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Research Article

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L^p estimates for maximal functions along surfaces of revolution on product spaces

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Abstract: This paper is concerned with establishing L^p estimates for a class of maximal operators associated to surfaces of revolution with kernels in $L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, $q > 1$. These estimates are used in extrapolation to obtain the L^p boundedness of the maximal operators and the related singular integral operators when their kernels are in the $L(logL)^\kappa(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ or in the block space $B_q^{0,\kappa-1}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Our results substantially improve and extend some known results.

Keywords: maximal functions, L^p boundedness, Rough kernels, surfaces of revolution, extrapolation

MSC: Primary 42B20; Secondary 40B25, 47G10

1 Introduction and main results

Let $n, m \geq 2$, and let $\mathbf{R}^N$ ($N = n$ or m) be the N-dimensional Euclidean space. Let $\mathbf{S}^{N-1}$ be the unit sphere in $\mathbf{R}^N$ equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$. Also, let $x' = x/|x|$ for $x \in \mathbf{R}^n \setminus \{0\}$, $y' = y/|y|$ for $y \in \mathbf{R}^m \setminus \{0\}$.

Let $K_{\Omega,h}(x, y) = \Omega(x', y')|x|^{-n}|y|^{-m}h(|x|, |y|)$, where h is a measurable function on $\mathbf{R}^+ \times \mathbf{R}^+$ and Ω is an integrable function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ that satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(x', .)d\sigma(x') = \int_{\mathbf{S}^{m-1}} \Omega(., y')d\sigma(y') = 0 \quad and \tag{1.1}$$

$$\Omega(rx, ty) = \Omega(x, y) \quad for \;\; all \;\; r, t > 0. \tag{1.2}$$

For suitable mappings $\phi, \psi : \mathbf{R}^+ \to \mathbf{R}$, consider the singular integral operator $T^{P_1,P_2}_{\Omega,h,\phi,\psi}$ defined, initially for $\mathcal{C}_0^\infty$ functions on $\mathbf{R}^{n+1} \times \mathbf{R}^{m+1}$, by

$$T^{P_1,P_2}_{\Omega,h,\phi,\psi}(f)(\overline{x}, \overline{y}) = p.v \int_{\mathbf{R}^n \times \mathbf{R}^m} e^{iP_1(u)+iP_2(v)} \times f(x-u, x_{n+1} - \phi(|u|), y-v, y_{m+1} - \psi(|v|))K_{\Omega,h}(u, v)dudv,$$

where $(\overline{x}, \overline{y}) = (x, x_{n+1}, y, y_{m+1}) \in \mathbf{R}^{n+1} \times \mathbf{R}^{m+1}$ and $P_1 : \mathbf{R}^n \to \mathbf{R}$, $P_2 : \mathbf{R}^m \to \mathbf{R}$ are two real-valued polynomials.

When $P_1(u) = 0$ and $P_2(v) = 0$, we denote $T^{P_1,P_2}_{\Omega,h,\phi,\psi}$ by $T_{\Omega,h,\phi,\psi}$. Also, when $\phi(t) = \psi(t) = t$, then $T_{\Omega,h,\phi,\psi}$ (denoted by $T_{\Omega,h}$) is just the classical singular integral operator introduced by Fefferman in [1] in which he

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obtained the L^p boundedness of $T_{\Omega,h}$ for all $1 < p < \infty$ whenever Ω satisfies some regularity conditions and $h \equiv 1$. As a matter of fact, the systematic study of such operator began by Fefferman in [1], and then it was elaborated very much by Fefferman and Stein in [2]. Subsequently, the investigation of the L^p boundedness of $T_{\Omega,h}$ under very various conditions on Ω and h has attracted the attention of many authors. For example, it was proved in [3] that $T_{\Omega,h}$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for $1 < p < \infty$ when $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and h satisfies certain integrability-size condition. Furthermore, the authors of [3] established the optimality of the condition in the sense that the space $L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ cannot be replaced by $L(\log L)^{2-\varepsilon}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for any $0 < \varepsilon < 2$. For more information about the importance and the recent advances on the study of such operators, the readers are refereed (for instance to [1–5], and the references therein).

On the other side, the study of the singular integrals on product spaces along surfaces of revolution has been started. For example, if ϕ and ψ are in $C^2([0,\infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$, then Al-Salman in [4] showed that $T_{\Omega,1,\phi,\psi}$ is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ $(1 < p < \infty)$ provided that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Recently, Al-Salman improved this result in [6]. In fact, when ϕ, ψ are given as in [4], he verified the L^p boundedness of $T_{\Omega,h,\phi,\psi}$ for all $p \in (1,\infty)$ under the conditions $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and $h \in L^2(\mathbf{R}^+ \times \mathbf{R}^+, \frac{drdt}{rt})$ with $\|h\|_{L^2(\mathbf{R}^+\times\mathbf{R}^+, \frac{drdt}{rt})} \leq 1$.

The maximal operator that related to our singular integral operator is $\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}$ that given by

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}(f)(\overline{x},\overline{y}) = \sup_{h\in U}\left|T^{P_1,P_2}_{\Omega,h,\phi,\psi}(f)(\overline{x},\overline{y})\right|,$$

where $U = \left\{h \in L^2(\mathbf{R}^+ \times \mathbf{R}^+, \frac{drdt}{rt});\ \|h\|_{L^2(\mathbf{R}^+\times\mathbf{R}^+, \frac{drdt}{rt})} \leq 1\right\}$.

Again, when $P_1(u) = 0$ and $P_2(v) = 0$, we denote $\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}$ by $\mathcal{M}_{\Omega,\phi,\psi}$. Also, when $\phi(t) = \psi(t) = t$, then $\mathcal{M}_{\Omega,\phi,\psi}$ reduces to the classical maximal operator denoted by $\mathcal{M}_\Omega$. Historically, The operator $\mathcal{M}_\Omega$ was introduced by Ding in [7] in which he proved the L^2 boundedness of $\mathcal{M}_\Omega$ whenever $\Omega \in L(\log L)^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$. This result was improved independently by Al-Qassem and Pan in [8] and by Al-Salman in [9]. Precisely, they showed that $\mathcal{M}_\Omega$ is of type (p,p) for all $p \geq 2$ provided that $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Moreover, they pointed out that the condition $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is optimal in the sense that the exponent 1 in $L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ cannot be replaced by any smaller positive number $\tau < 1$ so that $\mathcal{M}_\Omega$ is bounded on $L^2(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$. Also, an improvement of the result in [7] was obtained by Al-Qassem in [10]. Indeed, Al-Qassem established the $L^p (2 \leq p < \infty)$ estimates for the class $\mathcal{M}_\Omega$ whenever Ω belongs to the block space $B^{(0,0)}_q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$. Furthermore, he proved that the condition $\Omega \in B^{(0,0)}_q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is nearly optimal in the sense that the operator $\mathcal{M}_\Omega$ may lose the L^2 boundedness if Ω is assumed to be in the space $B^{(0,\varepsilon)}_q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $-1 < \varepsilon < 0$. Recently, it was found in [6] that the maximal operator $\mathcal{M}_{\Omega,\phi,\psi}$ is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for any $p \geq 2$ if $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, and ϕ, ψ are in $C^2([0,\infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$. Very recently, when $\phi(t) = \psi(t) = t$, Al-Dolat and et al. found in [11] that the L^p $(p \geq 2)$ boundedness of $\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}$ is obtained under the condition $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \cup B^{(0,0)}_q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ with $q > 1$. Subsequently, the investigation of the L^p boundedness of $\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}$ under weak conditions has received much attentions from many mathematicians. For the significance of considering the integral operators $\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}$, we refear the readers to consult [8] and [11–13], among others.

