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Irreducible modules with highest weight vectors over modular Witt and special Lie superalgebras

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Abstract: Let $\mathbb{F}$ be an arbitrary field of characteristic $p > 2$. In this paper we study irreducible modules with highest weight vectors over Witt and special Lie superalgebras of $\mathbb{F}$. The same irreducible modules of general and special linear Lie superalgebras, which are the 0-th part of Witt and special Lie superalgebras in certain $\mathbb{Z}$-grading, are also considered. Then we establish a certain connection called a P-expansion between these modules.

Keywords: modular Lie superalgebra, graded module, irreducibility, highest weight

MSC: 17B10, 17B50, 17B70

1 Introduction

Let $\mathbb{F}$ be an arbitrary field of characteristic $p > 2$ and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the residue class ring mod 2. Throughout this paper we have assumed that all vector spaces, linear mappings and tensor products are over the underlying base field $\mathbb{F}$. Assume that x is a $\mathbb{Z}_2$-homogeneous element and $d(x)$ is the $\mathbb{Z}_2$-degree of x, if $d(x)$ occurs in an expression.

In 1967, Rudakov and Shafarevich [1] described all the irreducible representations of $\mathfrak{sl}(2)$ over an algebraically closed field $\mathbb{F}$ of characteristic $p > 2$. They demonstrated that in addition to the p-representations known since 1930s, all of which possess a highest and lowest weight and are labeled by one integer, there are other representations that form a variety of dimension 3. They described the $\mathfrak{g}$-modules not possessing a p-structure for Lie algebras $\mathfrak{g}$ with Cartan matrix. In 1974, Rudakov [2] described irreducible $\mathfrak{g}$-modules, where $\mathfrak{g}$ is a simple Lie algebras of vector fields over $\mathbb{C}$, for modules dual to modules of (formal) tensor fields. For a review of similar results and the importance of this particular type of module, we refer the readers to the papers [3, 4]. In the 1980s, Krylyuk [5, 6] studied the highest weight modules over the algebras of vector fields of series W and S possessing a p-structure. Shu [7] discussed the representations of Cartan type Lie algebras in characteristic $p > 2$ from the viewpoint of reducing rank. Zhang [8] constructed the simple L-modules with nonsingular characters and some simple modules with singular characters, where L is a restricted simple Lie algebra of Cartan type.

Since the classification of all the finite-dimensional simple complex Lie superalgebras was done by Kac [9], the problems of constructing a unified representation theory for all the types of simple Lie superalgebras

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has become more important than ever. Kac obtained essential results for the highest weight representations of classical Lie superalgebras [10, 11]. Most of Kac's results can also be extended to the remaining classical series of Lie superalgebras [12–14], while the representations of Lie superalgebras of Cartan type have been studied in [15, 16]. Recent work on the representation theory of modular Lie superalgebras of Cartan type can also be found in [17–19].

The structure of gradation plays a critical role in the research of Lie algebras and superalgebras. Shen [20–22] introduced an important notion which is called the mixed product and realized the graded modules over Lie algebras of Cartan type. The method of the mixed product can also be applied to Lie superalgebras of Cartan type over fields of characteristic zero [23]. In the case of modular Lie superalgebras, Zhang [24] has obtained the $\mathbb{Z}$-graded modules over finite-dimensional Hamiltonian Lie superalgebras.

This paper generalizes some of Shen's results in [20–22]. A brief summary of the relevant concepts in generalized Witt and special modular Lie superalgebras is presented in Section 2. Section 3 gives some properties of the graded modules over modular Lie superalgebras. In Section 4, the certain connection which is called a P-expansion between irreducible highest weight representations of generalized Witt and special modular Lie superalgebras, and the same irreducible highest weight representations of general linear Lie superalgebras $\mathfrak{gl}(m, n)$ and special linear Lie superalgebras $\mathfrak{sl}(m, n)$, is established.

2 Generalized Witt and special modular Lie superalgebras

In addition to the standard notation $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ is used for the set of positive integers and the set of nonnegative integers, respectively. Generally, let m, n denote fixed integers in $\mathbb{N} \setminus \{1, 2\}$. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, we put $|\alpha| := \sum_{i=1}^m \alpha_i$. Following [25], let $\mathcal{O}(m)$ denote the *divided power algebra* over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{N}_0^m\}$. For $\varepsilon_i := (\delta_{i1}, \ldots, \delta_{im})$, we abbreviate $x^{(\varepsilon_i)}$ to x_i, $i = 1, 2, \ldots, m$, where δ_{ij} is Kronecker delta. Let $\Lambda(n)$ be the *exterior superalgebra* over $\mathbb{F}$ in n variables $x_{m+1}, x_{m+2}, \ldots, x_{m+n}$ and $\mathcal{O}(m, n)$ denote the tensor product $\mathcal{O}(m) \otimes_{\mathbb{F}} \Lambda(n)$. Clearly, $\mathcal{O}(m, n)$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathcal{O}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. Moreover, $\mathcal{O}(m, n)$ is super-commutative. For $g \in \mathcal{O}(m)$, $f \in \Lambda(n)$, we write gf for $g \otimes f$. The following formulae hold in $\mathcal{O}(m, n)$:

