DE GRUYTER Open Math. 2019; 17:1737–1749

Open Mathematics

Research Article

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Coupled system of a fractional order differential equations with weighted initial conditions

https://doi.org/10.1515/math-2019-0120
Received August 15, 2019; accepted October 1, 2019

Abstract: Here, a coupled system of nonlinear weighted Cauchy-type problem of a diffre-integral equations of fractional order will be considered. We study the existence of at least one integrable solution of this system by using Schauder fixed point Theorem. The continuous dependence of the uniqueness of the solution is proved.

Keywords: Riemann-Liouville differential operator; coupled system; weighted Cauchy type problem; continuous dependence

MSC 2010: 34A08, 34A12, 45D05

1 Preliminaries and introduction

The coupled system was studied by many authors (see [1] and [11]). Also, the weighted Cauchy-type problem (see [2]-[7]). In [5] the author studied the problem:

$$\begin{cases} D^{\alpha} u(t) = f(t,u) + \int_0^t g(t,s,u(s))\, ds, \quad t > 0, \\ t^{1-\alpha} u(t)|_{t=0} = b, \ 0 < \alpha < 1, \ b \in R, \end{cases} \tag{1}$$

such that the functions f and g satisfies the following assumptions

(1) $t^{1-\alpha} f(t,u)$ is continuous on $R^+ \times C^0_{1-\alpha}(R^+)$ and

$$|f(t,u)| \le t^{\mu} \varphi(t) |u|^{m_1}, \mu \ge 0, m_1 > 1,$$

(2) $s^{1-\alpha} g(t,s,u(s))$ is continuous on $D_{R^+} \times C^0_{1-\alpha}(R^+)$ where

$$D_{R^+} = \{(t,s) \in R^+ \times R^+, 0 \le s \le t\},$$

and

$$|g(t,s,u(s))| \le (t-s)^{\beta-1} s^{\sigma} \psi(s) |u|^{m_2}, 0 < \beta < 1, \sigma \ge 0, m_2 > 1,$$

where $\varphi(t)$ and $\psi(s)$ are such that

(3) $\varphi(t)$ is continuous and $t^{\mu-(1-\alpha)m_1} \varphi(t)$ is continuous in case

$$\mu - (1-\alpha) m_1 < 0,$$

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(4) $\psi(t)$ is continuous and $t^{\sigma-(1-\alpha)m_2}\psi(t)$ is continuous in case

$$\sigma - (1-\alpha)m_2 < 0.$$

Under these assumptions the author proved the existence of at least one local solution in the space $C^{\alpha}_{1-\alpha}([0,h])$, where for $h > 0$

$$C^{\alpha}_{1-\alpha}([0,h]) = \{v \in C^{0}_{1-\alpha}([0,h]) : \exists\, c \in R \text{ and } v^{\star} \in C^{0}_{1-\alpha}([0,h]) \text{ such that } v(t) = ct^{\alpha-1} + I^{\alpha}v^{\star}(t)\},$$

and

$$C^{0}_{r}([0,h]) = \{v \in C^{0}((0,h]) : \lim_{t\to 0^{+}} t^{r}v(t) \text{ exists and finite}\},$$

(the space $C^{0}((0,h])$ is the usual space of continuous functions on $[0,h]$).

In comparison with earlier results, we study the coupled system of weighted Cauchy-type problems of diffre-integral equations of fractional order

$$\begin{cases} D^{\alpha}\, u(t) \;=\; f_1\Big(t, v(t), \int_0^t g_1(t,s,v(s))ds\Big), \\ D^{\beta}\, v(t) \;=\; f_2\Big(t, u(t), \int_0^t g_2(t,s,u(s))ds\Big), \end{cases} \tag{2}$$

where $t \in I = [0,1]$ and $\alpha, \beta \in (0,1)$ with the initial conditions

$$t^{1-\alpha}u(t)|_{t=0} = k_1 \;\text{ and }\; t^{1-\beta}v(t)|_{t=0} = k_2,$$

such that the functions f_i and g_i, $i = 1, 2$ satisfy the following assumptions:

(i) $f_i : I \times R \times R \to R$ be a function with the following properties:
- (a) for each $t \in I$, $f_i(t,\cdot,\cdot)$ is continuous,
- (b) for each $(u,v) \in R \times R$, $f_i(\cdot,u,v)$ is measurable,
- (c) there exist a real function $t \to a(t)$, $a \in L_1(I)$ and a positive constants b_1 and b_2 such that

$$|f_i(t,u,v)| \;\le\; a(t) \;+\; b_1\,|u| \;+\; b_2\,|v|, \quad \text{for each } \; t \in I,\; (u,v) \;\in\; R \times R;$$

(ii) $g_i : I \times I \times R \to R$ be a function with the following properties:
- (a) for each $(t,s) \in I \times I$, $g_i(t,s,\cdot)$ is continuous,
- (b) for each $u \in R$, $g_i(\cdot,\cdot,u)$ is measurable,
- (c) there exist a real function $(t,s) \to k(t,s)$, $k \in L_1(I)$ and a positive constant b_3 such that

$$|g_i(t,s,u)| \;\le\; k(t,s) \;+\; b_3\,|u|, \quad \text{for each } \; (t,s) \in I \times I, \;\; u \in R;$$

(iii) $b_1 + b_2b_3 < \Gamma(1+\alpha)$ and $b_1 + b_2b_3 < \Gamma(1+\beta)$.

