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New error bounds for linear complementarity problems of Σ -SDD matrices and SB -matrices

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Abstract: A new error bound for the linear complementarity problem (LCP) of Σ -SDD matrices is given, which depends only on the entries of the involved matrices. Numerical examples are given to show that the new bound is better than that provided by García-Esnaola and Peña [Linear Algebra Appl., 2013, 438, 1339–1446] in some cases. Based on the obtained results, we also give an error bound for the LCP of SB -matrices. It is proved that the new bound is sharper than that provided by Dai et al. [Numer. Algor., 2012, 61, 121–139] under certain assumptions.

Keywords: linear complementarity problems, error bounds, Σ -SDD matrices, SB -matrices

MSC: 15A48, 65G50, 90C31, 90C33

1 Introduction

Let \mathbb{R}^n be the n dimensional real vector space, and $\mathbb{C}^{n \times n}$ ($\mathbb{R}^{n \times n}$) be the set of all $n \times n$ complex (real) matrices. The linear complementarity problem often arises from the various scientific computing, economics and engineering areas such as quadratic programs, Nash equilibrium points for bimatrix games, network equilibrium problems, contact problems, and free boundary problems for journal bearing, etc. for more details, see [1–3]. Here, the linear complementarity problem (LCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad (Mx + q)^T x = 0 \quad (1)$$

or to show that no such vector x exists, where $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. We denote the problem (1) and its solution by $\text{LCP}(M, q)$ and x^* , respectively. A real square matrix M is called a P -matrix if all its principal minors are positive, and the $\text{LCP}(M, q)$ has a unique solution for any $q \in \mathbb{R}^n$ if and only if M is a P -matrix [3].

An important topic for the $\text{LCP}(M, q)$ is to estimate the upper bounds for error $\|x - x^*\|_\infty$, since these bounds can often be used in convergence analysis of iterative algorithms [4]. For M being a P -matrix, Chen and Xiang present the following computable upper bound for $\|x - x^*\|_\infty$ [5]:

$$\|x - x^*\|_\infty \leq \max_{d \in [0, 1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty, \quad (2)$$

where $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$ for each $i \in N$, $d = [d_1, d_2, \dots, d_n]^T \in [0, 1]^n$, and $r(x) = \min\{x, Mx + q\}$ denotes the componentwise minimum of two vectors. Moreover, to avoid the high-cost computations of the

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inverse matrix from (2), several easily computable bounds for LCPs were derived for the different subclass of P -matrices, such as positively diagonal Nekrasov matrices [6, 7], S -Nekrasov matrices [8, 9], QN -matrices [10, 11], S - QN -matrices [12], B -matrices [13–15], DB -matrices [16], SB -matrices [17, 18], MB -matrices [2], B -Nekrasov matrices [7, 19, 20], B_π^R -matrices [21, 22], Dashnic-Zusmanovich type matrices [23], and weakly chained diagonally dominant B -matrices [24–26]. In [27], García-Esnaola and Peña present an error bound for the LCP(M, q) involved with Σ -SDD matrices, this bound involves a parameter and works only for Σ -SDD matrices but not strictly diagonally dominant matrices.

In this paper, we give a new error bound for linear complementarity problems when the involved matrices are Σ -SDD matrices, which is dependent only on the entries of the involved matrix. As an application, we provide a new error bound for linear complementarity problem with SB -matrices. Numerical examples are reported to show that the obtained bounds are better than those in [17], [18] and [27] in some cases.

2 New error bounds for LCPs of Σ -SDD matrices

Let us first introduce some basic notations. A matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ is a Z -matrix if all its off-diagonal entries are nonpositive, and a nonsingular M -matrix if M is a Z -matrix with M^{-1} being nonnegative [1]. Let $N := \{1, \dots, n\}$ and S denotes a proper nonempty subset of N , $\bar{S} := N \setminus S$ denotes its complement in N . For a given matrix $M = [m_{ij}] \in \mathbb{C}^{n \times n}$, denote

$$r_i(M) = \sum_{j \in N \setminus \{i\}} |m_{ij}|, \quad r_i^S(M) = \sum_{j \in S \setminus \{i\}} |m_{ij}|, \quad \text{and} \quad r_i^{\bar{S}}(M) = \sum_{j \in \bar{S} \setminus \{i\}} |m_{ij}|,$$

where

$$r_i(M) := r_i^S(M) + r_i^{\bar{S}}(M).$$

Definition 2.1. [1] A matrix $M = [m_{ij}] \in \mathbb{C}^{n \times n}$ is called an strictly diagonally dominant (SDD) matrix if $|m_{ii}| > r_i(M)$ for all $i \in N$.

Definition 2.2. [27] A matrix $M = [m_{ij}] \in \mathbb{C}^{n \times n}$ is called a Σ -SDD matrix if there exists a nonempty subset S of N such that the following conditions hold:

$$\begin{cases} |m_{ii}| > r_i^S(M), & \text{for all } i \in S, \\ (|m_{ii}| - r_i^S(M))(|m_{jj}| - r_j^{\bar{S}}(M)) > r_i^{\bar{S}}(M)r_j^S(M), & \text{for all } i \in S, j \in \bar{S}. \end{cases}$$

Remark here that Σ -SDD matrices were usually called S -strictly diagonally dominant matrices in [28].

In [27], García-Esnaola and Peña provide the following error bound for the linear complementarity problem involved with Σ -SDD matrices.

Theorem 2.3. [27, Proposition 3.1] Suppose that $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is a Σ -SDD matrix with positive diagonal entries and A is not SDD, and $W = \text{diag}(w_i)$ is a diagonal matrix such that AW is SDD, where $w_i = \gamma$ ($\neq 1$) if $i \in S$ and $w_i = 1$ if $i \in \bar{S}$. For each $i \in N$, let

$$\beta_i(\gamma) = a_{ii}w_i - \sum_{j \neq i} |a_{ij}|w_j$$

and

$$\beta(\gamma) := \min_{i \in N} \{\beta_i(\gamma)\}.$$

If $\gamma < 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq f_1(\gamma) := \max \left\{ \frac{1}{\beta(\gamma)}, \frac{1}{\gamma} \right\}, \quad (3)$$

and if $\gamma > 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq f_2(\gamma) := \max \left\{ \frac{\gamma}{\beta(\gamma)}, \gamma \right\}, \quad (4)$$

where

$$(0 <) \gamma \in I_S := \left(\max_{i \in S} \frac{r_i^{\bar{S}}(A)}{|a_{ii}| - r_i^S(A)}, \min_{j \in \bar{S}} \frac{|a_{jj}| - r_j^{\bar{S}}(A)}{r_j^S(A)} \right)$$

assuming that if $r_j^S(A) = 0$, then $\frac{|a_{jj}| - r_j^{\bar{S}}(A)}{r_j^S(A)} = \infty$.

Recently, Wang et al. [29] proved that the infimum of error bounds (3) and (4) as a function of the parameter γ exists, and also can be determined.

