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On a new convergence in topological spaces

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Abstract: In this paper, we introduce a new way-below relation in T_0 topological spaces based on cuts and give the concepts of SI_2 -continuous spaces and weakly irreducible topologies. It is proved that a space is SI_2 -continuous if and only if its weakly irreducible topology is completely distributive under inclusion order. Finally, we introduce the concept of \mathcal{D} -convergence and show that a space is SI_2 -continuous if and only if its \mathcal{D} -convergence with respect to the topology $\tau_{SI_2}(X)$ is topological. In general, a space is SI -continuous if and only if its \mathcal{D} -convergence with respect to the topology $\tau_{SI}(X)$ is topological.

Keywords: s_2 -continuous poset, weakly irreducible topology, SI_2 -continuous space, \mathcal{D} -convergence

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1 Introduction

Domain theory which arose from computer science and logic, started as an outgrowth of theories of order. Rapidly progress in this domain required many materials on topologies (see [1–3]). Conversely, it is well known that given a topological space one can also define order structures (see [3–6]). At the 6th International Symposium in Domain Theory, J.D. Lawson emphasized the need to develop the core of domain theory directly in T_0 topological spaces instead of posets. Moreover, it was pointed out that several results in domain theory can be lifted from the context of posets to T_0 topological spaces (see [5, 6]). In the absence of enough joins, Ern  introduced the concept of s_2 -continuous posets and the *weak Scott topology* by means of the cuts instead of joins (see [7]). The notion of s_2 -continuity admits to generalize most important characterizations of continuity from dcpos to general posets and has the advantage that not even the existence of directed joins has to be required. In [6], Ern  further proved that the weak Scott topology is the weakest monotone determined topology with a given specialization order. In [5], Zhao and Ho defined a new way-below relation and a new topology constructed from any given topology on a set using irreducible sets in a T_0 topological space replacing directed subsets and investigated the properties of this derived topology and k -bounded spaces. It was proved that a space X is SI -continuous if and only if $SI(X)$ is a C -space.

Many convergent classes in posets were studied in [3, 8–12]. By different convergences, not only many notions of continuity are characterized, but also they make order and topology across each other. In [3], the concept of \mathcal{S} -convergence for dcpos was introduced by Scott to characterize continuous domains. It was proved that for a dcpo, the \mathcal{S} -convergence is topological if and only if it is a continuous domain. The main purpose of this paper is to lift the notion of s_2 -continuous posets to topology context. By the manner of Ern  we introduce a new way-below relation in T_0 topological spaces based on cuts and give the concepts of SI_2 -continuous spaces and weakly irreducible topologies. It is proved that a space is SI_2 -continuous if and only if its weakly irreducible topology is completely distributive under inclusion order. Finally, we introduce the

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concept of \mathcal{D} -convergence and show that a space is SI_2 -continuous if and only if its \mathcal{D} -convergence with respect to the topology $\tau_{SI_2}(X)$ in a topological space is topological. Furthermore, a space is SI -continuous if and only if its \mathcal{D} -convergence with respect to the topology $\tau_{SI}(X)$ is topological. The work carry out here is another response to the call by J. D. Lawson to develop domain theory in the wider context of T_0 topological spaces instead of restricting to posets.

2 Preliminaries

Let P be a partially ordered set (poset, for short). A nonempty set $D \subseteq P$ is *directed* if for any d_1, d_2 in D there exists d in D above d_1 and d_2 . The *principal ideal* generated by $x \in P$ is $\downarrow x = \{y \in P : y \leq x\}$. $\downarrow A = \bigcup_{a \in A} \downarrow a$ is the *lower set* or *downset* generalized by $A \subseteq P$; The *principal filter* $\uparrow x$ and *upper set* $\uparrow A$ are defined dually. A^\uparrow and A^\downarrow denote the sets of all *upper* and *lower bounds* of A , respectively. A *cut* δ of A in P is defined by $A^\delta = (A^\uparrow)^\downarrow$ for every $A \subseteq P$. Notice that $x \in A^\delta$ means $x \leq \bigvee A$ whenever A has a join (supremum).

