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Research Article

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An optimal fourth-order family of modified Cauchy methods for finding solutions of nonlinear equations and their dynamical behavior

Abstract: In this paper, we derive and analyze a new one-parameter family of modified Cauchy method free from second derivative for obtaining simple roots of nonlinear equations by using Pad*é* approximant. The convergence analysis of the family is also considered, and the methods have convergence order three. Based on the family of third-order method, in order to increase the order of the convergence, a new optimal fourth-order family of modified Cauchy methods is obtained by using weight function. We also perform some numerical tests and the comparison with existing optimal fourth-order methods to show the high computational efficiency of the proposed scheme, which confirm our theoretical results. The basins of attraction of this optimal fourth-order family and existing fourth-order methods are presented and compared to illustrate some elements of the proposed family have equal or better stable behavior in many aspects. Furthermore, from the fractal graphics, with the increase of the value m of the series in iterative methods, the chaotic behaviors of the methods become more and more complex, which also reflected in some existing fourth-order methods.

Keywords: iterative methods, Newton's method, Cauchy's method, order of convergence, Pad*é* approximant

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1 Introduction

In this paper, we consider iterative methods to find a simple root α,i.e, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I is a scalar function.

Finding the simple root of the nonlinear equation (1) is a common and important problems in numerical analysis of science and engineering, and iterative methods are usually used to approximate a solution of these equations. We know that Newton's method is an important and basic approach for solving nonlinear

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equations [1, 2], and its formulation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$ (2)

this method converges quadratically.

The classical Cauchy's method [2] is expressed as

$$x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L_f(x_n)}} \frac{f(x_n)}{f'(x_n)},$$ (3)

where

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}.$$ (4)

This family methods given by (3) is a well-known third-order method. However, the method depends on the second derivatives in computing process, and therefore their practical applications are restricted rigorously. In recent years, several methods with free second derivatives have been developed, see [3–10] and references therein.

In this paper, we will improve the family defined by (3) and obtain third and optimal fourth order family of second-derivative-free variants of Cauchy's methods by using Padé approximant. The rest of the paper is organized as follows: In Section 2, we present a new third order family of modified Cauchy method and show the order of convergence of this family; In Section 3, different numerical tests confirm the theoretical results, and the new methods are comparable with other known methods and give better results in many cases; In Section 4, based on the family of third-order method, a new optimal fourth-order family of iterative methods is obtained by using weight function; In Section 5, numerical tests and the comparison with the existing optimal fourth-order methods are included to confirm our theoretical results; In Section 6, the basins of attraction of the existing optimal fourth-order methods and our methods are presented and compared to illustrate their performances. Finally, we infer some conclusions.

2 Development of the third order method and its convergence analysis

In order to avoid the evaluation of the second derivatives $f''(x_n)$ of Cauchy's method (3), we consider approximating it by the derivative $y''(x_n)$ of the following second degree Padé approximant:

$$y(t) = \frac{a_1 + a_2(t - w_n) + a_3(t - w_n)^2}{1 + a_4(t - w_n)},$$ (5)

where a_1, a_2, a_3 and a_4 are real parameters. We impose the tangency conditions

$$y(x_n) = f(x_n), y'(x_n) = f'(x_n), y(w_n) = f(w_n),$$ (6)

where x_n is nth iterate and

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$ (7)

By using the tangency conditions from (6), we obtain the value of a_1, a_2, a_4, and a_4 is determined in terms of a_3 in the following

$$a_1 = f(w_n), a_2 = f'(x_n) - \frac{2f'(x_n)f(w_n)}{f(x_n)}, a_4 = \frac{a_3}{f'(x_n)} - \frac{f'(x_n)f(w_n)}{f^2(x_n)}.$$ (8)

From (5), we also have

$$y''(t) = \frac{2[a_3 - a_2 a_4 + a_1 a_4^2]}{[1 + a_4(t - w_n)]^3}.$$ (9)

Substituting (8) into (9) yields

$$f''(x_n) \approx y''(x_n) = \frac{2f'^4(x_n)f(w_n)}{f(x_n)[f'^2(x_n)f(x_n) + a_3 f^2(x_n) - f'^2(x_n)f(w_n)]}. \tag{10}$$

Using (10) we can approximate

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)} \approx \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) + a_3 f^2(x_n) - f'^2(x_n)f(w_n)\right]}. \tag{11}$$

We define

$$L_{f,\,\mu}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) + \mu f^2(x_n) - f'^2(x_n)f(w_n)\right]}. \tag{12}$$

Using $L_{f,\,\mu}(x_n, w_n)$ instead of $L_f(x_n)$, we obtain a new one-parameter family of modified Cauchy method free from second derivative

$$x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L_{f,\,\mu}(x_n, w_n)}} \frac{f(x_n)}{f'(x_n)}, \tag{13}$$

where $\mu \in R$. Similar to the classical Cauchy's method, a square root is required in (13). However, this may cost expensively, even fail in the case $1 - 2L_{f,\,\mu}(x_n, w_n) < 0$. In order to avoid the calculation of the square roots, we will derive some forms free from square roots by Taylor approximation [4].

It is easy to know that Taylor approximation of $\sqrt{1 - 2L_{f,\,\mu}(x_n, w_n)}$ is

$$\sqrt{1 - 2L_{f,\,\mu}(x_n, w_n)} = \sum_{k \geq 0}^{m} \binom{\frac{1}{2}}{k}(-2L_{f,\,\mu}(x_n, w_n))^k, \tag{14}$$

where $m > 0$.

Using (14) in (13), we can obtain the following form

$$x_{n+1} = x_n - \frac{2}{1 + \sum_{k \geq 0}^{m} \binom{\frac{1}{2}}{k}(-2L_{f,\,\mu}(x_n, w_n))^k} \frac{f(x_n)}{f'(x_n)}, \tag{15}$$

where $\mu \in R$.

On the other hand, it is clear that

$$\frac{2}{1 + \sqrt{1 - 2L_{f,\,\mu}(x_n, w_n)}} = \frac{1 - \sqrt{1 - 2L_{f,\,\mu}(x_n, w_n)}}{L_{f,\,\mu}(x_n, w_n)} = \sum_{k \geq 0}^{m} \binom{\frac{1}{2}}{k+1}(-1)^k 2^{k+1} L_{f,\,\mu}(x_n, w_n)^k \tag{16}$$

Then, Using (16) in (13), we also can construct a new family of iterative methods as follows:

$$x_{n+1} = x_n - \left(\sum_{k \geq 0}^{m} \binom{\frac{1}{2}}{k+1}(-1)^k 2^{k+1} L_{f,\,\mu}(x_n, w_n)^k \right) \frac{f(x_n)}{f'(x_n)}, \tag{17}$$

where $\mu \in R, m > 0$.

We have the convergence analysis of the methods by (17).

Theorem 2.1. Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α, then the order of convergence of the methods defined by (17) is three, and the error equation

$$e_{n+1} = \left[\frac{c_2 \mu}{f'(\alpha)} - c_2^2 \right] e_n^3 + O(e_n^4). \tag{18}$$

Proof. Let $e_n = x_n - \alpha$, we use the following Taylor expansions:

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)], \tag{19}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$. Furthermore, we have

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5)].\tag{20}$$

Dividing (19) by (20),

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5).\tag{21}$$

From (21), we get

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2 c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5).\tag{22}$$

Expanding $f(w_n)$ in Taylor's Series about α and using (22), we get

$$\begin{aligned}
f(w_n) &= f'(\alpha)[w_n - \alpha + c_2(w_n - \alpha)^2 + c_3(w_n - \alpha)^3 + c_4(w_n - \alpha)^4 + \cdots]\\
&= f'(\alpha)[c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (5c_2^3 + 3c_4 - 7c_2 c_3)e_n^4 + O(e_n^5)].
\end{aligned}\tag{23}$$

Since (20), we obtain

$$\begin{aligned}
f'^2(x_n) =& f'^2(\alpha)[1 + 4c_2 e_n + (6c_3 + 4c_2^2)e_n^2 + (8c_4 + 12c_2 c_3)e_n^3\\
& + (10c_5 + 16c_2 c_4 + 9c_3^2)e_n^4 + O(e_n^5)].
\end{aligned}\tag{24}$$

Because of (19), we get

$$f^2(x_n) = f'^2(\alpha)[e_n^2 + 2c_2 e_n^3 + (2c_3 + c_2^2)e_n^4 + O(e_n^5)].\tag{25}$$

From (23) and (24), we get

$$\begin{aligned}
f'^2(x_n)f(w_n) =& f'^3(\alpha)[c_2 e_n^2 + (2c_3 + 2c_2^2)e_n^3 + (7c_2 c_3 + 3c_4 + c_2^3)e_n^4\\
& + (4c_5 + 4c_3 c_2^2 + 10c_2 c_4 + 6c_3^3)e_n^5 + O(e_n^6)].
\end{aligned}\tag{26}$$

From (20), (24), (25) and (26), we obtain

$$\begin{aligned}
L_{f,\mu}(x_n, w_n) =& \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) + \mu f^2(x_n) - f'^2(x_n)f(w_n)\right]} = 2c_2 e_n + \left(4c_3 - 4c_2^2 - \frac{2\mu c_2}{f'(\alpha)}\right)e_n^2\\
& + \left(6c_4 - 12c_2 c_3 + 6c_2^3 + \frac{8\mu c_2^2 - 4\mu c_3}{f'(\alpha)} + \frac{2\mu^2 c_2}{f'^2(\alpha)}\right)e_n^3 + \left(8c_5 + 22c_3 c_2^2 - 16c_2 c_4\right.\\
& \left. - 8c_3^2 - 6c_2^4 + \frac{26c_2 c_3\mu - 20\mu c_2^3 - 6c_4\mu}{f'(\alpha)} + \frac{4c_3\mu^2 - 12\mu^2 c_2^2}{f'^2(\alpha)} - \frac{2\mu^3 c_2}{f'^3(\alpha)}\right)e_n^4 + O(e_n^5).
\end{aligned}\tag{27}$$

Furthermore, from (27) we have

$$\begin{aligned}
\sum_{k\geq 0}^{m}\binom{\frac{1}{2}}{k+1}(-1)^k 2^{k+1}L_{f,\mu}(x_n, w_n)^k =& 1 + \frac{1}{2}L_{f,\mu}(x_n, w_n) + \frac{1}{2}L_{f,\mu}(x_n, w_n)^2 + \frac{5}{8}L_{f,\mu}(x_n, w_n)^3\\
& + \frac{7}{8}L_{f,\mu}(x_n, w_n)^4 + \cdots\\
=& 1 + c_2 e_n + \left(2c_3 - \frac{\mu c_2}{f'(\alpha)}\right)e_n^2 + \left(3c_4 + 2c_2 c_3 - \frac{2\mu c_3}{f'(\alpha)} + \frac{\mu^2 c_2}{f'^2(\alpha)}\right)e_n^3\\
& + \left(8c_3^2 - 10c_3 c_2^2 - \frac{16\mu c_2 c_3}{f'(\alpha)} - 10c_2^4 + \frac{9\mu c_2^3}{f'(\alpha)} + \frac{6\mu^2 c_2^2}{f'^2(\alpha)} + 12c_2 c_4\right)e_n^4\\
& + O(e_n^5).
\end{aligned}\tag{28}$$

Since (17) and (28), we have

$$x_{n+1} = x_n - \left(\sum_{k\geq 0}^{m}\binom{\frac{1}{2}}{k+1}(-1)^k 2^{k+1}L_{f,\mu}(x_n, w_n)^k\right)\frac{f(x_n)}{f'(x_n)}$$

$$= x_n - e_n - (c_2^2 - \frac{\mu c_2}{f'(\alpha)})e_n^3 + O(e_n^4), \tag{29}$$

from $e_{n+1} = x_{n+1} - \alpha$, we have

$$e_{n+1} = -\left(c_2^2 - \frac{\mu c_2}{f'(\alpha)}\right)e_n^3 + O(e_n^4). \tag{30}$$

Then the methods defined by (17) is shown to converge of the order three.

Similar to the proof of Theorem 2.1, we can prove that the methods defined by (12) and (15) are third-order methods.

Some special cases

1^0 : If $\mu = 0$, from (12) and (17) we obtain

$$L_{f,0}(x_n, w_n) = \frac{2f(w_n)}{\left[f(x_n) - f(w_n)\right]}, \tag{31}$$

$$x_{n+1} = x_n - \left(\sum_{k \geq 0}^{m} \binom{\frac{1}{2}}{k+1}(-1)^k 2^{k+1} L_{f,0}(x_n, w_n)^k\right)\frac{f(x_n)}{f'(x_n)}, \tag{32}$$

where $m > 0$. For $m = 2$, we obtain a third-order method(LM_1)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,0}(x_n, w_n) + \frac{1}{2}L_{f,0}(x_n, w_n)^2\right)\frac{f(x_n)}{f'(x_n)}. \tag{33}$$

For $m = 3$, we obtain from (17) a third-order method(LM_2)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,0}(x_n, w_n) + \frac{1}{2}L_{f,0}(x_n, w_n)^2 + \frac{5}{8}L_{f,0}(x_n, w_n)^3\right)\frac{f(x_n)}{f'(x_n)}. \tag{34}$$

2^0 : If $\mu = 1$, from (12) we obtain

$$L_{f,1}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) + f^2(x_n) - f'^2(x_n)f(w_n)\right]}. \tag{35}$$

For $m = 2$, we obtain from (17) a third-order method(LM_3)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,1}(x_n, w_n) + \frac{1}{2}L_{f,1}(x_n, w_n)^2\right)\frac{f(x_n)}{f'(x_n)}. \tag{36}$$

3^0 : If $\mu = -\frac{1}{2}$, from (12) we obtain

$$L_{f,-\frac{1}{2}}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) - \frac{1}{2}f^2(x_n) - f'^2(x_n)f(w_n)\right]}. \tag{37}$$

For $m = 2$, we obtain from (17) a third-order method(LM_4)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,-\frac{1}{2}}(x_n, w_n) + \frac{1}{2}L_{f,-\frac{1}{2}}(x_n, w_n)^2\right)\frac{f(x_n)}{f'(x_n)}. \tag{38}$$

4^0 : If $\mu = -1$, for $m = 2$, we obtain a third-order method (LM_5) from (12) and (17)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,-1}(x_n, w_n) + \frac{1}{2}L_{f,-1}(x_n, w_n)^2\right)\frac{f(x_n)}{f'(x_n)}. \tag{39}$$

5^0 : If $\mu = \frac{1}{2}$, from (12) and (15) we obtain some iterative methods as follows:

For $m = 1$, we obtain a third-order method

$$x_{n+1} = x_n - \frac{2}{2 - L_{f,\frac{1}{2}}(x_n, w_n)}\frac{f(x_n)}{f'(x_n)}. \tag{40}$$

For $m = 2$, we obtain a third-order method(LM_6)

$$x_{n+1} = x_n - \frac{4}{4 - 2L_{f,\frac{1}{2}}(x_n, w_n) - L_{f,\frac{1}{2}}(x_n, w_n)^2} \frac{f(x_n)}{f'(x_n)}. \qquad (41)$$

For $m = 3$, we obtain a third-order method

$$x_{n+1} = x_n - \frac{4}{4 - 2L_{f,\frac{1}{2}}(x_n, w_n) - L_{f,\frac{1}{2}}(x_n, w_n)^2 - L_{f,\frac{1}{2}}(x_n, w_n)^3} \frac{f(x_n)}{f'(x_n)}. \qquad (42)$$

3 Numerical examples of the third order methods

In this section, we present the results of numerical simulations in Table 2 to compare the efficiencies of the methods. The considered methods are Newton method (NM), the method of Weerakoon and Fernando [8] (WF), the method of Potra and Pták (PP) [9], Chebyshev's method (CHM) [11–12], Halley's method (HM) [11], and our new methods (33) (LM_1), (34) (LM_2), (36) (LM_3), (38) (LM_4), (39) (LM_5) and (41) (LM_6). Displayed in Table 2 are the number of iterations (IT), the number of function evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative, the absolute residual error of the corresponding function value ($|f(x_n)|$), the computing time (TIME, the unit of time is one second) and the distance of two consecutive approximations $\delta = |x_n - x_{n-1}|$. All computations were done using Matlab 7.1 environment with a ADM athlon(tm) II X2 250-3.01 GHz based PC. We accept an approximate solution rather than the exact root, depending on the precision ϵ of the computer. We use the following stopping criteria for computer programs: $|f(x_n)| < \epsilon$, we used the fixed stopping criterion $\epsilon = 10^{-15}$. " – " is divergence. We used the following test functions and display the computed approximate zero $x^\star$ in Table 1 [13].