Theorem 2.4. [29, Theorem 3] Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be Σ -SDD matrix and let S be any subset satisfying Definition 2.2. Suppose that A and W satisfy all assumptions of Theorem 2.3, and $I_S \subseteq (0, 1)$. Let

$$\beta_i(\gamma) = \begin{cases} (|a_{ii}| - r_i^S(A)) \cdot \gamma - r_i^{\bar{S}}(A), & \text{if } i \in S, \\ (|a_{ii}| - r_i^{\bar{S}}(A)) - r_i^S(A) \cdot \gamma, & \text{if } i \in \bar{S}, \end{cases}$$

$$P_S := \{\gamma : \beta_i(\gamma) = \beta_j(\gamma), i, j \in N, i \neq j, \gamma \in I_S\},$$

and

$$\tilde{P}_S := \{\gamma : \beta(\gamma) = \gamma, \gamma \in I_S\}.$$

Then

$$\max_{d \in [0, 1]^n} \|(I - D + DA)^{-1}\|_{\infty} \leq \inf_{\gamma \in I_S} f_1(\gamma) = \min_{\gamma \in (P_S \cup \tilde{P}_S \cup \{\gamma_0, \gamma_{|P_S|+1}\})} f_1(\gamma), \quad (5)$$

where

$$\gamma_0 := \max_{i \in S} \frac{r_i^{\bar{S}}(A)}{|a_{ii}| - r_i^S(A)} \text{ and } \gamma_{|P_S|+1} := \min_{j \in \bar{S}} \frac{|a_{jj}| - r_j^{\bar{S}}(A)}{r_j^S(A)}.$$

Theorem 2.5. [29, Theorem 4] Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be a Σ -SDD matrix and let S be any subset satisfying Definition 2.2. Suppose that A and W satisfy all assumptions of Theorem 2.3, and $I_S \subseteq (1, +\infty)$. Let

$$\beta'_i(\gamma) := \frac{\beta_i(\gamma)}{\gamma} = \begin{cases} (|a_{ii}| - r_i^S(A)) - r_i^{\bar{S}}(A) \cdot \frac{1}{\gamma}, & \text{if } i \in S, \\ (|a_{ii}| - r_i^{\bar{S}}(A)) \cdot \frac{1}{\gamma} - r_i^S(A), & \text{if } i \in \bar{S}, \end{cases}$$

$$P'_S := \{\gamma : \beta'_i(\gamma) = \beta'_j(\gamma), i, j \in N, i \neq j, \gamma \in I_S\},$$

and

$$\tilde{P}'_S := \{\gamma : \beta'(\gamma) = \frac{1}{\gamma}, \gamma \in I_S\}.$$

Then

$$\max_{d \in [0, 1]^n} \|(I - D + DA)^{-1}\|_{\infty} \leq \inf_{\gamma \in I_S} f_2(\gamma) = \min_{\gamma \in (P'_S \cup \tilde{P}'_S \cup \{\gamma_0, \gamma_{|P'_S|+1}\})} f_2(\gamma), \quad (6)$$

where

$$\gamma_0 := \max_{i \in S} \frac{r_i^{\bar{S}}(A)}{|a_{ii}| - r_i^S(A)} \text{ and } \gamma_{|P'_S|+1} := \min_{j \in \bar{S}} \frac{|a_{jj}| - r_j^{\bar{S}}(A)}{r_j^S(A)}.$$

Observe from Theorem 2.4 and Theorem 2.5 that if A is a large matrix, then the calculations of P_S and \tilde{P}'_S (or P'_S and \tilde{P}_S) in bounds (5) and (6) will be very complicated. On the other hand, for strictly diagonally dominant matrices, the bounds (5) and (6) become invalid. So it is interesting to find alternative error bounds depending only on the elements of the matrices for the LCP(M, q) when M is a Σ -SDD matrix. We next address this problem, before that some lemmas are listed.

Lemma 2.6. [14, Lemma 3] Let $\gamma > 0$ and $\eta \geq 0$. Then for any $x \in [0, 1]$,

$$\frac{1}{1-x+\gamma x} \leq \frac{1}{\min\{\gamma, 1\}}$$

and

$$\frac{\eta x}{1-x+\gamma x} \leq \frac{\eta}{\gamma}.$$

Lemma 2.7. [30, Theorem 3] Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be a Σ -SDD matrix and S be a nonempty proper subset of N . Then

$$\|A^{-1}\|_{\infty} \leq \max_{i \in S, j \in \bar{S}} \max \left\{ \rho_{ij}^S(A), \rho_{ji}^{\bar{S}}(A) \right\},$$

where

$$\rho_{ij}^S(A) = \frac{|a_{ii}| - r_i^S(A) + r_j^S(A)}{(|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\bar{S}}(A)) - r_i^{\bar{S}}(A)r_j^S(A)}$$

and

$$\rho_{ji}^{\bar{S}}(A) = \frac{|a_{jj}| - r_j^{\bar{S}}(A) + r_i^{\bar{S}}(A)}{(|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\bar{S}}(A)) - r_i^{\bar{S}}(A)r_j^S(A)}.$$

Lemma 2.8. Let S be a nonempty proper subset of N , and $M = [m_{ij}] \in \mathbb{C}^{n \times n}$ be a Σ -SDD matrix with $m_{ii} > 0$ for all $i \in N$. Let $M_D = I - D + DM = [\tilde{m}_{ij}]$, where $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$. Then M_D is a Σ -SDD matrix. Furthermore, for each $i \in N$,

$$r_i^S(M_D) = d_i r_i^S(M), \quad r_i^{\bar{S}}(M_D) = d_i r_i^{\bar{S}}(M),$$

and

$$\frac{r_i^S(M_D)}{\tilde{m}_{ii}} \leq \frac{r_i^S(M)}{m_{ii}}, \quad \frac{r_i^{\bar{S}}(M_D)}{\tilde{m}_{ii}} \leq \frac{r_i^{\bar{S}}(M)}{m_{ii}}.$$

Proof. Since

$$\tilde{m}_{ij} = \begin{cases} 1 - d_i + d_i m_{ij}, & i = j, \\ d_i m_{ij}, & i \neq j, \end{cases} \quad \text{and} \quad \frac{d_i}{1 - d_i + d_i m_{ii}} \leq \frac{1}{m_{ii}} \quad \text{for all } i, j \in N,$$

it follows that for each $i \in N$,

$$r_i^S(M_D) = \sum_{j \in S \setminus \{i\}} |\tilde{m}_{ij}| = \sum_{j \in S \setminus \{i\}} |d_i m_{ij}| = d_i \sum_{j \in S \setminus \{i\}} |m_{ij}| = d_i r_i^S(M), \quad (7)$$

and

$$r_i^{\bar{S}}(M_D) = \sum_{j \in \bar{S} \setminus \{i\}} |\tilde{m}_{ij}| = \sum_{j \in \bar{S} \setminus \{i\}} |d_i m_{ij}| = d_i \sum_{j \in \bar{S} \setminus \{i\}} |m_{ij}| = d_i r_i^{\bar{S}}(M). \quad (8)$$

By (7) and (8), we have that for each $i \in S$,

$$\tilde{m}_{ii} - r_i^S(M_D) = 1 - d_i + d_i m_{ii} - d_i r_i^S(M) \geq d_i (m_{ii} - r_i^S(M))$$

and

$$\frac{r_i^S(M_D)}{\tilde{m}_{ii}} = \frac{d_i r_i^S(M)}{1 - d_i + m_{ii} d_i} \leq \frac{r_i^S(M)}{m_{ii}} < 1. \quad (9)$$

Similarly, for each $j \in \bar{S}$,

$$\tilde{m}_{jj} - r_j^{\bar{S}}(M_D) = 1 - d_j + d_j m_{jj} - d_j r_j^{\bar{S}}(M) \geq d_j (m_{jj} - r_j^{\bar{S}}(M))$$

and

$$\frac{r_j^{\bar{S}}(M_D)}{\tilde{m}_{jj}} = \frac{d_j r_j^{\bar{S}}(M)}{1 - d_j + m_{jj} d_j} \leq \frac{r_j^{\bar{S}}(M)}{m_{jj}} < 1. \quad (10)$$