$$\begin{aligned} x^{(\alpha)}x^{(\beta)} &= \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)} \quad \text{for } \alpha, \beta \in \mathbb{N}_0^m, \\ x_i x_j &= -x_j x_i \quad \text{for } i, j = m+1, \ldots, m+n, \\ x^{(\alpha)} x_j &= x_j x^{(\alpha)} \quad \text{for } \alpha \in \mathbb{N}_0^m, j = m+1, \ldots, m+n, \end{aligned}$$

where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^m \binom{\alpha_i+\beta_i}{\alpha_i}$. Put $\mathrm{Y}_0 := \{1, 2, \ldots, m\}$, $\mathrm{Y}_1 := \{m+1, m+2, \ldots, m+n\}$ and $\mathrm{Y} := \mathrm{Y}_0 \cup \mathrm{Y}_1$. Set

$$\mathbb{B}_k := \{\langle i_1, i_2, \ldots, i_k \rangle \mid m+1 \leq i_1 < i_2 < \cdots < i_k \leq m+n\}$$

and $\mathbb{B} := \bigcup_{k=0}^n \mathbb{B}_k$, where $\mathbb{B}_0 := \emptyset$. For $u = \langle i_1, i_2, \ldots, i_k \rangle \in \mathbb{B}_k$, set $|u| := k$, $|\emptyset| := 0$, $x^{\emptyset} := 1$, $x^u := x_{i_1} x_{i_2} \cdots x_{i_k}$ and $x^E := x_{m+1} x_{m+2} \cdots x_{m+n}$. Clearly, $\left\{ x^{(\alpha)} x^u \mid \alpha \in \mathbb{N}_0^m, u \in \mathbb{B} \right\}$ constitutes an $\mathbb{F}$-basis of $\mathcal{O}(m, n)$. Let $D_1, D_2, \ldots, D_{m+n}$ be the linear transformations of $\mathcal{O}(m, n)$ such that

$$D_r(x^{(\alpha)} x^u) = \begin{cases} x^{(\alpha-\varepsilon_r)} x^u, & r \in \mathrm{Y}_0, \\ x^{(\alpha)} \partial(x^u)/\partial x_r, & r \in \mathrm{Y}_1, \end{cases}$$

where $\partial/\partial x_r$ is the superderivation of $\Lambda(n)$ such that $\partial x_s/\partial x_r = \delta_{rs}$ for $r, s \in \mathrm{Y}_1$. For more details on superderivations for Lie superalgebras, the reader is referred to [9, 26]. $D_1, D_2, \ldots, D_{m+n}$ are superderivations of the superalgebra $\mathcal{O}(m, n)$. Let

$$W(m, n) := \left\{ \sum_{r \in \mathrm{Y}} f_r D_r \;\middle|\; f_r \in \mathcal{O}(m, n) \right\}.$$

Then $W(m,n)$ is a Lie superalgebra, which is contained in $\mathrm{Der}(\mathcal{O}(m,n))$, where $\mathrm{Der}(\mathcal{O}(m,n))$ denotes the superderivation space of $\mathcal{O}(m,n)$. Obviously, $d(D_i)=\tau(i)$, where

$$\tau(i):=\begin{cases}\bar{0}, & i\in \mathrm{Y}_0,\\ \bar{1}, & i\in \mathrm{Y}_1.\end{cases}$$

An easy verification shows that

$$[fD,gE]=fD(g)E-(-1)^{d(fD)d(gE)}gE(f)D+(-1)^{d(D)d(g)}fg[D,E]$$

for $f,g\in\mathcal{O}(m,n)$, $D,E\in\mathrm{Der}(\mathcal{O}(m,n))$. In particular, the following formula holds in $W(m,n)$:

$$[fD_r,gD_s]=fD_r(g)D_s-(-1)^{d(fD_r)d(gD_s)}gD_s(f)D_r$$

for $f,g\in\mathcal{O}(m,n)$, $r,s\in\mathrm{Y}$.