Note that if $\alpha = \beta$, $f_1 = f_2$, $g_1 = g_2$, $k_1 = k_2 = b$ and $u(t) = v(t)$, then problem (2) will take the form

$$\begin{cases} D^{\alpha}\, u(t) \;=\; f\Big(t, u(t), \int_0^t g(t,s,u(s))ds\Big), \\ \qquad\qquad t^{1-\alpha}u(t)|_{t=0} = b, \end{cases}$$

which is the generalization of problem (1).

2 Main results

2.1 Integral representation

Lemma 2.1. *Let the assumptions (i-iii) be satisfied. If the solution of the coupled system (2) exists, then it can be represented by the coupled system of nonlinear integral equations of fractional order*

$$\begin{aligned} u(t) &= k_1\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))\, d\theta\right) ds, \\ v(t) &= k_2\, t^{\beta-1} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u(s), \int_0^s g_2(s,\theta,u(\theta))\, d\theta\right) ds. \end{aligned} \tag{3}$$

Proof. Let $u(t)$ be a solution of

$$D^\alpha\, u(t) = \frac{d}{dt} I^{1-\alpha} u(t) = f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))\, ds\right).$$

Integrate both sides, we get

$$I^{1-\alpha} u(t) - I^{1-\alpha} u(t)|_{t=0} = I\, f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))\, ds\right).$$

Operating by I^α on both sides of the last equation, we get

$$Iu(t) - I^\alpha\, C = I^{1+\alpha} f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))\, ds\right).$$

Differentiate both sides, we get

$$u(t) - C_1\, t^{\alpha-1} = I^\alpha f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))\, ds\right).$$

From the initial condition, we find that $C_1 = k_1$, then

$$u(t) = k_1\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))\, d\theta\right) ds.$$

Similarly, we can get

$$v(t) = k_2\, t^{\beta-1} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u(s), \int_0^s g_2(s,\theta,u(\theta))\, d\theta\right) ds.$$

Therefore, the solution (u, v) of system (2) can be represented by system (3).

2.2 Existence of solution

Let $L_1(I)$ be a class of Lebesgue integrable functions on the interval I, with the norm $||x|| = \int_I |x(t)| dt$.

Define the operator T by

$$T(u,v)(t) = (T_1 v(t), T_2 u(t)),$$

where

$$T_1 v(t) = k_1\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right) ds,$$

$$T_2 u(t) = k_2\, t^{\beta-1} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u(s), \int_0^s g_2(s,\theta,u(\theta))d\theta\right) ds.$$

It is clear that the fixed point of the operator T is the solution of system (3).

Theorem 2.1. *Assume that f_i and g_i satisfy the assumptions (i-iii). Then the coupled system of weighted Cauchy-type problems (2) has at least one solution $(u, v) \in L_1 \times L_1$.*

Proof. Define

$$X = \left\{(u(t), v(t))|(u(t), v(t)) \in L_1 \times L_1 \text{ and } ||(u,v)||_{L_1\times L_1} = ||u||_{L_1} + ||v||_{L_1} \leq r\right\}.$$

For $(u, v) \in X$, we have

$$\begin{aligned}
||T_1 v|| &= \int_0^1 |T_1 v(t)|\, dt \\
&\leq \int_0^1 |k_1\, t^{\alpha-1}|\, dt + \int_0^1 \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right) ds \right| dt \\
&\leq \left(\frac{k_1\, t^\alpha}{\alpha}\right)_0^1 + \int_0^1 \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, dt \left| f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right)\right| ds \\
&= \frac{k_1}{\alpha} + \int_0^1 \frac{(t-s)^\alpha}{\Gamma(1+\alpha)}\Big|_s^1 \left| f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right)\right| ds \\
&= \frac{k_1}{\alpha} + \int_0^1 \frac{(1-s)^\alpha}{\Gamma(1+\alpha)} \left| f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right)\right| ds \\
&\leq \frac{k_1}{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right)\right| ds \\
&\leq \frac{k_1}{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left[a(s) + b_1\,|v(s)| + b_2 \int_0^s \left|g_1(s,\theta,v(\theta))\right| d\theta\right] ds \\
&\leq \frac{k_1}{\alpha} + \frac{1}{\Gamma(1+\alpha)}\, ||a|| + \frac{b_1}{\Gamma(1+\alpha)}\, ||v||_{L_1} + \frac{b_2}{\Gamma(1+\alpha)} \int_0^1 \int_0^s \left[k(s,\theta) + b_3 |v(\theta)|\right] d\theta\, ds \\
&\leq \frac{k_1}{\alpha} + \frac{1}{\Gamma(1+\alpha)}\, ||a|| + \frac{b_1}{\Gamma(1+\alpha)}\, ||v||_{L_1} + \frac{b_2\, k^\star}{\Gamma(1+\alpha)} + \frac{b_2\, b_3}{\Gamma(1+\alpha)}\, ||v||_{L_1} \\
&\leq \frac{k_1}{\alpha} + \frac{1}{\Gamma(1+\alpha)}\, ||a|| + \frac{b_1 + b_2\, b_3}{\Gamma(1+\alpha)}\, r_1 + \frac{b_2\, k^\star}{\Gamma(1+\alpha)} \leq r_1,
\end{aligned}$$

where $k^\star = \int_0^1 \int_0^s k(s,\theta)\, d\theta\, ds$ and

$$r_1 = \frac{\frac{k_1}{\alpha} + \frac{||a||}{\Gamma(1+\alpha)} + \frac{b_2\, k^\star}{\Gamma(1+\alpha)}}{1 - \frac{b_1 + b_2\, b_3}{\Gamma(1+\alpha)}}.$$