Hence,

$$(\tilde{m}_{ii} - r_i^S(M_D))(\tilde{m}_{jj} - r_j^{\bar{S}}(M_D)) \geq d_i(m_{ii} - r_i^S(M)) \cdot d_j(m_{jj} - r_j^{\bar{S}}(M)), \text{ for all } i \in S, j \in \bar{S}.$$

If $d_k = 0$ for some $k \in N$, then from (9) and (10) we get

$$(\tilde{m}_{ii} - r_i^S(M_D))(\tilde{m}_{jj} - r_j^{\bar{S}}(M_D)) > d_i r_i^{\bar{S}}(M) \cdot d_j r_j^S(M) = r_i^{\bar{S}}(M_D) \cdot r_j^S(M_D), \text{ for } i = k \text{ or } j = k. \quad (11)$$

If $0 < d_k \leq 1$ for some $k \in N$, then from the fact that M is a Σ -SDD matrix we obtain

$$\begin{aligned} (\tilde{m}_{ii} - r_i^S(M_D))(\tilde{m}_{jj} - r_j^{\bar{S}}(M_D)) &\geq d_i(m_{ii} - r_i^S(M)) \cdot d_j(m_{jj} - r_j^{\bar{S}}(M)) \\ &> d_i r_i^{\bar{S}}(M) \cdot d_j r_j^S(M) \\ &= r_i^{\bar{S}}(M_D) \cdot r_j^S(M_D). \end{aligned} \quad (12)$$

Now the conclusion follows from (9), (10), (11) and (12). \square

By Lemma 2.6, Lemma 2.7, and Lemma 2.8, we establish the first main result of this paper.

Theorem 2.9. Let $M = [m_{ij}] \in \mathbb{C}^{n \times n}$ be a Σ -SDD matrix with positive diagonal entries for some nonempty subset $S \subset N$. Then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^S(M), \chi_{ji}^{\bar{S}}(M) \right\}, \quad (13)$$

where

$$\chi_{ij}^S(M) := \frac{\frac{(m_{ii} - r_i^S(M))(m_{jj} - r_j^{\bar{S}}(M))}{\min\{m_{jj} - r_j^{\bar{S}}(M), 1\}} + \frac{r_j^S(M)(m_{ii} - r_i^S(M))}{\min\{m_{ii} - r_i^S(M), 1\}}}{(m_{ii} - r_i^S(M))(m_{jj} - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^S(M)}$$

and

$$\chi_{ji}^{\bar{S}}(M) := \frac{\frac{(m_{jj} - r_j^{\bar{S}}(M))(m_{ii} - r_i^S(M))}{\min\{m_{ii} - r_i^S(M), 1\}} + \frac{r_i^{\bar{S}}(M)(m_{jj} - r_j^{\bar{S}}(M))}{\min\{m_{jj} - r_j^{\bar{S}}(M), 1\}}}{(m_{ii} - r_i^S(M))(m_{jj} - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^S(M)}.$$

Proof. Let $M_D = I - D + DM = [\tilde{m}_{ij}]$, where $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$. Since M is a Σ -SDD matrix, then by Lemma 2.7 and Lemma 2.8 we have that M_D is a Σ -SDD matrix, and

$$\|M_D^{-1}\|_\infty \leq \max_{i \in S, j \in \bar{S}} \left\{ \rho_{ij}^S(M_D), \rho_{ji}^{\bar{S}}(M_D) \right\}, \quad (14)$$

where

$$\rho_{ij}^S(M_D) = \frac{|\tilde{m}_{ii}| - r_i^S(M_D) + r_j^S(M_D)}{(|\tilde{m}_{ii}| - r_i^S(M_D))(|\tilde{m}_{jj}| - r_j^{\bar{S}}(M_D)) - r_i^{\bar{S}}(M_D)r_j^S(M_D)}$$

and

$$\rho_{ji}^{\bar{S}}(M_D) = \frac{|\tilde{m}_{jj}| - r_j^{\bar{S}}(M_D) + r_i^{\bar{S}}(M_D)}{(|\tilde{m}_{ii}| - r_i^S(M_D))(|\tilde{m}_{jj}| - r_j^{\bar{S}}(M_D)) - r_i^{\bar{S}}(M_D)r_j^S(M_D)}.$$

Note that

$$\tilde{m}_{ij} = \begin{cases} 1 - d_i + d_i m_{ij}, & i = j, \\ d_i m_{ij}, & i \neq j. \end{cases}$$

Then by Lemma 2.6 and Lemma 2.8, it follows that for each $i \in S$,

$$\frac{1}{|\tilde{m}_{ii}| - r_i^S(M_D)} = \frac{1}{1 - d_i + m_{ii}d_i - d_i r_i^S(M)} \leq \frac{1}{\min\{m_{ii} - r_i^S(M), 1\}}, \quad (15)$$

and

$$\frac{r_i^{\bar{S}}(M_D)}{|\tilde{m}_{ii}| - r_i^S(M_D)} = \frac{d_i r_i^{\bar{S}}(M)}{1 - d_i + d_i m_{ii} - d_i r_i^S(M)} \leq \frac{r_i^{\bar{S}}(M)}{m_{ii} - r_i^S(M)}. \quad (16)$$

Analogously, for each $j \in \bar{S}$, we have

$$\frac{1}{\tilde{m}_{jj} - r_j^{\bar{S}}(M_D)} \leq \frac{1}{\min \{m_{jj} - r_j^{\bar{S}}(M), 1\}}, \quad (17)$$

and

$$\frac{r_j^{\bar{S}}(M_D)}{|\tilde{m}_{jj}| - r_j^{\bar{S}}(M_D)} = \frac{d_j r_j^{\bar{S}}(M)}{1 - d_j + d_j m_{jj} - d_j r_j^{\bar{S}}(M)} \leq \frac{r_j^{\bar{S}}(M)}{m_{jj} - r_j^{\bar{S}}(M)}. \quad (18)$$

By (15), (16), (17) and (18), it follows that for each $i \in S, j \in \bar{S}$,

$$\begin{aligned} \rho_{ij}^{\bar{S}}(M_D) &= \frac{|\tilde{m}_{ii}| - r_i^{\bar{S}}(M_D) + r_j^{\bar{S}}(M_D)}{(|\tilde{m}_{ii}| - r_i^{\bar{S}}(M_D))(|\tilde{m}_{jj}| - r_j^{\bar{S}}(M_D)) - r_i^{\bar{S}}(M_D)r_j^{\bar{S}}(M_D)} \\ &= \frac{\frac{1}{|\tilde{m}_{jj}| - r_j^{\bar{S}}(M_D)} + \frac{r_j^{\bar{S}}(M_D)}{(|\tilde{m}_{ii}| - r_i^{\bar{S}}(M_D))(|\tilde{m}_{jj}| - r_j^{\bar{S}}(M_D))}}{1 - \frac{r_i^{\bar{S}}(M_D)r_j^{\bar{S}}(M_D)}{(|\tilde{m}_{ii}| - r_i^{\bar{S}}(M_D))(|\tilde{m}_{jj}| - r_j^{\bar{S}}(M_D))}} \\ &\leq \frac{\frac{1}{\min \{m_{jj} - r_j^{\bar{S}}(M), 1\}} + \frac{r_j^{\bar{S}}(M)}{\min \{m_{ii} - r_i^{\bar{S}}(M), 1\}} \cdot (m_{jj} - r_j^{\bar{S}}(M))}{1 - \frac{r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)}{(m_{ii} - r_i^{\bar{S}}(M))(m_{jj} - r_j^{\bar{S}}(M))}} \\ &= \frac{\frac{(m_{ii} - r_i^{\bar{S}}(M))(m_{jj} - r_j^{\bar{S}}(M))}{\min \{m_{jj} - r_j^{\bar{S}}(M), 1\}} + \frac{r_j^{\bar{S}}(M)(m_{ii} - r_i^{\bar{S}}(M))}{\min \{m_{ii} - r_i^{\bar{S}}(M), 1\}}}{(m_{ii} - r_i^{\bar{S}}(M))(m_{jj} - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)} \\ &=: \chi_{ij}^{\bar{S}}(M). \end{aligned} \quad (19)$$