On the one hand, given a poset P , we can generate some intrinsic topologies. The upper sets form the *Alexandroff upper topology* $\alpha(P)$. The *weak Scott topology* $\sigma_2(P)$ consists of all upper sets U such that $D^\delta \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$ for all directed sets D of P . In case P is a dcpo, the weak Scott topology coincides with the usual *Scott topology* $\sigma(P)$, which consists of all upper sets U such that $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ for all directed sets D in P . The *upper topology* generating by the complements of the principal ideals is denoted by $\nu(P)$. Clearly, $\nu(P) \subseteq \sigma_2(P) \subseteq \sigma(P) \subseteq \alpha(P)$.

On the other hand, for a topological space (X, τ) , the *specialization order* \leq on X is defined by $y \leq x$ if and only if $y \in cl\{x\}$. It is antisymmetric, hence a partial order, if and only if (X, τ) is T_0 . The specialization order on X is denoted by \leq_τ if there is need to emphasize the topology τ . Note that the specialization order of the Alexandroff upper topology on a poset coincides with the underlying order and $\downarrow x = cl\{x\}$.

If not otherwise stated, in topological contexts, lower sets, upper sets and related notions refer to the specialization order.

Remark 2.1. Let (X, τ) be a T_0 space.

- (1) If $D \subseteq X$ is a directed set with respect to the specialization order, then D is irreducible;
- (2) If $U \subseteq X$ is an open set, then U is an upper set; Similarly, if $F \subseteq X$ is a closed set, then F is a lower set.

Definition 2.1. [7] Let P be a poset.

- (1) For any $x, y \in P$, we say that x is way below y , written $x \ll y$ if for all directed sets $D \subseteq P$ with $y \in D^\delta$, there exists $d \in D$ such that $x \leq d$. The set $\{y \in P : y \ll x\}$ will be denoted by $\Downarrow x$ and $\{y \in P : x \ll y\}$ denoted by $\Uparrow x$;
- (2) P is called s_2 -continuous if for all $x \in P$, $x \in (\Downarrow x)^\delta$ and $\Downarrow x$ is directed.

Indeed, since $\Downarrow x \subseteq \downarrow x$ we have $x = \bigvee \Downarrow x$ if and only if $x \in (\Downarrow x)^\delta$.

Definition 2.2. [5] Let (X, τ) be a T_0 space. For $x, y \in X$, define $x \ll_{SI} y$ if for all irreducible sets F , $y \leq \bigvee F$ implies there exists $e \in F$ such that $x \leq e$ whenever $\bigvee F$ exists. The set $\{y \in X : y \ll_{SI} x\}$ is denoted by $\downarrow_{SI} x$ and the set $\{y \in X : x \ll_{SI} y\}$ by $\uparrow_{SI} x$.

Definition 2.3. [5] Let (X, τ) be a T_0 space. X is called *SI-continuous* if the following conditions are satisfied:

- (1) $\downarrow_{SI} x$ is directed for all $x \in X$;
- (2) $x = \bigvee \downarrow_{SI} x$ for all $x \in X$;
- (3) $\uparrow_{SI} x \in \tau$ for all $x \in X$.

Definition 2.4. [5] Let (X, τ) be a T_0 space. A subset $U \subseteq X$ is called *SI-open* if the following conditions are satisfied:

- (1) $U \in \tau$;
- (2) For all $F \in \text{Irr}_\tau(X)$, $\bigvee F \cap U \neq \emptyset$ implies $F \cap U \neq \emptyset$ whenever $\bigvee F$ exists.

The set of all SI-open sets of (X, τ) is denoted by $\tau_{SI}(X)$.

3 SI_2 -continuous spaces

In this section, we define a SI_2 -continuous space derived by the irreducible set of a topological space. Some properties of this derived SI_2 -continuity are investigated.

Let (X, τ) be a topological space. A nonempty subset $F \subseteq X$ is called *irreducible* if for every closed sets B and C , whenever $F \subseteq B \cup C$, one has either $F \subseteq B$ or $F \subseteq C$. The set of all irreducible sets of the topological space (X, τ) will be denoted by $\text{Irr}_\tau(X)$ or $\text{Irr}(X)$.

Definition 3.1. Let (X, τ) be a T_0 space and $x, y \in X$. Define $x \ll_r y$ if for every irreducible set E , $y \in E^\delta$ implies there exists $e \in E$ such that $x \leq e$. We denote the set $\{y \in X : y \ll_r x\}$ by $\downarrow_r x$ and the set $\{y \in X : x \ll_r y\}$ by $\uparrow_r x$.

Remark 3.1. Let (X, τ) be a T_0 space and $x, y, u, v \in X$.