Similarly, we get

$$||T_2 u|| \leq \frac{k_2}{\beta} + \frac{1}{\Gamma(1+\beta)} ||a|| + \frac{b_1 + b_2\, b_3}{\Gamma(1+\beta)} r_2 + \frac{b_2\, k^\star}{\Gamma(1+\beta)} \leq r_2,$$

where

$$r_2 = \frac{\frac{k_2}{\beta} + \frac{||a||}{\Gamma(1+\beta)} + \frac{b_2\, k^\star}{\Gamma(1+\beta)}}{1 - \frac{b_1 + b_2\, b_3}{\Gamma(1+\beta)}}.$$

Let

$$r = \max\{r_1 + r_2\}.$$

Then,

$$||T(u,v)(t)|| = ||T_1 v(t), T_2 u(t)|| = ||T_1 v(t)|| + ||T_2 u(t)|| \leq r_1 + r_2 \leq r.$$

Therefore, for $(u,v) \in X$, we get $T(u,v) \in X$ and hence $TX \in X$. Now, from the assumptions (i-a) - (ii-a), we deduce that T maps X into $L_1 \times L_1$ continuously. Moreover, we have

$$\begin{aligned}
||f_i|| &= \int_0^1 \left| f_i\left(t, w(t), \int_0^t g_i(t,s,w(s))ds\right)\right| dt \\
&\leq \int_0^1 \left(a(t) + b_1\, |w(t)| + b_2 \int_0^t \left| g_i(t,s,w(s))\right| ds \right) dt \\
&\leq ||a|| + b_1\, ||w|| + b_2 \int_0^1 \int_0^t \left[k(t,s) + b_3\, |w(s)| \right] ds\, dt \\
&\leq ||a|| + b_1\, ||w|| + b_2\, k^\star + b_2\, b_3\, ||w||.
\end{aligned}$$

This estimation shows that f_i in $L_1(I)$.

Now, we will use Kolmogorov compactness criterion (see [8]) to show that T is compact. So, let $\aleph$ be a bounded subset of L_1. Then $T_1(\aleph)$ is bounded in $L_1(I)$. Now we show that $(T_1 v)_h \to T_1 v$ in $L_1(I)$ as $h \to 0$, uniformly with respect to $T_1 v \in T\,\aleph$.
Indeed:

$$\begin{aligned}
||(T_1 v)_h - T_1 v|| &= \int_0^1 |(T_1 v)_h(t) - (T_1 v)(t)|\, dt \\
&= \int_0^1 \left| \frac{1}{h} \int_t^{t+h} (T_1 v)(s)\, ds - (T_1 v)(t) \right| dt \\
&\leq \int_0^1 \left(\frac{1}{h} \int_t^{t+h} |\, (T_1 v)(s) - (T_1 v)(t)\,|\, ds \right) dt \\
&\leq \int_0^1 \frac{1}{h} \int_t^{t+h} |k_1\, s^{\alpha-1} - k_1\, t^{\alpha-1}|\, ds\, dt \\
&\quad + \int_0^1 \frac{1}{h} \int_t^{t+h} \Bigg| I^\alpha f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right) \\
&\quad - I^\alpha f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))ds\right) \Bigg|\, ds\, dt,
\end{aligned}$$

since $f_1 \in L_1(I)$ we get that $I^\alpha f_1(.) \in L_1(I)$. Moreover $t^{\alpha-1} \in L_1(I)$. So, we have (see [10])

$$\frac{1}{h}\int_t^{t+h} |\, k_1\, s^{\alpha-1} \;-\; k_1\, t^{\alpha-1}\,|\, ds \;\to\; 0$$

and

$$\frac{1}{h}\int_t^{t+h} \left| I^\alpha f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right) - I^\alpha f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))ds\right)\right| ds \;\to\; 0$$

for a.e. $t \in I$. Therefore, $T_1(\aleph)$ is relatively compact, that is, T_1 is a compact operator, similarly T_2 is a compact operator. Hence T is a compact operator

Therefore, Schauder fixed point Theorem (see [9]) implies that T has a fixed point (u, v) which is a solution of the coupled system (3).

