^{\bar{p}'}(\Omega)$. This shows the validity of (2.17). Here

$$\varrho'_{p(\cdot)}(f) = \int_\Omega \frac{1}{p'(x)p(x)^{1/(p(x)-1)}} |f(x)|^{p'(x)} dx$$

defines the dual semimodular $\varrho'_{p(\cdot)}$ on $L^{p'(\cdot)}(\Omega)$ .

# 3 Orlicz space of range-varying type

Let $I = (0, b]$ for some $0 < b < \infty$ or $I = (0, \infty)$, on which there is a complete and regular Borel measure $\mu$, and let $\Lambda(I)$ be the collection of all the bounded and closed subinterval of $I$. Suppose that $\mathcal{A}$ is a BTL, and $\theta : I \to \mathcal{A}$ is an order-continuous map, that is for any nest of intervals $\{J_k \in \Lambda(I) : k = 1, 2, \cdots\}$ shrinking to $t$, limits

$$\lim_{k\to\infty} \theta^-_{J_k} = \lim_{k\to\infty} \theta^+_{J_k} = \theta(t)$$

always hold simultaneously, where $\theta_J^- = \inf_{t\in J}\theta(t)$ and $\theta_J^+ = \sup_{t\in J}\theta(t)$ according to the order. Given a regular BSN (II) $\{X_\alpha : \alpha \in \mathcal{A}\}$, define

$$L_-^0(I, X_{\theta(\cdot)}) = \left\{f \in L^0(I, X_{\alpha^-} + X_{\alpha^+}) : f|_J \in L^0(J, X_{\theta_J^-} + X_{\theta_J^+}) \text{ for all } J \in \Lambda(I)\right\},$$

and

$$L^0(I, X_{\theta(\cdot)}) = \{f \in L_-^0(I, X_{\theta(\cdot)}) : f(t) \in X_{\theta(t)} \text{ for a.e. } t \in I\}.$$

Obviously, both $L_-^0(I, X_{\theta(\cdot)})$ and $L^0(I, X_{\theta(\cdot)})$ are linear spaces according to the sum and scalar multiplication of abstract valued functions.

Using the norm-continuity of $\{X_\alpha : \alpha \in \mathcal{A}\}$ and the order-continuity of $\theta$, we can prove that (cf. [10] for a proof of the similar result)

**Proposition 3.1.** *For all $u \in L^0(I, X_{\theta(\cdot)})$, the norm function $t \mapsto \|u(t)\|_{\theta(t)}$ is measurable.*

Denote by $\mathcal{S}(I, X_{\alpha^-} \cap X_{\alpha^+})$ and $\mathcal{S}(I)$ the sets all $X_{\alpha^-} \cap X_{\alpha^+}$–valued and scalar simple functions respectively. Evidently $\mathcal{S}(I, X_{\alpha^-} \cap X_{\alpha^+})$ is contained in $L^0(I, X_{\theta(\cdot)})$, so for every $\varphi \in \mathcal{S}(I, X_{\alpha^-} \cap X_{\alpha^+})$, the norm function $t \mapsto \|\varphi(t)\|_{\theta(t)}$ is measurable.

Define the space of strongly measurable functions of range-varying type as follows:

$$\begin{aligned} L_+^0(I, X_{\theta(\cdot)}) = \Big\{f \in L^0(I, X_{\alpha^-} + X_{\alpha^+}) : {} & f(t) \in X_{\theta(t)} \text{ for a.e. } t \in I, \\ & \text{and there exists } \{s_k\} \subset \mathcal{S}(I, X_{\alpha^-} \cap X_{\alpha^+}) \text{ s.t.} \\ & \lim_{k\to\infty} \|s_k(t) - f(t)\|_{\theta(t)} = 0 \text{ a.e. on } I\Big\}. \end{aligned}$$

It is easy to see that, $L_+^0(I, X_{\theta(\cdot)})$ is a subspace of $L^0(I, X_{\theta(\cdot)})$.

Suppose that $\varphi : I \times [0, \infty) \to [0, \infty]$ is a generalized $\Phi$–function, which says, for a.e. $t \in I$, $\varphi(t, \cdot)$ is a convex and left-continuous function satisfying $\lim_{x\to 0}\varphi(t, x) = \varphi(t, 0) = 0$ and $\lim_{x\to\infty}\varphi(t, x) = \infty$, and for all $s \in [0, \infty]$, $\varphi(\cdot, s)$ is measurable. By the left-continuity of $\varphi(t, \cdot)$, for every measurable function $h : I \to [0, \infty)$, the composite function $t \mapsto \varphi(t, h(t))$ is measurable. Thus for all $u \in L^0(I, X_{\theta(\cdot)})$, the composite function $t \mapsto \varphi(t, \|u(t)\|_{\theta(t)})$ is also measurable, consequently the integral

$$\Phi^\varphi(u) = \int_I \varphi(t, \|u(t)\|_{\theta(t)})d\mu \tag{3.1}$$

makes sense, and it defines a semimodualr on $L^0(I, X_{\theta(\cdot)})$. Use $L^\varphi(I, X_{\theta(\cdot)})$ and $L_+^\varphi(I, X_{\theta(\cdot)})$ to denote the semimodular spaces induced by $\Phi^\varphi$ from $L^0(I, X_{\theta(\cdot)})$ and $L_+^0(I, X_{\theta(\cdot)})$ respectively, and use $\mathcal{S}_\varphi(I, X_{\alpha^-} \cap X_{\alpha^+})$ and $S_\varphi(I)$ to denote the subsets of $\mathcal{S}(I, X_{\alpha^-} \cap X_{\alpha^+})$ and $S(I)$ respectively comprised of compactly supported functions. Evidently $\mathcal{S}_\varphi(I, X_{\alpha^-} \cap X_{\alpha^+}) \subseteq L^\varphi(I, X_{\theta(\cdot)})$ provided $\mathcal{S}_\varphi(I) \subseteq L^\varphi(I)$. Following the same process as in the proof of [10, Theorem 2.5] with only $X_{\theta_J^-}$ replaced by $X_{\theta_J^-} + X_{\theta_J^+}$, we can derive that

**Theorem 3.2.** *According to the Luxemburg norm, $L^\varphi(I, X_{\theta(\cdot)})$ is a Banach space.*

For each $k \in \mathbb{N}$, divide $I$ into $2^k$ equal parts if $I = (0, b]$, or infinite many equal parts with the length $1/2^k$ of each part if $I = (0, \infty)$. Denote by $t_{k,j} = jb/2^k$ and $J_{k,j} = (t_{k,j}, t_{k,j+1}]$ for $j = 1, \cdots, 2^k - 1$, if $I = (0, b]$, or $t_{k,j} = j/2^k$ and $J_{k,j} = (t_{k,j}, t_{k,j+1}]$ for $j = 1, 2, \cdots$, if $I = (0, \infty)$. Let $I_k = (b/2^k, b]$ or $I_k = (1/2^k, k]$, and define $\theta_n^\pm(t) = \theta_{J_{k,j}}^\pm$ for $t \in J_{k,j}$ and $j = 1, \cdots, 2^k - 1$, or $j = 1, 2, \cdots$, then we obtain two step functions. Obviously $\theta_k^-(t) \le \theta_{k+1}^-(t)$, $\theta_k^+(t) \ge \theta_{k+1}^+(t)$, and

$$\lim_{k\to\infty}\theta_k^-(t) = \lim_{k\to\infty}\theta_k^+(t) = \theta(t)$$

for all $t \in I$ since $\theta$ is order-continuous. Set

$$\bar{X}_{\theta_k(t)} = X_{\theta_k^-(t)} + X_{\theta_k^+(t)} \text{ and } \hat{X}_{\theta_k(t)} = X_{\theta_k^-(t)} \cap X_{\theta_k^+(t)},$$

then we can define two function spaces $L^{\varphi}(I_k, \bar{X}_{\theta_k(\cdot)})$ and $L^{\varphi}(I_k, \hat{X}_{\theta_k(\cdot)})$ as the semimodular spaces derived from $L^0(I_k, \bar{X}_{\theta_k(\cdot)})$ and $L^0(I_k, \hat{X}_{\theta_k(\cdot)})$ by the semimodular (3.1) with $X_\theta(t)$ replaced by $\bar{X}_{\theta_k(t)}$ and $\hat{X}_{\theta_k(t)}$ respectively. By the uniform boundedness and successive property of $\{X_\alpha\}$, adjoint with the monotonicity of $\varphi$, we can derive that