- (1) $x \ll_r y$ implies $x \leq y$;
- (2) $u \leq x \ll_r y \leq v$ implies $u \ll_r v$.

Definition 3.2. Let (X, τ) be a T_0 space. A subset $U \subseteq X$ is called *weakly irreducibly open* if the following conditions are satisfied:

- (1) $U \in \tau$;
- (2) $F^\delta \cap U \neq \emptyset$ implies $F \cap U \neq \emptyset$ for all $F \in \text{Irr}_\tau(X)$.

The set of all weakly irreducibly open sets of (X, τ) is denoted by $\tau_{SI_2}(X)$. Complements of all weakly irreducibly open sets are called *weakly irreducibly closed sets*.

Lemma 3.1. Let (X, τ) be a T_0 space. Then $\tau_{SI_2}(X)$ is a topology on X .

Proof. (1) Clearly $\emptyset, X \in \tau_{SI_2}(X)$;

- (2) It claims that $U \cap V \in \tau_{SI_2}(X)$ for any $U, V \in \tau_{SI_2}(X)$. Indeed, for any $F \in \text{Irr}_\tau(X)$, if $F^\delta \cap (U \cap V) \neq \emptyset$, then $F^\delta \cap U \neq \emptyset$ and $F^\delta \cap V \neq \emptyset$. Note $U, V \in \tau_{SI_2}(X)$, we have $F \cap U \neq \emptyset$ and $F \cap V \neq \emptyset$. Since $F \in \text{Irr}_\tau(X)$, we have $F \cap (U \cap V) \neq \emptyset$. Clearly $U \cap V \in \tau$, so $U \cap V \in \tau_{SI_2}(X)$.

- (3) Assume that $\{U_i : i \in I\} \subseteq \tau_{SI_2}(X)$. Firstly, $\bigcup_{i \in I} U_i \in \tau$. Secondly, for any $F \in \text{Irr}_\tau(X)$, if $F^\delta \cap (\bigcup_{i \in I} U_i) = \bigcup_{i \in I} (F^\delta \cap U_i) \neq \emptyset$, then there exists some $i \in I$ such that $F^\delta \cap U_i \neq \emptyset$. Note that $F \in \text{Irr}_\tau(X)$ and $U_i \in \tau_{SI_2}(X)$, then $F \cap U_i \neq \emptyset$. Thus we have $F \cap (\bigcup_{i \in I} U_i) \neq \emptyset$. Therefore $\bigcup_{i \in I} U_i \in \tau_{SI_2}(X)$. \square

Remark 3.2. Let (X, τ) be a T_0 space. Then $\tau_{SI_2}(X)$ is always coarser than $\tau_{SI}(X)$, and if any irreducible set in X has a supremum, then both topologies coincide. In the following, the space $(X, \tau_{SI_2}(X))$ is also simply denoted by $SI_2(X)$.

The following example is due to Ern  (see [7, Example 2.5]).

Example 3.1. Let P be poset delineated by Figure 1 and $B = \{b_n : n = 0, 1, 2, \dots\}$, $C = \{c_k : k = 1, 2, \dots\}$. The order \leq on $P = B \cup C$ is defined as follows:

- $\downarrow a_0 = \{a_0\} \cup B$,
- $\downarrow a_1 = \{a_1, b_0\} \cup C$,
- $\downarrow a_2 = \{a_2, b_0, b_1\} \cup C$,
- $\downarrow a_n = \{a_n\} \cup \{b_m : m < n\} (n = 3, 4, \dots)$,

$\downarrow b_n = \{b_n\} (n = 0, 1, 2, \dots),$
 $\downarrow c_n = \{c_m : m \leq n\} (n = 1, 2, \dots).$
 $x \leq y \Leftrightarrow x \in \downarrow y.$ Endow P with the Alexandroff upper topology.

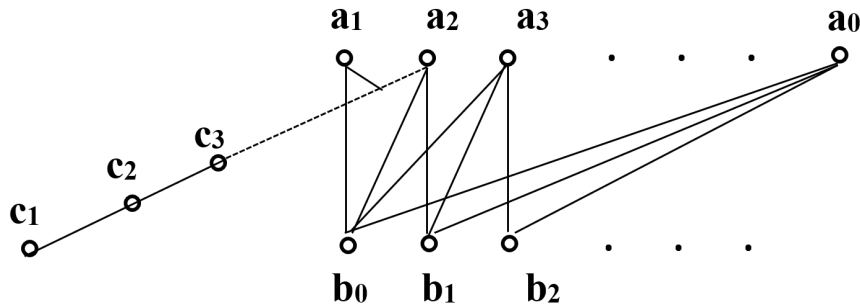


Figure 1: A set which is SI -open is not weakly irreducibly open.