4 Development of the optimal fourth order method and its convergence analysis

Corresponding to the well known Traub's method (see [14]), This scheme (17) with order of convergence three, is not optimal in the sense of Kung-Traub conjecture [14]. In this section, we introduce parametric weight functions and the well-known technique of undetermined coefficients to the family of iterative methods (17) to increase the order of convergence to four.

Table 1: Test functions and display the computed approximate zero $x^\star$.

Test functions	$x^\star$
$f_1(x) = x^3 + 4x^2 - 10$	1.3652300134140969
$f_2(x) = x^2 - e^x - 3x + 2$	0.25753028543986076
$f_3(x) = sin(x)e^x + ln(1 + x^2)$	0
$f_4(x) = (x - 1)^3 - 1$	2
$f_5(x) = cosx - x$	0.73908513321516067
$f_6(x) = sin^2x - x^2 + 1$	1.4044916482153411
$f_7(x) = e^{x^2+7x-30} - 1$	3

Table 2: Comparison of various third-order methods and Newton's method.

	IT	NFE	$|f(x_n)|$	TIME	δ
$f_1 : x_0 = 1$					
NM	5	10	0	0.046897	2.126987475037367e-011
WF	3	9	0	0.016892	2.284722713019605e-006
PP	4	12	0	0.033053	1.558753126573720e-013
CHM	4	12	0	0.033136	1.643130076445232e-014
HM	3	9	0	0.018876	3.698649917449615e-007
LM_1	3	9	0	0.017555	7.656778435505274e-006
LM_2	3	9	0	0.017310	6.519974116159233e-009
LM_3	3	9	0	0.016915	6.545952378145259e-006
LM_4	4	12	0	0.033269	2.220446049250313e-016
LM_5	4	12	0	0.032977	2.220446049250313e-016
LM_6	3	9	0	0.016579	1.582211815787105e-006
$f_1 : x_0 = 2$					
NM	5	10	0	0.048387	5.020497351182485e-010
WF	4	12	0	0.021366	4.440892098500626e-016
PP	4	12	0	0.033571	7.949196856316121e-014
CHM	4	12	0	0.033689	2.065014825802791e-014
HM	3	9	0	0.016968	3.107350415199051e-006
LM_1	3	9	0	0.017158	1.870204660026076e-007
LM_2	4	12	0	0.033365	2.220446049250313e-016
LM_3	3	9	0	0.016895	3.063923381674272e-008
LM_4	3	9	0	0.016990	3.636202978718472e-007
LM_5	3	9	0	0.017591	6.449765455052159e-007
LM_6	4	12	0	0.033007	2.220446049250313e-016
$f_2 : x_0 = 0$					
NM	4	8	0	0.026519	2.665312415217613e-012
WF	3	9	0	0.016351	7.801814749797131e-012
PP	3	9	0	0.016522	1.219191414492116e-012
CHM	3	9	0	0.018237	8.906764215055318e-013
HM	3	9	0	0.017423	7.374600929921371e-012
LM_1	3	9	0	0.017213	1.014743844507393e-013
LM_2	3	9	0	0.017442	1.497690860219336e-013
LM_3	3	9	0	0.017380	3.035904860837491e-013
LM_4	3	9	0	0.016920	2.620348382720295e-012
LM_5	3	9	0	0.016538	1.656591530618812e-011
LM_6	2	6	0	0.000403	1.015871229748111e-005
$f_2 : x_0 = 0.5$					
NM	4	8	0	0.040693	1.791899961745003e-013
WF	3	9	0	0.017298	6.424749621203318e-012
PP	3	9	0	0.016290	4.607425552194400e-014
CHM	3	9	0	0.017653	3.087480271446452e-011
HM	3	9	0	0.018192	4.208039472430869e-011
LM_1	3	9	0	0.017986	1.054711873393899e-015
LM_2	3	9	0	0.017973	7.216449660063518e-016
LM_3	3	9	0	0.017978	1.497135748707024e-013
LM_4	3	9	0	0.017530	9.942047185518277e-014
LM_5	3	9	0	0.016987	7.234768339969833e-013
LM_6	3	9	0	0.016680	1.887379141862766e-015

$f_3 : x_0 = 1$

NM	7	14	3.537126081266182e-024	0.076967	1.085848323840232e-012
WF	4	12	2.621304391538411e-016	0.031335	4.330310691887267e-006
PP	5	15	8.196910187379942e-033	0.050273	8.806888499109001e-012
CHM	5	15	6.352230116524407e-022	0.051221	2.520356663650445e-011
HM	5	15	7.257520328033309e-029	0.054265	8.459855063117184e-015
LM_1	4	12	8.271806125530277e-025	0.035949	3.479185746468363e-009
LM_2	4	12	0	0.034977	1.804501237019987e-013
LM_3	4	12	0	0.033510	3.732361177500448e-009
LM_4	4	12	0	0.033691	6.861191797005728e-010
LM_5	4	12	0	0.032995	3.650246134545045e-015
LM_6	4	12	0	0.033798	2.539428973634579e-010

$f_3 : x_0 = 0.5$

NM	6	12	5.905159674954809e-020	0.060337	1.402992074360412e-010
WF	4	12	1.764824578467612e-017	0.032156	4.200981459101664e-009
PP	4	12	1.694834561079519e-019	0.034667	2.767209186089879e-007
CHM	4	12	8.798671206634291e-017	0.033659	3.059650585008672e-007
HM	4	12	3.044907255908736e-017	0.035225	5.518067735074518e-009
LM_1	4	12	0	0.034032	4.539201791677570e-014
LM_2	4	12	0	0.035989	5.535242064507416e-014
LM_3	4	12	0	0.032770	2.221451036901571e-013
LM_4	3	9	1.163082115698561e-019	0.016919	2.852561908633870e-007
LM_5	4	12	0	0.034210	6.743878120329082e-014
LM_6	3	9	0	0.017372	1.053838990356347e-011

$f_4 : x_0 = 2.5$

NM	6	12	0	0.055351	1.154631945610163e-014
WF	4	12	0	0.031820	7.314593375440381e-012
PP	4	12	0	0.032233	4.221685223626537e-010
CHM	4	12	0	0.033641	9.853584614916144e-011
HM	4	12	0	0.033511	4.662936703425658e-014
LM_1	3	9	6.661338147750939e-016	0.017800	1.544537542308433e-008
LM_2	4	12	0	0.033585	2.244870955792067e-013
LM_3	3	9	0	0.016430	6.254473188249676e-007
LM_4	3	9	0	0.016473	1.067152234579538e-006
LM_5	4	12	0	0.032690	4.440892098500626e-016
LM_6	4	12	0	0.032958	2.042810365310288e-014

$f_4 : x_0 = 3.5$

NM	7	14	0	0.086241	2.877564853065451e-011
WF	5	15	0	0.049605	6.550315845288424e-013
PP	5	15	0	0.048335	4.512221707386743e-010
CHM	5	15	0	0.049367	4.188738245147761e-011
HM	4	12	0	0.033641	4.485352507632712e-006
LM_1	4	12	0	0.035118	1.079692646399622e-008
LM_2	4	12	0	0.034134	7.838174553853605e-013
LM_3	4	12	0	0.035020	3.379705404427114e-010
LM_4	4	12	0	0.033277	1.283696526854783e-008
LM_5	4	12	0	0.033268	8.547096808086963e-009
LM_6	4	12	0	0.032565	8.725183908708800e-009

$f_5 : x_0 = 0$

NM	5	10	0	0.049831	1.701233598438989e-010
WF	3	9	0	0.017922	7.792236328407753e-007
PP	4	12	0	0.032661	1.500558566291943e-010
CHM	4	12	0	0.033834	5.327979279989847e-009
HM	4	12	0	0.032642	1.121325254871408e-014
LM_1	4	12	0	0.033695	3.819167204710539e-014
LM_2	3	9	0	0.017802	8.247395144600489e-008
LM_3	4	12	0	0.037679	1.818811767861917e-011
LM_4	4	12	0	0.033281	8.344436253082677e-013
LM_5	4	12	0	0.033370	8.471505719143124e-010
LM_6	4	12	0	0.032211	2.348121697082206e-013

$f_5 : x_0 = 1$

NM	4	8	0	0.030217	1.701233598438989e-010
WF	2	6	4.440892098500626e-016	0.003077	2.674277017133964e-005
PP	3	9	0	0.016561	9.809075773858922e-011
CHM	3	9	0	0.018428	1.600380383770528e-009
HM	3	9	0	0.018336	6.624212289807474e-010
LM_1	3	9	0	0.017117	2.252753539266905e-012
LM_2	3	9	0	0.016831	4.671929509925121e-012
LM_3	3	9	0	0.018593	2.668459120336308e-009
LM_4	3	9	0	0.016778	1.749711486809247e-012
LM_5	3	9	0	0.016682	1.375262126401822e-010
LM_6	3	9	0	0.017519	3.148793448204401e-010

$f_6 : x_0 = 1$

NM	6	12	3.330669073875470e-016	0.064253	3.059774655866931e-013
WF	4	12	4.440892098500626e-016	0.032786	1.793023507445923e-010
PP	16	48	4.440892098500626e-016	0.246502	1.531728257564424e-007
CHM	5	15	4.440892098500626e-016	0.050039	6.883094094689568e-010
HM	4	12	4.440892098500626e-016	0.038356	2.686739719592879e-013
LM_1	4	12	3.330669073875470e-016	0.034670	1.042735342515755e-008
LM_2	4	12	3.330669073875470e-016	0.034475	7.038286398142191e-009
LM_3	4	12	4.440892098500626e-016	0.034466	3.420013050536852e-008
LM_4	4	12	3.330669073875470e-016	0.033590	1.918714076509787e-010
LM_5	4	12	4.440892098500626e-016	0.034618	1.002852445530778e-007
LM_6	4	12	3.330669073875470e-016	0.033031	6.483733550055604e-010

$f_6 : x_0 = 2.5$

NM	6	12	3.330669073875470e-016	0.060863	1.404654170755748e-012
WF	4	12	3.330669073875470e-016	0.033203	4.229505634611996e-012
PP	4	12	3.330669073875470e-016	0.033607	1.030850205196998e-008
CHM	4	12	3.330669073875470e-016	0.034233	1.475204565171140e-007
HM	4	12	4.440892098500626e-016	0.033904	9.462626682221753e-009
LM_1	4	12	4.440892098500626e-016	0.034679	1.265654248072679e-014
LM_2	3	9	3.330669073875470e-016	0.016425	1.176158348492606e-008
LM_3	4	12	4.440892098500626e-016	0.033903	4.662936703425658e-015
LM_4	4	12	3.330669073875470e-016	0.033077	1.501021529293212e-013
LM_5	4	12	4.440892098500626e-016	0.033523	8.837375276016246e-014
LM_6	4	12	3.330669073875470e-016	0.033035	2.375877272697835e-014

$f_7 : x_0 = 3.25$					
NM	8	16	0	0.102321	9.720393379097914e-010
WF	6	18	0	0.066815	1.691979889528739e-013
PP	6	18	0	0.068323	1.131490456884876e-010
CHM	6	18	0	0.069897	2.398081733190338e-014
HM	5	15	0	0.052332	3.082423205569285e-012
LM_1	4	12	0	0.034222	1.781058993621798e-007
LM_2	4	12	0	0.033930	5.758273724509877e-008
LM_3	4	12	0	0.033957	2.094845084066321e-007
LM_4	4	12	0	0.033676	1.635068742622536e-007
LM_5	4	12	0	0.033192	1.496285984003976e-007
LM_6	-	-	-	-	-
$f_7 : x_0 = 3.45$					
NM	11	22	0	0.152806	4.008793297316515e-011
WF	8	24	0	0.101062	7.105427357601002e-015
PP	8	24	0	0.102346	2.160227552394645e-010
CHM	7	21	0	0.096244	1.268533917908599e-006
HM	6	18	0	0.062830	1.694565332499565e-008
LM_1	6	18	0	0.067983	3.221867217462204e-012
LM_2	5	15	0	0.051391	7.682743330406083e-014
LM_3	6	18	0	0.066789	3.313793683901167e-012
LM_4	6	18	0	0.066753	3.173461493588548e-012
LM_5	6	18	0	0.068585	3.125055769714891e-012
LM_6	-	-	-	-	-

We consider using a weight function $H(\mu(x_n, w_n, \gamma_i))$ instead of μ in the operator (12), and consider the well-known technique of undetermined coefficients to design an new operator $L_{f,H,\tilde{\mu}}(x_n, w_n)$ as follows

$$L_{f,H,\tilde{\mu}}(x_n, w_n) = \frac{2\mu_1 f'^2(x_n)f(w_n)}{\left[\mu_2 f'^2(x_n)f(x_n) + \mu_3 H(\mu(x_n, w_n, \gamma_i))f^2(x_n) - \mu_4 f'^2(x_n)f(w_n)\right]}, \tag{43}$$

where $H(\mu(x_n, w_n, \gamma_i))$ is a function of real variable

$$\mu(x_n, w_n, \gamma_i) = \frac{\gamma_1 f(x_n) + \gamma_2 f(w_n)}{1 + \gamma_3 f(x_n) + \gamma_4 f(w_n)}, \tag{44}$$

$\gamma_i(i = 1, \ldots, 4)$ and $\mu_j(j = 1, \ldots, 4)$ are real parameters. Then, using (43) in (17), we also can construct two new optimal fourth-order family of modified Cauchy methods as follows:

$$x_{n+1} = x_n - \frac{2}{1 + \sum_{k\geq0}^{m} \binom{\frac{1}{2}}{k}(-2L_{f,H,\tilde{\mu}}(x_n, w_n))^k} \frac{f(x_n)}{f'(x_n)}, \tag{45}$$

and

$$x_{n+1} = x_n - \left(\sum_{k\geq0}^{m} \binom{\frac{1}{2}}{k+1}(-1)^k 2^{k+1} L_{f,H,\tilde{\mu}}(x_n, w_n)^k \right) \frac{f(x_n)}{f'(x_n)}, \tag{46}$$

where $m > 0$.