Let

$$\mathbf{t}:=(t_1,t_2,\ldots,t_m)\in\mathbb{N}^m,\quad \pi:=(\pi_1,\pi_2,\ldots,\pi_m),$$

where $\pi_i:=p^{t_i}-1$, $i\in\mathrm{Y}_0$. Let $\mathbb{A}:=\left\{\alpha\in\mathbb{N}_0^m\mid\alpha_i\leqslant\pi_i, i=1,2,\ldots,m\right\}$. Then

$$\mathcal{O}(m,n,\mathbf{t}):=\mathrm{span}_{\mathbb{F}}\left\{x^{(\alpha)}x^u\,\middle|\,\alpha\in\mathbb{A},u\in\mathbb{B}\right\}$$

is a finite-dimensional subalgebra of $\mathcal{O}(m,n)$ with a natural $\mathbb{Z}$-gradation $\mathcal{O}(m,n,\mathbf{t})=\oplus_{r=1}^{\xi}\mathcal{O}(m,n,\mathbf{t})_r$ by putting

$$\mathcal{O}(m,n,\mathbf{t})_r:=\mathrm{span}_{\mathbb{F}}\left\{x^{(\alpha)}x^u\,\middle|\,|\alpha|+|u|=r\right\},\quad \xi:=|\pi|+n.$$

Set

$$W(m,n,\mathbf{t}):=\left\{\sum_{r\in\mathrm{Y}}f_rD_r\,\middle|\,f_r\in\mathcal{O}(m,n,\mathbf{t})\right\}.$$

Then $W(m,n,\mathbf{t})$ is called the generalized Witt modular Lie superalgebra and it is a subalgebra of $W(m,n)$. In particular, it is a finite-dimensional simple Lie superalgebra (see [27]). Clearly, $W(m,n,\mathbf{t})$ is a free $\mathcal{O}(m,n,\mathbf{t})$-module with basis $\{D_r\mid r\in\mathrm{Y}\}$. Note that $W(m,n,\mathbf{t})$ possesses a standard $\mathbb{F}$-basis $\left\{x^{(\alpha)}x^uD_r\,\middle|\,\alpha\in\mathbb{A},u\in\mathbb{B},r\in\mathrm{Y}\right\}$.

Let $r,s\in\mathrm{Y}$ and $D_{rs}:\mathcal{O}(m,n,\mathbf{t})\to W(m,n,\mathbf{t})$ be a linear mapping such that

$$D_{rs}(f)=(-1)^{\tau(r)\tau(s)}D_r(f)D_s-(-1)^{(\tau(r)+\tau(s))d(f)}D_s(f)D_r, \tag{1}$$

where $f\in\mathcal{O}(m,n,\mathbf{t})$ and $r,s\in\mathrm{Y}$. Then the following equation holds:

$$[D_k,D_{rs}(f)]=(-1)^{\tau(k)\tau(r)}D_{rs}(D_k(f)),\quad f\in\mathcal{O}(m,n,\mathbf{t});k,r,s\in\mathrm{Y}.$$

Put

$$S(m,n,\mathbf{t}):=\mathrm{span}_{\mathbb{F}}\left\{D_{rs}(f)\mid f\in\mathcal{O}(m,n,\mathbf{t});r,s\in\mathrm{Y}\right\}.$$

Then $S(m,n,\mathbf{t})$ is called the special modular Lie superalgebra. $S(m,n,\mathbf{t})$ is also a finite-dimensional simple Lie superalgebra (see [27]).

Let $\mathrm{div}:W(m,n,\mathbf{t})\to\mathcal{O}(m,n,\mathbf{t})$ be the *divergence* such that

$$\mathrm{div}\left(\sum_{r\in\mathrm{Y}}f_rD_r\right)=\sum_{r\in\mathrm{Y}}(-1)^{\tau(r)d(f_r)}D_r(f_r).$$

It follows that

$$\mathrm{div}[D,E]=D\mathrm{div}(E)-(-1)^{d(D)d(E)}E\mathrm{div}(D)\quad\text{for any } D,E\in W(m,n,\mathbf{t}).$$

Then div is a superderivation from $W(m,n,\mathbf{t})$ to $\mathcal{O}(m,n,\mathbf{t})$. Following [27], put

$$\overline{S}(m,n,\mathbf{t}):=\left\{D\in W(m,n,\mathbf{t})\mid\mathrm{div}(D)=0\right\}.$$