The main result of this work is formulated as follows:

Theorem 1.1. *Let $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, $q > 1$ and satisfy the conditions (1.1)-(1.2) with $\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \leq 1$, and let $\mu = \mu_q(\Omega) = \log(e+\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})})$. Assume that ϕ, ψ are in $C^2([0,\infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$. Let $P_1 : \mathbf{R}^n \to \mathbf{R}$ and $P_2 : \mathbf{R}^m \to \mathbf{R}$ be two real-valued polynomials of degrees d_1, d_2, respectively. Then there exists a constant $C_{p,q} > 0$ such that*

$$\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}(f)\right\|_{L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})} \leq C_{p,q}\,(1+\mu)\,\|f\|_{L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})} \tag{1.3}$$

for all $p \geq 2$, where $C_{p,q} = \left(\frac{2^{1/q'}}{2^{1/q'}-1}\right)^2 C_p$ and C_p is a positive constant that may depend on the degrees of the polynomials P_1, P_2 but it is independent on Ω, ϕ, ψ, q, and the coefficients of the polynomials P_1, P_2.

We remark that by the result in Theorem 1.1 and using an extrapolation argument, we get that $\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}$ is bounded on $L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})$ for $2\le p<\infty$ if $\Omega\in L(\log L)(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\cup B^{(0,0)}_q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q>1$.

Here and henceforth, the letter C denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

2 Preliminary lemmas

In this section, we present and prove some lemmas used in the sequel. The first lemma can be derived by applying the same technique that Al-Qassam and Pan used in [14, pp. 64-65].

Lemma 2.1. *Let $\Omega\in L^q(\mathbf{S}^{N-1})$, $q>1$ be a homogeneous function of degree zero on $\mathbf{R}^N$ with $\|\Omega\|_{L^1(\mathbf{S}^{N-1})}\le 1$, and let $\phi:\mathbf{R}^+\to\mathbf{R}$ be a $C^2([0,\infty))$, convex and increasing function with $\phi(0)=0$. Consider the maximal function $\mathcal{N}_{\Omega,\phi}$ given by*

$$\mathcal{N}_{\Omega,\phi}f(\overline{z})=\sup_{\mathbf{j}\in\mathbf{Z}}\int\limits_{2^j\le|y|\le 2^{j+1}}\left|f(z-y,z_{N+1}-\phi(|y|))\right|\frac{|\Omega(y)|}{|y|^N}dy.$$

Then for $p>1$ and $f\in L^p(\mathbf{R}^{N+1})$ there exists a positive number C_p such that

$$\left\|\mathcal{N}_{\Omega,\phi}(f)\right\|_p\le C_p\|f\|_p .$$

Lemma 2.2. *Assume that ϕ,ψ are $C^2([0,\infty))$, convex and increasing functions with $\phi(0)=\psi(0)=0$. Let $\Omega\in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, $q>1$ and satisfy the conditions (1.1)-(1.2) with $\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\le 1$. Then for all $f\in L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})$ and $p>1$, the maximal function*

$$\mathcal{N}_{\Omega,\phi,\psi}f(\overline{x},\overline{y})=\sup_{i,j\in\mathbf{Z}}\iint\limits_{\Lambda_{i,j}}\left|f(x-u,x_{n+1}-\phi(|u|),y-v,y_{m+1}-\psi(|v|))\right|\frac{|\Omega(u,v)|}{|u|^n|v|^m}dudv$$

satisfies

$$\left\|\mathcal{N}_{\Omega,\phi,\psi}(f)\right\|_p\le C_p\|f\|_p ,$$

where $\Lambda_{i,j}=\left\{(u,v)\in\mathbf{R}^n\times\mathbf{R}^m:2^i\le|u|\le 2^{i+1},2^j\le|v|\le 2^{j+1}\right\}$ and the positive constant C_p is independent of the functions ϕ,ψ and Ω.

It is easy to prove the above lemma by using Lemma 2.1 and the inequality $\mathcal{N}_{\Omega,\phi,\psi}f(\overline{x},\overline{y})\le\mathcal{N}_{\Omega,\psi}\circ\mathcal{N}_{\Omega,\phi}f(\overline{x},\overline{y})$, where $\mathcal{N}_{\Omega,\phi}f(\overline{x},\overline{y})=\mathcal{N}_{\Omega,\phi}f(\cdot,\overline{y})(\overline{x})$, $\mathcal{N}_{\Omega,\psi}f(\overline{x},\overline{y})=\mathcal{N}_{\Omega,\psi}f(\overline{x},\cdot)(\overline{y})$, and $\circ$ denotes the composition of operators.

A significant step toward proving Theorem 1.1 is to estimate the following Fourier transform:

Lemma 2.3. *Let $\Omega\in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, $q>1$ and satisfy the conditions (1.1)-(1.2) with $\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\le 1$, and let $\mu=\mu_q(\Omega)=\log(e+\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})})$. Assume that ϕ,ψ are arbitrary functions on $\mathbf{R}^+$, and assume also that $P_1=\sum_{|\alpha|\le d_1}a_\alpha x^\alpha$ is a polynomial of degree $d_1\ge 1$ such that $|x|^{d_1}$ is not one of its terms and $\sum_{|\alpha|=d_1}|a_\alpha|=1$; and $P_2=\sum_{|\beta|\le d_2}b_\beta y^\beta$ is a polynomial of degree $d_2\ge 1$ such that $|y|^{d_2}$ is not one of its terms and $\sum_{|\beta|=d_2}|b_\beta|=1$. For $i,j\in\mathbf{Z}$, define $\mathcal{J}_{i,j,\Omega,\phi,\psi}:\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}\to\mathbf{R}$ by*

$$\mathcal{J}_{i,j,\Omega,\phi,\psi}(\overline{\xi},\overline{\eta})=\int_1^{2^{2\mu}}\int_1^{2^{2\mu}}\left|\int\limits_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega(u,v)\mathcal{A}_{i,\Omega,\phi}(r,u,\xi.u,\xi_{n+1})\mathcal{B}_{j,\Omega,\psi}(t,v,\eta.v,\eta_{m+1})d\sigma(u)d\sigma(v)\right|^2\frac{drdt}{rt},$$

where

$$\mathcal{A}_{i,\Omega,\phi}(r,u,\xi.u,\xi_{n+1})=e^{-i\left[P_1(2^{-(i+1)\mu}ru)+(2^{-(i+1)\mu})ru\cdot\xi+\phi(2^{-(i+1)\mu}r)\xi_{n+1}\right]}$$

and

$$\mathcal{B}_{j,\Omega,\psi}(t,v,\eta.v,\eta_{m+1})=e^{-i\left[P_2(2^{-(j+1)\mu}tv)+(2^{-(j+1)\mu})tv\cdot\eta+\psi(2^{-(j+1)\mu}t)\eta_{m+1}\right]}.$$