Then it is easy to see that $\uparrow b_0$ is open in $\tau_{SI}(P)$. $C = \{c_k : k = 1, 2, \dots\}$ is an irreducible set with $b_0 \in C^\delta \cap \uparrow b_0 \neq \emptyset$ while $C \cap \uparrow b_0 = \emptyset$, and whence $\uparrow b_0 \notin \tau_{SI_2}(\alpha(P))$. Thus $\tau_{SI_2}(\alpha(P))$ is proper contained in $\tau_{SI}(\alpha(P))$.

Definition 3.3. Let (X, τ) be a T_0 space. X is called SI_2 -continuous if the following conditions are satisfied:

- (1) $\downarrow_r x$ is directed for all $x \in X$;
- (2) $x \in (\downarrow_r x)^\delta$ for all $x \in X$;
- (3) $\uparrow_r x \in \tau$ for all $x \in X$.

Lemma 3.2. Let P be a poset. Then $SI_2(P, \alpha(P)) = (P, \sigma_2(P))$.

Remark 3.3. (1) Let P be a poset. Then P is an s_2 -continuous poset if and only if it is an SI_2 -continuous space with respect to the Alexandroff upper topology.
 (2) Let (X, τ) be a T_0 space. If X is an SI_2 -continuous space, then it is also an s_2 -continuous poset under the specialization order. But the converse may not be true.

Example 3.2. Let X be an infinite set with a cofinite topology τ . Then it is a T_1 space. Clearly it is an antichain under the specialization order, and hence it is an s_2 -continuous poset. But $\uparrow_r x = \{x\} \notin \tau$ for all $x \in X$, then (X, τ) is not an SI_2 -continuous space.

Let us note that an SI_2 -continuous space is SI -continuous space, but the converse may not be true:

Example 3.3. Consider the Euclidean plane $\mathbb{R} \times \mathbb{R}$ under the usual topology. It is an SI -continuous space, but it is not SI_2 -continuous, since every lower half-plane

$$E_a = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \leq a\}$$

is a directed set with $E_a^\delta = \mathbb{R} \times \mathbb{R}$ and $\bigcap \{E_a : a \in \mathbb{R}\} = \emptyset$, thus \ll_r is empty.

The following theorem shows that the SI_2 -continuity of the topological space has the interpolation property.

Theorem 3.1. Let X be an SI_2 -continuous space and $x, y \in X$. If $x \ll_r y$, then there exists $z \in X$ such that $x \ll_r z \ll_r y$.

Proof. Let X be an SI_2 -continuous space and $x \ll_r y$. Then we have $\downarrow_r y$ is directed and $y \in (\downarrow_r y)^\delta$. Since the union of a directed family of directed sets, $E = \bigcup \{\downarrow_r z : z \in \downarrow_r y\}$ is still a directed set (hence an irreducible set) and $y \in E^\delta$. So there exists $z \in \downarrow_r y$ such that $x \leq u \ll_r z$ for some $u \in X$. Thus $x \ll_r z \ll_r y$. \square

Remark 3.4. In Theorem 3.1, when we prove the interpolation property, we do not need the third condition in the definition of the SI_2 -continuous space.

Lemma 3.3. Let X be an SI_2 -continuous space. Then for any $x \in X$, $\uparrow_r x \in \tau_{SI_2}(X)$.

Proof. It follows from Definition 3.3 and Theorem 3.1. \square

Lemma 3.4. Let (X, τ) be a T_0 space and $y \in \text{int}_{\tau_{SI_2}(X)} \uparrow x$. Then $x \ll_r y$, where $\text{int}_{\tau_{SI_2}(X)} \uparrow x$ denotes the interior of $\uparrow x$ with respect to the topology $\tau_{SI_2}(X)$.