Then $S(m, n, \mathbf{t})$ is contained in $\overline{S}(m, n, \mathbf{t})$ and $\overline{S}(m, n, \mathbf{t})$ is a subalgebra of $W(m, n, \mathbf{t})$. The $\mathbb{Z}$-gradation of $\mathcal{O}(m, n, \mathbf{t})$ induces naturally $\mathbb{Z}$-gradation structures of $W(m, n, \mathbf{t}) = \oplus_{i=-1}^{\xi-1} W(m, n, \mathbf{t})_i$ and $S(m, n, \mathbf{t}) = \oplus_{i=-1}^{\xi-2} S(m, n, \mathbf{t})_i$, where

$$W(m, n, \mathbf{t})_i := \mathrm{span}_{\mathbb{F}} \left\{ fD_r \middle| r \in \mathrm{Y}, f \in \mathcal{O}(m, n, \mathbf{t})_{i+1} \right\},$$
$$S(m, n, \mathbf{t})_i := \mathrm{span}_{\mathbb{F}} \left\{ D_{rs}(f) \middle| r, s \in \mathrm{Y}, f \in \mathcal{O}(m, n, \mathbf{t})_{i+2} \right\}.$$

In addition, $\overline{S}(m, n, \mathbf{t})$ is also a $\mathbb{Z}$-graded subalgebra of $W(m, n, \mathbf{t})$. For convenience, $W(m, n, \underline{t})$, $S(m, n, \mathbf{t})$, $\overline{S}(m, n, \mathbf{t})$ and $\mathcal{O}(m, n, \mathbf{t})$ will be denoted by W, S, $\overline{S}$ and $\mathcal{O}$, respectively.

3 Graded modules over modular Lie superalgebras

Let $\mathfrak{gl}(m, n) = \mathfrak{gl}(m, n)_{\bar{0}} \oplus \mathfrak{gl}(m, n)_{\bar{1}}$ be the general linear Lie superalgebra of all $s \times s$ matrices over $\mathbb{F}$ (see [9]), where $s = m + n$. Set $\overline{\mathfrak{gl}}(m, n, \mathbf{t}) = \mathcal{O} \otimes \mathfrak{gl}(m, n)$. Define the operation [,] in $\overline{\mathfrak{gl}}(m, n, \mathbf{t})$ as follows:

$$[a \otimes x, b \otimes y] = (-1)^{d(x)d(b)} ab \otimes [x, y], \tag{2}$$

where $a, b \in \mathcal{O}$, $x, y \in \mathfrak{gl}(m, n)$. Then $\overline{\mathfrak{gl}}(m, n, \mathbf{t})$ is a Lie superalgebra. For $A \in W$, define $A \otimes 1 \in \mathrm{End}(\overline{\mathfrak{gl}}(m, n, \mathbf{t}))$ by

$$(A \otimes 1)(a \otimes x) = A(a) \otimes x, \quad a \in \mathcal{O}, x \in \mathfrak{gl}(m, n). \tag{3}$$

Let $P \in \mathfrak{gl}(m, n)_{\bar{0}}$ be an $s \times s$ invertible matrix. Suppose that $A = \sum_{i=1}^{s} a_i D_i \in W_\alpha$, where $\alpha \in \mathbb{Z}_2$. Let

$$\widetilde{A} = \sum_{k,j=1}^{s} (-1)^{\tau(k)\tau(j)+\tau(j)+\tau(k)\alpha} (D_k a_j) \otimes P^{-1} E_{kj} P, \tag{4}$$

where E_{kj} is an $s \times s$ matrix whose (i, l)-entry is $\delta_{ki}\delta_{jl}$. Then $\widetilde{A} \in \overline{\mathfrak{gl}}(m, n, \mathbf{t})_\alpha$. By virtue of the definition of superderivation we have

$$D_k(ab) = (D_k a)b + (-1)^{\tau(k)d(a)} a(D_k b), \tag{5}$$

where $a, b \in \mathcal{O}$, $k = 1, \ldots, s$.

Let $A = \sum_{i=1}^{s} a_i D_i \in W_\alpha$ and $B = \sum_{j=1}^{s} b_j D_j \in W_\beta$, where $\alpha, \beta \in \mathbb{Z}_2$. Then

$$[A, B] = \sum_{j=1}^{s} q_j D_j, \text{ where } q_j = \sum_{i=1}^{s} (a_i(D_i b_j) - (-1)^{\alpha\beta}(b_i(D_i a_j)). \tag{6}$$

Using the formulae from (2) to (6), a direct calculation shows the following proposition.