Then, a positive constant C exists such that

$$\sup_{(\overline{\xi},\overline{\eta})\in\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}} \mathcal{J}_{i,j,\Omega,\phi,\psi}(\overline{\xi},\overline{\eta}) \le C\mu^2 2^{(i+j+2)/4q'}.$$

Proof. On one hand, it is trivial to get that

$$\begin{aligned}\mathcal{J}_{i,j,\Omega,\phi,\psi}(\overline{\xi},\overline{\eta}) &\le C\int_1^{2^{2\mu}}\int_1^{2^{2\mu}}\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|\Omega(u,v)|\,d\sigma(u)d\sigma(v)\right)^2\frac{drdt}{rt}\\ &\le C\mu^2\,\|\Omega\|_1^2 \le C\mu^2.\end{aligned} \tag{2.1}$$

Also, it is easy to see that

$$P_1(\gamma ru)-P_1(\gamma rz)+\gamma ru\cdot\xi-\gamma rz\cdot\xi=(\gamma r)^{d_1}\left(\sum_{|\alpha|=d_1}a_\alpha u^\alpha-\sum_{|\alpha|=d_1}a_\alpha z^\alpha\right)+\gamma r(u-z)\cdot\xi+H(u,z,r,\xi),$$

with $\frac{d^{d_1}}{dr^{d_1}}H(u,z,r,\xi)=0$ and $\gamma=2^{-(i+1)\mu}$. Without lossing of generality, we may assume that $d_1>1$. Hence, by Van der-Corput Lemma, we obtain

$$\left|\int_1^{2^{2\mu}}\mathcal{A}_{i,\Omega,\phi}(r,u,\xi.u,\xi_{n+1})\overline{\mathcal{A}_{i,\Omega,\phi}(r,z,\xi.z,\xi_{n+1})}\frac{dr}{r}\right| \le C\left|\gamma^{d_1}\sum_{|\alpha|=d_1}a_\alpha\left(u^\alpha-z^\alpha\right)\right|^{-1/d_1}.$$

Combine the last inequality with the trivial estimates

$$\left|\int_1^{2^{2\mu}}\mathcal{A}_{i,\Omega,\phi}(r,u,\xi.u,\xi_{n+1})\overline{\mathcal{A}_{i,\Omega,\phi}(r,z,\xi.z,\xi_{n+1})}\frac{dr}{r}\right| \le C\mu,$$

we deduce

$$\left|\int_1^{2^{2\mu}}\mathcal{A}_{i,\Omega,\phi}(r,u,\xi.u,\xi_{n+1})\overline{\mathcal{A}_{i,\Omega,\phi}(r,z,\xi.z,\xi_{n+1})}\frac{dr}{r}\right| \le C\mu^{1-\theta}\left|\gamma^{d_1}\sum_{|\alpha|=d_1}a_\alpha\left(u^\alpha-z^\alpha\right)\right|^{-\theta/d_1}$$

for any $0<\theta<1$. In the same manner, we derive

$$\left|\int_1^{2^{2\mu}}\mathcal{B}_{j,\Omega,\psi}(t,v,\eta.v,\eta_{m+1})\overline{\mathcal{B}_{j,\Omega,\psi}(t,w,\eta.w,\eta_{m+1})}\frac{dt}{t}\right| \le C\mu^{1-\theta}\left|2^{-(j+1)\mu d_2}\sum_{|\beta|=d_2}b_\beta\left(v^\beta-w^\beta\right)\right|^{-\theta/d_2}.$$

Thus, using Hölder's inequality leads to

$$\begin{aligned}\left(\mathcal{J}_{i,j,\Omega,\phi,\psi}(\overline{\xi},\overline{\eta})\right)^{q'} \le \|\Omega\|_q^{2q'}\iint_{(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})^2}&\left|\int_1^{2^{2\mu}}\int_1^{2^{2\mu}}\mathcal{A}_{i,\Omega,\phi}(r,u,\xi.u,\xi_{n+1})\overline{\mathcal{A}_{i,\Omega,\phi}(r,z,\xi.z,\xi_{n+1})}\frac{dr}{r}\right.\\ &\left.\times\,\mathcal{B}_{j,\Omega,\psi}(t,v,\eta.v,\eta_{m+1})\overline{\mathcal{B}_{j,\Omega,\psi}(t,w,\eta.w,\eta_{m+1})}\frac{dt}{t}\right|^{q'}\sigma(u)d\sigma(z)\sigma(v)d\sigma(w).\end{aligned} \tag{2.2}$$

Since $\sum_{|\alpha|=d_1}|a_\alpha|=\sum_{|\beta|=d_2}|b_\beta|=1$, then by taking $\theta=1/4\mu q'$, we have

$$\mathcal{J}_{i,j,\Omega,\phi,\psi}(\overline{\xi},\overline{\eta}) \le \|\Omega\|_q^2\,2^{(i+1)/4q'}2^{(j+1)/4q'}\mu^{2-1/2\mu q'} \le C\mu^2 2^{(i+j+2)/4q'}. \tag{2.3}$$

□

We shall need the following Lemma which can be acquired by using the arguments employed in the proof of [6, Theorem 4.1] as well as [15, Theorem 1.6].

Lemma 2.4. *Let $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, $q > 1$ and satisfy the conditions (1.1)-(1.2) with $\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \le 1$. Assume that ϕ, ψ and μ are given as in Theorem 1.1. Then there exists a constant $C_{p,q} > 0$ such that*

$$\left\|\mathcal{M}_{\Omega,\phi,\psi}(f)\right\|_{L^p(\mathbf{R}^{n+1}\times\mathbf{S}^{m-1})} \le C_{p,q}\,(1+\mu)\,\|f\|_{L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})} \tag{2.4}$$

for $2 \le p < \infty$.