To complete the proof, let $(u(t), v(t))$ be a solution of

$$\begin{aligned} u(t) &= k_1\, t^{\alpha-1} \;+\; I^\alpha f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))ds\right), \\ v(t) &= k_2\, t^{\beta-1} \;+\; I^\beta f_2\left(t, u(t), \int_0^t g_2(t,s,u(s))ds\right), \end{aligned} \tag{4}$$

which gives

$$t^{1-\alpha}u(t)|_{t=0} \;=\; k_1,\;\; t^{1-\beta}v(t)|_{t=0} \;=\; k_2.$$

Operating on both sides of the first and second equations in (4) by $I^{1-\alpha}$ and $I^{1-\beta}$ respectively, we get

$$I^{1-\alpha}u(t) = k_1 \;+\; I^{1-\alpha}\, I^\alpha f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))ds\right),$$

$$I^{1-\beta}v(t) = k_2 \;+\; I^{1-\beta}\, I^\beta f_2\left(t, u(t), \int_0^t g_2(t,s,u(s))ds\right).$$

Differentiate both sides, we obtain

$$\begin{cases} D^\alpha\, u(t) \;=\; f_1\left(t, v(t), \int_0^t g_1(t,s,v(s))ds\right), \\ D^\beta\, v(t) \;=\; f_2\left(t, u(t), \int_0^t g_2(t,s,u(s))ds\right). \end{cases}$$

2.3 Uniqueness of the solution

For the uniqueness of the solution we have the following theorem:

Theorem 2.2. *Suppose that the functions f_i and g_i satisfy conditions (i-b), (ii-b) and (iii) of Theorem 2.1 in addition to the following assumptions:*

$$\left| f_i(t,u_1,v_1) \;-\; f_i(t,u_2,v_2)\right| \;\le\; b_1\,|u_1 \;-\; u_2| \;+\; b_2\,|v_1 \;-\; v_2|,\; i \;=\; 1,\; 2 \tag{5}$$

and

$$|g_i(t,s,v_1) \;-\; g_i(t,s,v_2)| \;\le\; b_3\,|v_1 \;-\; v_2|,\; i \;=\; 1,\; 2. \tag{6}$$

Then the coupled system of weighted Cauchy-type problems (2) has a unique solution.

Proof. From assumption (5), we get

$$|f_i(t,u,v) - f_i(t,0,0)| \le b_1\,|u| + b_2\,|v|.$$

But since

$$|f_i(t,u,v)| - |f_i(t,0,0)| \le |f_i(t,u,v) - f_i(t,0,0)| \le b_1\,|u| + b_2\,|v|,$$

therefore

$$|f_i(t,u,v)| \le |f_i(t,0,0)| + b_1\,|u| + b_2\,|v|,$$

i.e. assumptions $(i-a)$ and $(i-c)$ of theorem 2.1 are satisfied, similarly assumptions $(ii-a)$ and $(ii-c)$ of Theorem 2.1 are satisfied.

Then from Theorem 2.1 the solution exists. Now we prove the uniqueness of this solution:

Let (u_1,v_1) and (u_2,v_2) be two solutions of (3). Then

$$\begin{aligned} u_2(t) - u_1(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[f_1\left(s, v_2(s), \int_0^s g_1(s,\theta,v_2(\theta))d\theta\right) - f_1\left(s, v_1(s), \int_0^s g_1(s,\theta,v_1(\theta))d\theta\right)\right] ds, \\ &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{b_1\,|v_2(s)-v_1(s)| + b_2 \int_0^s \Big|g_1(s,\theta,v_2(\theta)) - g_1(s,\theta,v_1(\theta))\Big| d\theta\right\} ds. \end{aligned}$$

Therefore

$$\int_0^1 |u_2(t) - u_1(t)|\,dt \le \int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{b_1\,|v_2(s) - v_1(s)| + b_2 \int_0^s \Big|g_1(s,\theta,v_2(\theta)) - g_1(s,\theta,v_1(\theta))\Big|\,d\theta\right\} ds\,dt,$$

$$\begin{aligned} ||u_2 - u_1||_{L_1} &\le \int_0^1\int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,dt \left\{b_1\,|v_2(s) - v_1(s)| + b_2 \int_0^s \Big|g_1(s,\theta,v_2(\theta)) - g_1(s,\theta,v_1(\theta))\Big|\,d\theta\right\} ds \\ &= \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(1+\alpha)} \left\{b_1\,|v_2(s) - v_1(s)| + b_2 \int_0^s \Big|g_1(s,\theta,v_2(\theta)) - g_1(s,\theta,v_1(\theta))\Big|\,d\theta\right\} ds \\ &\le \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left\{b_1|v_2(s) - v_1(s)| + b_2\,b_3 \int_0^s \Big|v_2(\theta) - v_1(\theta)\Big|\,d\theta\right\} ds \\ &\le \frac{b_1 + b_2\,b_3}{\Gamma(1+\alpha)}\,||v_2 - v_1||_{L_1}. \end{aligned}$$

Similarly

$$||v_2 - v_1||_{L_1} \le \frac{b_1 + b_2\,b_3}{\Gamma(1+\beta)}\,||u_2 - u_1||_{L_1}.$$

Therefore

$$\begin{aligned} ||(u_2,v_2) - (u_1,v_1)|| &= ||u_2 - u_1||_{L_1} + ||v_2 - v_1||_{L_1} \\ &\le \frac{b_1 + b_2\,b_3}{\Gamma(1+\alpha)}\,||v_2 - v_1||_{L_1} + \frac{b_1 + b_2\,b_3}{\Gamma(1+\beta)}\,||u_2 - u_1||_{L_1} \\ &\le ||u_2 - u_1||_{L_1} + ||v_2 - v_1||_{L_1} \\ &= ||(u_2,v_2) - (u_1,v_1)||, \end{aligned}$$

which implies

$$||(u_2,v_2) - (u_1,v_1)|| = 0 \Rightarrow (u_2,v_2) = (u_1,v_1).$$

This completes the proof.