**Theorem 3.3.** *For all $k \in \mathbb{N}$, the following imbeddings*

$$L^{\varphi}(I_k, \hat{X}_{\theta_k(\cdot)}) \hookrightarrow L^{\varphi}(I_k, X_{\theta(\cdot)}) \hookrightarrow L^{\varphi}(I_k, \bar{X}_{\theta_k(\cdot)}). \tag{3.2}$$

*hold. Moreover, if $f \in L^{\varphi}(I_k, \bar{X}_{\theta_k(\cdot)})$ for all $k \in \mathbb{N}$, and*

$$C = \sup_{k \in \mathbb{N}} \|f\|_{L^{\varphi}(I_k, \bar{X}_{\theta_k(\cdot)})} < \infty,$$

*then $f \in L^{\varphi}(I, X_{\theta(\cdot)})$, and $\|f\|_{L^{\varphi}(I, X_{\theta(\cdot)})} \le C$.*

Concerning the space $L^{\varphi}_{+}(I, X_{\theta(\cdot)})$, we have

**Theorem 3.4.** *Assume that $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a dense BSN (II), i.e. $X_{\alpha^-} \cap X_{\alpha^+}$ is dense in $X_\alpha$ for all $\alpha \in \mathcal{A}$, and $\mathcal{S}_\varphi(I) \subseteq L^{\varphi}(I)$, then $L^{\varphi}_{+}(I, X_{\theta(\cdot)})$ is a closed subspace of $L^{\varphi}(I, X_{\theta(\cdot)})$. Moreover, for every $u \in L^{\varphi}_{+}(I, X_{\theta(\cdot)})$, there is a sequence of $\mathcal{S}_\varphi(I, X_{\alpha^-} \cap X_{\alpha^+})$ converging to $u$ in $L^{\varphi}(I, X_{\theta(\cdot)})$.*

The above two results are much similar to those obtained in [11], and here we omit the whole proofs.

Let $X$ be a Banach space, and $L^{\varphi}(I, X)$ be the abstract-valued Orlicz space of range-fixed type. Define the associate function space

$$L^{\varphi}(I, X)' = \Big\{\xi \in L^0(I, X^\star) : \int_I |\langle \xi(t), f(t)\rangle| d\mu < \infty \text{ for all } f \in L^{\varphi}(I, X)\Big\},$$

with the norm

$$\|\xi\|_{L^{\varphi}(I,X)'} := \sup\Big\{\frac{\int_I |\langle \xi(t), f(t)\rangle| d\mu}{\|f\|_{L^{\varphi}(I,X)}} : f \in L^{\varphi}(I, X), f \neq 0\Big\}.$$

One can easily check that, according to $\|\cdot\|_{L^{\varphi}(I,X)'}$, $L^{\varphi}(I, X)'$ becomes a Banach spaces, and the following equality

$$\langle\langle T\xi, f\rangle\rangle = \int_I \langle \xi(t), f(t)\rangle d\mu,\ f \in L^{\varphi}(I, X),\ \xi \in L^{\varphi}(I, X)' \tag{3.3}$$

defines a linear imbedding map $T : L^{\varphi}(I, X)' \to L^{\varphi}(I, X)^\star$ with

$$\|T\xi\|_{L^{\varphi}(I,X)^\star} \le \|\xi\|_{L^{\varphi}(I,X)'},$$

where $L^{\varphi}(I, X)^\star$ represents the dual space of $L^{\varphi}(I, X)$. For the relation between $L^{\varphi}(I, X)'$ and $L^{\varphi'}(I, X^\star)$, we have $L^{\varphi'}(I, X^\star) \hookrightarrow L^{\varphi}(I, X)'$ with the estimates

$$\|\xi\|_{L^{\varphi}(I,X)'} \le 2\|\xi\|_{L^{\varphi'}(I,X^\star)}.$$

Here $\varphi'(t, \cdot)$ stands for the conjugate function of $\varphi(t, \cdot)$, i.e.

$$\varphi'(t, \varsigma) = \sup_{\tau > 0}\{\varsigma\tau - \varphi(t, \tau)\} \text{ for a.e. } t \in I.$$

All these properties are natural extensions of those for the Orlicz space of scalar type (please compare to [5, §2.7]). Furthermore, we have

**Theorem 3.5.** *Suppose that $\varphi$ is locally integrable, i.e. $\int_E \varphi(t, \lambda) d\mu < \infty$ for all $\lambda > 0$ and all commpact subsets $E$ of $I$, then $L^{\varphi}(I, X)'$ is equal to $L^{\varphi'}(I, X^\star)$ with the equivalent norms, i.e.*

$$\|\xi\|_{L^{\varphi'}(I,X^\star)} \le \|\xi\|_{L^{\varphi}(I,X)'} \le 2\|\xi\|_{L^{\varphi'}(I,X^\star)}.$$

**Proof** : Pick $\xi \in L^{\varphi}(I,X)'$. Without loss of generality, assume that $\|\xi\|_{L^{\varphi}(I,X)'} \le 1$. Then $T\xi \in L^{\varphi}(I,X)^{\star}$, and $\left(\Phi^{\varphi}\right)^{\star}(T\xi) \le \|T\xi\|_{L^{\varphi}(I,X)^{\star}} \le 1$. Since $\xi$ is strongly measurable, there exits a sequence $\{\eta_k\}$ in $S(I,X^{\star})$ such that $\eta_k(t) \to \xi(t)$ in $X^{\star}$ and $\|\eta_k(t)\|_{X^{\star}} \le \|\xi(t)\|_{X^{\star}}$ a.e. on $I$. Similar to the arguments in [5, §2.7], let $\{r_j\}$ be the collection of all rational numbers in $[0,\infty)$ with $r_1 = 0$. For each $j \in \mathbb{N}^+$, construct a function as follows:

$$\omega_j(t) = \chi_{I_j} \max\{r_i\|\xi(t)\|_{X^{\star}} - \varphi(t,r_i) : i = 1,2,\cdots,j\}.$$

Evidently, $\{\omega_j\}$ is an increasing sequence of nonnegative and measurable functions, and

$$\lim_{j\to\infty} \omega_j(t) = \varphi'(t,\|\xi(t)\|_{X^{\star}}) \tag{3.4}$$

for a.e. $t \in I$. Moreover for every $j \in \mathbb{N}^+$, there is correspondingly a nonnegative scalar simple function $s_j$ with $\mathrm{supp} s_j \subset I_j$, such that

$$\omega_j(t) = s_j(t)\|\xi(t)\|_{X^{\star}} - \varphi(t,s_j(t)).$$

Since $\varphi$ is locally integrable, for every $\lambda > 0$, we have $\int_I \varphi(t,\lambda^{-1}s_j(t))d\mu < \infty$. Fix $0 < \lambda < 1/4$, then by the absolute convergence of the integral, there is $\delta_j > 0$, such that $\int_D \varphi(t,\lambda^{-1}s_j(t))d\mu < 1/j$ for all measurable subsets $D$ of $I_j$ with $\mu(D) < \delta_j$. For each $j \in \mathbb{N}^+$, by the Egrov's theorem, there is a measurable set $E_j \subseteq I_j$ with $\mu(I_j\backslash E_j) < \delta_j$ such that $\eta_k(t) \to \xi(t)$ in $X^{\star}$ uniformly on $E_j$ as $k \to \infty$. Thus for sufficiently large integer $k_j$, we have

$$\|\eta_{k_j}(t) - \xi(t)\|_{X^{\star}} < \frac{1}{(1+s_j(t))j^2},\ \forall\, t \in E_j.$$

Notice that $\eta_{k_j}(t)$ takes only finite many values in $X^{\star}$, so there is another function $w_j \in S_{\varphi}(I,X)$ satisfying $\|w_j(t)\|_X = 1$ and

$$\langle \eta_{k_j}(t), w_j(t)\rangle \ge \|\eta_{k_j}(t)\|_{X^{\star}} - \frac{1}{(1+s_j(t))j^2}$$

for all $t \in I_j$.