Proof. Choose collections of functions $\{\Phi_i\}_{i\in\mathbf{Z}}$ and $\{\Psi_j\}_{j\in\mathbf{Z}}$ defined on $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively with the following properties:

(i) $\widehat{\Phi}_i$ *is supported in* $\left\{\xi \in \mathbf{R}^n : |\xi| \in \mathcal{I}_{i,\mu} = \left[2^{-(i+1)\mu}, 2^{-(i-1)\mu}\right]\right\}$;

(ii) $\widehat{\Psi}_j$ *is supported in* $\left\{\eta \in \mathbf{R}^m : |\eta| \in \mathcal{I}_{j,\mu}\right\}$;

(iii) $0 \le \widehat{\Phi}_i,\ \widehat{\Psi}_j \le 1$;

(iv) $\sum_{i\in\mathbf{Z}}\left(\widehat{\Phi}_i\right)^2(\xi) = \sum_{j\in\mathbf{Z}}\left(\widehat{\Psi}_j\right)^2(\eta) = 1.$

Define the multiplier operators $S_{j,i}$ in $\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}$ via the Fourier transform given by

$$\widehat{S_{j,i}}(\overline{\xi},\overline{\eta}) = \widehat{\Phi}_i(|\xi|)\widehat{\Psi}_j(|\eta|)).$$

Hence, for any $f \in \mathcal{C}_0^\infty(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})$, we have

$$\mathcal{M}_{\Omega,\phi,\psi}(f)(\overline{x},\overline{y}) \le \sum_{j,i\in\mathbf{Z}} \mathcal{T}_{\Omega,\phi,\psi,j,i}(f)(\overline{x},\overline{y}), \tag{2.5}$$

where

$$\mathcal{T}_{\Omega,\phi,\psi,j,i}(f)(\overline{x},\overline{y}) = \left(\iint_{\mathbf{R}^+\times\mathbf{R}^+}\left|\mathcal{W}_{\Omega,\phi,\psi,j,i}(f)(\overline{x},\overline{y})\right|^2\frac{drdt}{rt}\right)^{1/2},$$

$$\left(\mathcal{W}_{\Omega,\phi,\psi,j,i}(f)\right)(\overline{x},\overline{y}) = \sum_{s,l\in\mathbf{Z}}\iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} S_{j+l,i+s}(f)(x-ru, x_{n+1}-\phi(r), y-tv, y_{m+1}-\psi(t))\chi_{\mathcal{I}_{s,\mu}\times\mathcal{I}_{l,\mu}}\Omega(u,v)d\sigma(u)\sigma(v).$$

Therefore, by using [6, Theorem 4.1], we get

$$\left\|\mathcal{T}_{\Omega,\phi,\psi,j,i}(f)\right\|_p \le C_{p,q}\mu 2^{-\varepsilon_1|j|}2^{-\varepsilon_2|i|}\,\|f\|_p \tag{2.6}$$

for some constants $0 < \varepsilon_1, \varepsilon_2 < 1$ and for all $2 \le p < \infty$. Consequently, the inequality (2.4) follows by using (2.5) and (2.6). □

3 Proof of Theorem 1.1

The proof of Theorem 1.1 mainly depends on the approaches employed in the proof of [11, Theorem 1.1], which have their roots in [16]. Precisly, we argue the mathematical induction on the degrees of the polynomials P_1 and P_2.

If $d_1 = d_2 = 0$, then by Lemma 2.4, we directly attain

$$\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}(f)\right\|_p \le C_{p,q}\ (1+\mu)\,\|f\|_p \tag{3.1}$$

for all $p \ge 2$. Also, if $d_1 = 0$ or $d_2 = 0$, then by [17, Theorem 1.1], it is easy to satisfy the inequality (1.3) for all $p \ge 2$.

Now, assume that (1.3) is true for any polynomial P_1 of degree less than or equal to d_1 and for any polynomial P_2 of degree d_2. We need to show that (1.3) is still true if $degree(P_1) = d_1 + 1$, and $degree(P_2) = d_2$. Without loss of generality, we may assume $P_1(x) = \sum_{|\alpha|\leq d_1+1} a_\alpha x^\alpha$ is a polynomial of degree $d_1 + 1$ such that $\sum_{|\alpha|=d_1+1} |a_\alpha| = 1$ and does not contain $|x|^{d_1+1}$ as one of its terms. Also, we may assume $P_2(y) = \sum_{|\beta|\leq d_2} b_\beta y^\beta$ is a given polynomial of degree d_2 such that $\sum_{|\beta|=d_2} |b_\beta| = 1$ and does not contain $|y|^{d_2}$ as one of its terms. By duality and a simple change of variables, we have

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}(f)(\bar{x},\bar{y}) = \left(\iint_{\mathbf{R}^+\times\mathbf{R}^+} |\,\mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t)|^2 \frac{drdt}{rt}\right)^{1/2},$$

where

$$\mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} e^{iP_1(ru)+iP_2(tv)} f(x-ru, x_{n+1}-\phi(r), y-tv, y_{m+1}-\psi(t))\Omega(u,v)d\sigma(u)d\sigma(v).$$

Choose two collections of $\mathcal{C}^\infty$ functions $\{Y_i\}_{i\in\mathbf{Z}}$ and $\{\Gamma_j\}_{j\in\mathbf{Z}}$ on $(0,\infty)$, that satisfying the following conditions:

$$\operatorname{supp} Y_i \subseteq \mathcal{I}_{i,\mu} = \left[2^{-(i+1)\mu}, 2^{-(i-1)\mu}\right];\ \operatorname{supp}\Gamma_j \subseteq \mathcal{I}_{j,\mu}\ ;$$
$$0 \leq Y_i, \Gamma_j \leq 1;\quad and \sum_{i\in\mathbf{Z}} Y_i(u) = \sum_{j\in\mathbf{Z}} \Gamma_j(v) = 1.$$

Define the multiplier operators $S_{j,i}$ in $\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}$ by

$$\widehat{(S_{j,i}f)}(\bar{\xi},\bar{\eta}) = Y_i(|\xi|)\Gamma_j(|\eta|)\hat{f}(\xi,\eta)\quad for\ \ (\bar{\xi},\bar{\eta}) = (\xi,\xi_{n+1},\eta,\eta_{m+1}) \in \mathbf{R}^{n+1}\times\mathbf{R}^{m+1}.$$

Set

$$A_\infty(u) = \sum_{i=-\infty}^{0} Y_i(u),\quad A_0(u) = \sum_{i=1}^{\infty} Y_i(u),\quad B_\infty(v) = \sum_{j=-\infty}^{0} \Gamma_j(v),\quad and\ B_0(v) = \sum_{j=1}^{\infty} \Gamma_j(v).$$

Thanks to Minkowski's inequality, we have

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi}(f)(\bar{x},\bar{y}) \leq \mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty}(f)(\bar{x},\bar{y}) + \mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,0}(f)(\bar{x},\bar{y}) + \mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,0,\infty}(f)(\bar{x},\bar{y}) + \mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,0,0}(f)(\bar{x},\bar{y}), \tag{3.2}$$

where

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty}(f)(\bar{x},\bar{y}) = \left(\int_{2^{-\mu}}^{\infty}\int_{2^{-\mu}}^{\infty} |\,A_\infty(r)B_\infty(t)\mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t)|^2 \frac{drdt}{rt}\right)^{1/2},$$