Proof. Let $y \in \text{int}_{\tau_{SI_2}(X)} \uparrow x$. For every irreducible set E with $y \in E^\delta$, we have $y \in E^\delta \cap \text{int}_{\tau_{SI_2}(X)} \uparrow x \neq \emptyset$, and hence $\text{int}_{\tau_{SI_2}(X)} \uparrow x \cap E \neq \emptyset$. Thus there exists $e \in \text{int}_{\tau_{SI_2}(X)} \uparrow x \cap E$. Thus we have $x \leq e$ and $e \in E$. This shows $x \ll_r y$. \square

Theorem 3.2. Let (X, τ) be a T_0 space. Then the following statements are equivalent:

- (1) X is an SI_2 -continuous space;
- (2) For all $U \in \tau_{SI_2}(X)$ and $x \in U$, there exists $y \in U$ such that $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow y \subseteq \uparrow y \subseteq U$;
- (3) $(\tau_{SI_2}(X), \subseteq)$ is a completely distributive lattice.

Proof. (1) \Rightarrow (2) Let X be an SI_2 -continuous space and $U \in \tau_{SI_2}(X)$ with $x \in U$. Since $x \in (\downarrow_r x)^\delta \cap U \neq \emptyset$, note that $\downarrow_r x$ is directed (hence irreducible), then we have $\downarrow_r x \cap U \neq \emptyset$. Thus there exists $y \ll_r x$ such that $y \in U$. By Lemma 3.3, $\uparrow_r y \in \tau_{SI_2}(X)$, we have that $x \in \uparrow_r y \subseteq \uparrow y \subseteq U$, that is, $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow y \subseteq \uparrow y \subseteq U$.

(2) \Rightarrow (1) For $x \in X$, consider the set $E = \{y \in X : x \in \text{int}_{\tau_{SI_2}(X)} \uparrow y\}$. Let $y_1, y_2 \in E$. Then $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow y_1 \cap \text{int}_{\tau_{SI_2}(X)} \uparrow y_2$. By (2), there exists $y \in \text{int}_{\tau_{SI_2}(X)} \uparrow y_1 \cap \text{int}_{\tau_{SI_2}(X)} \uparrow y_2$ such that $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow y \subseteq \uparrow y \subseteq \text{int}_{\tau_{SI_2}(X)} \uparrow y_1 \cap \text{int}_{\tau_{SI_2}(X)} \uparrow y_2$, so $y \in E$ and $y_1, y_2 \leq y$. This shows that E is a directed set. It is not hard to show $x \in E^\delta$. By Lemma 3.4, we have $E \subseteq \downarrow_r x$. Thus we have that $\downarrow_r x$ is directed (hence irreducible) and $x \in (\downarrow_r x)^\delta$, so we also have $\downarrow_r x \subseteq \downarrow E$. Thus $\downarrow_r x = \downarrow E$. From the above discussion we can derive that $\uparrow_r x = \bigcup_{y \in \uparrow x} \text{int}_{\tau_{SI_2}(X)} \uparrow y$, which is open in τ . Hence X is an SI_2 -continuous space.

(2) \Leftrightarrow (3) See [4]. \square

Corollary 3.1. ([7]) Let P be a poset. Then the following conditions are equivalent:

- (1) P is s_2 -continuous;
- (2) For all $U \in \sigma_2(P)$ and $x \in U$, there exists $y \in U$ such that $x \in \text{int}_{\sigma_2(P)} \uparrow y \subseteq \uparrow y \subseteq U$;
- (3) $(\sigma_2(P), \subseteq)$ is a completely distributive lattice.

Corollary 3.2. ([5]) Let (X, τ) be a T_0 space. Then the following statements are equivalent:

- (1) X is an SI -continuous space;
- (2) For all $U \in \tau_{SI}(X)$ and $x \in U$, there exists $y \in U$ such that $x \in \text{int}_{\tau_{SI}(X)} \uparrow y \subseteq \uparrow y \subseteq U$;
- (3) $(\tau_{SI}(X), \subseteq)$ is a completely distributive lattice.

4 \mathcal{D} -convergence in SI_2 -continuous spaces

In this section, the concept of \mathcal{D} -convergence in a topological space is introduced. It is proved that a space X is SI_2 -continuous if and only if the \mathcal{D} -convergence with respect to the topology $\tau_{SI_2}(X)$ in X is topological. In

general, a space X is SI -continuous if and only if its \mathcal{D} -convergence with respect to the topology $\tau_{SI}(X)$ in X is topological.