Let $f_j = s_j w_j$, then we obtain a member of $S(I_j,X)$ satisfying

$$\begin{aligned}\langle \xi(t), f_j(t)\rangle &\ge \left(\langle \eta_{k_j}(t), w_j(t)\rangle - \|\eta_{k_j}(t)-\xi(t)\|_{X^{\star}}\right)s_j(t)\\ &\ge \left(\|\eta_{k_j}(t)\|_{X^{\star}} - \frac{1}{(1+s_j(t))j^2} - \|\eta_{k_j}(t)-\xi(t)\|_{X^{\star}}\right)s_j(t)\\ &\ge \left(\|\eta_{k_j}(t)\|_{X^{\star}} - \|\eta_{k_j}(t)-\xi(t)\|_{X^{\star}}\right)s_j(t) - \frac{\chi_{I_j}}{j^2}\\ &\ge \left(\|\xi(t)\|_{X^{\star}} - 2\|\eta_{k_j}(t)-\xi(t)\|_{X^{\star}}\right)s_j(t) - \frac{\chi_{I_j}}{j^2}.\end{aligned}$$

Consequently,

$$\begin{aligned}\left(\Phi^{\varphi}\right)^{\star}(T\xi) &\ge \int_I \left\{\langle \xi(t), f_j(t)\rangle - \varphi(t,f_j(t))\right\}d\mu\\ &\ge \int_I \omega_j(t)d\mu - 2\int_{E_j} \frac{s_{j(t)}}{(1+s_j(t))j^2}d\mu - 4\int_{I_j\backslash E_j} s_j(t)\|\xi(t)\|_{X^{\star}}d\mu - \frac{1}{j}\\ &\ge \int_I \omega_j(t)d\mu - 4\Big(\int_{I_j\backslash E_j} \varphi(t,\lambda^{-1}s_j(t))d\mu + \lambda\int_I \varphi'(t,\|\xi(t)\|_{X^{\star}})d\mu\Big) - \frac{3}{j}\\ &\ge \int_I \omega_j(t)d\mu - 4\lambda\int_I \varphi'(t,\|\xi(t)\|_{X^{\star}})d\mu - \frac{7}{j}.\end{aligned}$$

Let $j \to \infty$, and use (3.4), we obtain

$$\Phi^{\varphi'}(\eta) \le \frac{1}{1-4\lambda}\left(\Phi^{\varphi}\right)^{\star}(T\eta) \le \frac{1}{1-4\lambda}.$$

Thus $\xi \in L^{\varphi'}(I,X^{\star})$ and $\Phi^{\varphi'}(\eta) \le 1$ by the arbitrariness of $\lambda \in (0,1/4)$. Finally by the scaling arguments, we reach the desired conclusion. □

*Remark* 3.6. Different to the scalar case (please compare to [5, Theorem 2.7.4]), for the function $\xi \in L^{\varphi}(I, X)'$, we could not find a sequence of $X$–valued simple functions, say $\{h_k\}$, verifying

$$\varphi'(t, \|\xi(t)\|_{X^*}) = \lim_{k\to\infty} \{\langle \xi(t), h_k(t)\rangle - \varphi(t, \|h_k(t)\|_X)\},$$

unless $X$ is separable. To derive the inclusion $\xi \in L^{\varphi'}(I, X^*)$, we introduce the multiplier $\lambda \in (0, 1)$, along which, the absolute convergence of the integral, and the Egrov's theorem are applied together. Due to these differences, Theorem 3.5 is not a parallel extension of [5, Theorem 2.7.4] from the scalar case to the vector-valued case.

**Theorem 3.7.** *Suppose that $S_{\varphi}(I) \subseteq L^{\varphi}(I)$, and the dual space $X^*$ satisfies the Radon-Nikodym's property w.r.t. every bounded subinterval of I. Then the map T is an isometrical isomorphism between $L^{\varphi}(I, X)'$ and $L^{\varphi}(I, X)^*$.*

**Proof** : It suffices to show that for every $\Xi \in L^{\varphi}(I, X)^*$, there is only one function $\xi \in L^{\varphi}(I, X)'$ such that $T\xi = \Xi$ in the sense of (3.3). If $\Xi = 0$, then take $\xi = 0$ and there is nothing to do. If $\Xi \neq 0$, then the proof can be made by the scaling arguments. So we can assume $\|\Xi\|_{L^{\varphi}(I,X)^*} = 1$. Since $S_{\varphi}(I) \subseteq L^{\varphi}(I)$, for every compact subset $E$ of $I$ and $u \in X$, the function $\chi_E u$ belongs to $L^{\varphi}(I, X)$, and

$$\|\chi_E u\|_{L^{\varphi}(I,X)} \leq \|\chi_E\|_{L^{\varphi}(I)}\|u\| < \infty. \tag{3.5}$$

Fix $k \in \mathbb{N}$, and consider the $X^*$–valued function $\mu_k$:

$$\langle \mu_k(E), u\rangle = \langle\langle \Xi, \chi_E u\rangle\rangle, \quad u \in X,$$

acting on the collection of all measurable subsets of $I_k$. By (3.5), we can easily show that $\mu_k$ is a totally bounded $X^*$–valued measure on $I_k$ with the total variation no more than $\|\Xi\|_{L^{\varphi}(I,X)^*}\|\chi_{I_k}\|_{L^{\varphi}(I)}$. Hence under the Radon-Nikodym's assumption of $X^*$, we can find a unique function $\xi_k \in L^1(I_k, X^*)$ satisfying

$$\langle\langle \Xi, \chi_E u\rangle\rangle = \langle \mu_k(E), u\rangle = \int_{I_k} \langle \xi_k(t), \chi_E u\rangle d\mu$$

for all measurable subsets $E$ of $I_k$. By the uniqueness of $\xi_k$, it is easy to check that $\xi_{k+1}(t) = \xi_k(t)$ a.e. on $I_k$. So if we let $\xi(t) = \xi_n(t)$ for $t \in I_n$, then we obtain a strongly measurable $X^*$–valued function on $I$ satisfying

$$\langle\langle \Xi, f\rangle\rangle = \int_I \langle \xi(t), f(t)\rangle d\mu \tag{3.6}$$

for the function $f = u\chi_E$ with $E$ compact and $u \in X$. By the linearity of $\Xi$ and the integration, we can easily check that (3.6) is also satisfied for all $f \in \mathbb{S}_{\varphi}(I, X)$.

As for $f \in L^{\varphi}(I, X)$, there exits a sequence $\{w_k\}$ in $S(I, X)$ such that $w_k(t) \to f(t)$ in $X$ and $\|w_k(t)\|_X \leq \|f(t)\|_X$ a.e. on $I$. Let $\tilde{w}_k = w_k\chi_{I_k}\mathrm{sgn}(\langle \xi, w_k\rangle)$, then $\tilde{w}_k$ is also a simple function satisfying $\tilde{w}_k \in S_{\varphi}(I, X)$, $\|\tilde{w}_k\|_{L^{\varphi}(I,X)} \leq \|f\|_{L^{\varphi}(I,X)}$, and $\langle \xi(t), \tilde{w}_k(t)\rangle \to |\langle \xi(t), f\rangle|$ a.e. on $I$ as $k \to \infty$. Thus using $\tilde{w}_k$ to replace $f$ in (3.6), and letting $k \to \infty$, we have

$$\begin{aligned}\int_I |\langle \xi(t), f\rangle| d\mu &\leq \liminf_{k\to\infty} \int_I \langle \xi(t), \tilde{w}_k(t)\rangle d\mu = \liminf_{k\to\infty} \langle\langle \Xi, \tilde{w}_k\rangle\rangle \\ &\leq \limsup_{k\to\infty} \|\Xi\|_{L^{\varphi}(I,X)^*}\|\tilde{w}_k\|_{L^{\varphi}(I,X)} \leq \|\Xi\|_{L^{\varphi}(I,X)^*}\|f\|_{L^{\varphi}(I,X)}.\end{aligned}$$