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,0}(f)(\bar{x},\bar{y}) = \left(\int_{2^{-\mu}}^{\infty}\int_{0}^{1} |\,A_\infty(r)B_0(t)\mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t)|^2 \frac{drdt}{rt}\right)^{1/2},$$

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,0,\infty}(f)(\bar{x},\bar{y}) = \left(\int_{0}^{1}\int_{2^{-\mu}}^{\infty} |\,A_0(r)B_\infty(t)\mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t)|^2 \frac{drdt}{rt}\right)^{1/2},$$

and

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,0,0}(f)(\bar{x},\bar{y}) = \left(\int_{0}^{1}\int_{0}^{1} |\,A_0(r)B_0(t)\mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t)|^2 \frac{drdt}{rt}\right)^{1/2}.$$

Let us first estimate the L^p-norm of $\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty}(f)$. Define

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty,i,j}(f)(\bar{x},\bar{y}) = \left(\int_{2^{-(i-1)\mu}}^{2^{-(i+1)\mu}}\int_{2^{-(j-1)\mu}}^{2^{-(j+1)\mu}} \left| \mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t)\right|^2 \frac{drdt}{rt}\right)^{1/2}.$$

Hence, by generalized Minkowski's inequality, it is easy to reach

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty}(f)(\bar{x},\bar{y}) \le \sum_{i,j=-\infty}^{0} \mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty,i,j}(f)(\bar{x},\bar{y}). \tag{3.3}$$

If $p = 2$, then by a simple change of variables, Plancherel's theorem, Fubini's theorem, and Lemma 2.3, we get that

$$\begin{aligned}\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty,i,j}(f)\right\|_2 &= \left(\int_{\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}} \left|\widehat{f}(\bar{\xi},\bar{\eta})\right|^2 \mathcal{J}_{i,j,\Omega,\phi,\psi}(\bar{\xi},\bar{\eta})d\bar{\xi}d\bar{\eta}\right)^{1/2}\\ &\le C2^{\frac{(i+j+2)}{8q'}}(1+\mu)\,\|f\|_2\,.\end{aligned} \tag{3.4}$$

However, if $p > 2$, then by the duality, there exists $b \in L^{(p/2)'}(\mathbf{R}^{n+1}\times\mathbf{R}^{n+1})$ with $\|b\|_{L^{(p/2)'}(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})} = 1$ such that

$$\begin{aligned}\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty,i,j}(f)\right\|_p^2 = &\iint_{\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}}\int_1^{2^{2\mu}}\int_1^{2^{2\mu}}\Bigg|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega(u,v)\mathcal{A}_{i,\Omega,\phi}(r,u,0,0)\mathcal{B}_{j,\Omega,\psi}(t,v,0,0)\\ &\times f(x-2^{-(i+1)\mu}ru, x_{n+1}-\phi(2^{-(i+1)\mu}r), y-2^{-(j+1)\mu}tu, y_{m+1}-\psi(2^{-(j+1)\mu}t))d\sigma(u)d\sigma(v)\Bigg|^2\\ &\times\frac{drdt}{rt}\left|b(\bar{x},\bar{y})\right|d\bar{x}d\bar{y}.\end{aligned}$$

So, by Hölder's inequality and Lemma 2.2, we conclude that

$$\begin{aligned}\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty,i,j}(f)\|_p^2 &\le C\iint_{\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}}\left|f(\bar{z},\bar{w})\right|^2\int_1^{2^{2\mu}}\int_1^{2^{2\mu}}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\left|\Omega(u,v)\right|\\ &\times\left|b(z+2^{-(i+1)\mu}ru, z_{n+1}+\phi(2^{-(i+1)\mu}r), w+2^{-(j+1)\mu}tu, w_{m+1}+\psi(2^{-(j+1)\mu}t))\right|d\sigma(u)d\sigma(v)\frac{drdt}{rt}d\bar{z}d\bar{w}\\ &\le C\,(1+\mu)^2\left\||f|^2\right\|_{(p/2)}\left\|\mathcal{N}_{\Omega,\phi,\psi}(\widetilde{b})\right\|_{(p/2)'}\\ &\le C_p\,(1+\mu)^2\,\|f\|_p^2\left\|\widetilde{b}\right\|_{(p/2)'}\|\Omega\|_1\,,\end{aligned}$$

where $\widetilde{b}(\bar{z},\bar{w}) = b(-\bar{z},-\bar{w})$. Thus,

$$\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty,i,j}(f)\right\|_p \le C_p\,(1+\mu)\,\|f\|_p\,,$$

which when Combined with (3.4) gives that there is $\epsilon \in (0,1)$ so that

$$\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty,i,j}(f)\right\|_p \le C_p 2^{\frac{\epsilon(i+j+2)}{8q'}}(1+\mu)\,\|f\|_p \tag{3.5}$$

for all $p \ge 2$. Therefore, by (3.3) and (3.5), we obtain

$$\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,\infty}(f)\right\|_p \le C_{p,q}\,(1+\mu)\,\|f\|_p \tag{3.6}$$

for all $p \geq 2$. Now, let us estimate the L^p-norm of $\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,0,0}(f)$. Take $Q_1(x) = \sum_{|\alpha|\leq d_1} a_\alpha x^\alpha$, and define $\mathcal{M}^{Q_1,P_2}_{\Omega,\phi,\psi,0,0}(f)$ and $\mathcal{M}^{P_1,P_2,Q}_{\Omega,\phi,\psi,0,0}(f)$ by

$$\mathcal{M}^{Q_1,P_2}_{\Omega,\phi,\psi,0,0}(f)(\overline{x},\overline{y}) = \left(\int_0^1\int_0^1 \left|\mathcal{G}_{Q_1,P_2,\phi,\psi,\Omega}(f)(\overline{x},\overline{y},r,t)\right|^2 \frac{drdt}{rt}\right)^{1/2},$$

and

$$\mathcal{M}^{P_1,P_2,Q}_{\Omega,\phi,\psi,0,0}(f)(\overline{x},\overline{y}) = \left(\int_0^1\int_0^1 \left|\mathcal{H}^{Q}_{P_1,P_2,\phi,\psi,\Omega}(f)(\overline{x},\overline{y},r,t)\right|^2 \frac{drdt}{rt}\right)^{1/2},$$

where

$$\mathcal{H}^{Q}_{P_1,P_2,\phi,\psi,\Omega}(f)(\overline{x},\overline{y},r,t) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \left(e^{iP_1(ru)+iP_2(tv)} - e^{iQ_1(ru)+iP_2(tv)}\right)$$
$$\times f(x-ru, x_{n+1}-\phi(r), y-tv, y_{m+1}-\psi(t))\Omega(u,v)d\sigma(u)d\sigma(v).$$