Definition 4.1. Let (X, τ) be a T_0 space. A net $(x_j)_{j \in J}$ in X is said to converge to $x \in X$ if there exists a directed set $D \subseteq X$ with respect to the specialization order such that

- (1) $x \in D^\delta$;
- (2) For all $d \in D$, $d \leq x_j$ holds eventually.

In this case we write $x \equiv_{\mathcal{D}} \lim x_j$.

Let \mathcal{D} denote the class of those pairs $((x_j)_{j \in J}, x)$ with $x \equiv_{\mathcal{D}} \lim x_j$. Then $\mathcal{O}(\mathcal{D}) = \{U \subseteq X : \text{whenever } ((x_j)_{j \in J}, x) \in \mathcal{D} \text{ and } x \in U, \text{ then eventually } x_j \in U\}$ is a topology.

Example 4.1. Let $P = \{a_j : j \in \mathbb{N}\} \cup \{a, b\}$, where \mathbb{N} denotes the set of all natural numbers. The order on P is defined by $a_1 \leq a_2 \leq \dots \leq a$, $b \leq a$. Endow P with the Alexandroff upper topology. If $x_j = a_j$ for all $j \in \mathbb{N}$, then $(x_j)_{j \in \mathbb{N}}$ is a net. Take $D = \{a_j : j \in \mathbb{N}\}$, and then D is a directed subset of P with $b \in D^\delta$. Moreover, for all $d \in D$, $d \leq x_j$ holds eventually, hence the net $(x_j)_{j \in \mathbb{N}}$ converges to b .

Proposition 4.1. Let X be an SI_2 -continuous space. Then $x \equiv_{\mathcal{D}} \lim x_j$ if and only if the net $(x_j)_{j \in J}$ converges to the element x with respect to the topology $\tau_{SI_2}(X)$. That is, the \mathcal{D} -convergence is topological.

Proof. Firstly, suppose that a net $(x_j)_{j \in J}$ in X converges to $x \in X$ and $x \in U \in \tau_{SI_2}(X)$. Then there exists a directed set $D \subseteq X$ (hence an irreducible set) such that $d \leq x_j$ holds eventually for all $d \in D$ and $x \in D^\delta$, and hence $U \cap D \neq \emptyset$, that is, there exists $d \in U \cap D$. Clearly, the net $x_j \in U$ holds eventually as U is an upper set. Conversely, assume that the net $(x_j)_{j \in J}$ converges to an element x with respect to the topology $\tau_{SI_2}(X)$. For all $y \in \downarrow_r x$, then one has $x \in \uparrow_r y \in \tau_{SI_2}(X)$ by Lemma 3.3. Thus there exists $k \in J$ such that $x_j \in \uparrow_r y$ for any $j \geq k$. By SI_2 -continuity of X , we have that $x \in (\downarrow_r x)^\delta$ and $\downarrow_r x$ is directed. Thus $((x_j)_{j \in J}, x) \in \mathcal{D}$, that is, $x \equiv_{\mathcal{D}} \lim x_j$. \square

Proposition 4.2. Let (X, τ) be a T_0 space. If the \mathcal{D} -convergence with respect to the topology $\tau_{SI_2}(X)$ is topological, then X is SI_2 -continuous.