In the following result, we present the conditions that the weight function $H(\mu(x_n, w_n, \gamma_i))$ and the parameters must satisfy for obtaining two families of iterative methods with fourth-order of convergence, which becoming optimal schemes by Kung-Traub conjecture.

Theorem 3.1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open interval I, such that $\alpha \in I$ is a simple solution of the nonlinear equation $f(x) = 0$. Let $H : \mathbb{R} \to \mathbb{R}$ be any sufficiently differentiable function satisfying $H(0) = 0$, $| H'(0) | < \infty$, $| H''(0) | < \infty$. If x_0 is close enough to α, $\mu_1 = \mu_2 \neq 0$, and $\mu_4 = 0$, then the method defined by (46) has fourth-order of convergence and its error equation is:

$$e_{n+1} = \frac{1}{\mu_2} c_2 \left(\mu_3 H'(0)\gamma_1 - \mu_2 c_3 \right) e_n^4 + O(e_n^5). \tag{47}$$

Proof. Let $e_n = x_n - \alpha$, because of the Taylor series expansions of $f(x_n)$ and $f(w_n)$, we have

$$
\begin{aligned}
\mu(x_n, w_n, \gamma_i) = &\frac{\gamma_1 f(x_n) + \gamma_2 f(w_n)}{1 + \gamma_3 f(x_n) + \gamma_4 f(w_n)} \\
= &f'(\alpha)\gamma_1 e_n + \left(f'(\alpha)(\gamma_1 c_2 + \gamma_2 c_2) - f'^2(\alpha)\gamma_1\gamma_3 \right) e_n^2 + \Big(\frac{1}{2} f'(\alpha)(2\gamma_1 c_3 - 4\gamma_2 c_2^2 + 4\gamma_2 c_3) \\
&- f'^2(\alpha)(\gamma_1 c_2 + \gamma_2 c_2)\gamma_3 + f'^3(\alpha)\gamma_1\gamma_3^2 - \frac{1}{2}f'(\alpha)\gamma_1(2\gamma_3 f'(\alpha)c_2 + 2\gamma_4 f'(\alpha)c_2) \Big) e_n^3 \\
&+ \Big(-\frac{1}{6}f'(\alpha)\gamma_1(6\gamma_3 f'(\alpha)c_3 + 12\gamma_4 f'(\alpha)c_3 - 12\gamma_4 f'(\alpha)c_2^2) + \frac{1}{6}f'(\alpha)(30\gamma_2 c_2^3 + 6\gamma_1 c_4 \\
&- 42\gamma_2 c_2 c_3 + 18\gamma_2 c_4) - \frac{1}{2}f'^2(\alpha)(2\gamma_1 c_3 - 4\gamma_2 c_2^2 + 4\gamma_2 c_3)\gamma_3 + f'^3(\alpha)(\gamma_1 c_2 + \gamma_2 c_2)\gamma_3^2 \\
&- f'^4(\alpha)\gamma_1\gamma_3^3 + f'^2(\alpha)\gamma_1\gamma_3(2\gamma_3 f'(\alpha)c_2 + 2\gamma_4 f'(\alpha)c_2) - \frac{1}{2}f'(\alpha)(\gamma_1 c_2 + \gamma_2 c_2)(2\gamma_3 f'(\alpha)c_2 \\
&+ 2\gamma_4 f'(\alpha)c_2) \Big) e_n^4 + O(e_n^5). \tag{48}
\end{aligned}
$$

Taking into account the expansion of $\mu(x_n, w_n, \gamma_i)$, and by using Taylor series expansion of $H(\mu(x_n, w_n, \gamma_i))$ around 0, we obtain

$$
\begin{aligned}
H(\mu(x_n, w_n, \gamma_i)) = &H(0) + H'(0)\mu(x_n, w_n, \gamma_i) + \frac{H''(0)}{2!}\mu^2(x_n, w_n, \gamma_i) + \frac{H'''(0)}{3!}\mu^3(x_n, w_n, \gamma_i) + O(\mu^4(x_n, w_n, \gamma_i)) \\
= &H(0) + f'(\alpha)H'(0)\gamma_1 e_n + \Big(H'(0)f'(\alpha)\gamma_1 c_2 + \frac{1}{2}H''(0)f'^2(\alpha)\gamma_1^2 + H'(0)f'(\alpha)\gamma_2 c_2 \\
&- H'(0)f'^2(\alpha)\gamma_1\gamma_3 \Big) e_n^2 + \Big(H'(0)f'^3(\alpha)\gamma_1\gamma_3^2 - 2H'(0)f'(\alpha)\gamma_2 c_2^2 + H'(0)f'(\alpha)\gamma_1 c_3 \\
&+ H''(0)f'^2(\alpha)\gamma_2 c_2\gamma_1 - 2H'(0)f'^2(\alpha)\gamma_1\gamma_3 c_2 - H'(0)f'^2(\alpha)\gamma_3\gamma_2 c_2 - H'(0)f'^2(\alpha)\gamma_1\gamma_4 c_2 \\
&+ H''(0)f'^2(\alpha)\gamma_1^2 c_2 + \frac{1}{6}H'''(0)f'^3(\alpha)\gamma_1^3 + 2H'(0)f'(\alpha)\gamma_2 c_3 - H''(0)f'^3(\alpha)\gamma_1^2\gamma_3 \Big) e_n^3 + O(e_n^4). \tag{49}
\end{aligned}
$$

From (20), (24), (25), (26) and (43), we obtain

$$
\begin{aligned}
L_{f,H,\tilde{\mu}}(x_n, w_n) = &\frac{2\mu_1 f'^2(x_n)f(w_n)}{\left[\mu_2 f'^2(x_n)f(x_n) + \mu_3 H(\mu(x_n, w_n, \gamma_i))f^2(x_n) - \mu_4 f'^2(x_n)f(w_n) \right]} \\
= &2\frac{\mu_1}{\mu_2}c_2 e_n + \Big(8\frac{\mu_1}{\mu_2}c_2^2 + 2\frac{\mu_1}{\mu_2}(2c_3 - 2c_2^2) + \frac{1}{3}\frac{\mu_1}{f'(\alpha)\mu_2^2}c_2(6f'(\alpha)\mu_4 c_2 - 30\mu_2 f'(\alpha)c_2 \\
&- 6\mu_3 H(0)) \Big) e_n^2 + \Big(8\frac{\mu_1}{\mu_2}c_2^3 + 8\frac{\mu_1}{\mu_2}(2c_3 - 2c_2^2)c_2 + \frac{4}{3}\frac{\mu_1}{f'(\alpha)\mu_2^2}c_2^2(6f'(\alpha)\mu_4 c_2 - 30\mu_2 f'(\alpha)c_2 \\
&- 6\mu_3 H(0)) + 12\frac{\mu_1}{\mu_2}c_2 c_3 + \frac{\mu_1}{\mu_2}(10c_2^3 - 14c_2 c_3 + 6c_4) + \frac{1}{3}\frac{\mu_1}{f'(\alpha)\mu_2^2}(2c_3 - 2c_2^2)(6f'(\alpha)\mu_4 c_2 \\
&- 30\mu_2 f'(\alpha)c_2 - 6\mu_3 H(0)) + \frac{1}{18}\frac{\mu_1}{f'^2(\alpha)\mu_2^3}c_2(6f'(\alpha)\mu_4 c_2 - 30\mu_2 f'(\alpha)c_2 - 6\mu_3 H(0))^2 \\
&+ \frac{1}{6}\frac{\mu_1}{f'(\alpha)\mu_2^2}c_2(-12f'(\alpha)\mu_3 H'(0)\gamma_1 - 84f'(\alpha)\mu_2 c_3 - 24\mu_3 H(0)c_2 + 24f'(\alpha)\mu_4 c_2^2 \\
&- 96f'(\alpha)\mu_2 c_2^2 + 24f'(\alpha)\mu_4 c_3) \Big) e_n^3 + O(e_n^4). \tag{50}
\end{aligned}
$$

Substituting (50) into (46), we have

$$x_{n+1} = x_n - \left(\sum_{k \geq 0}^{m} \binom{\frac{1}{2}}{k+1}(-1)^k 2^{k+1} L_{f,H,\tilde{\mu}}(x_n, w_n)^k \right) \frac{f(x_n)}{f'(x_n)}$$

$$
= x_n - \left(1 + \frac{1}{2}L_{f,H,\tilde{\mu}}(x_n, w_n) + \frac{1}{2}L_{f,H,\tilde{\mu}}(x_n, w_n)^2 + \frac{5}{8}L_{f,H,\tilde{\mu}}(x_n, w_n)^3 + \frac{7}{8}L_{f,H,\tilde{\mu}}(x_n, w_n)^4 + \cdots\right)\frac{f(x_n)}{f'(x_n)}
$$

$$
= x_n - e_n - \left(\frac{\mu_1}{\mu_2}c_2 - c_2\right)e_n^2 - \left(3\frac{\mu_1}{\mu_2}c_2^2 + \frac{\mu_1}{\mu_2}(2c_3 - 2c_2^2) + \frac{1}{6}\frac{\mu_1}{f'(\alpha)\mu_2^2}c_2(6f'(\alpha)\mu_4 c_2 - 30f'(\alpha)\mu_2 c_2\right.
$$

$$
- 6\mu_3 H(0)) + 2\frac{\mu_1^2}{\mu_2^2}c_2^2 - 2c_3 + 2c_2^2\bigg)e_n^3 - \frac{1}{f'^2(\alpha)\mu_2^3}\left(c_2^3 f'^2(\alpha)(13\mu_1\mu_2^2 + \mu_1\mu_4^2 - 14\mu_1^2\mu_2 + 4\mu_1^2\mu_4 - 4\mu_2^3\right.
$$

$$
+ 5\mu_1^3 - 7\mu_1\mu_2\mu_4) + f'^2(\alpha)\mu_2 c_2 c_3(7\mu_2^2 - 14\mu_1\mu_2 + 4\mu_1\mu_4 + 8\mu_1^2) + 3c_4 f'(\alpha)^2\mu_2^2(-\mu_2 + \mu_1)
$$

$$
+ \mu_1 c_2\mu_3(\mu_3 H(0)^2 - f'^2(\alpha)\mu_2 H'(0)\gamma_1) - \mu_1 c_2^2 f'(\alpha)\mu_3 H(0)(-7\mu_2 + 4\mu_1 + 2\mu_4)
$$

$$
- 2\mu_1 f'(\alpha)\mu_2 c_3\mu_3 H(0)\bigg)e_n^4 + O(e_n^5). \tag{51}
$$

From $e_{n+1} = x_{n+1} - \alpha$, we consider that if $H(0) = 0$, $\mu_1 = \mu_2$, $\mu_4 = 0$, Then, we obtain the error equation of (46) in the form:

$$
e_{n+1} = \frac{1}{\mu_2}c_2\left(\mu_3 H'(0)\gamma_1 - \mu_2 c_3\right)e_n^4 + O(e_n^5). \tag{52}
$$

Then the methods defined by (46) is shown to converge of the order four. □

Similar to the proof of Theorem 3.1, we can prove that the methods defined by (43) and (45) are fourth-order methods.

When $H(0) = 0$, $\mu_1 = \mu_2$, $\mu_4 = 0$, we obtain

$$
L_{f,H,\tilde{\mu}}(x_n, w_n) = \frac{2\mu_1 f'^2(x_n)f(w_n)}{\left[\mu_1 f'^2(x_n)f(x_n) + \mu_3 H(\mu(x_n, w_n, \gamma_i))f^2(x_n)\right]}. \tag{53}
$$

Let $\lambda = \frac{\mu_3}{\mu_1}$ in (53), we have

$$
L_{f,H,\lambda}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) + \lambda H(\mu(x_n, w_n, \gamma_i))f^2(x_n)\right]}. \tag{54}
$$

From the expansion of $L_{f,H,\lambda}(x_n, w_n)$ in (54), we can obtain following members of family (45) and (46).