2.4 Continuous dependence on initial data

Now we show that the solution of the coupled system (2) is depending continuously on initial data.

Theorem 2.3. *Let the assumptions of Theorem 2.2 be satisfied. Then the solution of the weighted Cauchy-type problem (2) is depending continuously on initial data,*

Proof. Let $(u(t), v(t))$ be a solution of the couple

$$u(t) = k_1\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))\, d\theta\right) ds,$$

$$v(t) = k_2\, t^{\beta-1} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u(s), \int_0^s g_2(s,\theta,u(\theta))\, d\theta\right) ds$$

and let $(\widetilde{u}(t), \widetilde{v}(t))$ be a solution of the above coupled system such that $t^{1-\alpha}\, \widetilde{u}(t)|_{t=0} = \widetilde{k_1}$ and $t^{1-\beta}\, \widetilde{v}(t)|_{t=0} = \widetilde{k_2}$. Then

$$u(t) - \widetilde{u}(t) = (k_1 - \widetilde{k_1})\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))\, d\theta\right) - f_1\left(s, \widetilde{v}(s), \int_0^s g_1(s,\theta,\widetilde{v}(\theta))\, d\theta\right)\right] ds,$$

$$|u(t) - \widetilde{u}(t)| \leq |k_1 - \widetilde{k_1}|\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ b_1\, |v(s) - \widetilde{v}(s)| + b_2 \int_0^s \left| g_1(s,\theta,v(\theta)) - g_1(s,\theta,\widetilde{v}(\theta))\right| d\theta \right\} ds.$$

Therefore

$$\int_0^1 |u(t) - \widetilde{u}(t)|\, dt \leq \frac{1}{\alpha} |k_1 - \widetilde{k_1}| + \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ b_1\, |v(s) - \widetilde{v}(s)| + b_2 \int_0^s \left| g_1(s,\theta,v(\theta)) - g_1(s,\theta,\widetilde{v}(\theta))\right| d\theta \right\} ds\, dt,$$

$$\begin{aligned} ||u - \widetilde{u}||_{L_1} &\leq \frac{1}{\alpha} |k_1 - \widetilde{k_1}| + \int_0^1 \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, dt \left\{ b_1\, |v(s) - \widetilde{v}(s)| + b_2 \int_0^s \left| g_1(s,\theta,v(\theta)) - g_1(s,\theta,\widetilde{v}(\theta))\right| d\theta \right\} ds \\ &= \frac{1}{\alpha} |k_1 - \widetilde{k_1}| + \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(1+\alpha)} \left\{ b_1\, |v(s) - \widetilde{v}(s)| + b_2 \int_0^s \left| g_1(s,\theta,v(\theta)) - g_1(s,\theta,\widetilde{v}(\theta))\right| d\theta \right\} ds \end{aligned}$$

$$\leq \frac{1}{\alpha}|k_1-\widetilde{k_1}|+\frac{1}{\Gamma(1+\alpha)}\int_0^1\left\{b_1|v(s)-\widetilde{v}(s)|+b_2\,b_3\int_0^s\left|v(\theta)-\widetilde{v}(\theta)\right|d\theta\right\}ds$$
$$\leq \frac{1}{\alpha}|k_1-\widetilde{k_1}|+\frac{b_1+b_2\,b_3}{\Gamma(1+\alpha)}||v-\widetilde{v}||_{L_1}.$$

Similarly

$$||v-\widetilde{v}||_{L_1}\leq\frac{1}{\beta}|k_2-\widetilde{k_2}|+\frac{b_1+b_2\,b_3}{\Gamma(1+\beta)}||u-\widetilde{u}||_{L_1}.$$

Therefore

$$\begin{aligned}||(u,v)-(\widetilde{u},\widetilde{v})||_{L_1}&=||u-\widetilde{u}||_{L_1}+||v-\widetilde{v}||_{L_1}\\&\leq\frac{1}{\alpha}|k_1-\widetilde{k_1}|+\frac{b_1+b_2\,b_3}{\Gamma(1+\alpha)}||v-\widetilde{v}||_{L_1}\\&\quad+\frac{1}{\beta}|k_2-\widetilde{k_2}|+\frac{b_1+b_2\,b_3}{\Gamma(1+\beta)}||u-\widetilde{u}||_{L_1}\\&\leq\max\left\{\frac{1}{\alpha},\frac{1}{\beta}\right\}\left(|k_1-\widetilde{k_1}|+|k_2-\widetilde{k_2}|\right)\\&\quad+(b_1+b_2\,b_3)\max\left\{\frac{1}{\Gamma(1+\alpha)},\frac{1}{\Gamma(1+\beta)}\right\}\left(||v-\widetilde{v}||_{L_1}+||u-\widetilde{u}||_{L_1}\right)\\&\leq M^\star\left(|k_1-\widetilde{k_1}|+|k_2-\widetilde{k_2}|\right)\\&\quad+N^\star\,(b_1+b_2\,b_3)\left(||v-\widetilde{v}||_{L_1}+||u-\widetilde{u}||_{L_1}\right)\\&=M^\star\left(|k_1-\widetilde{k_1}|+|k_2-\widetilde{k_2}|\right)\\&\quad+N^\star\,(b_1+b_2\,b_3)\,||(u,v)-(\widetilde{u},\widetilde{v})||_{L_1},\end{aligned}$$

where $M^\star=\max\left\{\frac{1}{\alpha},\frac{1}{\beta}\right\}$ and $N^\star=\max\left\{\frac{1}{\Gamma(1+\alpha)},\frac{1}{\Gamma(1+\beta)}\right\}$.