Proof. Suppose that the \mathcal{D} -convergence with respect to the topology $\tau_{SI_2}(X)$ is topological. For all $x \in X$, take $J = \{(U, n, a) \in N(x) \times \mathbb{N} \times X : a \in U\}$, where $N(x)$ consists of all weakly irreducibly open sets which contain x , and define an order on J as follows: $(U, m, a) \leq (V, n, b)$ if and only if V is proper subset of U or $U = V$ and $m \leq n$. Obviously, J is directed. Let $x_j = a$ for all $j = (U, m, a) \in J$. Then it is not hard to show that the net $(x_j)_{j \in J}$ converges to x with respect to the weakly irreducible topology $\tau_{SI_2}(X)$. By the condition, one has $x \equiv_{\mathcal{D}} \lim x_j$. Thus there exists a directed set D with respect to the specialization order such that $x \in D^\delta$ and for all $d \in D$, $d \leq x_j$ holds eventually. Then there exists $k = (U, m, a) \in J$ such that $(V, n, b) = j \geq k$ implies $d \leq x_j = b$ for all $d \in D$. Especially one has $(U, m+1, b) \geq (U, m, a) = k$ for all $b \in U$. So $x \in U \subseteq \uparrow d$. It follows that $D \subseteq \downarrow x$ and $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow d$. By Lemma 3.4, $d \ll_r x$, and then $D \subseteq \downarrow_r x$. Thus $x \in D^\delta \subseteq (\downarrow_r x)^\delta$. It is easy to show that $\downarrow_r x$ is directed. Finally, it follows that $\uparrow_r x \in \tau$ for all $x \in X$. Indeed, if $y \in \uparrow_r x$, then there exists $z \in X$ such that $x \ll_r z \ll_r y$ by Remark 3.4. From the above argument, as long as we replace $J = \{(U, n, a) \in N(x) \times \mathbb{N} \times X : a \in U\}$ with $I = \{(U, n, a) \in N(y) \times \mathbb{N} \times X : a \in U\}$, where $N(y)$ consists of all weakly irreducibly open sets containing y , similarly we have that $(x_i)_{i \in I}$ converges to y with respect to the topology $\tau_{SI_2}(X)$, and then there exists a directed set D_1 of eventual lower bounds of the net $(x_i)_{i \in I}$ such that $y \in D_1^\delta$. Note $z \ll_r y$. Thus there exists $d_1 \in D_1$ such that $z \leq d_1$. For $d_1 \in D_1$, there exists $i_0 = (U_1, m, a)$ such that $d_1 \leq x_i = b$ for all $i \geq i_0$, where $U_1 \in \tau_{SI_2}(X)$. In particular, we have $(U_1, m+1, u) \geq (U_1, m, a) = i_0$ for all $u \in U_1$. We conclude that $d_1 \leq u$, and hence $y \in U_1 \subseteq \uparrow d_1$. Since $d_1 \leq u$ for $u \in U_1$ and $x \ll_r z$ with $z \leq d_1$, we have $U_1 \subseteq \uparrow_r x$. This shows that $\uparrow_r x \in \tau$. Hence X is SI_2 -continuous. \square

From Propositions 4.1 and 4.2, we immediately have:

Theorem 4.1. *Let (X, τ) be a T_0 space. Then the following statements are equivalent:*

- (1) X is SI_2 -continuous;
- (2) The \mathcal{D} -convergence with respect to the topology $\tau_{SI_2}(X)$ is topological.

Similarly, we also have:

Theorem 4.2. *Let (X, τ) be a T_0 space. Then the following statements are equivalent:*

- (1) X is SI -continuous;
- (2) The \mathcal{D} -convergence with respect to the topology $\tau_{SI}(X)$ is topological.

Corollary 4.1. ([17]) *Let P be a poset. Then the following statements are equivalent:*

- (1) P is s_2 -continuous;
- (2) The \mathcal{S} -convergence in P is topological for the weak Scott topology, that is, for all $x \in P$ and all nets $(x_j)_{j \in J}$ in P , $x \equiv_{\mathcal{S}} \lim x_j$ if and only if $(x_j)_{j \in J}$ converges to the element x with respect to the weak Scott topology.

Corollary 4.2. ([3]) *Let P be a dcpo. Then the following statements are equivalent:*

- (1) P is a domain;
- (2) The \mathcal{S} -convergence in P is topological for the Scott topology, that is, for all $x \in P$ and all nets $(x_j)_{j \in J}$ in P , $x \equiv_{\mathcal{S}} \lim x_j$ if and only if $(x_j)_{j \in J}$ converges to the element x with respect to the Scott topology.

5 Conclusion

At the Sixth International Symposium on Domain Theory, J.D. Lawson encouraged the domain theory community to consider the scientific program of developing domain theory in the wider context of T_0 spaces instead of restricting to posets. In this paper, we introduce a new way-below relation in T_0 topological spaces based on the cuts and give the concepts of SI_2 -continuous spaces and weakly irreducible topologies. It is proved that a space is SI_2 -continuous if and only if its weakly irreducible topology is completely distributive under inclusion order. Finally, we introduce the concept of \mathcal{D} -convergence and show that a space is SI_2 -continuous if and only if its \mathcal{D} -convergence with respect to the topology $\tau_{SI_2}(X)$ is topological. In general, a space is SI -continuous if and only if its \mathcal{D} -convergence with respect to the topology $\tau_{SI}(X)$ is topological. The present paper can be seen as one of the some works towards the new direction, which may deserve further investigation. Indeed there are some questions to which we possess no answers. The following is such one.

In the first condition of the definition of the \mathcal{D} -convergence, whether we can change directed set into irreducible set?

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