Some special cases

1^0 : If we consider the following weight function $H_1 = H(\mu(x_n, w_n, \gamma_i)) = 0$, from (46) and (54) we obtain

$$
L_{f,H_1,\lambda} = \frac{2f(w_n)}{f(x_n)}, \tag{55}
$$

$$
x_{n+1} = x_n - \left(\sum_{k \geq 0}^{m} \binom{\frac{1}{2}}{k+1}(-1)^k 2^{k+1}L_{f,H_1,\lambda}(x_n, w_n)^k\right)\frac{f(x_n)}{f'(x_n)}, \tag{56}
$$

where $m > 0$. For $m = 2$, we obtain a recently developed fourth-order method by Khattri et al. (KM$_1$) [13]

$$
x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,H_1,\lambda}(x_n, w_n) + \frac{1}{2}L_{f,H_1,\lambda}(x_n, w_n)^2\right)\frac{f(x_n)}{f'(x_n)}
$$

$$
= x_n - \left(1 + \frac{f(w_n)}{f(x_n)} + \frac{2f^2(w_n)}{f^2(x_n)}\right)\frac{f(x_n)}{f'(x_n)}. \tag{57}
$$

For $m = 3$, we also get the existing optimal fourth-order method by Khattri et al. (KM$_2$) [13]

$$
x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,H_1,\lambda}(x_n, w_n) + \frac{1}{2}L_{f,H_1,\lambda}(x_n, w_n)^2 + \frac{5}{8}L_{f,H_1,\lambda}(x_n, w_n)^3\right)\frac{f(x_n)}{f'(x_n)}
$$

$$
= x_n - \left(1 + \frac{f(w_n)}{f(x_n)} + \frac{2f^2(w_n)}{f^2(x_n)} + \frac{5f^3(w_n)}{f^3(x_n)}\right)\frac{f(x_n)}{f'(x_n)}. \tag{58}
$$

For $m = 4$, we obtain the developed fourth-order method by Khattri et al. (KM$_3$) [13], which is given by

$$
x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,H_1,\lambda}(x_n, w_n) + \frac{1}{2}L_{f,H_1,\lambda}(x_n, w_n)^2 + \frac{5}{8}L_{f,H_1,\lambda}(x_n, w_n)^3 + \frac{7}{8}L_{f,H_1,\lambda}(x_n, w_n)^4\right)\frac{f(x_n)}{f'(x_n)}
$$

$$= x_n - \left(1 + \frac{f(w_n)}{f(x_n)} + \frac{2f^2(w_n)}{f^2(x_n)} + \frac{5f^3(w_n)}{f^3(x_n)} + \frac{14f^4(w_n)}{f^4(x_n)}\right)\frac{f(x_n)}{f'(x_n)}. \tag{59}$$

2^0 : Now, we consider the following weight function, which also satisfies all the conditions of Theorem 3.1. If $H_2 = H(\mu(x_n, w_n, \gamma_i)) = \frac{\mu(x_n, w_n, \gamma_i)^2}{1 - \mu(x_n, w_n, \gamma_i)}$, $\gamma_1 = 0$, $\gamma_2 = \gamma_3 = \gamma_4 = 1$ and $\lambda = 1$, from (43) and (54), we have

$$L_{f,H_2,1}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) + H_2 f^2(x_n)\right]}. \tag{60}$$

For $m = 2$, from (46) and (60) we obtain a new fourth-order method (LTM$_1$)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,H_2,1}(x_n, w_n) + \frac{1}{2}L_{f,H_2,1}(x_n, w_n)^2\right)\frac{f(x_n)}{f'(x_n)}. \tag{61}$$

For $m = 3$, we obtain a new fourth-order method (LTM$_2$)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,H_2,1}(x_n, w_n) + \frac{1}{2}L_{f,H_2,1}(x_n, w_n)^2 + \frac{5}{8}L_{f,H_2,1}(x_n, w_n)^3\right)\frac{f(x_n)}{f'(x_n)}. \tag{62}$$

For $m = 4$, we obtain a new fourth-order method

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,H_2,1}(x_n, w_n) + \frac{1}{2}L_{f,H_2,1}(x_n, w_n)^2 + \frac{5}{8}L_{f,H_2,1}(x_n, w_n)^3 + \frac{7}{8}L_{f,H_2,1}(x_n, w_n)^4\right)\frac{f(x_n)}{f'(x_n)}. \tag{63}$$

For $m = 2$, from (44) and (60) we obtain a new fourth-order method (LTM$_3$)

$$x_{n+1} = x_n - \frac{4}{4 - 2L_{f,H_2,1}(x_n, w_n) - L_{f,H_2,1}(x_n, w_n)^2}\frac{f(x_n)}{f'(x_n)}. \tag{64}$$

For $m = 3$, we obtain a new fourth-order method (LTM$_4$)

$$x_{n+1} = x_n - \frac{4}{4 - 2L_{f,H_2,1}(x_n, w_n) - L_{f,H_2,1}(x_n, w_n)^2 - L_{f,H_2,1}(x_n, w_n)^3}\frac{f(x_n)}{f'(x_n)}. \tag{65}$$

3^0 : If $H_3 = H(\mu(x_n, w_n, \gamma_i)) = \mu(x_n, w_n, \gamma_i)$, $\gamma_1 = -1$, $\gamma_2 = 1$, $\gamma_3 = -1$, $\gamma_4 = 1$ and $\lambda = -\frac{2}{3}$, from (43) and (54), we have

$$L_{f,H_3,-\frac{2}{3}}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) - \frac{2}{3}H_3 f^2(x_n)\right]}. \tag{66}$$

For $m = 2$, from (45) and (66) we obtain a new fourth-order method (LTM$_5$)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,H_3,-\frac{2}{3}}(x_n, w_n) + \frac{1}{2}L_{f,H_3,-\frac{2}{3}}(x_n, w_n)^2\right)\frac{f(x_n)}{f'(x_n)}. \tag{67}$$

For $m = 3$, we obtain a new fourth-order method (LTM$_6$)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f,H_3,-\frac{2}{3}}(x_n, w_n) + \frac{1}{2}L_{f,H_3,-\frac{2}{3}}(x_n, w_n)^2 + \frac{5}{8}L_{f,H_3,-\frac{2}{3}}(x_n, w_n)^3\right)\frac{f(x_n)}{f'(x_n)}. \tag{68}$$

For $m = 2$, from (44) and (66) we obtain a new fourth-order method

$$x_{n+1} = x_n - \frac{4}{4 - 2L_{f,H_3,-\frac{2}{3}}(x_n, w_n) - L_{f,H_3,-\frac{2}{3}}(x_n, w_n)^2}\frac{f(x_n)}{f'(x_n)}. \tag{69}$$

For $m = 3$, we obtain a new fourth-order method

$$x_{n+1} = x_n - \frac{4}{4 - 2L_{f,H_3,-\frac{2}{3}}(x_n, w_n) - L_{f,H_3,-\frac{2}{3}}(x_n, w_n)^2 - L_{f,H_3,-\frac{2}{3}}(x_n, w_n)^3}\frac{f(x_n)}{f'(x_n)}. \tag{70}$$

4^0 : If $H_4 = H(\mu(x_n, w_n, \gamma_i)) = \frac{\mu(x_n, w_n, \gamma_i)}{1 - 2\mu(x_n, w_n, \gamma_i)}$, $\gamma_1 = -1, \gamma_2 = 1, \gamma_3 = -\frac{4}{5}, \gamma_4 = 2$ and $\lambda = \frac{1}{2}$, from (43) and (54), we have

$$L_{f, H_4, \frac{1}{2}}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{\left[f'^2(x_n)f(x_n) + \frac{1}{2}H_4 f^2(x_n)\right]}. \tag{71}$$

For $m = 2$, from (46) and (71) we obtain a new fourth-order method

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f, H_4, \frac{1}{2}}(x_n, w_n) + \frac{1}{2}L_{f, H_4, \frac{1}{2}}(x_n, w_n)^2\right)\frac{f(x_n)}{f'(x_n)}. \tag{72}$$

For $m = 3$, we obtain a new fourth-order method (LTM$_7$)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f, H_4, \frac{1}{2}}(x_n, w_n) + \frac{1}{2}L_{f, H_4, \frac{1}{2}}(x_n, w_n)^2 + \frac{5}{8}L_{f, H_4, \frac{1}{2}}(x_n, w_n)^3\right)\frac{f(x_n)}{f'(x_n)}. \tag{73}$$

For $m = 2$, from (45) and (71) we obtain a new fourth-order method (LTM$_8$)

$$x_{n+1} = x_n - \frac{4}{4 - 2L_{f, H_4, \frac{1}{2}}(x_n, w_n) - L_{f, H_4, \frac{1}{2}}(x_n, w_n)^2}\frac{f(x_n)}{f'(x_n)}. \tag{74}$$

For $m = 3$, we obtain a new fourth-order method

$$x_{n+1} = x_n - \frac{4}{4 - 2L_{f, H_4, \frac{1}{2}}(x_n, w_n) - L_{f, H_4, \frac{1}{2}}(x_n, w_n)^2 - L_{f, H_4, \frac{1}{2}}(x_n, w_n)^3}\frac{f(x_n)}{f'(x_n)}. \tag{75}$$

5 Numerical examples of the optimal fourth order methods

This section is devoted to verify the validity and effectiveness of our theoretical results which we have proposed earlier. We present the numerical results obtained by applying the proposed methods on some scalar equations. We are going to compare LTM$_1$-LTM$_8$ with some fourth-order known methods as Kung-Traub scheme [14]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f^2(x_n)}{(f(x_n) - f(y_n))^2}\frac{f(y_n)}{f'(x_n)}, \tag{76}$$

denoted by KTM; three methods of Khattri et al. [13], denoted by KM$_1$, KM$_2$ and KM$_3$ (see (57), (58) and (59) in special cases of fourth order methods); the fourth-order method by Chun [15] denoted by CM$_1$; two fourth-order methods by Chun and Ham [16] denoted by CM$_2$, CM$_3$; the fourth-order method by Kou et al. [17] denoted by NSPP; the fourth-order method by Sharma and Bahl [18] denoted by SBM, which the method are applied to solve systems of nonlinear equations by Sharma et al. [19]. The CM$_1$ is given as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - 2\frac{f^2(x_n)}{f'(x_n)[f(x_n) - f(y_n)]} + \frac{f(x_n)}{f'(x_n)} + \frac{f'(x_n)f(y_n)}{f^2(x_n) + f'^2(x_n)}. \tag{77}$$

The CM$_2$ is defined as

$$y_n = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) - f(x_n)},$$

$$L_n = f'^2(x_n) - f(x_n),$$

$$x_{n+1} = x_n - \left[\frac{(f(y_n) - f(x_n))L_n^2 - f^2(x_n)f'^2(x_n)}{2(f(y_n) - f(x_n))L_n^2 + f(x_n)f'^2(x_n)(f'^2(x_n) - 2f(x_n))} \right] \frac{f(x_n)}{f'(x_n)}, \quad (78)$$

and CM_3 is

$$y_n = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) + 100f(x_n)},$$

$$L_n = f'^2(x_n) + 100f(x_n),$$

$$x_{n+1} = x_n - \left[\frac{(f(y_n) - f(x_n))L_n^2 + 100f^2(x_n)f'^2(x_n)}{2(f(y_n) - f(x_n))L_n^2 + f(x_n)f'^2(x_n)(f'^2(x_n) + 200f(x_n))} \right] \frac{f(x_n)}{f'(x_n)}. \quad (79)$$

The NSPP is given as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f^2(x_n) + f^2(y_n)}{f'(x_n)(f(x_n) - f(y_n))}. \quad (80)$$

The SBM is defined as

$$y_n = x_n - \frac{2}{3} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \left[\frac{-1}{2} + \frac{9}{8} \frac{f'(x_n)}{f'(y_n)} + \frac{3}{8} \frac{f'(y_n)}{f'(x_n)} \right] \frac{f(x_n)}{f'(x_n)}. \quad (81)$$

Displayed in Table 3 are the number of iterations (IT), the number of function evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative, the absolute residual error of the corresponding function value ($|f(x_n)|$), the distance of two consecutive approximations $\delta = |x_n - x_{n-1}|$, and the computational approximate order of convergence ρ, which is an approximation of the theoretical order of convergence, introduced in [20] as

$$p \approx \rho = \frac{ln(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}{ln(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|)}. \quad (82)$$

In this section, the computations were done using Matlab 7.1 environment. We accept an approximate solution rather than the exact root, depending on the precision ϵ of the computer. We use the following stopping criteria for computer programs: $|f(x_n)| < \epsilon$, we used the fixed stopping criterion $\epsilon = 10^{-15}$. "–" is divergence. "~" means that it converges to other solutions. We used the test functions and display the computed approximate zero x^* in Table 1.

From Table 3, it is clear that CM_1, CM_2, LTM_4, SBM, LTM_8, NSPP and LTM_7 require less number of iterations (IT) and function evaluations (NFE) in the corresponding test function $f_1(x)$ compared with the other fourth-order methods, especially the method CM_1 performs best in terms of convergence.

In test function $f_2(x)$, the methods KM_3, CM_1 and LTM_3 have better performances. The numerical results also show that KM_3 have smaller residual error in the corresponding function $|f_2(x_n)|$ compared with LTM_3. The existing method CM_3 fails in convergence for the case $f_2(x)$.

Regarding the results of test function $f_3(x)$, we claim that our methods and the existing fourth-order methods have almost similar performance.

From the results of the test function $f_4(x)$, our methods LTM_2, LTM_4, LTM_6, LTM_7, LTM_8 and the existing method KM_2, KM_3, CM_1, CM_2, CM_3 require less number of iterations (IT) and function evaluations (NFE) than other methods, which demonstrate that several of our methods converge faster than some existing ones.

In test function $f_5(x)$, the methods KM_2, CM_2, CM_3 and our method LTM_2 have better performances in terms of the speed of convergence. The existing methods KM_3 and SBM fails in convergence for the case $f_5(x)$. The results show that our fourth-order method LTM_2 can compete with KTM, KM_1, CM_1, NSPP, SBM and KM_3.

In test function $f_6(x)$, in terms of convergence, the methods KM_1, SBM, LTM_1, LTM_5, LTM_6 perform slightly worse, while the method CM_3 performs the worst.

In test function $f_7(x)$, we also check the effectiveness of our methods when we consider the same nonlinear equation with same initial approximation. Then, we find that the methods KM_3, CM_3 and our methods LTM_2, LTM_3, LTM_4, LTM_6, LTM_7, LTM_8 perform better than KTM, KM_1, KM_2, CM_1, CM_2, NSPP and

Table 3: Comparison of some different fourth-order methods.