$$\left(1-N^\star\,(b_1+b_2\,b_3)\right)||(u,v)-(\widetilde{u},\widetilde{v})||_{L_1}\leq M^\star\left(|k_1-\widetilde{k_1}|+|k_2-\widetilde{k_2}|\right)$$
$$\Rightarrow\ ||(u,v)-(\widetilde{u},\widetilde{v})||_{L_1}\leq\left(\frac{M^\star}{1-N^\star\,(b_1+b_2\,b_3)}\right)\left(|k_1-\widetilde{k_1}|+|k_2-\widetilde{k_2}|\right).$$

Therefore, if $|k_1-\widetilde{k_1}|<\frac{\delta(\varepsilon)}{2}$ and $|k_2-\widetilde{k_2}|<\frac{\delta(\varepsilon)}{2}$, then $||(u,v)-(\widetilde{u},\widetilde{v})||_{L_1}<\varepsilon$. Now from the equivalence we get that the solution of the weighted Cauchy-type problem (2) is depending continuously on initial data.

3 Solution in $C_{1-\alpha}([0,T])$

Now, define the space $C_{1-\alpha}([0,T])$ by

$$C_{1-\alpha}([0,T])=\left\{u:t^{1-\alpha}u(t)\in C([0,T])\right\},$$

with norm

$$||u||_{C_{1-\alpha}}=||t^{1-\alpha}\,u||_C$$

and $C([0,T])$ is the space of continuous functions defined on $[0,T]$ with norm

$$||u||_C=\sup_{t\in[0,T]}|u(t)|.$$

Corollary 3.1. *Let the assumptions of Theorem 2.1 satisfied. Then the coupled system of weighted Cauchy-type problems (2) has a solution* $(u, v) \in C_{1-\alpha}([0, T]) \times C_{1-\alpha}([0, T])$.

Proof. Define

$$Y = \{(u(t), v(t)) | (u(t), v(t)) \in C_{1-\alpha} \times C_{1-\alpha} : ||(u, v)||_{C_{1-\alpha}\times C_{1-\alpha}} = \max\left(||u||_{C_{1-\alpha}}, ||v||_{C_{1-\alpha}}\right) \le r'\},$$

and define the subset Q_r by

$$Q_r = \{(u(t), v(t)) \in Y : ||(u(t), v(t))||_Y \le r'\},$$

where $r' = \max\{r'_1, r'_2\}$, (r'_1 and r'_2 will be indicated in the proof). The set Q_r is nonempty, closed and convex. Let $T : Q_r \to Q_r$. For $(u, v) \in Q_r$, T is a continuous operator: indeed, if $\{(u_n(t), v_n(t)\}$ is a sequence in Q_r which converges to $(u(t), v(t))$ for every $t \in [0, T]$. Then

$$\lim_{n\to\infty} T_1 v_n(t) = k_1\, t^{\alpha-1} + \lim_{n\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v_n(s), \int_0^s g_1(s, \theta, v_n(\theta)) d\theta\right) ds,$$

from the assumptions and Lebesgue dominated convergence theorem, we get that

$$\lim_{n\to\infty} T_1 v_n(t) = T_1 v(t).$$

Similarly

$$\lim_{n\to\infty} T_2 u_n(t) = T_2 u(t).$$

Then

$$\lim_{n\to\infty} T(u_n, v_n)(t) = T(u, v)(t).$$

For $(u, v) \in Q_r$, we have

$$\begin{aligned}
|t^{1-\alpha}\, T_1 v(t)| &\le k_1 + t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f_1\left(s, v(s), \int_0^s g_1(s, \theta, v(\theta))\, d\theta\right)\right| ds \\
&\le k_1 + t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[a(s) + b_1\, |v(s)| + b_2 \int_0^s \left|g_1(s, \theta, v(\theta))\right| d\theta\right] ds \\
&\le k_1 + t^{1-\alpha}\, I^\alpha |a(t)| + b_1\, t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|\, ds \\
&\quad + b_2\, t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \left[k(s, \theta) + b_3 |v(\theta)|\right] d\theta\, ds \\
&= k_1 + t^{1-\alpha}\, I^{\alpha-\gamma} I^\gamma |a(t)| + b_1\, t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, s^{\alpha-1}\, s^{1-\alpha}\, |v(s)|\, ds \\
&\quad + b_2\, t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s k(s, \theta)\, d\theta\, ds \\
&\quad + b_2\, b_3\, t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \theta^{\alpha-1} \theta^{1-\alpha}\, |v(\theta)|\, d\theta\, ds \\
&\le k_1 + T^{1-\alpha}\, I^{\alpha-\gamma}\, a^\star + b_1\, T^{1-\alpha}\, ||v||_{C_{1-\alpha}} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, s^{\alpha-1}\, ds
\end{aligned}$$