In a similar way, we can prove that for each $i \in S, j \in \bar{S}$,

$$\rho_{ji}^{\bar{S}}(M_D) \leq \chi_{ji}^{\bar{S}}(M). \quad (20)$$

The conclusion follows from (14), (19) and (20). \square

Remark 2.10. Observe that bound (13) in Theorem 2.9 only depends on the elements of M , and it is easy to implement. For a set S with finite elements, we use $|S|$ to denote the number of elements in the set S . From bound (13), we obtain the number of the basic arithmetic operations of bound (13) is $|S| \cdot |\bar{S}| \cdot (2n + 14)$ (requiring $|S| \cdot |\bar{S}| \cdot [2(n - 1) + 4]$ additions and $12 \cdot |S| \cdot |\bar{S}|$ comparisons, multiplications and divisions of numbers). Furthermore, it follows from $|S| < n$ and $|\bar{S}| < n$ that $|S| \cdot |\bar{S}| \cdot (2n + 14) < n^2(2n + 14)$. Thus, the bound (13) of Theorem 2.9 can be performed in polynomial time.

By Theorem 2.9, we can easily obtain the following result.

Corollary 2.11. Let $M = [m_{ij}] \in \mathbb{C}^{n \times n}$ be a Σ -SDD matrix with positive diagonal entries for some nonempty subset S of N . For each $i \in S, j \in \bar{S}$,

(i) if $m_{ii} - r_i^{\bar{S}}(M) \leq 1$ and $m_{jj} - r_j^{\bar{S}}(M) \leq 1$, then

$$\max_{d \in [0, 1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \max_{i \in S, j \in \bar{S}} \max \{ \chi_{ij}^{\bar{S}}(M), \chi_{ji}^{\bar{S}}(M) \},$$

where

$$\chi_{ij}^{\bar{S}}(M) := \frac{|m_{ii}| - r_i^{\bar{S}}(M) + r_j^{\bar{S}}(M)}{(|m_{ii}| - r_i^{\bar{S}}(M))(|m_{jj}| - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)}$$

and

$$\chi_{ji}^{\bar{S}}(M) := \frac{|m_{jj}| - r_j^{\bar{S}}(M) + r_i^{\bar{S}}(M)}{(|m_{ii}| - r_i^{\bar{S}}(M))(|m_{jj}| - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)};$$

(ii) if $m_{ii} - r_i^{\bar{S}}(M) > 1$ and $m_{jj} - r_j^{\bar{S}}(M) \leq 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^{\bar{S}}(M), \chi_{ji}^{\bar{S}}(M) \right\},$$

where

$$\chi_{ij}^{\bar{S}}(M) := \frac{(1 + r_j^{\bar{S}}(M))(|m_{ii}| - r_i^{\bar{S}}(M))}{(|m_{ii}| - r_i^{\bar{S}}(M))(|m_{jj}| - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)}$$

and

$$\chi_{ji}^{\bar{S}}(M) := \frac{(m_{jj} - r_j^{\bar{S}}(M))(m_{ii} - r_i^{\bar{S}}(M)) + r_i^{\bar{S}}(M)}{(m_{ii} - r_i^{\bar{S}}(M))(m_{jj} - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)};$$

(iii) if $m_{ii} - r_i^{\bar{S}}(M) \leq 1$ and $m_{jj} - r_j^{\bar{S}}(M) > 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^{\bar{S}}(M), \chi_{ji}^{\bar{S}}(M) \right\},$$

where

$$\chi_{ij}^{\bar{S}}(M) := \frac{(m_{ii} - r_i^{\bar{S}}(M))(m_{jj} - r_j^{\bar{S}}(M)) + r_j^{\bar{S}}(M)}{(m_{ii} - r_i^{\bar{S}}(M))(m_{jj} - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)}$$

and

$$\chi_{ji}^{\bar{S}}(M) := \frac{(1 + r_i^{\bar{S}}(M))(|m_{ii}| - r_i^{\bar{S}}(M))}{(|m_{ii}| - r_i^{\bar{S}}(M))(|m_{jj}| - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)};$$

(v) if $m_{ii} - r_i^{\bar{S}}(M) > 1$ and $m_{jj} - r_j^{\bar{S}}(M) > 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^{\bar{S}}(M), \chi_{ji}^{\bar{S}}(M) \right\},$$

where

$$\chi_{ij}^{\bar{S}}(M) := \frac{(|m_{ii}| - r_i^{\bar{S}}(M))(|m_{jj}| - r_j^{\bar{S}}(M)) + r_j^{\bar{S}}(M)}{(|m_{ii}| - r_i^{\bar{S}}(M))(|m_{jj}| - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)}$$

and

$$\chi_{ji}^{\bar{S}}(M) := \frac{(|m_{ii}| - r_i^{\bar{S}}(M))(|m_{jj}| - r_j^{\bar{S}}(M)) + r_i^{\bar{S}}(M)}{(|m_{ii}| - r_i^{\bar{S}}(M))(|m_{jj}| - r_j^{\bar{S}}(M)) - r_i^{\bar{S}}(M)r_j^{\bar{S}}(M)}.$$

Next, three examples are given to show the advantage of the bound (13) in Theorem 2.9. Before that, a well-known result which will be used later is given.

Theorem 2.12. [31, Remark 2.4] Let $M = [m_{ij}] \in \mathbb{C}^{n \times n}$ is an SDD matrix with positive diagonal entries. Then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \max \left\{ \frac{1}{\beta}, 1 \right\},$$

where

$$\beta := \min_{i \in N} \left\{ m_{ii} - \sum_{j \neq i} |m_{ij}| \right\}.$$

Example 2.13. Consider the family of SDD matrices in [14], where

$$M_k = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ -0.8 & 1 & 0 & -0.1 \\ 0 & -0.1\frac{k}{k+1} - 0.8 & 1 & -0.1 \\ -0.8 & -0.1 & 0 & 1 \end{bmatrix} \text{ with } k \geq 1.$$

Since M_k does not satisfy the assumptions of Theorem 2.3, Theorem 2.4, and Theorem 2.5, so we cannot use bounds (3), (4), (5), and (6) to estimate $\max_{d \in [0,1]^4} \|(I - D + DM_k)^{-1}\|_\infty$. However, according to an SDD matrix is a Σ -SDD matrix for any nonempty $S \subseteq N$, taking the set $S = \{1, 2\}$ and $\bar{S} = \{3, 4\}$, by bound (13) of Theorem 2.9, we obtain

$$\max_{d \in [0,1]^4} \|(I - D + DM_k)^{-1}\|_\infty \leq \max_{i \in S, j \in \bar{S}} \left\{ \chi_{ij}^S(M_k), \chi_{ji}^{\bar{S}}(M_k) \right\} = \frac{1 + 0.1\frac{k}{k+1}}{0.18 - 0.1[0.1\frac{k}{k+1} + 0.8]} < \frac{110}{9}.$$