Thus, by Minkowski's inequality, we deduce

$$\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,0,0}(f)(\overline{x},\overline{y}) \leq \mathcal{M}^{Q_1,P_2}_{\Omega,\phi,\psi,0,0}(f)(\overline{x},\overline{y}) + \mathcal{M}^{P_1,P_2,Q}_{\Omega,\phi,\psi,0,0}(f)(\overline{x},\overline{y}). \tag{3.7}$$

On one hand, since $\deg(Q_1) \leq d_1$, then by induction step we have

$$\left\|\mathcal{M}^{Q_1,P_2}_{\Omega,\phi,\psi,0,0}(f)\right\|_p \leq C_{p,q}\ (1+\mu)\,\|f\|_p \tag{3.8}$$

for all $p \geq 2$. On the other hand, it is easy to check that

$$\left|e^{iP_1(ru)} - e^{iQ_1(ru)}\right| \leq r^{(d_1+1)}\left|\sum_{|\alpha|=d_1+1} a_\alpha u^\alpha\right| \leq r^{d_1+1}.$$

So, by following a similar argument as in [18] and by Cauchy-Schwartz inequality, we have that

$$\begin{aligned}
\mathcal{M}^{P_1,P_2,Q}_{\Omega,\phi,\psi,0,0}(f)(\overline{x},\overline{y}) &\leq C\Bigg(\int_0^1\int_0^1\Bigg|\iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} e^{iP_2(tv)}f(x-ru,x_{n+1}-\phi(r),y-tv,y_{m+1}-\psi(t))\\
&\quad\times\ \Omega(u,v)d\sigma(u)d\sigma(v)\Bigg|^2 \left|e^{iP_1(ru)}-e^{iQ_1(ru)}\right|^2\frac{drdt}{rt}\Bigg)^{1/2}\\
&\leq C\Bigg(\int_0^1 r^{d_1}\int_{\mathbf{S}^{n-1}}\int_0^1\Bigg|\int_{\mathbf{S}^{m-1}} e^{iP_2(tv)}f(x-ru,x_{n+1}-\phi(r),y-tv,y_{m+1}-\psi(t))\\
&\quad\times\ \Omega(u,v)d\sigma(v)\Bigg|^2\frac{dt}{t}d\sigma(u)dr\Bigg)^{1/2}\\
&\leq C\Bigg(\int_{\mathbf{S}^{n-1}}\sum_{j=1}^{\infty}\left(2^{-jd_1}\right)\int_{2^{-j}}^{2^{-j+1}}\int_0^1\Bigg|\int_{\mathbf{S}^{m-1}} e^{iP_2(tv)}f(x-ru,x_{n+1}-\phi(r),y-tv,y_{m+1}-\psi(t))\\
&\quad\times\ \Omega(u,v)d\sigma(v)\Bigg|^2\frac{dt}{t}drd\sigma(u)\Bigg)^{1/2}\\
&\leq C\left(\sum_{j=1}^{\infty}\left(2^{-jd_1}\right)\int_{2^{-j}}^{2^{-j+1}}\int_{\mathbf{S}^{n-1}}\left(\mathcal{M}^{(2)}_{P_1,\Omega_n,\phi}(f_r(\overline{x},\overline{y}))\right)^2 d\sigma(u)dr\right)^{1/2}
\end{aligned}$$

$$\leq C\left(\mathcal{N}_{\Omega,\psi}\circ\left(\mathcal{M}^{(2)}_{P_1,\Omega_n,\phi}(f_r(\bar{x},\bar{y}))\right)^2\right)^{1/2},$$

where $\circ$ denotes the composition of operators, $\mathcal{N}_{\Omega,\psi}f(\bar{x},\bar{y})=\mathcal{N}_{\Omega,\psi}f(\cdot,\bar{y})(\bar{x})$ is the maximal function defined as in Lemma 2.1; and $\mathcal{M}^{(2)}_{P_1,\Omega_n,\phi}(f_r(\bar{x},\bar{y})=\mathcal{M}^{(2)}_{P_1,\Omega_n,\phi}(f_r(\bar{x},\cdot)(\bar{y})$ is the maximal operator in the one parameter setting defined as in [17, Eq. (1.2)]. Hence, by following a similar argument as in [18, p. 607] together with [17] and Lemma 2.1, we get

$$\left\|\mathcal{M}^{P_1,P_2,Q}_{\Omega,\phi,\psi,0,0}(f)\right\|_p\leq C_{p,q}\ (1+\mu)\,\|f\|_p \tag{3.9}$$

for all $p\geq 2$. Therefore, by (3.7)-(3.9), we obtain that for all $p\geq 2$,

$$\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,0,0}(f)\right\|_p\leq C_{p,q}\ (1+\mu)\,\|f\|_p\,. \tag{3.10}$$

In the same manner, we can derive that

$$\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,\infty,0}(f)\right\|_p\leq C_{p,q}\ (1+\mu)\,\|f\|_p \tag{3.11}$$

and

$$\left\|\mathcal{M}^{P_1,P_2}_{\Omega,\phi,\psi,0,\infty}(f)\right\|_p\leq C_{p,q}\ (1+\mu)\,\|f\|_p \tag{3.12}$$

for all $p\geq 2$. Consequently, by (3.2), (3.6) and (3.10)-(3.12), we satisfy the inequality (1.3) for any polynomial P_1 of degree d_1+1 and for any polynomial P_2 of degree d_2. Similarly, we can show that the inequality (1.3) holds for any polynomial P_2 of degree d_2+1 and for any polynomial P_1 of degree d_1. This completes the proof of Theorem 1.1.

4 Further results

For $\gamma>1$, define $\Delta_\gamma\left(\mathbf{R}^+\times\mathbf{R}^+\right)$ to be the set of all measurable functions h on $\mathbf{R}^+\times\mathbf{R}^+$ satisfying the condition

$$\sup_{R_1,R_2>0}\left(\frac{1}{R_1R_2}\int_0^{R_1}\int_0^{R_2}|h(t,r)|^\gamma\,dtdr\right)^{1/\gamma}<\infty$$

and define $\Delta_\infty\left(\mathbf{R}^+\times\mathbf{R}^+\right)=L^\infty\left(\mathbf{R}^+\times\mathbf{R}^+\right)$. Also, for $1\leq\gamma<\infty$, define $\mathfrak{L}^\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ to be the set of all measurable functions $h:\mathbf{R}^+\times\mathbf{R}^+\to\mathbf{R}$ that satisfy the condition $\|h\|_{L^\gamma(\mathbf{R}^+\times\mathbf{R}^+,\frac{drdt}{rt})}=\left(\int_0^\infty\int_0^\infty|h(r,t)|^\gamma\,\frac{drdt}{rt}\right)^{1/\gamma}\leq 1$ and define $\mathfrak{L}^\infty(\mathbf{R}^+\times\mathbf{R}^+)=L^\infty(\mathbf{R}^+\times\mathbf{R}^+,\frac{drdt}{rt})$.