$$+b_2\, T^{1-\alpha} \frac{k^\star}{\Gamma(\alpha)} + b_2\, b_3\, T^{1-\alpha}\, ||v||_{C_{1-\alpha}} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \theta^{\alpha-1}\, d\theta\, ds$$

$$\leq k_1 + T^{1-\alpha} \frac{a^\star}{\Gamma(\alpha-\gamma+1)} T^{\alpha-\gamma} + b_1\, T^{1-\alpha}\, ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{2\alpha-1}$$

$$+b_2\, T^{1-\alpha} \frac{k^\star}{\Gamma(\alpha)} + b_2\, b_3\, T^{1-\alpha}\, ||v||_{C_{1-\alpha}} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{s^\alpha}{\alpha}\, ds$$

$$= k_1 + \frac{a^\star}{\Gamma(\alpha-\gamma+1)} T^{1-\gamma} + b_1\, ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha}$$

$$+ b_2\, T^{1-\alpha} \frac{k^\star}{\Gamma(\alpha)} + b_2\, b_3\, T^{1-\alpha}\, ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)}{\Gamma(2\alpha+1)} T^{2\alpha}$$

$$= k_1 + \frac{a^\star}{\Gamma(\alpha-\gamma+1)} T^{1-\gamma} + b_1\, ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha}$$

$$+b_2\, T^{1-\alpha} \frac{k^\star}{\Gamma(\alpha)} + b_2\, b_3\, ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)}{\Gamma(2\alpha+1)} T^{\alpha+1}$$

$$\leq k_1 + \frac{a^\star}{\Gamma(\alpha-\gamma+1)} T^{1-\gamma} + b_1\, r_1' \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha}$$

$$+b_2\, T^{1-\alpha} \frac{k^\star}{\Gamma(\alpha)} + b_2\, b_3\, r_1' \frac{\Gamma(\alpha)}{\Gamma(2\alpha+1)} T^{\alpha+1},$$

where $a^\star = \sup_{t\in[0,T]} I^\gamma |a(t)|$. Then

$$||T_1\, v||_{C_{1-\alpha}} \leq r_1',$$

where

$$r_1' = \frac{k_1 + \frac{a^\star}{\Gamma(\alpha-\gamma+1)} T^{1-\gamma} + \frac{b_2\, k^\star}{\Gamma(\alpha)} T^{1-\alpha}}{1 - \left(\frac{b_1\, \Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha} + \frac{b_2\, b_3\, \Gamma(\alpha)}{\Gamma(2\alpha+1)} T^{\alpha+1}\right)}, \quad 0 < \gamma < \alpha.$$

Similarly

$$|t^{1-\beta}\, T_2 u(t)| \leq k_2 + \frac{a^\star}{\Gamma(\beta-\gamma+1)} T^{1-\gamma} + b_1\, r_2' \frac{\Gamma(\beta)}{\Gamma(2\beta)} T^{\beta} + b_2\, T^{1-\beta} \frac{k^\star}{\Gamma(\beta)} + b_2\, b_3\, r_2' \frac{\Gamma(\beta)}{\Gamma(2\beta+1)} T^{\beta+1},$$

$$||T_2\, u||_{C_{1-\alpha}} \leq r_2',$$

where

$$r_2' = \frac{k_2 + \frac{a^\star}{\Gamma(\beta-\gamma+1)} T^{1-\gamma} + \frac{b_2\, k^\star}{\Gamma(\beta)} T^{1-\beta}}{1 - \left(\frac{b_1\, \Gamma(\beta)}{\Gamma(2\beta)} T^{\beta} + \frac{b_2\, b_3\, \Gamma(\beta)}{\Gamma(2\beta+1)} T^{\beta+1}\right)}, \quad 0 < \gamma < \beta.$$

Then $T_1 v(t)$ is uniformly bounded in Q_r, similarly $T_2 u(t)$ is uniformly bounded in Q_r.

Since

$$||T(u,v)(t)|| = ||T_1 v(t), T_2 u(t)|| = \max\left(||T_1 v||_{C_{1-\alpha}}, ||T_2 u||_{C_{1-\alpha}}\right) \leq \max(r_1', r_2') \leq r'.$$

Therefore, T is uniformly bounded in Q_r.