Therefore $\xi(t) \in L^{\varphi}(I, X)'$, $T\xi = \Xi$ and $\|\xi\|_{L^{\varphi}(I,X)'} \leq \|\Xi\|_{L^{\varphi}(I,X)^*}$. □

Combining Theorem 3.5 and 3.7, we obtain

**Theorem 3.8.** *Suppose that $\varphi$ is locally integrable, X is a Banach space, which dual $X^*$ satisfies the Radon-Nikodym's property w.r.t. every bounded subinterval of I. Then*

$$L^{\varphi}(I, X)^* = L^{\varphi'}(I, X^*)$$

*in the sense of isomorphism.*

Recall that every reflexive space satisfies the Radon-Nikodym's property with respect to every complete and finite measure space, thus from the above theorem we can derive that

**Corollary 3.9.** *Suppose that $X$ is reflexive, both $\varphi$ and $\varphi'$ are locally integrable. Then the abstract-valued Orlicz space with fixed range $L^{\varphi}(I,X)$ is reflexive.*

Concerning the abstract-valued Orlicz space of range-varying type, we have

**Theorem 3.10.** *Assume the following hypotheses:*

1. *$\{X_\alpha : \alpha \in \mathcal{A}\}$ is a dense, regular BSN (II),*
2. *the dual family $\{X_\alpha^\star : \alpha \in \mathcal{A}\}$ constitute another continuous and successive BSN (II),*
3. *for all $\alpha, \gamma \in \mathcal{A}$, $X_\alpha^\star + X_\gamma^\star$ satisfies the Radon-Nikodym's property w.r.t. every finite subinterval of $I$, and*
4. *the generalized $\Phi$–function $\varphi$ is locally integrable.*

*Then the integral*

$$\int_I \langle \xi(t), f(t)\rangle d\mu =: \langle\langle T\xi, f\rangle\rangle,\ f \in L_+^{\varphi}(I, X_{\theta(\cdot)}),\ \xi \in L^{\varphi'}(I, X_{\theta(\cdot)}^\star) \tag{3.7}$$

*defines an linear isomorphism $T$ from $L^{\varphi'}(I, X_{\theta(\cdot)}^\star)$ onto $L_+^{\varphi}(I, X_{\theta(\cdot)})^\star$.*

**Proof :** The linearity and continuity of $T$ is easy to check from the definition (3.7) with the estimate

$$\|T\xi\|_{L_+^{\varphi}(I,X_{\theta(\cdot)})^\star} \le 2\|\xi\|_{L^{\varphi'}(I,X_{\theta(\cdot)}^\star)}$$

by the Young's inequality for $\varphi$.

Conversely, for any $\Xi \in L^{\varphi}(I, X_{\theta(\cdot)})^\star$, our goal is to prove the existence of a function $\xi \in L^{\varphi'}(I, X_{\theta(\cdot)}^\star)$ satisfying $\Xi = T\xi$ and

$$\|\xi\|_{L^{\varphi'}(I,X_{\theta(\cdot)}^\star)} \le \|\Xi\|_{L_+^{\varphi}(I,X_{\theta(\cdot)})^\star}.$$

Firstly, since the sum spaces $X_{\alpha^+}^\star + X_{\alpha^-}^\star$ and $X_{\theta_J^+}^\star + X_{\theta_J^-}^\star$ for $J \in \Lambda(I)$ satisfy the Radon-Nikodym's property, by invoking Theorem 3.8, we can find a function $\xi \in L^{\varphi'}(I, X_{\alpha^+}^\star + X_{\alpha^-}^\star)$ satisfying $\xi|_J \in L^{\varphi'}(J, X_{\theta_J^+}^\star \cap X_{\theta_J^-}^\star)$ for all $J \in \Lambda(I)$, and

$$\langle\langle \Xi|_{L^{\varphi}(J,X_{\theta_J^+}\cap X_{\theta_J^-})}, f\rangle\rangle = \int_J \langle \xi(t), f(t)\rangle d\mu$$

for all $f \in L^{\varphi}(J, X_{\theta_J^+} \cap X_{\theta_J^-})$. Thus for each $k \in \mathbb{N}$, we have

$$\langle\langle \Xi|_{L^{\varphi}(I_k,\hat{X}_{\theta_k(\cdot)})}, f\rangle\rangle = \int_{I_k} \langle \xi(t), f(t)\rangle d\mu,\ \forall f \in L^{\varphi}(J, \hat{X}_{\theta_k(\cdot)}), \tag{3.8}$$

which, combined with (3.2), and conclusions of Theorem 3.5, 3.7, yields

$$\begin{aligned}
\|\xi\|_{L^{\varphi'}(I_k,\bar{X}_{\theta_k(\cdot)}^\star)} &\le \|\xi\|_{L^{\varphi}(I_k,\bar{X}_{\theta_k(\cdot)})'} \\
&= \sup\Big\{\frac{\langle\langle \Xi|_{L^{\varphi}(I_k,\hat{X}_{\theta_k(\cdot)})}, f\rangle\rangle}{\|f\|_{L^{\varphi}(I_k,\hat{X}_{\theta_k(\cdot)})}} : f \in L^{\varphi}(I_k, \hat{X}_{\theta_k(\cdot)}), f \neq 0\Big\} \\
&\le C\sup\Big\{\frac{\langle\langle \Xi|_{L^{\varphi}(I_k,X_{\theta(\cdot)})}, f\rangle\rangle}{\|f\|_{L^{\varphi}(I,X_{\theta(\cdot)})}} : f \in L^{\varphi}(I_k, \hat{X}_{\theta_k(\cdot)}), f \neq 0\Big\} \\
&\le C\|\Xi\|_{L^{\varphi}(I_k,X_{\theta(\cdot)})^\star}.
\end{aligned}$$

Since the above constant $C > 0$ is independent of $k$, by applying the second conclusion of Theorem 3.4, we conclude that $\xi \in L^{\varphi'}(I, X_{\theta(\cdot)}^\star)$, and

$$\|\xi\|_{L^{\varphi'}(I,X_{\theta(\cdot)}^\star)} \le C\|\Xi\|_{L^{\varphi}(I,X_{\theta(\cdot)})^\star}.$$

Finally, using the density of $L^\varphi(I, \hat{X}_{\theta_k(\cdot)})$ into $L^\varphi(I, X_{\theta(\cdot)})$, we can prove the validity of (3.8) for all $f \in L^\varphi(I, X_{\theta(\cdot)})$. Thus $\Xi = T\xi$, and we have completed the proof.

Under the condition $L^0_+(I, X_{\theta(\cdot)}) = L^0(I, X_{\theta(\cdot)})$, it is easy to see that $L^\varphi_+(I, X_{\theta(\cdot)}) = L^\varphi(I, X_{\theta(\cdot)})$. Thus we have

**Theorem 3.11.** *Suppose that the following hypotheses are all satisfied:*

1. *Both $\{X_\alpha : \alpha \in \mathcal{A}\}$ and $\{X^\star_\alpha : \alpha \in \mathcal{A}\}$ are regular and dense BSNs (II),*
2. *$L^0_+(I, X_{\theta(\cdot)}) = L^0(I, X_{\theta(\cdot)})$ and $L^0_+(I, X^\star_{\theta(\cdot)}) = L^0(I, X^\star_{\theta(\cdot)})$,*
3. *for all $\alpha \in \mathcal{A}$, $X_\alpha$ is reflexive, and*
4. *the generalized $\Phi$–function $\varphi$ and its conjugate $\varphi'$ are both locally integrable.*

*Then the function space $L^\varphi(I, X_{\theta(\cdot)})$ is reflexive.*

*Remark* 3.12. It is easy to check that Theorem 3.10 and 3.11 still hold respectively for the space $L^\varphi_+(I, X_{\theta(\cdot)})^\star$ and $L^\varphi(I, X_{\theta(\cdot)})$ in case that $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a dense, regular *BSN* of type (I). Compare to [11, Theorem 3.12], here $\varphi$ is merely a general generalized $\Phi$–function with local integrability assumption, and the extra hypothesis that $X^\star_\alpha$ is norm-attainable for every $\alpha \in \mathcal{A}$ is no longer needed. In this sense, Theorem 3.10 and 3.11 can be viewed as the improvements of Theorem 3.12 and Corollary 3.15 in [11] respectively.