	Method	IT	NFE	$|f(x_n)|$	δ	ρ
$f_1(x) = 0$	KTM	19	57	0	1.007297327770829e-006	3.57
$x_0 = -0.1$	KM_1	64	192	0	1.477826749862743e-009	3.76
	KM_2	45	135	0	4.900428897136600e-005	3.72
	KM_3	102	306	0	1.091067414571434e-005	4.42
	CM_1	6	18	0	9.497031339122941e-007	3.15
	CM_2	9	27	0	1.145895600629387e-009	3.80
	CM_3	45	135	0	2.819966482547898e-014	3.49
	NSPP	14	42	0	2.220446049250313e-016	1.98
	SBM	12	36	0	2.602835724729857e-010	3.81
	LTM_1	49	147	0	1.015936601511669e-008	3.71
	LTM_2	21	63	0	1.554312234475219e-015	4.09
	LTM_3	17	51	0	1.743050148661496e-013	3.90
	LTM_4	11	33	0	3.606588179216885e-007	4.10
	LTM_5	21	63	0	4.322075137341841e-006	4.31
	LTM_6	26	78	0	2.886579864025407e-015	3.62
	LTM_7	16	48	0	5.212047367475492e-006	4.00
	LTM_8	14	42	0	6.794564910705958e-014	4.18
$f_2(x) = 0$	KTM	5	15	0	7.216449660063518e-016	3.89
$x_0 = 5$	KM_1	5	15	0	1.105560087921731e-011	4.13
	KM_2	5	15	0	1.152966611073225e-013	3.95
	KM_3	4	12	0	9.358037653284246e-006	4.12
	CM_1	4	12	8.881784197001252e-016	1.997754338076696e-004	6.61
	CM_2	5	15	0	2.425837308805967e-014	3.91
	CM_3	-	-	-	-	-
	NSPP	5	15	0	2.473671267821942e-011	4.20
	SBM	5	15	4.440892098500626e-016	9.436895709313831e-016	3.78
	LTM_1	5	15	0	5.860528728973691e-011	4.24
	LTM_2	5	15	0	6.933342788784103e-014	3.94
	LTM_3	4	12	4.440892098500626e-016	1.094604859936954e-004	5.29
	LTM_4	5	15	0	2.942091015256665e-015	3.96
	LTM_5	5	15	0	1.326020566683184e-007	3.54
	LTM_6	5	15	0	6.679101716144942e-013	3.51
	LTM_7	5	15	0	5.632716515435732e-013	3.54
	LTM_8	5	15	0	7.216449660063518e-016	3.64

	Method	IT	NFE	$|f(x_n)|$	δ	ρ
$f_3(x) = 0$	KTM	4	12	2.698906001294071e-024	1.217853159082307e-008	3.62
$x_0 = 1.9$	KM_1	4	12	0	2.148368599356326e-007	3.45
	KM_2	4	12	0	9.592177833278592e-011	4.65
	KM_3	4	12	0	1.846342610051233e-011	4.20
	CM_1	4	12	0	1.119123721644855e-010	3.75
	CM_2	5	15	0	1.656923030498442e-007	3.50
	CM_3	5	15	5.048709793414476e-029	2.684470484445867e-013	3.61
	NSPP	4	12	1.985233470127266e-023	4.289648665705555e-008	3.55
	SBM	4	12	2.418605025271162e-021	4.917930580759401e-011	3.78
	LTM_1	4	12	5.293955920339377e-023	2.170159571681513e-007	3.45
	LTM_2	4	12	0	9.881239316361645e-011	4.65
	LTM_3	4	12	6.617444900424221e-024	9.954034684185604e-009	3.61
	LTM_4	4	12	1.615587133892632e-027	1.428516766682764e-011	4.27
	LTM_5	4	12	3.308722450212111e-024	2.045966094113560e-008	3.59
	LTM_6	4	12	3.231174267785264e-027	2.309911940542863e-011	4.33
	LTM_7	4	12	0	7.432071191110223e-011	4.26
	LTM_8	4	12	4.135903062765138e-025	2.239952597536676e-009	3.65
$f_4(x) = 0$	KTM	5	15	0	1.776356839400251e-015	3.95
$x_0 = 5$	KM_1	5	15	0	6.295657328792004e-010	3.78
	KM_2	4	12	0	5.425764586330928e-005	3.72
	KM_3	4	12	0	1.135485330872044e-006	3.82
	CM_1	4	12	0	1.661871202074394e-005	3.34
	CM_2	4	12	0	4.525455072901252e-006	3.51
	CM_3	4	12	0	1.644155478430776e-009	2.87
	NSPP	5	15	0	7.429612480791548e-013	3.89
	SBM	5	15	0	1.021405182655144e-014	3.93
	LTM_1	5	15	0	6.829425913679188e-010	3.78
	LTM_2	4	12	6.661338147750939e-016	5.805743723130696e-005	3.71
	LTM_3	5	15	0	4.440892098500626e-016	3.86
	LTM_4	4	12	6.661338147750939e-016	1.499435946517025e-007	3.89
	LTM_5	5	15	0	9.895031460871451e-010	3.86
	LTM_6	4	12	0	4.299091316228854e-005	3.24
	LTM_7	4	12	0	1.045922523612575e-005	4.31
	LTM_8	4	12	0	5.399981822362676e-005	3.37

	Method	IT	NFE	$|f(x_n)|$	δ	ρ
$f_5(x) = 0$	KTM	9	27	0	1.288968931589807e-013	3.80
$x_0 = 5.8$	KM$_1$	5	15	0	5.195843755245733e-014	3.30
	KM$_2$	4	12	0	1.789890458070431e-011	3.60
	KM$_3$	-	-	-		-
	CM$_1$	5	15	1.110223024625157e-016	6.661338147750939e-016	3.97
	CM$_2$	4	12	0	4.142503895465666e-009	3.84
	CM$_3$	4	12	1.110223024625157e-016	8.643999676372083e-006	2.93
	NSPP	18	54	0	1.818157153921085e-005	4.93
	SBM	-	-	-		-
	LTM$_1$	5	15	0	1.394815624942147e-004	5.37
	LTM$_2$	4	12	1.110223024625157e-016	1.033639332229663e-004	2.87
	LTM$_3$	9	27	0	1.776356839400251e-015	4.10
	LTM$_4$	16	48	0	1.953010203822325e-004	5.98
	LTM$_5$	6	18	0	9.492295838242626e-011	3.68
	LTM$_6$	5	15	1.110223024625157e-016	2.079826453160738e-005	4.89
	LTM$_7$	6	18	0	7.094251031070087e-006	3.25
	LTM$_8$	6	18	1.110223024625157e-016	4.067230241489028e-007	4.21
$f_6(x) = 0$	KTM	5	15	4.440892098500626e-016	2.725704772088555e-005	3.05
$x_0 = 30$	KM$_1$	6	18	4.440892098500626e-016	2.819966482547898e-014	3.90
	KM$_2$	5	15	3.330669073875470e-016	3.551058632700332e-006	3.61
	KM$_3$	5	15	3.330669073875470e-016	2.469841886565405e-009	4.11
	CM$_1$	5	15	4.440892098500626e-016	2.900702567032454e-008	4.28
	CM$_2$	5	15	3.330669073875470e-016	5.619937435419331e-006	3.47
	CM$_3$	~	~	~	~	~
	NSPP	5	15	8.881784197001252e-016	1.200137685715141e-004	2.70
	SBM	6	18	4.440892098500626e-016	4.440892098500626e-016	3.97
	LTM$_1$	6	18	4.440892098500626e-016	2.287059430727823e-014	3.91
	LTM$_2$	5	15	3.330669073875470e-016	1.494727950301922e-006	3.74
	LTM$_3$	5	15	4.440892098500626e-016	7.169287327135621e-006	3.21
	LTM$_4$	5	15	4.440892098500626e-016	1.957256579032674e-011	4.06
	LTM$_5$	6	18	3.330669073875470e-016	2.093081263865315e-011	3.82
	LTM$_6$	6	18	4.440892098500626e-016	3.996802888650564e-015	4.03
	LTM$_7$	5	15	3.33066907387547e-016	6.713331008989520e-005	3.12
	LTM$_8$	5	15	4.440892098500626e-016	1.567170138416785e-005	3.77

$f_7(x) = 0$	KTM	5	15	0	8.881784197001252e-016	3.96
$x_0 = 3.25$	KM_1	5	15	0	2.532513754260890e-009	3.80
	KM_2	5	15	0	4.440892098500626e-016	3.35
	KM_3	4	12	0	3.625560864861654e-007	3.72
	CM_1	5	15	0	1.288302797775032e-012	3.92
	CM_2	5	15	0	4.440892098500626e-016	1.39
	CM_3	4	12	0	8.305973686617563e-010	3.67
	NSPP	5	15	0	1.444178110432404e-012	3.92
	SBM	5	15	0	1.243449787580175e-014	3.95
	LTM_1	5	15	0	2.535360366096029e-009	3.80
	LTM_2	4	12	0	1.436867444937207e-005	4.49
	LTM_3	4	12	0	1.407228477168232e-005	3.71
	LTM_4	4	12	0	2.142859223397409e-010	3.57
	LTM_5	6	18	0	4.440892098500626e-016	1.19
	LTM_6	4	12	0	1.473763922721361e-005	4.46
	LTM_7	4	12	0	1.399004200797194e-005	4.51
	LTM_8	4	12	0	1.387654194484611e-005	3.71

SBM in terms of speed of convergence for solving the nonlinear equations. However, in this particular case the method LTM_5 don't perform better than other methods.

Consequently, our fourth-order methods can compete with some fourth-order known methods, such as KTM, KM_1, KM_2, KM_3, CM_1, CM_2, CM_3, NSPP and SBM, especially the present methods LTM_2, LTM_3, LTM_4, LTM_7 and LTM_8 perform equal or better than some existing methods in many aspects.

Application to a physical problem

We consider Planck's radiation law problem [21–22]

$$\Phi(\widetilde{\lambda}) = \frac{8\pi c P \widetilde{\lambda}^{-5}}{e^{\frac{cP}{\widetilde{\lambda}BT}} - 1},$$

(83)

which calculates the energy density within an isothermal blackbody. In the expression of formula (83), $\widetilde{\lambda}$ is the wavelength of the radiation, T is the absolute temperature of the blackbody, B is Boltzmann's constant, P is the Planck's constant and c is the speed of light. In some cases, due to the needs of the application, it is often necessary to determine wavelength $\widetilde{\lambda}$ which corresponds to maximum energy density $\Phi(\widetilde{\lambda})$. To find the critical points, we use the Chain Rule to differentiate the function of equation (83), and obtain

$$\Phi'(\widetilde{\lambda}) = \left(\frac{8\pi c P \widetilde{\lambda}^{-6}}{e^{\frac{cP}{\widetilde{\lambda}BT}} - 1}\right) \left(\frac{\frac{cP}{\widetilde{\lambda}BT} e^{\frac{cP}{\widetilde{\lambda}BT}}}{e^{\frac{cP}{\widetilde{\lambda}BT}} - 1} - 5\right),$$

(84)

so to find the critical number of Φ for the maxima, we solve the equation

$$\frac{\frac{cP}{\widetilde{\lambda}BT} e^{\frac{cP}{\widetilde{\lambda}BT}}}{e^{\frac{cP}{\widetilde{\lambda}BT}} - 1} - 5 = 0.$$

(85)

Consider the relationship between variables, if $X = \frac{cP}{\widetilde{\lambda}BT}$, then the equation (85) is converted into the following nonlinear equation

$$F(X) = e^{-X} + \frac{X}{5} - 1 = 0.$$

(86)

The function (86) is continuous, and it has a solution $X = 0$, which is what we do not interest. We want to obtain positive roots of the nonlinear function, so that requires us to apply iterative method to get

Table 4: Comparison of fourth-order methods for the physical problem.

	Method	IT	NFE	$\|F(X_n)\|$	δ	TIME
$F(X) = 0$	KTM	3	9	0	2.709308333237459e-010	0.022328
$X_0 = 3$	KM_1	3	9	0	6.467101520968299e-009	0.028262
	KM_2	3	9	0	2.692956968530780e-012	0.022563
	KM_3	3	9	2.220446049250313e-016	3.547384608282300e-010	0.022609
	CM_1	3	9	0	3.070780428959807e-004	0.023050
	CM_2	3	9	2.220446049250313e-016	5.524792026534442e-006	0.022492
	CM_3	3	9	2.220446049250313e-016	1.943731808839999e-005	0.023956
	NSPP	3	9	0	9.101039921688425e-010	0.022708
	SBM	3	9	0	1.925377191014377e-009	0.023061
	LTM_1	3	9	0	3.959470085135308e-009	0.022057
	LTM_2	3	9	0	9.615003904173136e-009	0.020803
	LTM_3	3	9	2.220446049250313e-016	1.243449787580175e-014	0.020864
	LTM_4	3	9	0	4.251994312198804e-010	0.021211
	LTM_5	3	9	0	4.753649580013786e-005	0.021086
	LTM_6	3	9	2.220446049250313e-016	4.648025434228487e-005	0.022313
	LTM_7	3	9	0	3.362910103312800e-004	0.020591
	LTM_8	3	9	0	3.350666193249197e-004	0.021832

approximate solution of this equation. Here, our desired root is $X^\star = 4.96511423174428$. Keeping in view this fact, we apply KTM, KM_1, KM_2, KM_3, CM_1, CM_2, CM_3, NSPP, SBM, and our methods LTM_1-LTM_8 to the nonlinear equation (86) and compare. Displayed in Table 4 are the number of iterations (IT), the number of function evaluations (NFE), the absolute residual error of the corresponding function value ($\|F(X_n)\|$), the computing time (TIME, the unit of time is one second) and the distance of two consecutive approximations $\delta = \|X_n - X_{n-1}\|$, where " – " is divergence. We use the following stopping criteria for computer programs: $\|F(X_n)\| < \epsilon = 10^{-15}$. Note that in Table 4, in terms of iterations number (IT) and function evaluations (NFE), the fourth-order methods have the same performance. KTM, KM_1, KM_2, CM_1, NSPP, SBM, and our methods LTM_1, LTM_2, LTM_4, LTM_5, LTM_7, LTM_8 have smaller residual error in the nonlinear function as compared to the other methods of fourth-order. Our methods is slightly better at computing time. Consequently, the roots of $F(X) = 0$ give the maximum wavelength of radiation $\widetilde{\lambda}$ by means of the following relation:

$$\widetilde{\lambda} \approx \frac{cP}{X^\star BT} = \frac{cP}{4.96511423174428BT}. \tag{87}$$

6 Basin of attractions

In this section, we study some dynamical properties of the family of iterative methods (45) and (46) based on their basins of attraction when they are applied to the complex polynomial $P(z)$. We investigate the structure of the basins of attraction for comparing convergence and stability of the family of iterative methods. Here we briefly introduce some necessary dynamical concepts and basic results to be used later. Most of them can be found in the classic works such as [20, 23–30] and references therein. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map on the Riemann sphere. The orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as the set $\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}$. A point $z_0 \in \hat{\mathbb{C}}$ is a fixed point of the rational function R if satisfy $R(z_0) = z_0$. A periodic point z_0 of period $m > 1$ is a point such that $R^m(z_0) = z_0$, where m is the smallest such integer. A point z_0 is called attracting if satisfy $\|R'(z_0)\| < 1$, repelling if satisfy $\|R'(z_0)\| > 1$, and neutral if satisfy $\|R'(z_0)\| = 1$. Moreover, if satisfy $\|R'(z_0)\| = 0$, the fixed point is super attracting.

Let $z_f^\star$ be an attracting fixed point of the rational function R. The basin of attraction of the fixed point $z_f^\star$ is defined

$$\mathcal{A}(z_f^\star) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z_f^\star, n \to \infty\}. \tag{88}$$

The set of points whose orbits tends to an attracting fixed point $z_f^\star$ is defined as the Fatou set, $\mathcal{F}(R)$. The complementary set, the Julia $\mathcal{J}(R)$, is the closure of the set consisting of its repelling fixed points, and establishes the borders between the basins of attraction.

Some known and existing fourth-order methods and our fourth-order methods are considered, they are KTM (76), KM_1 (57), KM_2 (58), KM_3 (59), CM_1 (77), CM_2 (78), CM_3 (79), NSPP (80), SBM (81), LTM_1 (61), LTM_2 (62), LTM_3 (64), LTM_4 (65), LTM_5 (67), LTM_6 (68), LTM_7 (73) and LTM_8 (74). In our experiments, we take a square region $D = [-2, 2] \times [-2, 2]$ of the complex plane, with 400×400 points, and we apply the iterative methods starting in very z_0 in the square. The iterative methods can converge to the root or, eventually, diverges. As an illustration, we consider the stopping criterium for convergence to be less than a tolerance $\epsilon = 10^{-7}$ and a maximum of 200 iterations. If a sequence $\{z_n\}$ with the residual $|P(z_n)| < \epsilon$, generated by the iterative method for the initial guess z_0 within the maximum iteration, then we decide the iterative method converges for z_0, otherwise we consider the method to be divergent. We take black color for denoting lack of convergence to any of the roots or convergence to the infinity.