In fact,

$$\chi_{13}^S(M_k) = \frac{1.8 + 0.1\frac{k}{k+1}}{0.9 - 0.1[0.1\frac{k}{k+1} + 0.8]}, \chi_{14}^S(M_k) = \frac{190}{91}, \chi_{23}^S(M_k) = \frac{1 + 0.1\frac{k}{k+1}}{0.18 - 0.1[0.1\frac{k}{k+1} + 0.8]}, \chi_{24}^S(M_k) = 10;$$

and

$$\chi_{31}^{\bar{S}}(M_k) = \frac{1}{0.9 - 0.1[0.1\frac{k}{k+1} + 0.8]}, \chi_{41}^{\bar{S}}(M_k) = \frac{110}{91}, \chi_{32}^{\bar{S}}(M_k) = \frac{1}{0.18 - 0.1[0.1\frac{k}{k+1} + 0.8]}, \chi_{42}^{\bar{S}}(M_k) = 10.$$

In contrast, by Theorem 2.12 (Remark 2.4 of [31]), we have

$$\max_{d \in [0,1]^4} \|(I - D + DM_k)^{-1}\|_\infty \leq \max \left\{ \frac{1}{\beta}, 1 \right\} = 10(k+1).$$

It is obvious that

$$10(k+1) \rightarrow +\infty, \text{ when } k \rightarrow +\infty.$$

Since a Σ -SDD matrix is an S-Nekrasov matrix, the bound (2.14) of Theorem 2.2 in [8] for S-Nekrasov matrices can also be used to estimate $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ when M is a Σ -SDD matrix. The following example shows that the bound (13) given in Theorem 2.9 is sharper than the bound (2.14) of Theorem 2.2 in [8].

Example 2.14. The LCP(M, q) has often been used to discuss formulation and solution of traffic equilibrium problems [32, 33]. Consider the matrix $M \in \mathbb{R}^{5 \times 5}$ arising from a simple traffic network problem [32]:

$$M = \begin{bmatrix} 10 & 0 & 0 & 5 & 0 \\ 0 & 15 & 0 & 0 & 5 \\ 0 & 0 & 20 & 0 & 0 \\ 2 & 0 & 0 & 20 & 0 \\ 0 & 1 & 0 & 0 & 25 \end{bmatrix}.$$

It is easy to check that M is a Σ -SDD matrix for any nonempty $S \subseteq N$. Since M does not satisfy the assumptions of Theorem 2.3, Theorem 2.4, and Theorem 2.5, so we cannot use bounds (3), (4), (5), and (6) to estimate $\max_{d \in [0,1]^5} \|(I - D + DM)^{-1}\|_\infty$. However, taking $S = \{2, 3, 5\}$, by bound (13) of Theorem 2.9, we obtain

$$\max_{d \in [0,1]^5} \|(I - D + DM)^{-1}\|_\infty \leq 1.$$

In contrast, bound (2.14) of Theorem 2.2 in [8] for S-Nekrasov matrices gives the following estimation:

$$\max_{d \in [0,1]^5} \|(I - D + DM)^{-1}\|_\infty \leq 1.5526.$$

It is also shown by Figure 1, in which the first 1000 matrices D are given by the following MATLAB code, that 1 is the exact value of $\max_{d \in [0,1]^5} \|(I - D + DM)^{-1}\|_\infty$.

MATLAB code : for $i = 1 : 1000$; $D = \text{diag}(\text{rand}(5, 1))$; end.

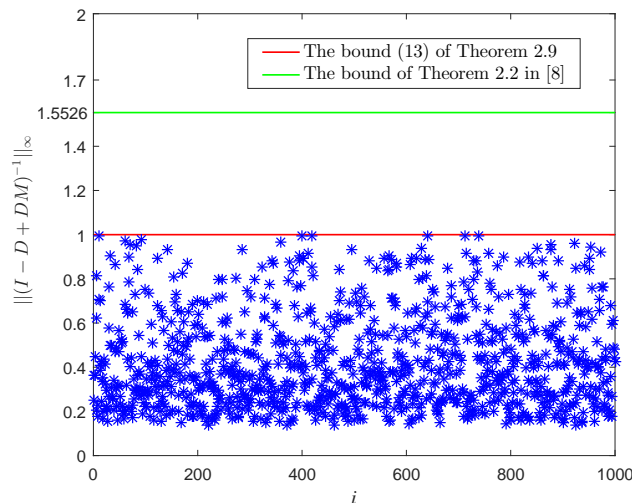


Figure 1: $\|(I - D + DM)^{-1}\|_{\infty}$ for the first 1000 matrices D generated by $\text{diag}(\text{rand}(5,1))$.

Example 2.15. Consider the following matrix

$$M = \begin{bmatrix} 1 & -0.6 & -0.2 & 0 \\ -0.25 & 1 & -0.25 & -0.5 \\ -0.2 & -0.4 & 1 & -0.4 \\ -1.2 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that M is not an SDD matrix, but it is a Σ -SDD matrix for $S = \{1\}$. In fact, by calculation, we have

$$r_1^S(M) = 0, r_2^S(M) = 0.25, r_3^S(M) = 0.2, r_4^S(M) = 1.2, \bar{r}_1^S(M) = 0.8, \bar{r}_2^S(M) = 0.75, \bar{r}_3^S(M) = 0.8, \text{ and } \bar{r}_4^S(M) = 0,$$

which satisfy the conditions of Definition 2.2. Then $I_S = (4/5, 5/6)$. So by Theorem 2.3 we can get the bound (3) with $\gamma \in I_S$ for $\max_{d \in [0,1]^4} \|(I - D + DM)^{-1}\|_{\infty}$, which is drawn in Figure 2, and its infimum can be determined by Theorem 2.4 in the following way. Since

$$\beta_1(\gamma) = \gamma - 0.8, \beta_2(\gamma) = 0.25(1 - \gamma), \beta_3(\gamma) = 0.2(1 - \gamma), \beta_4(\gamma) = 1 - 1.2\gamma,$$

$$P_S := \{\gamma : \beta_i(\gamma) = \beta_j(\gamma), i, j \in N, i \neq j, \gamma \in I_S\},$$

and

$$\tilde{P}_S := \{\gamma : \beta(\gamma) = \gamma, \gamma \in I_S\},$$

it follows that $P_S = \{9/11\}$,

$$\beta(\gamma) = \begin{cases} \beta_1(\gamma), & 4/5 < \gamma \leq 9/11, \\ \beta_4(\gamma), & 9/11 < \gamma \leq 5/6, \end{cases}$$

and $\tilde{P}_S = \emptyset$. Each $\beta_i(\gamma)$ and γ are drawn in Figure 3. By the bound (5) of Theorem 2.4, we have

$$\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_{\infty} \leq \inf_{\gamma \in (4/5, 5/6)} f_1(\gamma) = \min_{\gamma \in (P_S \cup \tilde{P}_S \cup \{4/5, 5/6\})} \max \left\{ \frac{1}{\beta(\gamma)}, \frac{1}{\gamma} \right\} = 55.$$

In addition, by the bound (13) of Theorem 2.9, we can see that

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^S(M), \chi_{ji}^{\bar{S}}(M) \right\} = 55.$$

It should be pointed out that the bound (13) is computationally much easier than the bound in Theorem 2.4, because it only depends on the elements of the matrix M .