# 4 Application in real interpolation spaces

Given an interpolation couple $(X_0, X_1)$ as in Example 2.2, suppose that $X_0 \cap X_1$ is dense in $X_i$, $i = 1, 2$. For each $t \in (0, \infty)$, let $\hat{X}_t = X_0 \cap X_1$, $\hat{X}^\star_t = X^\star_0 \cap X^\star_1$ endowed with the norms $\|u\|_{\hat{X}_t} = J(t, u)$, $\|\xi\|_{\hat{X}^\star_t} = J^\star(t, \xi)$, and let $\bar{X}_t = X_0 + X_1$, $\bar{X}^\star_t = X^\star_0 + X^\star_1$ endowed with the norm $\|u\|_{\bar{X}_t} = K(t, u)$, $\|\xi\|_{\bar{X}^\star_t} = K^\star(t, \xi)$. Here $J^\star(t, \xi)$ and $K^\star(t, \xi)$ are the $J$–functional and $K$–functional corresponding to the dual couple $(X^\star_0, X^\star_1)$. Denote by $\hat{X}_0 = \bar{X}_0 = X_0$, $\hat{X}_\infty = \bar{X}_\infty = X_1$, $\hat{X}^\star_0 = \bar{X}^\star_0 = X^\star_0$ and $\hat{X}^\star_\infty = \bar{X}^\star_\infty = X^\star_1$. If we take $\mathcal{A} = [0, \infty]$ equipped with the natural topology and order, then $\mathcal{A}$ is a *BTL*, attached to which, $\{\hat{X}_t : t \in [0, \infty]\}$, $\{\bar{X}_t : t \in [0, \infty]\}$, $\{\bar{X}^\star_t : t \in [0, \infty]\}$ and $\{\hat{X}^\star_t : t \in [0, \infty]\}$ are all regular and dense *BSN* (II), and the latter two ones are dual nets of the former two ones respectively. Let $I = (0, \infty)$, $d\mu = dt/t$ and $\theta(t) = t$, $\vartheta(t) = t^{-1}$, then we obtain a complete and regular Borel measure, two order-continuous maps, attached to which, four function spaces: $L^0(I, \hat{X}_{\theta(\cdot)})$, $L^0(I, \hat{X}_{\vartheta(\cdot)})$, $L^0(I, \bar{X}_{\theta(\cdot)})$, $L^0(I, \bar{X}_{\vartheta(\cdot)})$, and other four ones for the dual couple $(X^\star_0, X^\star_1)$ are well defined. All of them are equal to the strongly measurable ones, for example, $L^0(I, \hat{X}_{\theta(\cdot)}) = L^0_+(I, \hat{X}_{\theta(\cdot)})$.

Suppose that $\varphi : I \times [0, \infty) \to [0, \infty)$ is a generalized $\Phi$–function, satisfying $\varphi(t, \tau) > 0$ for all $\tau > 0$. Introduce two indices of $\varphi$,

$$\bar{p}_\varphi := \sup_{t>0} \sup_{\tau>0} \frac{\tau \partial^+ \varphi(t, \tau)}{\varphi(t, \tau)}, \ \underline{p}_\varphi := \inf_{t>0} \inf_{\tau>0} \frac{\tau \partial^+ \varphi(t, \tau)}{\varphi(t, \tau)},$$

where $\partial^+ \varphi(t, \tau)$ denotes the right derivative of $\varphi$ w.r.t. the variable $\tau$. A straight calculation shows that $\varphi(t, C\tau) \le C^{\bar{p}_\varphi} \varphi(t, \tau)$ for all $C > 1$ and all $t, \tau > 0$ if and only if $\bar{p}_\varphi < \infty$. In this case, $\varphi$ satisfies the $\Delta_2$–condition uniformly for all $t \in I$. If $\underline{p}_\varphi > 1$, then $\varphi(t, \cdot)$ is strictly convex. Furthermore, if $1 < \underline{p}_\varphi \le \bar{p}_\varphi < \infty$, then $\tau \mapsto \varphi(t, \tau)/\tau^{\underline{p}_\varphi}$ is increasing, while $\tau \mapsto \varphi(t, \tau)/\tau^{\bar{p}_\varphi}$ is decreasing (refer to [23]), from which, we can derive that

$$\varphi(t, \tau) \le \begin{cases} t^{\underline{p}_\varphi} \varphi(t, 1), & 0 < \tau < 1, \\ t^{\bar{p}_\varphi} \varphi(t, 1), & \tau \ge 1, \end{cases} \tag{4.1}$$

and

$$\lim_{\tau \to 0} \frac{\varphi(t, \tau)}{\tau} = 0, \ \lim_{\tau \to \infty} \frac{\varphi(t, \tau)}{\tau} = \infty.$$

Under this situation, $\varphi$ is called a generalized $N$–function. Denote by $\psi(t,\cdot)$ the conjugate function of $\varphi(t,\cdot)$ for each $t \in I$, we obtain another generalized $N$–function which can be constructed by

$$\psi(t,\tau) = \int_0^\tau \partial^+\psi(t,\zeta)d\zeta,$$

where $\partial^+\psi(t,\cdot)$ is the right-continuous inverse of $\partial^+\varphi(t,\cdot)$, i.e.

$$\partial^+\psi(t,\zeta) = \sup\{\tau \geq 0 : \partial^+\varphi(t,\tau) \leq \zeta\}.$$

For each $t \in I$, define

$$E_t = \{\tau > 0 : \tau = \partial^+\psi(t,\zeta) \ \text{for some } \zeta > 0\},$$

and

$$E_t' = \{\zeta > 0 : \zeta = \partial^+\varphi(t,\tau) \ \text{for some } \tau > 0\}.$$

The following properties of $E_t$ and $E_t'$ are easy to verified:

1. $\tau \in E_t$ iff there is not any $\tau_1 > \tau$ such that $\partial^+\varphi(t,\tau_1) = \partial^+\varphi(t,\tau)$;
   $\zeta \in E_t'$ iff there is not any $\zeta_1 > \zeta$ such that $\partial^+\psi(t,\tau_1) = \partial^+\psi(t,\tau)$.
2. $\tau \in E_t$ iff $\partial^+\varphi(t,\tau) \in E_t'$; $\zeta \in E_t'$ iff $\partial^+\psi(t,\zeta) \in E_t$.
3. $\zeta\tau = \varphi(t,\tau) + \psi(t,\zeta)$ in case that $\tau \in E_t$ or $\zeta \in E_t'$.

Based on these properties, we can deduce that

$$\begin{aligned}\inf_{\zeta>0} \frac{\psi(t,\zeta)}{\zeta\partial^+\psi(t,\zeta)} &= \inf\Big\{\frac{\tau\zeta - \varphi(t,\tau)}{\tau\zeta} : \zeta \in E_t', \tau = \partial^+\psi(t,\zeta)\Big\}\\ &= 1 - \sup\Big\{\frac{\varphi(t,\tau)}{\tau\zeta} : \zeta \in E_t', \tau = \partial^+\psi(t,\zeta)\Big\}\\ &= 1 - \sup\Big\{\frac{\varphi(t,\tau)}{\tau\zeta} : \tau \in E_t, \zeta = \partial^+\varphi(t,\tau)\Big\}\\ &= 1 - \sup_{\tau>0} \frac{\varphi(t,\tau)}{\tau\partial^+\varphi(t,\tau)},\end{aligned}$$

and analogously,

$$\sup_{\zeta>0} \frac{\psi(t,\zeta)}{\zeta\partial^+\psi(t,\zeta)} = 1 - \inf_{\tau>0} \frac{\varphi(t,\tau)}{\tau\partial^+\varphi(t,\tau)},$$

which tells us the relations

$$\underline{p}_\psi = \inf_{t>0}\inf_{\zeta>0} \frac{\zeta\partial^+\psi(t,\zeta)}{\psi(t,\zeta)} = \bar{p}_\varphi', \ \bar{p}_\psi = \sup_{t>0}\sup_{\zeta>0} \frac{\zeta\partial^+\psi(t,\zeta)}{\psi(t,\zeta)} = \underline{p}_\varphi',$$

where $\bar{p}_\varphi'$ and $\underline{p}_\varphi'$ are the conjugate exponents of $\bar{p}_\varphi$ and $\underline{p}_\varphi$ respectively.