Test problem 1. Let $P_1(z) = z^3 - 1$ having three simple zeros $\{-\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, 1\}$. Based on the Figure 1-3, we observe that the method CM_3 is the best method in terms of less chaotic behavior on the boundary points, the methods KTM, CM_2, NSPP, SBM, LTM_5 and LTM_8 are better. From the three Figures, we find that, with the increase of the value m of (56), the chaotic behaviors of the methods KM_1, KM_2 and KM_3 become more and more complex, which the feature of attraction basins is also reflected in our methods LTM_1, LTM_2, LTM_3 and LTM_4. In the next we have taken polynomials of increasing degree.

Test problem 2. Let $P_2(z) = z^5 + z$ having five simple zeros $\{-0.7071067812 \pm 0.7071067812i, 0, 0.707$ $1067812 \pm 0.7071067812i\}$. We conclude based on Figure 4-6 that the methods CM_2 and LTM_3 outperform all the others, and the methods LTM_8, KTM and SBM are better in terms of less chaotic behavior than other methods. However, the fractal picture of the method LTM_8 has some non convergent points. The method CM_3 has the most divergence points in Figure 5, so it performs worst in this test problem. Since the value of m is bigger, the method KM_3 has the most complex behavior on the boundary points.

Test problem 3. Let $P_3(z) = z^6 + z - 1$ having six simple zeros $\{0.7780895987, -1.1347241384, -0.451$ $0551586 \pm 1.0023645716i, 0.6293724285 \pm 0.7357559530i\}$. In the fractal pictures from Figure 7-9, it is clear that the methods KTM, CM_2, NSPP, SBM and our methods LTM_1, LTM_2 have the largest basins of attraction as compared to the other methods. In addition, although LTM_3, LTM_8 have a small amount of no convergence points, the two methods have less chaotic behavior on the boundary points than other methods, including the known and existing fourth-order methods KTM, KM_1, KM_2, KM_3, CM_1, NSPP and SBM. In terms of the dynamical behavior on the boundary points, the method KM_3 is most complex, followed by the methods LTM_6 and KM_2. From Figures 6(a), the method LTM_4 has more non-convergence point regions than other methods.

7 Conclusions

In this paper, we have designed and studied a new one-parameter family of modified Cauchy method free from second derivative for obtaining simple roots of nonlinear equations by using Padé approximant. The convergence analysis of the methods was also considered, and the methods have convergence order three. Based on the family of third-order method, a new optimal fourth-order family of iterative methods (in the sense of Kung-Traub's conjecture) is obtained by using weight function. We observed from numerical study that the proposed methods are efficient and demonstrate equal or better performance as compared with other well-known fourth-order methods. Finally, the dynamical analysis of this optimal fourth-order family and existing fourth-order methods have been made on some different polynomials, showing some elements of

the proposed family have equal or better stable behavior in many aspects. Furthermore, the fractal graphics show the chaotic behaviors of our methods become more and more complex with the increase of the value m of the series in iterative methods, which also reflected in the existing fourth-order methods KM_1, KM_2 and KM_3.

References

[1] Ostrowski A.M., Solution of equations in Euclidean and Banach space, Academic Press, New York, 1973.

[2] Traub J.F., Iterative Methods for Solution of Equations, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[3] Kou J., Some variants of Cauchy's method with accelerated fourth-order convergence, J. Comput. Appl. Math., 2008, 213, 71–78.

[4] Kou J., Fourth-order variants of Cauchy's method for solving nonlinear equations, Appl. Math. Comput., 2007, 192, 113–119.

[5] Chun C., Some second-derivative-free variants of Chebyshev-Halley methods, Appl. Math. Comput., 2007, 191, 410–414.

[6] Liu T., Li H., Some new variants of Cauchy's methods for solving nonlinear equations, J. Appl. Math., 2012, Article ID 927450, DOI: 10.1155/2012/927450.

[7] Zhou X., Modified Chebyshev-Halley methods free from second derivative, Appl. Math. Comput., 2008, 203, 824–827.

[8] Weerakoon S., Fernando T.G.I., A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett., 2000, 13, 87–93.

[9] Potra F.A., Pták V., Nondiscrete induction and iterative processes, Research Notes in Mathematics, Pitman, Boston, 1984, 103.

[10] Noor M.A., Some iterative methods free from second derivatives for nonlinear equations, Appl. Math. Comput., 2007, 192, 101–106.

[11] Gutiérrez J.M., Hernández M.A., A family of Chebyshev-Halley type methods in Banach spaces, Bull. Austr. Math. Soc., 1997, 55, 113–130.

[12] Kou J., Li Y., Wang X., On a family of second-derivative-free variants of Chebyshevs method, Appl. Math. Comput., 2006, 181, 982–987.

[13] Khattri S.K., Noor M.A., Al-Said E., Unifying fourth-order family of iterative methods, Appl. Math. Lett., 2011, 24, 1295–1300.

[14] Kung H.T., Traub J.F., Optimal order of one-point and multipoint iteration, J. Assoc. Comput. Math., 1974, 21, 634–651.

[15] Chun C., A family of composite fourth-order iterative methods for solving nonlinear equations, Appl. Math. Comput., 2007, 187(2), 951–956.

[16] Chun C., Ham Y., A one-parameter fourth-order family of iterative methods for nonlinear equations, Appl. Math. Comput., 2007, 189, 610–614.

[17] Kou J., Li Y., Wang X., A composite fourth-order iterative method for solving non-linear equations, Appl. Math. Comput., 2007, 184, 471–475.

[18] Sharma R., Bahl A., An optimal fourth order iterative method for solving nonlinear equations and its dynamics, Journal of Complex Analysis, 2015, Article ID 259167.

[19] Sharma J.R., Guha R.K., Sharma R., An efficient fourth order weighted Newton method for systems of nonlinear equations, Numer. Algorithms, 2013, 62(2), 307–323.

[20] Cordero A., Fardi M., Ghasemi M., Torregrosa J.R., Accelerated iterative methods for finding solutions of nonlinear equations and their dynamical behavior, Calcolo., 2012, 51, 17–30.

[21] Jain D., Families of Newton-like method with fourth-order convergence, Int. J. Comput. Math., 2013, 90, 1072–1082.

[22] Maroju P., Magreñán Á.A., Motsa S.S., Sarría Í., Second derivative free sixth order continuation method for solving nonlinear equations with applications, J. Math. Chem., 2018, 56, 2099–2116.

[23] Mayer S., Schleicher D., Immediate and virtual basins of Newton's method for entire functions, Ann. Inst. Fourier., 2006, 56, 325–336.

[24] Beardon A.F., Iteration of Rational Functions, Springer-Verlag, New York, 1991.

[25] Curry J., Garnet L., Sullivan D., On the iteration of a rational function: computer experiments with Newton's method, Comm. Math. Phys., 1983, 91, 267–277.

[26] Kneisl K., Julia sets for the super-Newton method, Cauchy's method and Halley's method, Chaos., 2001, 11, 359–370.

[27] Chun C., Lee M., A new optimal eighth-order family of iterative methods for the solution of nonlinear equations, Appl. Math. Comput., 2013, 223, 506–519.

[28] Wang X., Zhang T., Qin Y., Efficient two-step derivative-free iterative methods with memory and their dynamics, Int. J. Comput. Math., 2016, 93(8), 1423–1446, DOI: 10.1080/00207160.2015.1056168.

[29] Neta B., Scott M., Chun C., Basins of attraction for several methods to find simple roots of nonlinear equations, Appl. Math. Comput., 2012, 218(21), 10548–10556.

[30] Cordero A., Gutiérrez J.M., Magreñán Á., Torregrosa J.R., Stability analysis of a parametric family of iterative methods for solving nonlinear models, Appl. Math. Comput., 2016, 285, 26–40.

(a) KTM

(b) KM_1

(c) KM_2

(d) KM_3

(e) CM_1

(f) CM_2
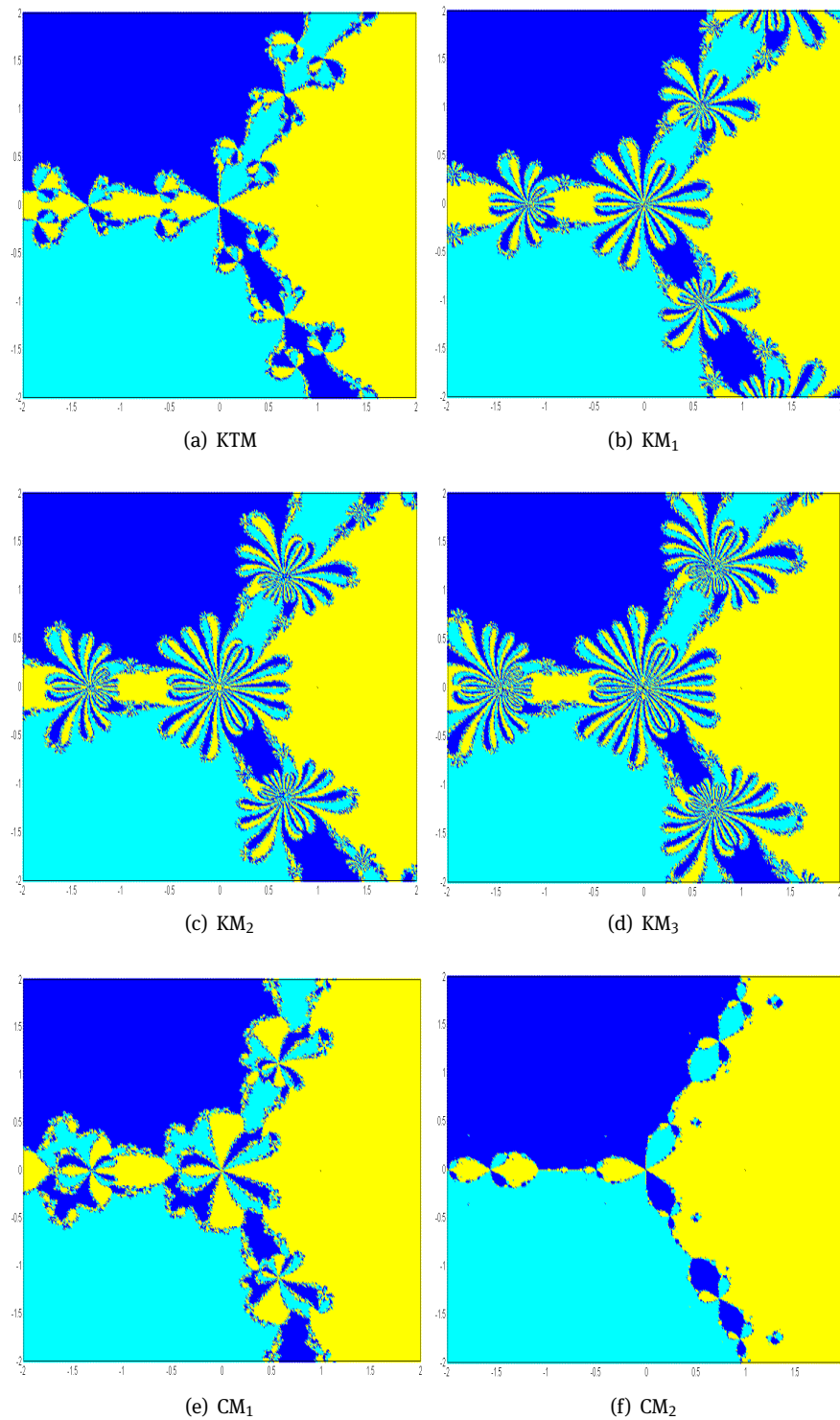
Figure 1: Basins of attraction of the methods KTM, KM_1, KM_2, KM_3, CM_1 and CM_2 respectively for $P_1(z) = z^3 - 1$.