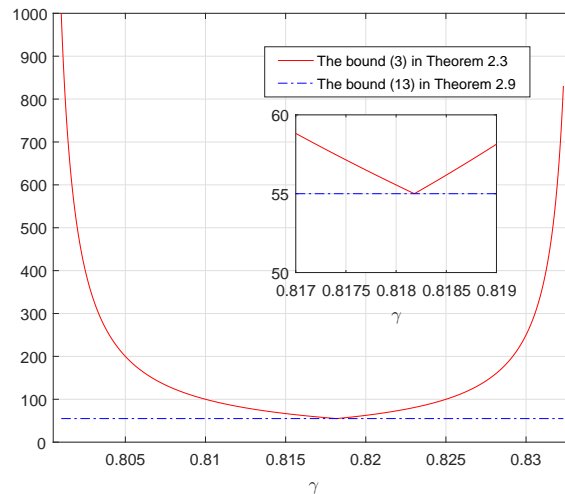
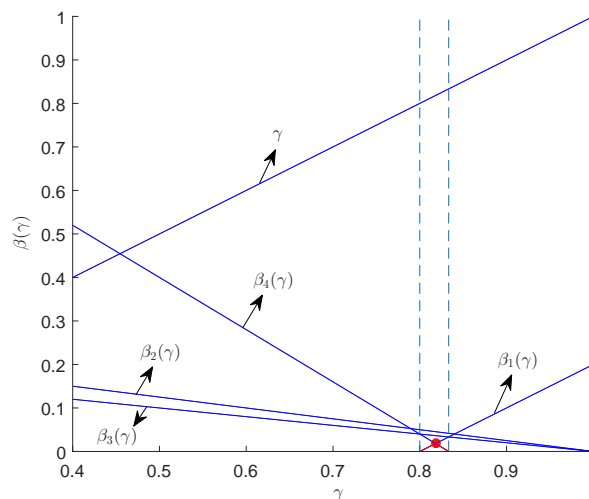


Figure 2: The bounds (3) and (13) in Theorem 2.3 and Theorem 2.9.

Figure 3: $\beta_i(\gamma)$, $\beta(\gamma)$, and γ .

3 New error bounds for LCPs of *SB*-matrices

Based on Theorem 2.9, we in this section present a new error bound for linear complementarity problems associated with *SB*-matrices. For a real matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$, we can write it as

$$M = B + C, \quad (21)$$

where

$$B = \begin{bmatrix} m_{11} - r_1 & m_{12} - r_1 & \cdots & m_{1n} - r_1 \\ m_{21} - r_2 & m_{22} - r_2 & \cdots & m_{2n} - r_2 \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} - r_n & m_{n2} - r_n & \cdots & m_{nn} - r_n \end{bmatrix}, \quad C = \begin{bmatrix} r_1 & r_1 & \cdots & r_1 \\ r_2 & r_2 & \cdots & r_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_n & \cdots & r_n \end{bmatrix}$$

with

$$r_i = \max\{0, m_{ij} | j \neq i\}.$$

Obviously, B is a Z -matrix and C is a nonnegative matrix of rank 1.

Let us recall the definition of SB-matrices which is proposed by Li et al. in [34] as a subclass of P -matrices.

Definition 3.1. A real matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ is called an S -strictly dominant B -matrix (SB-matrix) if it can be written in form (21) with B being a Σ -SDD matrix for some nonempty proper subset S of N whose diagonal entries are all positive.

By Theorem 2.9, we give an upper bound for $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ when M is an SB-matrix.

Theorem 3.2. Let $M \in \mathbb{R}^{n \times n}$ be an SB-matrix for some nonempty subset S of N , and $B = [b_{ij}]$ be the matrix of (21). Then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq (n-1) \cdot \max_{i \in S, j \in \bar{S}} \left\{ \chi_{ij}^S(B), \chi_{ji}^{\bar{S}}(B) \right\}, \quad (22)$$

where

$$\chi_{ij}^S(B) := \frac{\frac{(b_{ii} - r_i^S(B))(b_{jj} - r_j^{\bar{S}}(B))}{\min\{b_{jj} - r_j^{\bar{S}}(B), 1\}} + \frac{r_j^S(B)(b_{ii} - r_i^S(B))}{\min\{b_{ii} - r_i^S(B), 1\}}}{(b_{ii} - r_i^S(B))(b_{jj} - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}$$

and

$$\chi_{ji}^{\bar{S}}(B) := \frac{\frac{(b_{jj} - r_j^{\bar{S}}(B))(b_{ii} - r_i^S(B))}{\min\{b_{ii} - r_i^S(B), 1\}} + \frac{r_i^{\bar{S}}(B)(b_{jj} - r_j^{\bar{S}}(B))}{\min\{b_{jj} - r_j^{\bar{S}}(B), 1\}}}{(b_{ii} - r_i^S(B))(b_{jj} - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}.$$

Proof. Let $M_D = I - D + DM$, where $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$. Since M is an SB-matrix, then we can write $M = B + C$ as in (21) where B is a Σ -SDD matrix with positive diagonal entries, which yields that

$$M_D = I - D + DM = (I - D + DB) + DC = B_D + C_D,$$

where $B_D = I - D + DB$ and $C_D = DC$. Note that B is a Σ -SDD matrix with positive diagonal entries. Then by Lemma 2.8, B_D is also a Σ -SDD with positive diagonal entries. Obviously, B_D is a Z -matrix. Hence, we have that B_D is a nonsingular M -matrix. Similarly to the proof of Theorem 2 in [19], we have

$$\|M_D^{-1}\|_\infty \leq \|(I + (B_D)^{-1}C_D)^{-1}\|_\infty \|(B_D)^{-1}\|_\infty \leq (n-1) \|(B_D)^{-1}\|_\infty. \quad (23)$$

Next, we give an upper bound for $\|(B_D)^{-1}\|_\infty$. Since B is a Σ -SDD matrix, we have from Theorem 2.9 that

$$\|(B_D)^{-1}\|_\infty \leq \max_{i \in S, j \in \bar{S}} \max \left\{ \rho_{ij}^S(B_D), \rho_{ji}^{\bar{S}}(B_D) \right\} \leq \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^S(B), \chi_{ji}^{\bar{S}}(B) \right\}. \quad (24)$$

The conclusion follows from (23) and (24). \square

Corollary 3.3. Let $M \in \mathbb{R}^{n \times n}$ be an SB-matrix for some nonempty proper subset S of N , and $B = [b_{ij}]$ be the matrix of (21). Then for $i \in S, j \in \bar{S}$,

(i) if $b_{ii} - r_i^S(B) \leq 1$ and $b_{jj} - r_j^{\bar{S}}(B) \leq 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq (n-1) \cdot \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^S(B), \chi_{ji}^{\bar{S}}(B) \right\},$$

where

$$\chi_{ij}^S(B) := \frac{|b_{ii}| - r_i^S(B) + r_j^S(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}$$

and

$$\chi_{ji}^{\bar{S}}(B) := \frac{|b_{jj}| - r_j^{\bar{S}}(B) + r_i^{\bar{S}}(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)};$$

(ii) if $b_{ii} - r_i^S(B) > 1$ and $b_{jj} - r_j^{\bar{S}}(B) \leq 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq (n-1) \cdot \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^S(B), \chi_{ji}^{\bar{S}}(B) \right\},$$

where

$$\chi_{ij}^S(B) := \frac{(1 + r_j^S(B))(|b_{ii}| - r_i^S(B))}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}$$

and

$$\chi_{ji}^{\bar{S}}(B) := \frac{(b_{jj} - r_j^{\bar{S}}(B))(b_{ii} - r_i^S(B)) + r_i^{\bar{S}}(B)}{(b_{ii} - r_i^S(B))(b_{jj} - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)};$$

(iii) if $b_{ii} - r_i^S(B) \leq 1$ and $b_{jj} - r_j^{\bar{S}}(B) > 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq (n-1) \cdot \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^S(B), \chi_{ji}^{\bar{S}}(B) \right\},$$

where

$$\chi_{ij}^S(B) := \frac{(b_{ii} - r_i^S(B))(b_{jj} - r_j^{\bar{S}}(B)) + r_j^S(B)}{(b_{ii} - r_i^S(B))(b_{jj} - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}$$

and

$$\chi_{ji}^{\bar{S}}(B) := \frac{(1 + r_i^{\bar{S}}(B))(|b_{ii}| - r_i^S(B))}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)};$$

(v) if $b_{ii} - r_i^S(B) > 1$ and $b_{jj} - r_j^{\bar{S}}(B) > 1$, then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq (n-1) \cdot \max_{i \in S, j \in \bar{S}} \max \left\{ \chi_{ij}^S(B), \chi_{ji}^{\bar{S}}(B) \right\},$$

where

$$\chi_{ij}^S(B) := \frac{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) + r_j^S(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}$$

and

$$\chi_{ji}^{\bar{S}}(B) := \frac{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) + r_i^{\bar{S}}(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}.$$

To compare our error bound with those of [17] and [18], we first recall the following results given by Dai et al. of [17] and [18].