It is obvious that $\mathfrak{L}^\gamma(\mathbf{R}^+\times\mathbf{R}^+)\subset\Delta_\gamma\left(\mathbf{R}^+\times\mathbf{R}^+\right)$ for $1<\gamma<\infty$, $\Delta_{\gamma_1}\left(\mathbf{R}^+\times\mathbf{R}^+\right)\subset\Delta_{\gamma_2}\left(\mathbf{R}^+\times\mathbf{R}^+\right)$ for $\gamma_1>\gamma_2$ and $\Delta_\infty\left(\mathbf{R}^+\times\mathbf{R}^+\right)=\mathfrak{L}^\infty(\mathbf{R}^+\times\mathbf{R}^+)$.

The purpose of this section is to study the L^p boundedness of the singular inegral operator $T^{P_1,P_2}_{\Omega,h,\phi,\psi}(f)(\bar{x},\bar{y})$ and the maximal operator $\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega,\phi,\psi}$ under weaker conditions, where $\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega,\phi,\psi}$ is defined, initially for $f\in\mathcal{C}_0^\infty(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})$, by

$$\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega,\phi,\psi}(f)(\bar{x},\bar{y})=\sup_{h\in\mathfrak{L}^\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\left|T^{P_1,P_2}_{\Omega,h,\phi,\psi}(f)(\bar{x},\bar{y})\right|.$$

The first result of this section is the following:

Theorem 4.1. *Suppose that $\Omega\in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, $q>1$ and satisfy the conditions (1.1)-(1.2) with $\|\Omega\|_1\leq 1$. Assume that ϕ, ψ, μ, P_1, and P_2 are given as in Theorem 1.1. Then there exists a constant $C_{p,q}>0$ such that*

$$\left\|\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega,\phi,\psi}(f)\right\|_p\leq C_{p,q}\,(1+\mu)^{2/\gamma'}\,\|f\|_p \tag{4.1}$$

for all $\gamma' \le p < \infty$ *with* $1 < \gamma \le 2$*; and*

$$\left\|\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(1)}(f)\right\|_{\infty} \le C\,\|f\|_{\infty}\,. \tag{4.2}$$

Proof. It is clear that if $\gamma = 2$, then we have $\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(\gamma)} = \mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2}$. So, by Theorem 1.1, the inequality (4.1) holds for all $p \ge 2$. However, if $\gamma = 1$; we assume that $h \in L^1(\mathbf{R}^+ \times \mathbf{R}^+, \frac{drdt}{rt})$ and $f \in L^\infty(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$. Then for all $(\bar{x}, \bar{y}) \in \mathbf{R}^{n+1} \times \mathbf{R}^{m+1}$, we have

$$\left|\iint_{\mathbf{R}^+\times\mathbf{R}^+} h(r,t)\, \mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t)\frac{drdt}{rt}\right| \le C\,\|f\|_{\infty}\,\|h\|_1\,.$$

Hence, by taking the supremum on both sides over all h with $\|h\|_1 \le 1$, we reach

$$\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(1)}(f)(\bar{x},\bar{y}) \le C\,\|f\|_{\infty}$$

for almost every where $(\bar{x}, \bar{y}) \in \mathbf{R}^{n+1} \times \mathbf{R}^{m+1}$, which leads to

$$\left\|\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(1)}(f)\right\|_{\infty} \le C\,\|f\|_{\infty}\,. \tag{4.3}$$

Finally, if $1 < \gamma \le 2$. We follow a similar approach as in [15]. By duality, we get

$$\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(\gamma)}(f)(\bar{x},\bar{y}) = \left(\iint_{\mathbf{R}^+\times\mathbf{R}^+} \left|\mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)(\bar{x},\bar{y},r,t)\right|^{\gamma'} \frac{drdt}{rt}\right)^{1/\gamma'},$$

which gives

$$\left\|\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(\gamma)}(f)\right\|_{p} = \left\|\mathcal{G}_{P_1,P_2,\phi,\psi,\Omega}(f)\right\|_{L^p(L^{\gamma'}(\mathbf{R}^+\times\mathbf{R}^+,\frac{drdt}{rt}),\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})}\,. \tag{4.4}$$

Therefore, by applying the interpolation theorem for the Lebesgue mixed normed spaces to the inequalities (1.3) and (4.3), we directly obtain

$$\left\|\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(\gamma)}(f)\right\|_{p} \le C_{p,q}(1+\mu)^{2/\gamma'}\,\|f\|_p \tag{4.5}$$

for $\gamma' \le p < \infty$ with $1 < \gamma \le 2$; and $\left\|\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(1)}(f)\right\|_{\infty} \le C\,\|f\|_{\infty}$. This completes the proof. □

It is worth mentioning that when $\phi(t) = \psi(t) = t$ and $P_1(u) = P_2(v) = 0$, Al-Qassem and Pan in [8] extended the results of Theorem 4.1. In fact, they established the L^p boundedness of $\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(\gamma)}$ provided that $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for $\gamma' \le p < \infty$ with $1 < \gamma \le 2$.

By the conclusion in Theorem 4.1 and applying an extrapolation argument (see [16, 19, 20]), we shall improve and extend the corresponding results in [4, 6, 8, 11, 13]. Precisely, we obtain the following:

Theorem 4.2. *Suppose that P_1, P_2, ϕ, and ψ are given as in Theorem 1.1. Assume that $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \cup B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ with $q > 1$. Then $\mathcal{M}_{\Omega,\phi,\psi}^{P_1,P_2,(\gamma)}(f)$ is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for $\gamma' \le p < \infty$ with $1 < \gamma \le 2$; and it is bounded on $L^\infty(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for $\gamma = 1$.*

Proof. The idea of proving Theorem 4.2 is taken form [17], which has its roots in [16] as well as in [19]. When $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ with $1 < \gamma \le 2$ and Ω satisfies the conditions (1.1)-(1.2), then Ω can be decomposed as a sum of functions in $L^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (see [21]). In fact, we have

$$\Omega = \sum_{k=0}^{\infty} \Omega_k, \tag{4.6}$$

where

$$\int_{\mathbf{S}^{n-1}} \Omega_k(x',.)d\sigma(x') = \int_{\mathbf{S}^{m-1}} \Omega_k(.,y')d\sigma(y') = 0,$$

$$\Omega_0 \in L^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}), \quad \|\Omega_k\|_\infty \le C2^{4k}, \quad \|\Omega_k\|_1 \le C,$$

and

$$\sum_{i=1}^{\infty} k^{2/\gamma'} \|\Omega_k\|_1 \le C\,\|\Omega\|_{L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \; for\ k = 0,1,2,\cdots .$$