(a) CM_3

(b) NSPP

(c) SBM

(d) LTM_1

(e) LTM_2

(f) LTM_3
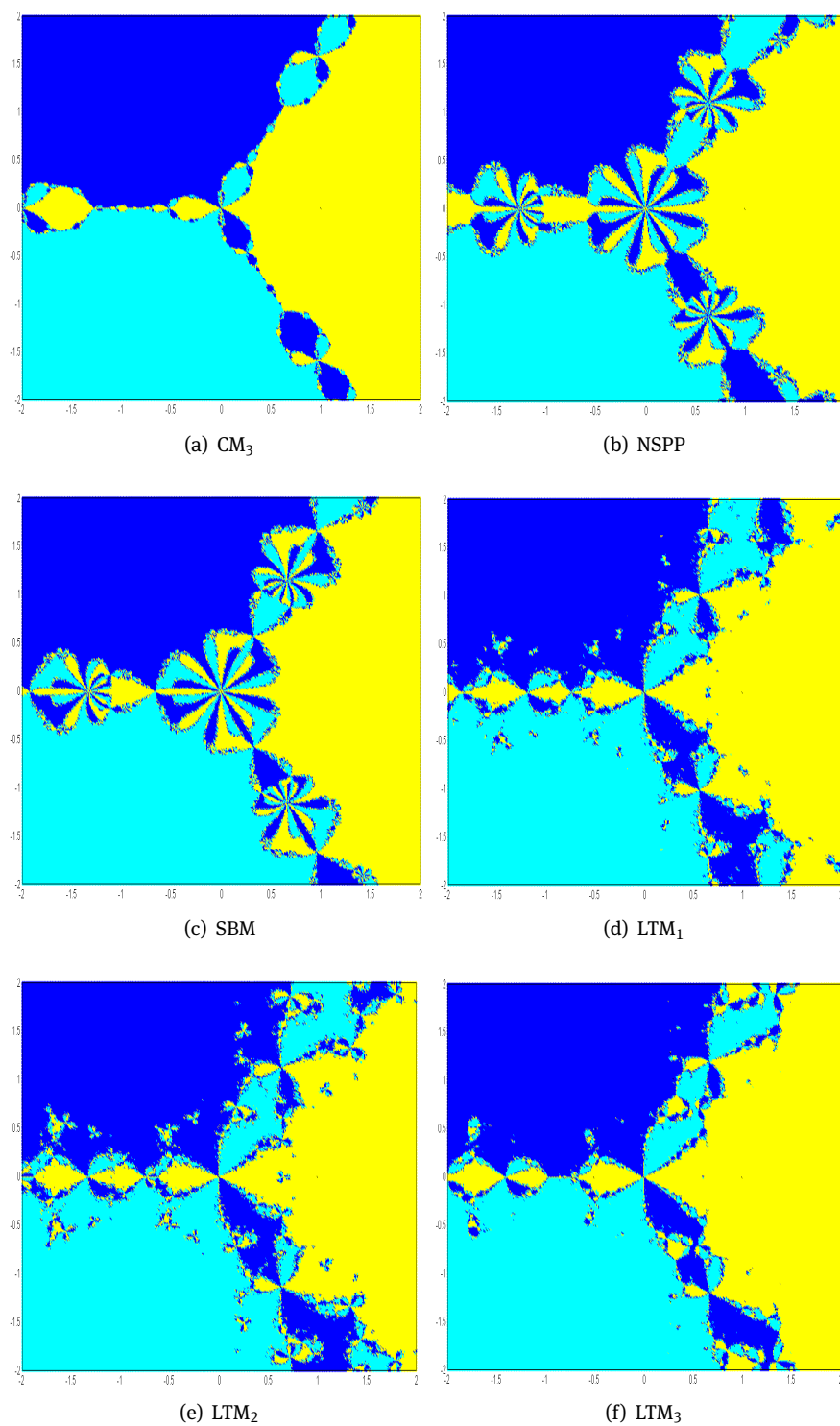
Figure 2: Basins of attraction of the methods CM_3, NSPP, SBM, LTM_1, LTM_2 and LTM_3 respectively for $P_1(z) = z^3 - 1$.
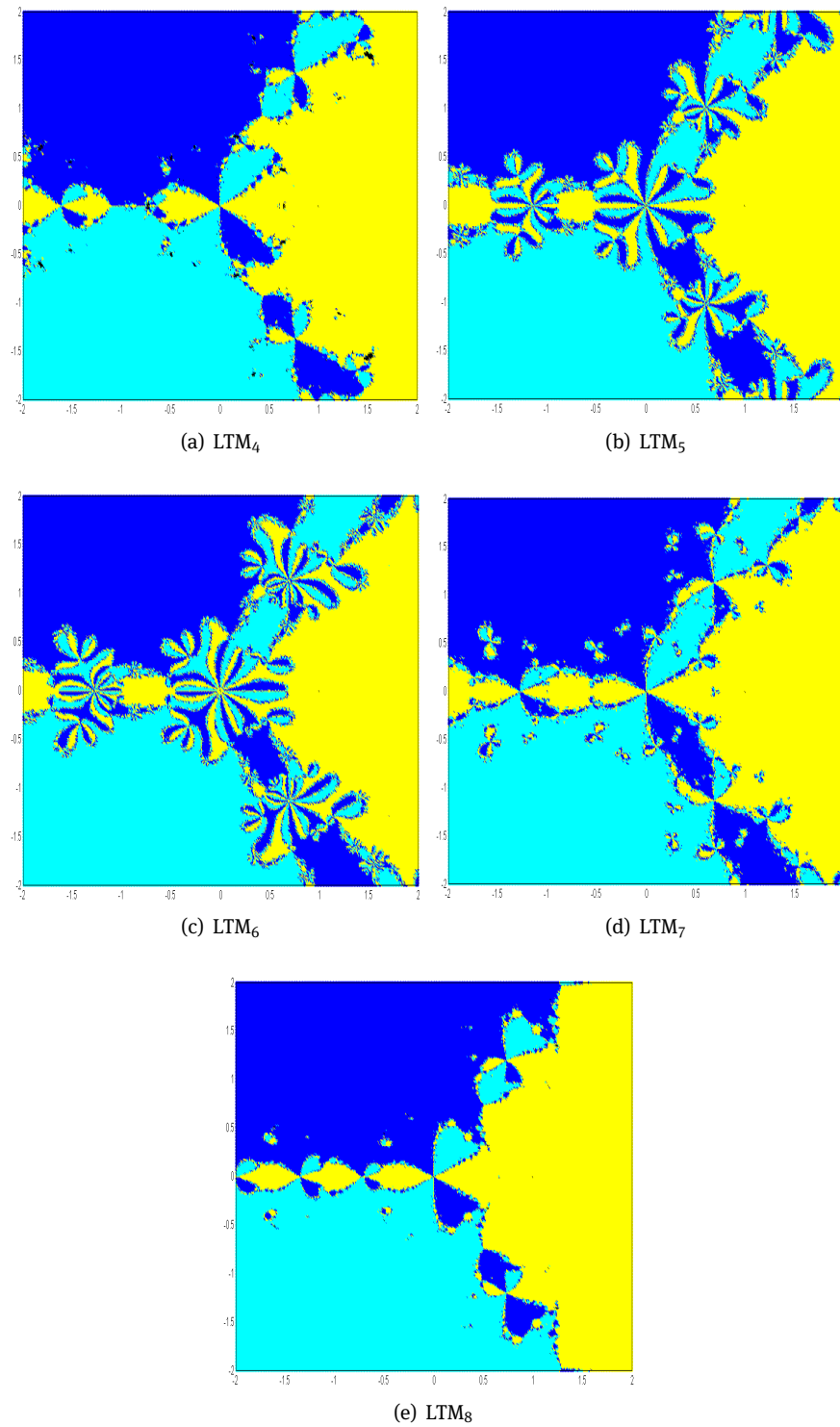
(a) LTM_4

(b) LTM_5

(c) LTM_6

(d) LTM_7

(e) LTM_8

Figure 3: Basins of attraction of the methods LTM_4, LTM_5, LTM_6, LTM_7 and LTM_8 respectively for $P_1(z) = z^3 - 1$.
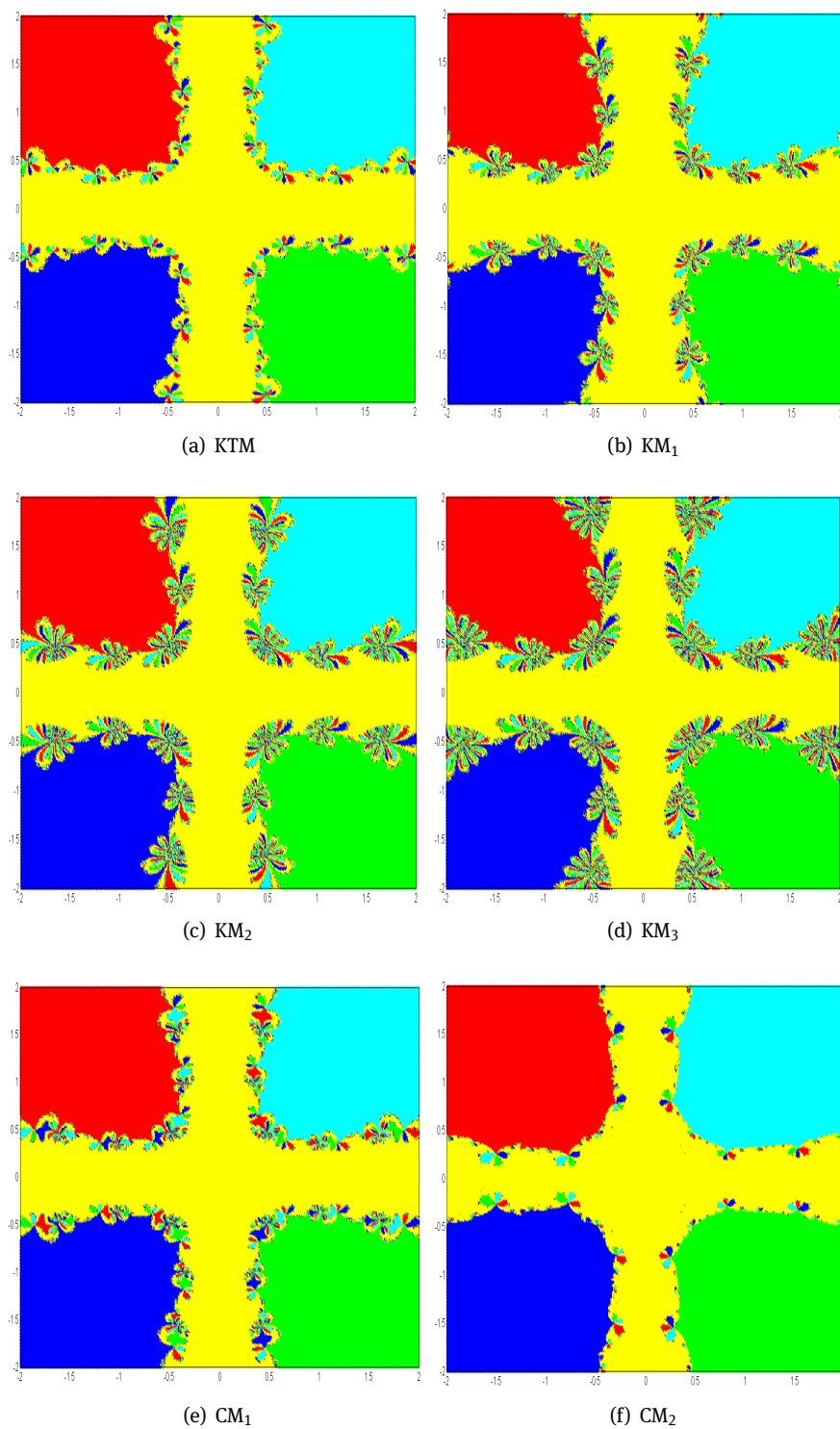
(a) KTM

(b) KM_1

(c) KM_2

(d) KM_3

(e) CM_1

(f) CM_2

Figure 4: Basins of attraction of the methods KTM, KM_1, KM_2, KM_3, CM_1 and CM_2 respectively for $P_2(z) = z^5 + z$.
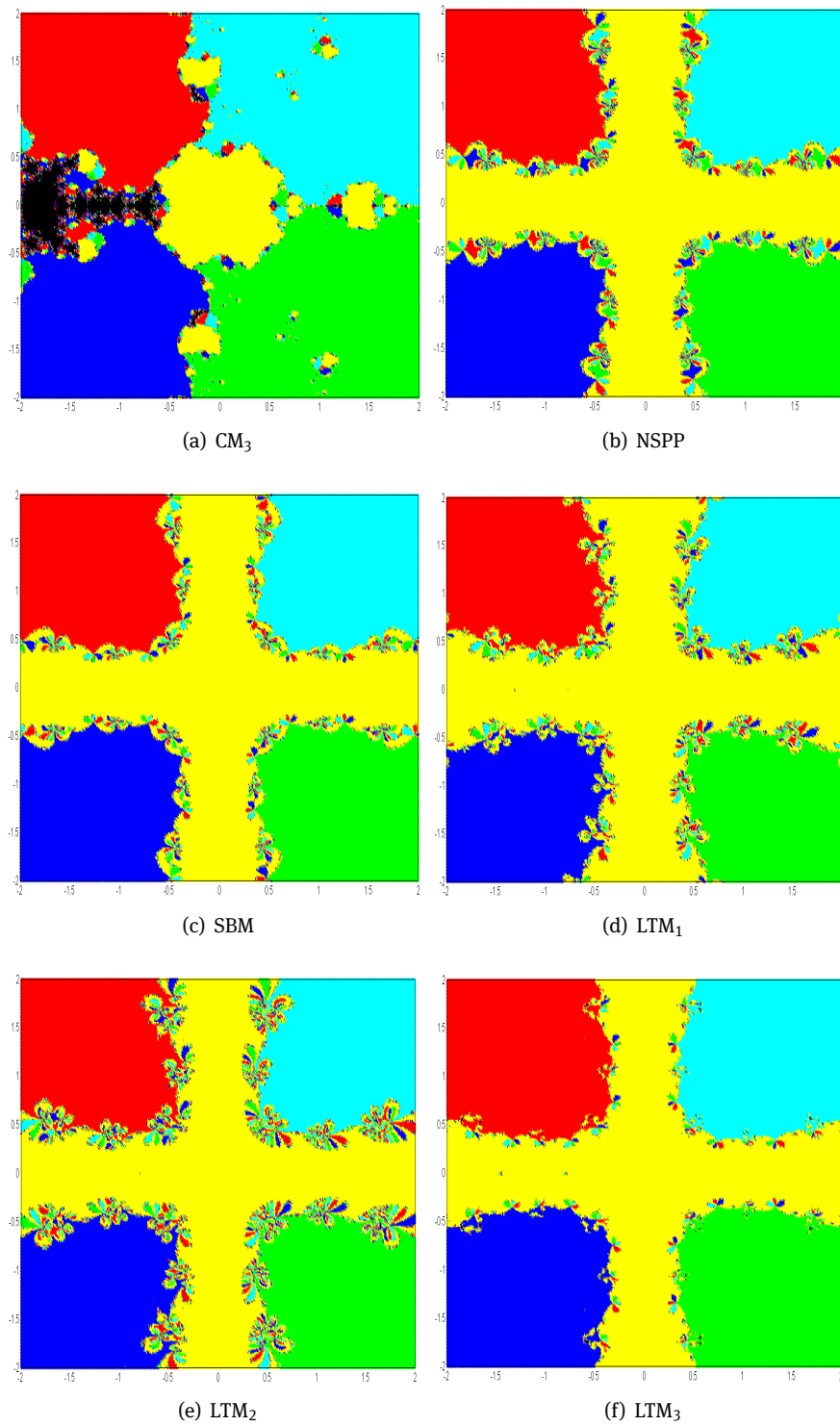
(a) CM$_3$