Theorem 3.4. [17, Theorem 3.1] Let $M \in \mathbb{R}^{n \times n}$ be an SB-matrix for some nonempty proper subset S of N , and $B = [b_{ij}]$ be the matrix of (21). Then

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq (n-1) \cdot \max\{\eta_1, \eta_2\}, \quad (25)$$

where

$$\begin{aligned} \eta_1 = & \max \left\{ \max_{i \in S, j \in \bar{S}} \frac{|b_{jj}| - r_j^{\bar{S}}(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}, 1 \right\} \\ & + \max \left\{ \max_{i \in S, j \in \bar{S}} \frac{r_i^{\bar{S}}(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}, \max_{i \in S} \frac{r_i^{\bar{S}}(B)}{|b_{ii}| - r_i^S(B)} \right\}, \end{aligned}$$

and

$$\eta_2 = \max \left\{ \max_{i \in S, j \in \bar{S}} \frac{|b_{ii}| - r_i^S(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}, 1 \right\} \\ + \max \left\{ \max_{i \in S, j \in \bar{S}} \frac{r_j^S(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}, \max_{j \in \bar{S}} \frac{r_j^S(B)}{|b_{jj}| - r_j^{\bar{S}}(B)} \right\}.$$

Theorem 3.5. [18, Theorem 2.4] Let $M \in \mathbb{R}^{n \times n}$ be an SB matrix for some nonempty proper subset S of N , $B = [b_{ij}]$ be the matrix of (21), and $W = \text{diag}(w_i)$ be a diagonal matrix such that BW is SDD, where $w_i = 1$ if $i \in S$ and $w_i = \eta$ if $i \in \bar{S}$. Let $N_S := \text{card}(S)$, $N_{\bar{S}} := \text{card}(\bar{S})$,

$$\alpha := \max \left\{ N_S + \eta N_{\bar{S}}, \frac{1}{\eta} N_S + N_{\bar{S}} \right\},$$

and

$$\xi_i := \begin{cases} b_{ii} - \sum_{j \in S \setminus \{i\}} |b_{ij}| - \eta \sum_{j \in \bar{S} \setminus \{i\}} |b_{ij}|, & i \in S, \\ \eta b_{ii} - \sum_{j \in S \setminus \{i\}} |b_{ij}| - \eta \sum_{j \in \bar{S} \setminus \{i\}} |b_{ij}|, & i \in \bar{S}, \end{cases}$$

where

$$(0 <) \eta \in \left(\max_{j \in \bar{S}} \frac{r_j^S(B)}{|b_{jj}| - r_j^{\bar{S}}(B)}, \min_{i \in S} \frac{|b_{ii}| - r_i^S(B)}{r_i^{\bar{S}}(B)} \right)$$

assuming that if $r_i^{\bar{S}}(B) = 0$, then $\frac{|b_{ii}| - r_i^S(B)}{r_i^{\bar{S}}(B)} = \infty$. Denote $\xi := \min_i \{\xi_i\}$. Then

$$\max_{d \in [0, 1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \frac{(\alpha - 1) \max\{\eta, 1\}}{\min\{\xi, \eta, 1\}}. \quad (26)$$

By Corollary 3.3, we easily obtain the following comparison result, which shows that the bound (22) of Theorem 3.2 improves the bound (25) of Theorem 3.4 under certain conditions.

Theorem 3.6. Let $M \in \mathbb{R}^{n \times n}$ be an SB-matrix for some nonempty proper subset S of N , and $B = [b_{ij}]$ be the matrix of (21). If $b_{ii} - r_i^S(B) \leq 1$ and $b_{jj} - r_j^{\bar{S}}(B) \leq 1$ for $i \in S, j \in \bar{S}$, then

$$\max_{i \in S, j \in \bar{S}} \left\{ \chi_{ij}^S(B), \chi_{ji}^{\bar{S}}(B) \right\} \leq \max\{\eta_1, \eta_2\},$$

where $\chi_{ij}^S(B)$ and $\chi_{ji}^{\bar{S}}(B)$ are given by Theorem 3.2, η_1 and η_2 are given by Theorem 3.4.

Proof. Since $b_{ii} - r_i^S(B) \leq 1$ and $b_{jj} - r_j^{\bar{S}}(B) \leq 1$ for $i \in S, j \in \bar{S}$, it follows from Corollary 3.3 that

$$\chi_{ij}^S(B) = \frac{|b_{ii}| - r_i^S(B) + r_j^S(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)},$$

and

$$\chi_{ji}^{\bar{S}}(B) = \frac{|b_{jj}| - r_j^{\bar{S}}(B) + r_i^{\bar{S}}(B)}{(|b_{ii}| - r_i^S(B))(|b_{jj}| - r_j^{\bar{S}}(B)) - r_i^{\bar{S}}(B)r_j^S(B)}.$$

Obviously,

$$\max_{i \in S, j \in \bar{S}} \left\{ \chi_{ji}^{\bar{S}}(B) \right\} \leq \eta_1, \quad \max_{i \in S, j \in \bar{S}} \left\{ \chi_{ij}^S(B) \right\} \leq \eta_2.$$

This completes the proof. \square

Moreover, the following example shows that the bound (22) of Theorem 3.2 is sharper than the bound (26) of Theorem 3.5 in some cases.

Example 3.7. Consider the following matrix

$$M = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{2}{5} & 1 & \frac{1}{5} & -\frac{2}{5} \\ -1 & 0 & 1 & -\frac{1}{6} \\ \frac{3}{4} & \frac{3}{4} & \frac{21}{40} & 1 \end{bmatrix}.$$

Obverse that M does not have one strictly diagonally dominant row in all rows, so it is not an H -matrix, consequently, not a Σ -SDD matrix. Furthermore, M can be written as $M = B + C$ with

$$B = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 & -\frac{3}{5} \\ -1 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & -\frac{9}{40} & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \end{bmatrix}.$$

Obviously, B is not SDD and so M is not a B -matrix. However, it is easy to check that B is a Σ -SDD matrix for $S = \{1\}$, which implies that M is an SB -matrix for $S = \{1\}$. Thus, by the bound (22) of Theorem 3.2, we can see that

$$\max_{d \in [0,1]^4} \|(I - D + DM)^{-1}\|_{\infty} \leq 120.$$

In contrast, by Theorem 3.5 we can get the bound (26) involved with $\eta \in (3, +\infty)$ for $\max_{d \in [0,1]^4} \|(I - D + DM)^{-1}\|_{\infty}$, which is drawn in Figure 4. It can be seen from Figure 4 that the bound (22) in Theorem 3.2 is smaller than the bound (26) in Theorem 3.5 (Theorem 2.4 in [18]).