Now, we show that T is a completely continuous operator.
Indeed, let $\tau_1, \tau_2 \in [0,T]$, $\tau_1 < \tau_2$ such that $|\tau_2 - \tau_1| < \delta$, we have

$$\tau_2^{1-\alpha} T_1 v(\tau_2) - \tau_1^{1-\alpha} T_1 v(\tau_1) = {\tau_2}^{1-\alpha} \int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right) ds$$

$$- {\tau_1}^{1-\alpha} \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), \int_0^s g_1(s,\theta,v(\theta))d\theta\right) ds$$

$$
\begin{aligned}
&= \tau_2{}^{1-\alpha}\int_0^{\tau_1}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1\left(s,v(s),\int_0^s g_1(s,\theta,v(\theta))d\theta\right)ds\\
&\quad+\tau_2{}^{1-\alpha}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1\left(s,v(s),\int_0^s g_1(s,\theta,v(\theta))d\theta\right)ds\\
&\quad-\tau_1{}^{1-\alpha}\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_1\left(s,v(s),\int_0^s g_1(s,\theta,v(\theta))d\theta\right)ds\\
&\le\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_1\left(s,v(s),\int_0^s g_1(s,\theta,v(\theta))d\theta\right)ds\\
&\quad+\tau_2{}^{1-\alpha}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1\left(s,v(s),\int_0^s g_1(s,\theta,v(\theta))d\theta\right)ds,
\end{aligned}
$$

$$
\begin{aligned}
\left|\tau_2^{1-\alpha}T_1v(\tau_2)-\tau_1^{1-\alpha}T_1v(\tau_1)\right| &\le\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_1\left(s,v(s),\int_0^s g_1(s,\theta,v(\theta))d\theta\right)\right|ds\\
&\quad+\tau_2{}^{1-\alpha}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_1\left(s,v(s),\int_0^s g_1(s,\theta,v(\theta))d\theta\right)\right|ds\\
&\le\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[a(s)+b_1|v(s)|\\
&\quad+b_2\int_0^s\left|g_1(s,\theta,v(\theta))\right|d\theta\Big]ds\\
&\quad+\tau_2{}^{1-\alpha}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left[a(s)+b_1|v(s)|+b_2\int_0^s\left|g_1(s,\theta,v(\theta))\right|d\theta\right]ds\\
&\le\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)I^{\alpha-\gamma}I^{\gamma}\,a(\tau_1)\\
&\quad+b_1\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{\alpha-1}s^{1-\alpha}|v(s)|ds\\
&\quad+b_2\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\left[k(s,\theta)\;+\;b_3|v(\theta)|\right]d\theta\,ds\\
&\quad+\tau_2^{1-\alpha}I_{\tau_1}^{\alpha-\gamma}I_{\tau_1}^{\gamma}a(\tau_2)+b_1\tau_2^{1-\alpha}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{\alpha-1}s^{1-\alpha}|v(s)|ds\\
&\quad+b_2\tau_2^{1-\alpha}\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\left[k(s,\theta)\;+\;b_3|v(\theta)|\right]d\theta\,ds\\
&\le\frac{a^*}{\Gamma(\alpha-\gamma+1)}\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)\tau_1^{\alpha-\gamma}\\
&\quad+b_1\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)||v||_{C_{1-\alpha}}\frac{\Gamma(\alpha)\tau_1^{2\alpha-1}}{\Gamma(2\alpha)}\\
&\quad+b_2k^*\frac{1}{\Gamma(\alpha)}\left(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}\right)
\end{aligned}
$$

$$+b_2 b_3 \left(\tau_2^{1-\alpha} - \tau_1^{1-\alpha}\right) ||v||_{C_{1-\alpha}} \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \theta^{\alpha-1} d\theta\, ds$$

$$+\frac{a^\star(\tau_2 - \tau_1)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \tau_2^{1-\alpha} + b_1 \tau_2^{1-\alpha} ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)(\tau_2-\tau_1)^{2\alpha-1}}{\Gamma(2\alpha)}$$

$$+b_2\, k^\star \frac{1}{\Gamma(\alpha)} \tau_2^{1-\alpha} + b_2 b_3 \tau_2^{1-\alpha} ||v||_{C_{1-\alpha}} \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \theta^{\alpha-1} d\theta\, ds$$

$$\le \frac{a^\star}{\Gamma(\alpha-\gamma+1)} \left(\tau_2^{1-\alpha} - \tau_1^{1-\alpha}\right) \tau_1^{\alpha-\gamma}$$

$$+b_1 \left(\tau_2^{1-\alpha} - \tau_1^{1-\alpha}\right) ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)\tau_1^{2\alpha-1}}{\Gamma(2\alpha)}$$

$$+b_2\, k^\star \frac{1}{\Gamma(\alpha)} \left(\tau_2^{1-\alpha} - \tau_1^{1-\alpha}\right)$$

$$+b_2 b_3 \left(\tau_2^{1-\alpha} - \tau_1^{1-\alpha}\right) ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)\, \tau_1^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$+\frac{a^\star(\tau_2 - \tau_1)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \tau_2^{1-\alpha} + b_1 \tau_2^{1-\alpha} ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)(\tau_2-\tau_1)^{2\alpha-1}}{\Gamma(2\alpha)}$$

$$+b_2 k^\star \frac{1}{\Gamma(\alpha)} \tau_2^{1-\alpha} + b_2 b_3 \tau_2^{1-\alpha} ||v||_{C_{1-\alpha}} \frac{\Gamma(\alpha)\,(\tau_2-\tau_1)^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Therefore $\{T(u,v)(t)\}$ is equi-continuous. By Arzela-Ascoli Theorem then $\{T(u,v)(t)\}$ is relatively compact. Therefore, the conditions of the Schauder fixed point Theorem hold, which implies that T has a fixed point in Q_r. Then (2) has a solution $(u,v) \in C_{1-\alpha}([0,T]) \times C_{1-\alpha}([0,T])$.

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