In the sequel, we always assume that $1 < \underline{p}_\varphi \leq \bar{p}_\varphi < \infty$, and both $\varphi(\cdot,1)$ and $\psi(\cdot,1)$ lie in $L^\infty(I)$ with the same essential upper bounds $C_0$. Under this situation, for fixed index $s \in (0,1)$, the new four generalized $N$–functions $\varphi(t,t^{-s}\tau)$, $\varphi(t,t^s\tau)$, $\psi(t,t^{-s}\zeta)$ and $\psi(t,t^s\zeta)$ are all locally integrable. Moreover, $\psi(t,t^s\zeta)$ is the conjugate of $\varphi(t,t^{-s}\tau)$, $\psi(t,t^{-s}\zeta)$ is the conjugate of $\varphi(t,t^s\tau)$. Using $\varphi(t,t^{-s}\tau)$ and $\varphi(t,t^s\tau)$, we can construct the following four modulars

$$\Phi^\varphi_{s,t,J}(f) = \int_0^\infty \varphi(t,t^{-s}J(t,f(t)))\frac{dt}{t}, \ f \in L^0(I;\hat{X}_{\theta(\cdot)}),$$

$$\Phi^\varphi_{s,t^{-1},J}(f) = \int_0^\infty \varphi(t,t^sJ(t^{-1},f(t)))\frac{dt}{t}, \ f \in L^0(I;\hat{X}_{\vartheta(\cdot)}),$$

$$\Phi^{\varphi}_{s,t,K}(f) = \int_0^{\infty} \varphi(t, t^{-s}K(t, f(t)))\frac{dt}{t},\ f \in L^0(I; \bar{X}_{\theta(\cdot)})$$

and

$$\Phi^{\varphi}_{s,t^{-1},K}(f) = \int_0^{\infty} \varphi(t, t^{s}K(t^{-1}, f(t)))\frac{dt}{t},\ f \in L^0(I; \bar{X}_{\vartheta(\cdot)})$$

to produce four Banach spaces $L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})$, $L^{\varphi}_s(I, \hat{X}_{\vartheta(\cdot)})$, $L^{\varphi}_s(I, \bar{X}_{\theta(\cdot)})$ and $L^{\varphi}_s(I, \bar{X}_{\vartheta(\cdot)})$ respectively. Similarly, using $\psi(t, t^{-s}\zeta)$ and $\psi(t, t^{s}\zeta)$, we can construct other four modulars $\Phi^{\psi}_{s,t,J^\star}$, $\Phi^{\psi}_{s,t^{-1},J^\star}$, $\Phi^{\psi}_{s,t,K^\star}$ and $\Phi^{\psi}_{s,t^{-1},K^\star}$ to produce respectively four spaces $L^{\psi}_s(I, \hat{X}^\star_{\theta(\cdot)})$, $L^{\psi}_s(I, \hat{X}^\star_{\vartheta(\cdot)})$, $L^{\psi}_s(I, \bar{X}^\star_{\theta(\cdot)})$ and $L^{\psi}_s(I, \bar{X}^\star_{\vartheta(\cdot)})$, w.r.t. the dual couple $(X^\star_0, X^\star_1)$. Notice that ([20, §3.1])

$$K^\star(t, \eta) = \sup_{u\in\hat{X}_1, u\neq 0} \frac{|\langle \eta, u\rangle|}{J(t^{-1}, u)},\ \forall\, \eta \in \bar{X}^\star_1,$$

and

$$J^\star(t, \eta) = \sup_{u\in\bar{X}_1, u\neq 0} \frac{|\langle \eta, u\rangle|}{K(t^{-1}, u)},\ \forall\, \eta \in \hat{X}^\star_1,$$

so under the reflexive assumption of $X_0$ and $X_1$, by invoking Theorem 3.10, 3.11, we can find that $L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})$, $L^{\varphi}_s(I, \hat{X}_{\vartheta(\cdot)})$, $L^{\varphi}_s(I, \bar{X}_{\theta(\cdot)})$ and $L^{\varphi}_s(I, \bar{X}_{\vartheta(\cdot)})$ are all reflexive, and

$$L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})^\star \cong L^{\psi}_s(I, \bar{X}^\star_{\vartheta(\cdot)}),\ \ L^{\varphi}_s(I, \hat{X}_{\vartheta(\cdot)})^\star \cong L^{\psi}_s(I, \bar{X}^\star_{\theta(\cdot)}), \tag{4.2}$$

and

$$L^{\varphi}_s(I, \bar{X}_{\theta(\cdot)})^\star \cong L^{\psi}_s(I, \hat{X}^\star_{\vartheta(\cdot)}),\ \ L^{\varphi}_s(I, \bar{X}_{\vartheta(\cdot)})^\star \cong L^{\psi}_s(I, \hat{X}^\star_{\theta(\cdot)}). \tag{4.3}$$

Denote by

$$M = \Big\{g \in L^{\varphi}(I, \hat{X}_t) : \int_I g(t)\frac{dt}{t} = 0 \text{ in } \bar{X}_1\Big\}.$$

Evidently, $M$ is a closed subspace of $L^{\varphi}(I, \hat{X}_{\theta(\cdot)})$, consequently the quotient set

$$L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})/M = \big\{[f] = \{f + g : \ g \in M\} : f \in L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})\big\}$$

is also a Banach space endowed with the norm

$$|||[f]||| = \inf\big\{\|f + g\|_{L^{\varphi}_s(I,\hat{X}_{\theta(\cdot)})} : g \in M\big\}.$$

Thus we can define $X_{s,\varphi,\theta,J}$ as a version of the quotient $L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})/M$, that is

$$X_{s,\varphi,\theta,J} = \Big\{u \in \bar{X}_1 : u = \int_I f(t)\frac{dt}{t} \text{ in } \bar{X}_1 \text{ for some } f \in L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})\Big\} \tag{4.4}$$

with the norm $\|u\|_{s,\varphi,\theta,J} = |||[f]|||$. We can also define $X_{s,\varphi,\theta,K}$ as a closed subspace of $L^{\varphi}_s(I, \bar{X}_{\theta(\cdot)})$, i.e.

$$X_{s,\varphi,\theta,K} = \{u \in \bar{X}^\star_1 : u\chi_I \in L^{\varphi}_s(I, \bar{X}_{\theta(\cdot)})\} \tag{4.5}$$

with the norm $\|u\|_{s,\varphi,\theta,K} = \|u\chi_I\|_{L^{\varphi}_s(I,\bar{X}_{\theta(\cdot)})}$. Analogously, spaces $X_{s,\varphi,\vartheta,J}$ and $X_{s,\varphi,\vartheta,K}$ can be defined only with $L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})$ in (4.4) and $L^{\varphi}_s(I, \bar{X}_{\theta(\cdot)})$ in (4.5) replaced by $L^{\varphi}_s(I, \hat{X}_{\vartheta(\cdot)})$ and $L^{\varphi}_s(I, \bar{X}_{\vartheta(\cdot)})$ respectively. All the four spaces defined above are called the intermediate spaces between $X_0$ and $X_1$.