(b) NSPP

(c) SBM

(d) LTM$_1$

(e) LTM$_2$

(f) LTM$_3$

Figure 5: Basins of attraction of the methods CM$_3$, NSPP, SBM, LTM$_1$, LTM$_2$ and LTM$_3$ respectively for $P_2(z) = z^5 + z$.
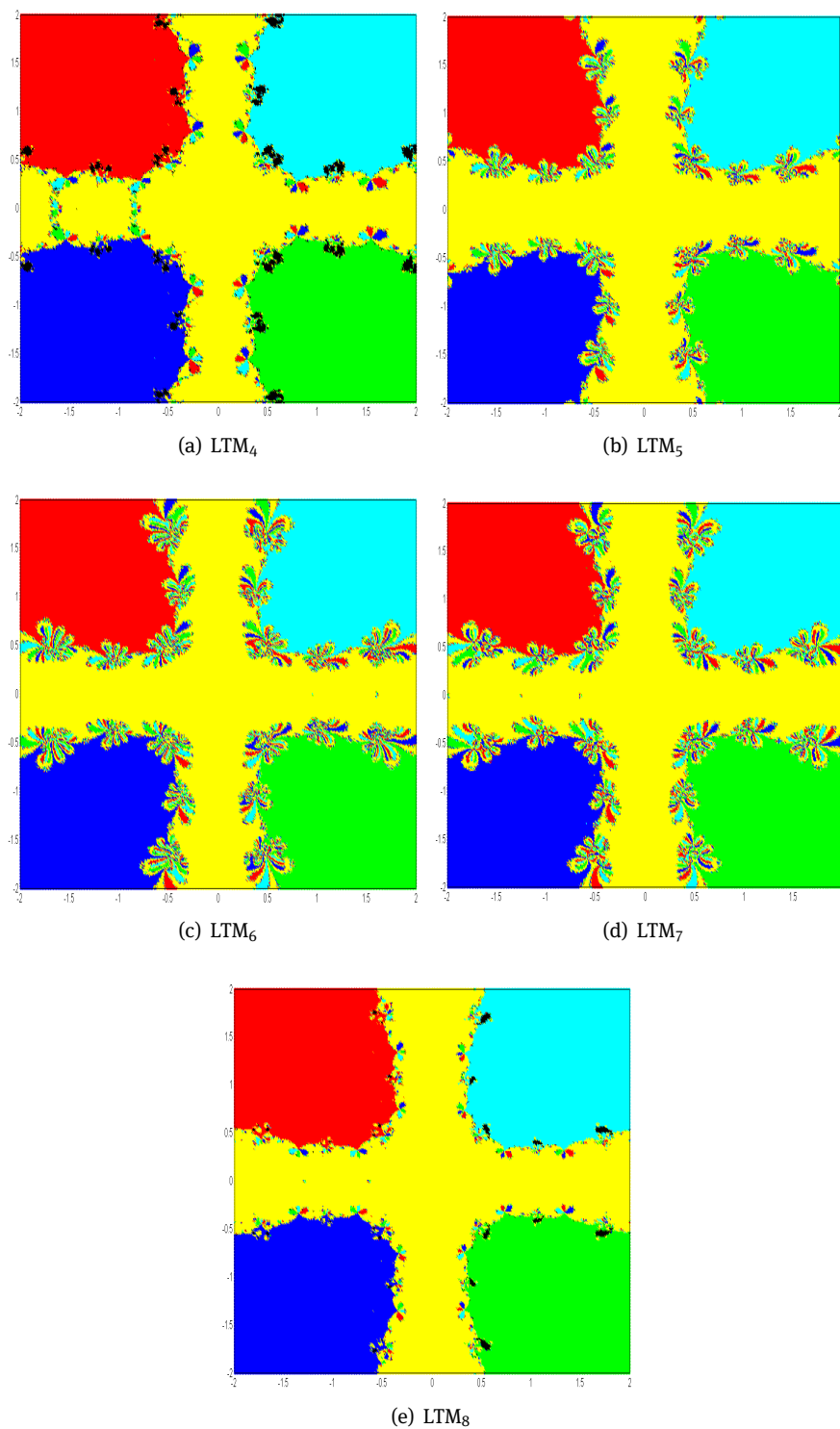
(a) LTM$_4$

(b) LTM$_5$

(c) LTM$_6$

(d) LTM$_7$

(e) LTM$_8$

Figure 6: Basins of attraction of the methods LTM$_4$, LTM$_5$, LTM$_6$, LTM$_7$ and LTM$_8$ respectively for $P_2(z) = z^5 + z$.
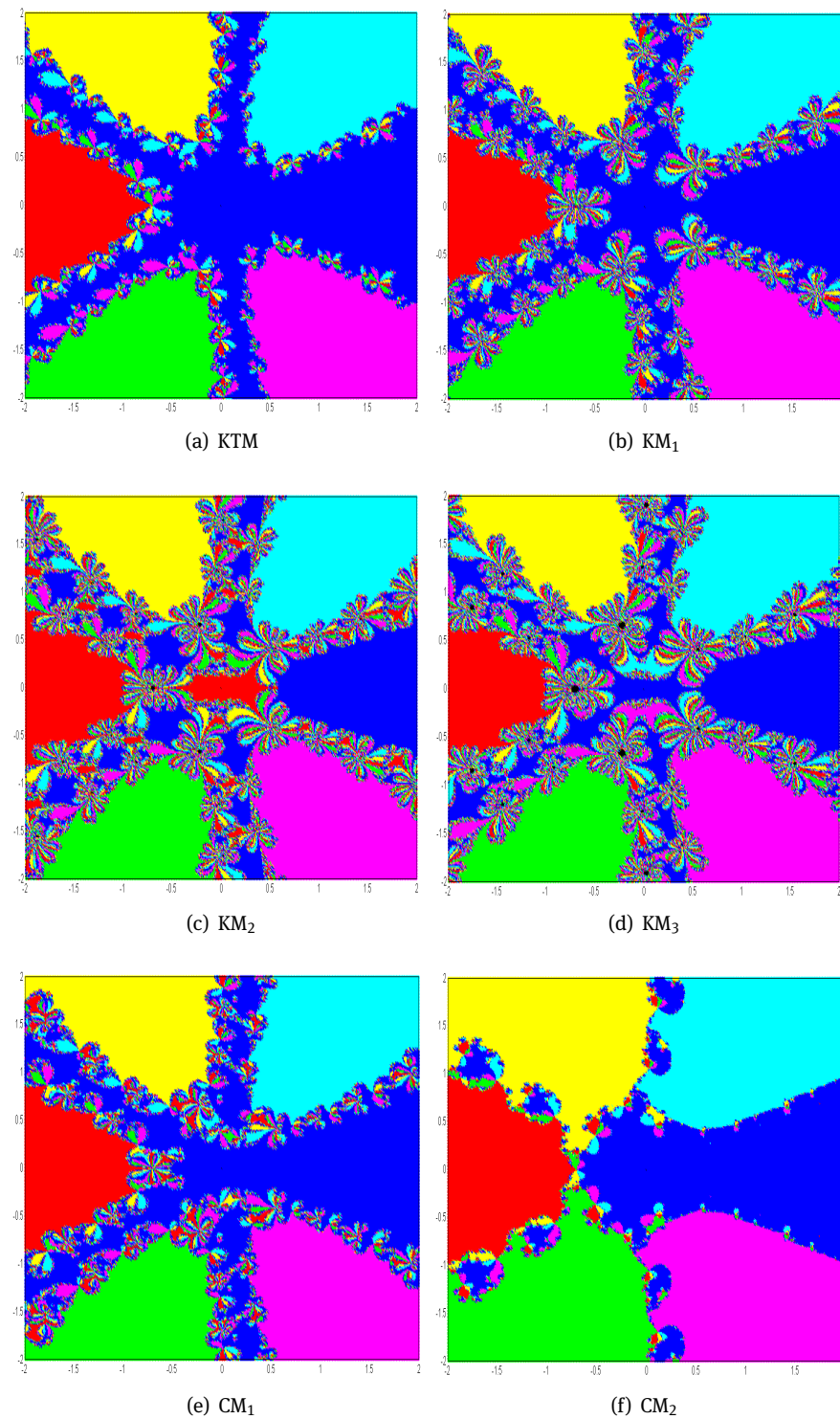
(a) KTM

(b) KM_1

(c) KM_2

(d) KM_3

(e) CM_1

(f) CM_2

Figure 7: Basins of attraction of the methods KTM, KM_1, KM_2, KM_3, CM_1 and CM_2 respectively for $P_3(z) = z^6 + z - 1$.

(a) CM$_3$

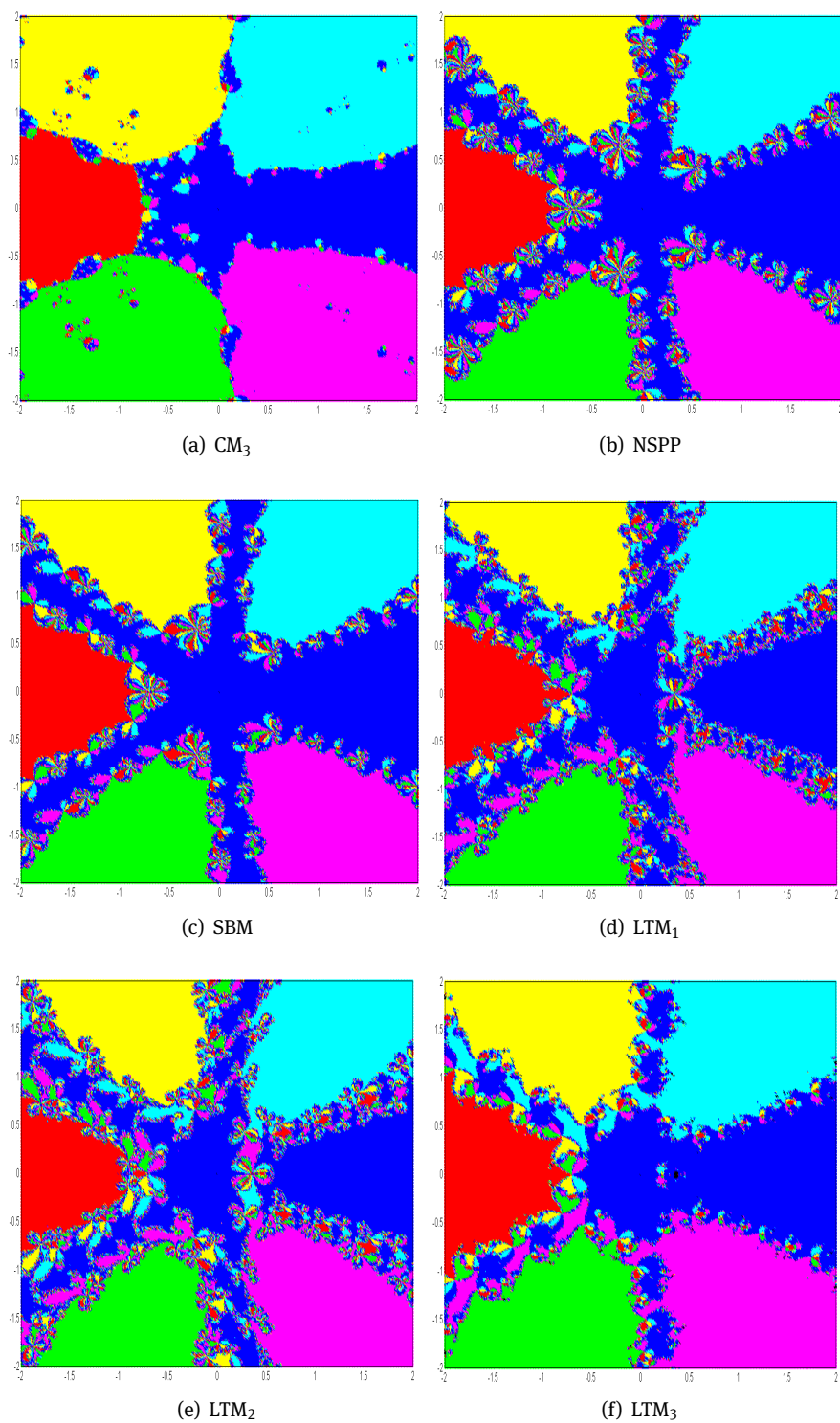
(b) NSPP

(c) SBM

(d) LTM$_1$

(e) LTM$_2$

(f) LTM$_3$

Figure 8: Basins of attraction of the methods CM$_3$, NSPP, SBM, LTM$_1$, LTM$_2$ and LTM$_3$ respectively for $P_3(z) = z^6 + z - 1$.

(a) LTM$_4$

(b) LTM$_5$

(c) LTM$_6$
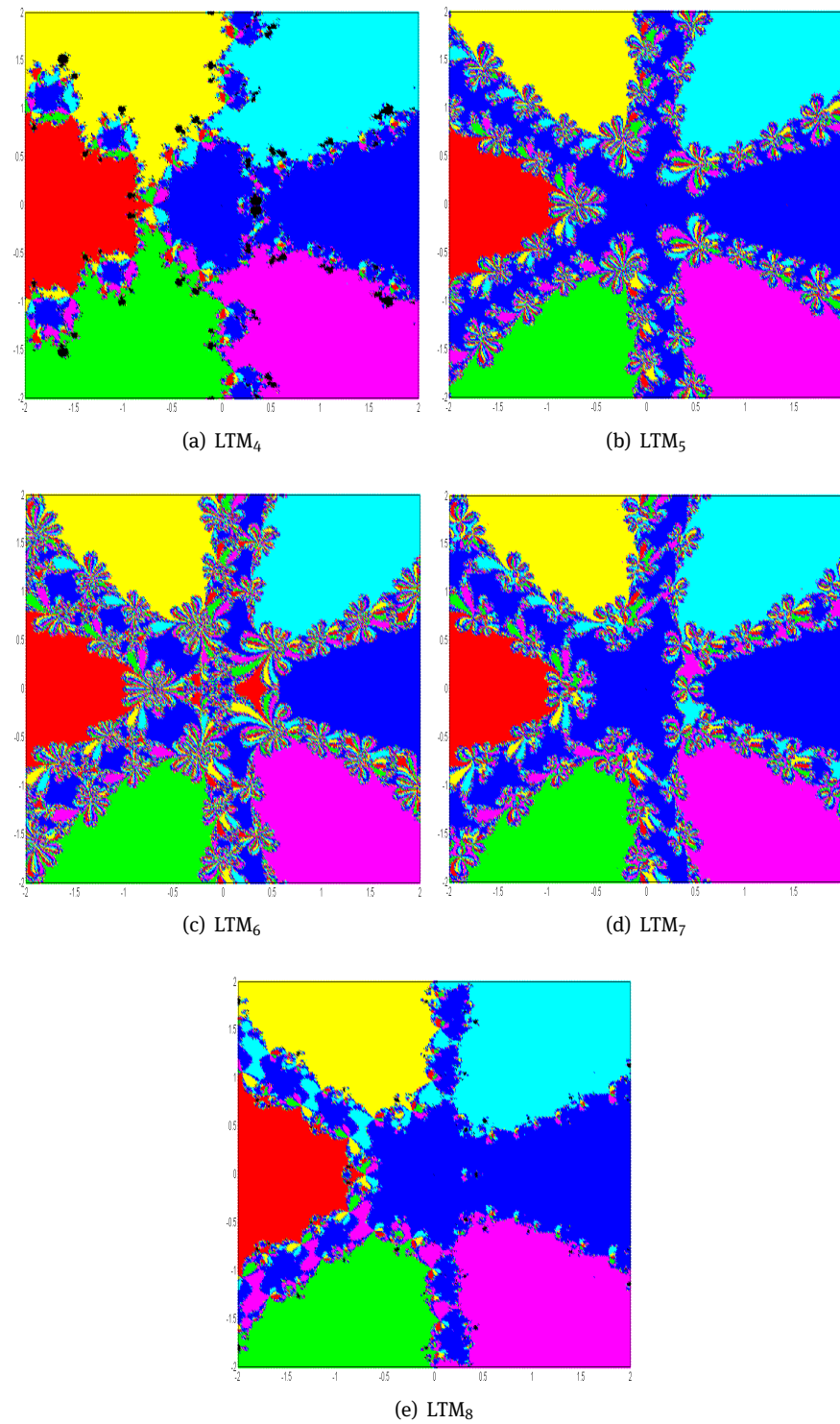

(d) LTM$_7$

(e) LTM$_8$

Figure 9: Basins of attraction of the methods LTM$_4$, LTM$_5$, LTM$_6$, LTM$_7$ and LTM$_8$ respectively for $P_3(z) = z^6 + z - 1$.