Hence, it is easy to see that

$$\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega,\phi,\psi}(f)(\overline{x},\overline{y}) \le \mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega_0,\phi,\psi}(f)(\overline{x},\overline{y}) + \sum_{k=1}^{\infty}\|\Omega_k\|_1\,\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega_k,\phi,\psi}(f)(\overline{x},\overline{y}) \tag{4.7}$$

and

$$\left(1+\log^{2/\gamma'}(e+\|\Omega_k\|_\infty)\right) \le \left(1+\log^{2/\gamma'}(e+C2^{4k})\right) \le Ck^{2/\gamma'}. \tag{4.8}$$

As $\Omega_0 \in L^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, then by Thorem 4.1 we get

$$\left\|\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega_0,\phi,\psi}\right\|_p \le C_p\left(1+\log^{2/\gamma'}(e+\|\Omega_0\|_2)\right)\|f\|_p \tag{4.9}$$

for $\gamma' \le p < \infty$. Therefore, by Minkoswski's inequality and (4.7)-(4.9), we obtain that

$$\begin{aligned}\left\|\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega,\phi,\psi}(f)\right\|_p &\le \left\|\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega_0,\phi,\psi}(f)\right\|_p + \sum_{k=1}^{\infty}\|\Omega_k\|_1\left\|\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega_k,\phi,\psi}(f)\right\|_p \\ &\le C_p\left(1+\sum_{k=1}^{\infty}\|\Omega_k\|_1\,k^{2/\gamma'}\right)\|f\|_p \\ &\le C_p\,\|\Omega\|_{L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_p \le C_p\,\|f\|_p\,.\end{aligned}$$

However, when $\Omega \in B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ with $q>1$, $1<\gamma\le 2$ and Ω satisfies the conditions (1.1)-(1.2), then Ω can be written as

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu, \tag{4.10}$$

where each c_μ is a complex number, each b_μ is a q-block supported in an interval I_μ on $(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and

$$M_q^{(0,2/\gamma'-1)}(\{c_\mu\}) = \sum_{\mu=1}^{\infty}|c_\mu|\left(1+\log^{2/\gamma'}(|I_\mu|^{-1})\right) < \infty.$$

For each μ, define the blocklike function $\tilde{b_\mu}$ by

$$\tilde{b_\mu}(x,y) = b_\mu(x,y) - \int_{\mathbf{S}^{n-1}} b_\mu(u,y)d\sigma(u) - \int_{\mathbf{S}^{m-1}} b_\mu(x,v)d\sigma(v) + \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} b_\mu(u,v)d\sigma(u)d\sigma(v). \tag{4.11}$$

It is clear that each $\tilde{b_\mu}(x,y)$ satisfies the following:

$$\int_{\mathbf{S}^{n-1}} \tilde{b_\mu}(u,\cdot)\,d\sigma(u) = \int_{\mathbf{S}^{m-1}} \tilde{b_\mu}(\cdot,v)\,d\sigma(v) = 0, \tag{4.12}$$

$$\left\|\tilde{b_\mu}\right\|_q \le C\,|I|^{-1/q'}, \quad and \quad \left\|\tilde{b_\mu}\right\|_1 \le C. \tag{4.13}$$

Without loss of generality, we may assume that $|I_\mu| < 1$. Therefore, by Minkoswski's inequality, Theorem 4.1 and (4.10)-(4.13), we obtain that

$$\begin{aligned}\left\|\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega,\phi,\psi}(f)\right\|_p &\le \sum_{\mu=1}^{\infty}|c_\mu|\left\|\mathcal{M}^{P_1,P_2,(\gamma)}_{\tilde{b_\mu},\phi,\psi}(f)\right\|_p\\ &\le C_{p,q}\sum_{\mu=1}^{\infty}|c_\mu|\left(1+\log^{2/\gamma'}(e+|I_\mu|^{-1})\right)\|f\|_p\\ &\le C_{p,q}\|f\|_p\end{aligned}$$

for all $p \ge \gamma'$. □

We point out that under the assumptions Ω belongs to the block space $B^{(0,1)}_q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, $h \in \Delta_\gamma\left(\mathbf{R}^+\times\mathbf{R}^+\right)$ for some $q, \gamma > 1$, and when ϕ, ψ are $C^2([0,\infty))$, convex increasing functions with $\phi(0) = \psi(0) = 0$, the author of [22] proved that for every p satisfying $|1/p - 1/2| < \min\left\{1/2, 1/\gamma'\right\}$, there exists a constant C_p such that

$$\left\|T_{\Omega,h,\phi,\psi}(f)\right\|_p \le C_p\|f\|_p$$

for every $f \in L^p\left(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}\right)$. By this result, it is clear that the range of p is the full range $(1,\infty)$ whenever $h \in \mathfrak{L}^\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ with $\gamma \ge 2$. But what is about the L^p boundedness of $T_{\Omega,h,\phi,\psi}$ when $h \in \mathfrak{L}^\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ for $1 < \gamma < 2$. We shall obtain an answer to this question in the affirmative as described in the following theorem.

Theorem 4.3. *Assume that $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) \cup B^{(0,2/\gamma'-1)}_q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, $q > 1$, and satisfying the conditions (1.1)-(1.2). Let $h \in \mathfrak{L}^\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ for some $1 < \gamma \le 2$, and let ϕ, ψ be given as in Theorem 1.1. Then the singular integral operator $T^{P_1,P_2}_{\Omega,h,\phi,\psi}(f)(\bar{x},\bar{y})$ is bounded on $L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})$ for all $1 < p < \infty$.*

Proof. As a direct consequence of Theorem 4.2 and the statement that

$$\left|T^{P_1,P_2}_{\Omega,h,\phi,\psi}(f)(\bar{x},\bar{y})\right| \le \|h\|_{L^\gamma(\mathbf{R}^+\times\mathbf{R}^+,\frac{drdt}{rt})}\mathcal{M}^{P_1,P_2,(\gamma)}_{\Omega,\phi,\psi}(f)(\bar{x},\bar{y}), \tag{4.14}$$

we acheive that $T^{P_1,P_2}_{\Omega,h,\phi,\psi}$ is bounded on $L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})$ for $\gamma' \le p < \infty$ with $1 < \gamma \le 2$. Moreover, by a standard duality argument, we can show that $T^{P_1,P_2}_{\Omega,h,\phi,\psi}$ is bounded on L^p for $1 < p \le \gamma$ with $1 < \gamma \le 2$. So, if $\gamma = 2$, then we are done. However, if $1 < \gamma < 2$, then we apply the real interpolation theorem to acquire the L^p boundedness of $T^{P_1,P_2}_{\Omega,h,\phi,\psi}$ for $(\gamma < p < \gamma')$. This completes the proof. □

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