Now let us make some investigations on the relations among these spaces. First of all, for each $f \in L^0(I; \hat{X}_{\theta(\cdot)})$, let $E = \{t \in I : J(t, f(t)) \le t^s\}$, $f_1 = f\chi_E$ and $f_2 = f - f_1$, then on account of (4.1), we have

$$\int_0^{\infty} (t^{-s}J(t, f_1(t)))^{\bar{p}_\varphi}\frac{dt}{t} + \int_0^{\infty} (t^{-s}J(t, f_2(t)))^{\underline{p}_\varphi}\frac{dt}{t} \le \Phi^{\varphi}_{s,t,J}(f) \le \int_0^{\infty} (t^{-s}J(t, f_1(t)))^{\underline{p}_\varphi}\frac{dt}{t} + \int_0^{\infty} (t^{-s}J(t, f_2(t)))^{\bar{p}_\varphi}\frac{dt}{t}$$

$$\leq \int_0^\infty (t^{-s}J(t,f(t)))^{\underline{p}_\varphi}\frac{dt}{t} + \int_0^\infty (t^{-s}J(t,f(t)))^{\bar{p}_\varphi}\frac{dt}{t},$$

which, with the aid of the imbedding $X_{s,\underline{p}_\varphi} \hookrightarrow X_{s,\bar{p}_\varphi}$, produce

$$X_{s,\underline{p}_\varphi} \hookrightarrow X_{s,\varphi,\theta,J} \hookrightarrow X_{s,\bar{p}_\varphi}. \tag{4.6}$$

With slight revisions, we can also reproduce (4.6) for $X_{s,\varphi,\vartheta,J}$, $X_{s,\varphi,\theta,K}$ and $X_{s,\varphi,\vartheta,K}$.

Secondly, take any $u \in X_{s,\varphi,\theta,J}$ and $f \in L_s^\varphi(I,\hat{X}_{\theta(\cdot)})$ satisfying (4.4). Since

$$\begin{aligned} t^{-s}K(t,u) &\leq \int_0^\infty t^{-s}\min\{1,\frac{t}{\tau}\}J(\tau,f(\tau)))\frac{d\tau}{\tau} \\ &= \int_0^\infty \tau^{-s}\min\{1,\tau\}\big(\frac{t}{\tau}\big)^{-s}J\big(\frac{t}{\tau},f\big(\frac{t}{\tau}\big)\big)\frac{d\tau}{\tau}, \end{aligned}$$

we can use the norm conjugate formula (cf. [5, §2.7]) to derive that

$$\begin{aligned} \|t^{-s}K(t,u)\|_{L^\varphi(I,dt/t)} &\leq 2\sup\Big\{\int_0^\infty |h(t)|\int_0^\infty \tau^{-s}\min\{1,\tau\}\big(\frac{t}{\tau}\big)^{-s}J\big(\frac{t}{\tau},f\big(\frac{t}{\tau}\big)\big)\frac{d\tau}{\tau}\frac{dt}{t} : \|h\|_{L^\psi(I,dt/t)} \leq 1\Big\} \\ &= 2\sup\Big\{\int_0^\infty \tau^{-s}\min\{1,\tau\}\int_0^\infty |h(t)|\big(\frac{t}{\tau}\big)^{-s}J\big(\frac{t}{\tau},f\big(\frac{t}{\tau}\big)\big)\frac{dt}{t}\frac{d\tau}{\tau} : \|h\|_{L^\psi(I,dt/t)} \leq 1\Big\} \\ &\leq 4\sup\Big\{\int_0^\infty \tau^{-s}\min\{1,\tau\}\frac{d\tau}{\tau} \\ &\quad \cdot\|h\|_{L^\psi(I,dt/t)}\|t^{-s}J(t,f(t)))\|_{L^\varphi(I,dt/t)} : \|h\|_{L^\psi(I,dt/t)} \leq 1\Big\} \\ &\leq \frac{4}{s(1-s)}\|f\|_{L_s^\varphi(I,\hat{X}_{\theta(\cdot)})}, \end{aligned}$$

which in turns yields the imbedding $X_{s,\varphi,\theta,J} \hookrightarrow X_{s,\varphi,\theta,K}$ with the constant $4/s(1-s)$.

Conversely, take any $u \in X_{s,\varphi,\theta,K}$. Then for each $k \in \mathbb{Z}$, there is a splitting $u = u_k + v_k$ in $\bar{X}_1$ with

$$\|u_k\|_0 + 2^k\|v_k\|_1 \leq 2K(2^k,u). \tag{4.7}$$

Thus applying imbedding (4.6), and the discrete version of the $K$–method for $X_{s,\bar{p}_\varphi}$ (refer to [22, §7.16]), we obtain

$$u_{k+1} - u_k = v_k - v_{k+1}, \tag{4.8}$$

and

$$\sum_{k=-1}^{-\infty}(u_{k+1}-u_k) = u_0 \text{ in } X_0, \ \sum_{k=0}^{\infty}(v_k - v_{k+1}) = v_0 \text{ in } X_1.$$

Let

$$f(t) = \frac{1}{\ln 2}\Big[\sum_{k=-1}^{-\infty}(u_{k+1}-u_k)\chi_{(2^k,2^{k+1}]} + \sum_{k=0}^{\infty}(v_k - v_{k+1})\chi_{(2^k,2^{k+1}]}\Big],$$

then $f(t) \in \hat{X}_1$ for all $t \in I$, and

$$\int_0^\infty f(t)\frac{dt}{t} = \frac{1}{\ln 2}\Big[\sum_{k=-1}^{-\infty}\int_{2^k}^{2^{k+1}}(u_{k+1}-u_k)\frac{dt}{t} + \sum_{k=0}^{\infty}\int_{2^k}^{2^{k+1}}(v_k - v_{k+1})\frac{dt}{t}\Big] = u_0 + v_0 = u.$$

Furthermore, by using (4.7) and (4.8), we can derive that

$$
\begin{aligned}
\Phi^{\varphi}_{s,t,J}(f) &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \varphi(t, t^{-s}f(t))\frac{dt}{t} \\
&\le \sum_{k=-1}^{-\infty} \int_{2^k}^{2^{k+1}} \varphi\Big(t, \frac{2^{-ks}}{\ln 2}J(2^{k+1}, u_{k+1}-u_k)\Big)\frac{dt}{t} \\
&\quad + \sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} \varphi\Big(t, \frac{2^{-ks}}{\ln 2}J(2^{k+1}, v_k - v_{k+1})\Big)\frac{dt}{t} \\
&\le \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \varphi\Big(t, \frac{2^{-ks}}{\ln 2}\max\{\|u_{k+1}-u_k\|_0, 2^{k+1}\|v_k - v_{k+1}\|_1\}\Big)\frac{dt}{t} \\
&\le \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \varphi\Big(t, \frac{2^{-ks+2}}{\ln 2}K(2^k, u)\Big)\frac{dt}{t} \\
&\le \int_0^{\infty} \varphi\Big(t, \frac{2^{2-s}}{\ln 2}t^{-s}K(t,u)\Big)\frac{dt}{t} \le C(2,s)\Phi^{\varphi}_{s,t,K}(u\chi_I),
\end{aligned}
$$

where $C(2, s)$ is the uniform $\Delta_2$–constant of $\varphi$ w.r.t. the coefficient $2^{2-s}/\ln 2$. The above inequality leads to the imbedding $X_{s,\varphi,\theta,K} \hookrightarrow X_{s,\varphi,\theta,J}$ with the constant $C(2, s)$. Summing up, we have

**Proposition 4.1.** *$X_{s,\varphi,\theta,J} \cong X_{s,\varphi,\theta,K}$ and $X_{s,\varphi,\vartheta,J} \cong X_{s,\varphi,\vartheta,K}$.*

The second equivalence can be verified in the same way.