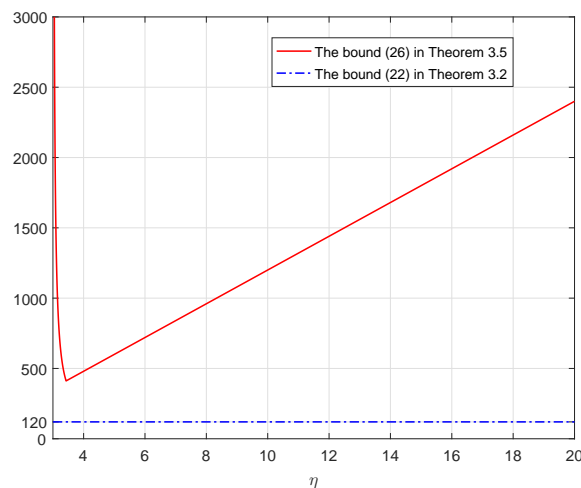


Figure 4: The bounds (22) and (26) in Theorem 3.2 and Theorem 3.5.

4 Conclusions

In this paper, for the linear complementarity problems with a Σ -SDD matrix M , we first give an alternative error bound for the $LCP(M, q)$ which depends only on the entries of M . Then, by this new result, a new error bound for the $LCP(M, q)$ with SB -matrices is provided. We also illustrate the results by numerical examples, where we improve bounds obtained in [17] and [18].

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References

- [1] Berman A., Plemmons R.J., *Nonnegative Matrix in the Mathematical Sciences*, SIAM Publisher: Philadelphia, USA, 1994.
- [2] Chen T.T., Li W., Wu X., Vong S., Error bounds for linear complementarity problems of MB -matrices, *Numer. Algor.*, 2015, 70, 341–356.
- [3] Cottle R.W., Pang J.S., Stone R.E., *The Linear Complementarity Problem*, SIAM Publisher: Philadelphia, USA, 2009.
- [4] Luo Z.Q., Tseng P., Error bound and convergence analysis of matrix splitting algorithms for the affine variational inequality problem, *SIAM J. Optim.*, 1992, 2, 43–54.
- [5] Chen X.J., Xiang S.H., Computation of error bounds for P -matrix linear complementarity problems, *Math. Program., Ser. A*, 2006, 106, 513–525.
- [6] García-Esnaola M., Peña J.M., Error bounds for linear complementarity problems of Nekrasov matrices, *Numer. Algor.*, 2014, 67, 655–667.
- [7] Li C.Q., Dai P.F., Li Y.T., New error bounds for linear complementarity problems of Nekrasov matrices and B -Nekrasov matrices, *Numer. Algor.*, 2017, 74, 997–1009.
- [8] Dai P.F., Li J.C., Bai J., Dong L., New error bounds for linear complementarity problems of S -Nekrasov matrices and B - S -Nekrasov matrices, *Comput. Appl. Math.*, 2019, DOI: 10.1007/s40314-019-0818-4.
- [9] Gao L., Wang Y.Q., Li C.Q., Li Y.T., Error bounds for the linear complementarity problem of S -Nekrasov matrices and B - S -Nekrasov matrices, *J. Comput. Appl. Math.*, 2018, 336, 147–159.
- [10] Dai P.F., Li J.C., Li Y.T., Zhang C.Y., Error bounds for the linear complementarity problem of QN -matrices, *Calcolo*, 2016, 53, 647–657.
- [11] Gao L., Wang Y., Li C.Q., New error bounds for the linear complementarity problem of QN -matrices, *Numer. Algor.*, 2018, 77, 229–242.
- [12] Li J., Li G., Error bounds for linear complementarity problems of S - QN matrices, *Numer. Algor.*, 2019, DOI: 10.1007/s11075-019-00710-0.
- [13] García-Esnaola M., Peña J.M., Error bounds for linear complementarity problems for B -matrices, *Appl. Math. Lett.*, 2009, 22, 1071–1075.
- [14] Li C.Q., Li Y.T., Note on error bounds for linear complementarity problems for B -matrices, *Appl. Math. Lett.*, 2016, 57, 108–113.
- [15] García-Esnaola M., Peña J.M., On the asymptotic optimality of error bounds for some linear complementarity problems, *Numer. Algor.*, 2019, 80, 521–532.
- [16] Dai P.F., Error bounds for linear complementarity problems of DB -matrices, *Linear Algebra Appl.*, 2011, 434, 830–840.
- [17] Dai P.F., Li Y.T., Lu C.J., Error bounds for linear complementarity problems for SB -matrices, *Numer. Algor.*, 2012, 61, 121–139.
- [18] Dai P.F., Lu C.J., Li Y.T., New error bounds for the linear complementarity problem with an SB -matrix, *Numer. Algor.*, 2013, 64, 741–757.
- [19] García-Esnaola M., Peña J.M., B -Nekrasov matrices and error bounds for linear complementarity problems, *Numer. Algor.*, 2016, 72, 435–445.
- [20] Li C.Q., Yang S., Huang H., Li Y.T., Wei Y.M., Note on error bounds for linear complementarity problems of Nekrasov matrices, *Numer. Algor.*, 2019, DOI: 10.1007/s11075-019-00685-y.
- [21] García-Esnaola M., Peña J.M., B_{π}^R -matrices and error bounds for linear complementarity problems, *Calcolo*, 2017, 54, 813–822.
- [22] Gao L., Li C., Li Y., Parameterized error bounds for linear complementarity problems of B_{π}^R -matrices and their optimal values, *Calcolo*, 2019, DOI: 10.1007/s10092-019-0328-1.
- [23] Li C.Q., Cvetković L., Wei Y.M., Zhao J.X., An infinity norm bound for the inverse of Dashnic–Zusmanovich type matrices with applications, *Linear Algebra Appl.*, 2019, 565, 99–122.
- [24] Li C.Q., Li Y.T., Weakly chained diagonally dominant B -matrices and error bounds for linear complementarity problems, *Numer. Algor.*, 2016, 73, 985–998.
- [25] Wang F., Error bounds for linear complementarity problems of weakly chained diagonally dominant B -matrices, *J. Inequal. Appl.*, 2017, 2017:33.

- [26] Sang C., Chen Z., A new error bound for linear complementarity problems of weakly chained diagonally dominant B -matrices, *Linear Multilinear A.*, 2019, DOI: 10.1080/03081087.2019.1649995.
- [27] García-Esnaola M., Peña J.M., Error bounds for the linear complementarity problem with a Σ -SDD matrix, *Linear Algebra Appl.*, 2013, 438, 1339–1346.
- [28] Cvetković L., Kostić V., Varga R.S., A new Geršgorin-type eigenvalue inclusion set, *Electron. Trans. Numer. Anal.*, 2004, 18, 73–80.
- [29] Wang Z.F., Li C.Q., Li Y.T., Infimum of error bounds for linear complementarity problems of Σ -SDD and Σ_1 -SSD matrices, *Linear Algebra Appl.*, 2019, 581, 285–303.
- [30] Morača N., Upper bounds for the infinity norm of the inverse of SDD and S -SDD matrices, *J. Comput. Appl. Math.*, 2007, 206, 666–678.
- [31] García-Esnaola M., Peña J.M., A comparison of error bounds for linear complementarity problems of H -matrices, *Linear Algebra Appl.*, 2010, 433, 956–964.
- [32] Dafermos S., Traffic equilibrium and variational inequalities, *Transport. Sci.*, 1980, 14, 42–54.
- [33] Ferris M.C., Pang J.S., Engineering and economic applications of complementarity problems, *SIAM Rev.*, 1997, 39, 669–713.
- [34] Li H.B., Huang T.Z., Li H., On some subclasses of P -matrices, *Numer. Linear Algebra*, 2007, 14, 391–405.