Thirdly, recall the equivalent representation of the dual space of the quotient:

$$(L^{\varphi}_s(I, \hat{X}_{\theta(\cdot)})/M)^{\star} \cong {}^{\perp}M := \Big\{\xi \in L^{\psi}_s(I, \bar{X}^{\star}_{\vartheta(\cdot)}) : \langle\langle \xi, g\rangle\rangle = \int_I \langle \xi(t), g(t)\rangle \frac{dt}{t} = 0 \text{ for all } g \in M\Big\}$$

**Theorem 4.2.** *Under present situations, we have*

$${}^{\perp}M = X^{\star}_{s,\varphi,\vartheta,K}$$

*in the sense of isomorphism.*

**Proof** : The inclusion $X^{\star}_{s,\varphi,\vartheta,K} \subseteq {}^{\perp}M$ is evident. For the inverse inclusion, take any $\xi \in {}^{\perp}M$. For all $0 < a < b < \infty$, $\alpha > 0$ and $u \in \hat{X}_1$, let

$$g(t) = \Big(e^{-t}\min\{1, t\}\chi_{[a,b]} - e^{-t/\alpha}\min\Big\{1, \frac{t}{\alpha}\Big\}\chi_{[\alpha a,\alpha b]}\Big)u.$$

A simple calculation shows that $g \in M$, consequently $\langle\langle \xi, g\rangle\rangle = 0$, i.e.

$$\int_a^b \langle \xi(t) - \xi(\alpha t), u\rangle e^{-t}\min\{1, t\}\frac{dt}{t} = 0.$$

By the arbitrariness of $0 < a < b < \infty$, we have $\langle \xi(t), u\rangle = \langle \xi(\alpha t), u\rangle$ for a.e. $t \in (0, \infty)$. Thus by the arbitrariness of $\alpha > 0$, we assert that $\langle \xi(t), u\rangle$ is equal to a constant a.e. on $I$. Finally taking $u$ over $\hat{X}_1$, we reach the desired conclusion: $\xi(t)$ is almost a constant function. □

As a straight consequence of the above theorem and the representations (4.2), (4.3), we have

**Corollary 4.3.** *$(X_{s,\varphi,\theta,J})^\star \cong X^\star_{s,\varphi,\vartheta,K}$ and $(X_{s,\varphi,\vartheta,K})^\star \cong X^\star_{s,\varphi,\theta,J}$, consequently, both $X_{s,\varphi,\theta,J}$ and $X_{s,\varphi,\vartheta,K}$ are reflexive.*

Following the same process, we can also deduce that $(X_{s,\varphi,\vartheta,J})^\star \cong X^\star_{s,\varphi,\theta,K}$ and $(X_{s,\varphi,\theta,K})^\star \cong X^\star_{s,\varphi,\vartheta,J}$, consequently, $X_{s,\varphi,\vartheta,J}$ and $X_{s,\varphi,\theta,K}$ are both reflexive.

At the end of the paper, we will show the interpolative property of the four spaces we introduced. For the sake of convenience, we only select $X_{s,\varphi,\theta,J}$ as the example to deal with. Let $(X_0, X_1)$ and $(Y_0, Y_1)$ be two interpolation couples, and $B : \bar{X}_1 \to \bar{Y}_1$ be a linear operator such that $B \in \mathcal{L}(X_i, Y_i)$ with the bounds $M_i$, $i = 1, 2$. Without loss of generality, assume that $M_i > 0$, $i = 1, 2$. Denote by $\tilde{\varphi}(t, \lambda) = \varphi(M_1 t/M_0, \lambda)$. Take $u \in X_{s,\varphi,\theta,J}$ with $\|u\|_{s,\varphi,\theta,J} \leq M_0^{s-1} M_1^{-s}$. Then for arbitrary $\varepsilon > 0$, there exists $f \in L^\varphi_s(I, \hat{X}_{\theta(\cdot)})$ such that $u = \int_0^\infty f(t)\frac{dt}{t}$ in $\bar{X}_1$ and $\Phi^\varphi_{s,t,J}(M_0^{1-s}M_1^s f) \leq 1 + \varepsilon$. Let $\tilde{f}(t) = f(M_1 t/M_0)$, we also have $u = \int_0^\infty \tilde{f}(t)\frac{dt}{t}$ in $\bar{X}_1$, consequently $Bu = \int_0^\infty B\tilde{f}(t)\frac{dt}{t}$ in $\bar{Y}_1$, and

$$\begin{aligned}\Phi^{\tilde{\varphi}}_{s,t,J}(B\tilde{f}) &\leq \int_0^\infty \varphi\Big(\frac{M_1}{M_0}t, t^{-s}M_0 J\Big(\frac{M_1}{M_0}t, \tilde{f}(t)\Big)\Big)\frac{dt}{t}\\ &= \int_0^\infty \varphi\big(t, t^{-s}J\big(t, M_0^{1-s}M_1^s f(t)\big)\big)\frac{dt}{t} \leq 1 + \varepsilon.\end{aligned}$$

Thus $\|Bu\|_{s,\tilde{\varphi},\theta,J} \leq \|B\tilde{f}\|_{L^{\tilde{\varphi}}_s(I,\hat{Y}_{\theta(\cdot)})} \leq 1+\varepsilon$, which yields $\|Bu\|_{s,\tilde{\varphi},\theta,J} \leq 1$ by the arbitrariness of $\varepsilon > 0$. Furthermore, by the scaling arguments, we can deduce that

$$\|Bu\|_{s,\tilde{\varphi},\theta,J} \leq M_0^{1-s}M_1^s\|u\|_{s,\varphi,\theta,J}. \tag{4.9}$$

*Remark* 4.4. Inequality (4.9) means that, if $\varphi(t,\cdot) = \varphi(\cdot)$ is an $N$–function, then $\tilde{\varphi} = \varphi$, and $X_{s,\varphi,\theta,J}$ is indeed an interpolation space between $X_0$ and $X_1$.

For the general case, we can derive a weaker result. Suppose that $u \in \hat{X}_1$ and $u \neq 0$. Set

$$g(t) = C\min\{t, t^{-1}\}, \text{ and } \tilde{g}(t) = g\Big(\frac{\|u\|_1}{\|u\|_0}t\Big),\ t > 0.$$

where $C > 0$ selected to make $\int_0^\infty g(t)dt/t = 1$. One can easily check that, $\tilde{g} \in L^1(0,\infty; dt/t)$, and $\int_0^\infty \tilde{g}(t)dt/t = 1$. As for the $\hat{X}_1$–valued function $\tilde{g}(t)u$, by (4.1), we have

$$\begin{aligned}\int_0^\infty \varphi\Big(t, t^{-s}J\Big(t, \frac{\tilde{g}(t)u}{\|u\|_0^{1-s}\|u\|_1^s}\Big)\Big)\frac{dt}{t} &= \int_0^\infty \varphi\Big(\frac{\|u\|_0}{\|u\|_1}t, t^{-s}g(t)\max\{1,t\}\Big)\frac{dt}{t}\\ &\leq CC_0\Big(\int_0^1 t^{(1-s)\underline{p}_\varphi}\frac{dt}{t} + \int_1^\infty t^{-s\underline{p}_\varphi}\frac{dt}{t}\Big) = \frac{CC_0}{s(1-s)\underline{p}_\varphi},\end{aligned}$$

which in turn yields

$$\|u\|_{s,\varphi,\theta,J} \leq \max\Big\{1, \frac{CC_0}{s(1-s)\underline{p}_\varphi}\Big\}\|u\|_0^{1-s}\|u\|_1^s$$

by scaling arguments. In this sense, we can also call $X_{s,\varphi,\theta,J}$ the interpolation space between $X_0$ and $X_1$ with the index $s$.

*Remark* 4.5. In this section, we construct four intermediate spaces between the spaces $X_0$ and $X_1$ using a generalized $N$–function $\varphi$. These spaces are proved to have the similar properties as the common interpolation spaces have. In concrete examples, we can take $\varphi(t,\tau) = \tau^{p(t)}$ and $X_1 = \mathcal{D}(A)$ for a sectorial operator $A$ to deal with the trace of the function space of maximal regularity: $W_0^{1,p(\cdot)}(I,X) \cap L^{p(\cdot)}(I,\mathcal{D}(A))$. This shows the further applications of the spaces we have investigated.

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