Proposition 3.1 ([24], Formula (7)). *Suppose that $A \in W_\alpha$ and $B \in W_\beta$, where $\alpha, \beta \in \mathbb{Z}_2$. If $C = [A, B]$, then*

$$\widetilde{C} = [\widetilde{A}, \widetilde{B}] + (A \otimes 1)(\widetilde{B}) - (-1)^{\alpha\beta}(B \otimes 1)(\widetilde{A}). \tag{7}$$

Suppose that L is a subalgebra of $\mathfrak{gl}(m, n)$ and $L(P) = \{P^{-1}AP \mid A \in L\}$. Then $L(P)$ is a subalgebra of $\mathfrak{gl}(m, n)$. Let $\Omega = \Omega_{\bar{0}} \oplus \Omega_{\bar{1}}$, where

$$\Omega_\alpha = \{A \in W_\alpha \mid \widetilde{A} \in \mathcal{O} \otimes L(P)\}, \quad \alpha \in \mathbb{Z}_2.$$

If $A, B \in \Omega$, then $[\widetilde{A}, \widetilde{B}] \in \Omega$. The formula (7) shows that Ω is a subalgebra of modular Witt Lie superalgebras W. The subalgebra Ω is called the P-expansion of L into W. Then the P-expansion of $\mathfrak{gl}(m, n)$ into W is exactly W.

The special linear Lie superalgebra $\mathfrak{sl}(m, n) = \{A \in \mathfrak{gl}(m, n) \mid \mathrm{str}(A) = 0\}$ is a subalgebra of $\mathfrak{gl}(m, n)$ (see [9]). Let Ω is the P-expansion of $\mathfrak{sl}(m, n)$ into W. If $A = \sum_{i=1}^{s} a_i D_i \in W$, then, for $\alpha \in \mathbb{Z}_2$,

$$A \in \Omega_\alpha \Leftrightarrow \widetilde{A} \in (\mathcal{O} \otimes \mathfrak{sl}(m, n))_\alpha$$

$$\Leftrightarrow \sum_{k=1}^{m}\sum_{j=1}^{s}(-1)^{\tau(j)}(D_k a_j)\otimes E_{kj} - (-1)^{\alpha}\sum_{k=m+1}^{s}\sum_{j=1}^{s}(D_k a_j)\otimes E_{kj} \in (\mathcal{O}\otimes \mathfrak{sl}(m,n))_{\alpha}$$
$$\Leftrightarrow \sum_{k=1}^{m}(D_k a_k)\otimes E_{kk} - (-1)^{\alpha}\sum_{k=m+1}^{s}(D_k a_k)\otimes E_{kk} \in (\mathcal{O}\otimes \mathfrak{sl}(m,n))_{\alpha}$$
$$\Leftrightarrow (\sum_{k=1}^{s}(-1)^{\tau(k)d(a_k)}(D_k a_k))\otimes E_{11} \in (\mathcal{O}\otimes \mathfrak{sl}(m,n))_{\alpha}$$
$$\Leftrightarrow \sum_{k=1}^{s}(-1)^{\tau(k)d(a_k)}(D_k a_k) = 0 \text{ and } D_k a_k \in \mathcal{O}_{\alpha}$$
$$\Leftrightarrow A \in \overline{S}_{\alpha}.$$

Hence $\Omega_\alpha = \overline{S}_\alpha$. It follows that $\Omega = \overline{S}$.

Let ρ be a representation of $L(P)$ on $\mathbb{Z}_2$-graded space V. Then ρ can be expanded to a representation ρ_1 of $\mathcal{O}\otimes L(P)$ on the space $\mathcal{O}\otimes V$, defined by

$$\rho_1(a\otimes x)(b\otimes v) = (-1)^{d(x)d(b)}ab\otimes \rho(x)(v), \tag{8}$$

where $a, b \in \mathcal{O}$, $x \in L(P)$, $v \in V$.

Proposition 3.2 ([24], Proposition 2)**.** *Let Ω be the P-expansion of L into W. Then*

$$\widetilde{\rho}(A) = \rho_1(A) + A\otimes 1, A \in \Omega \tag{9}$$

defines a representation $\widetilde{\rho}$ of Ω on $\mathbb{Z}_2$-graded space $\mathcal{O}\otimes V$.

By Proposition 3.2, $\mathcal{O}\otimes V$ which will be denoted by $\widetilde{V}$ is a Ω-module. In [20] the module $\widetilde{V}$ is called the mixed product of $\mathcal{O}$ and the module V.

A $\mathbb{Z}$-graded module V of X is called *positively graded* if $V = \bigoplus_{i\geq 0} V_i$ and $L_j \cdot V_i \subseteq V_{i+j}$, where X is a $\mathbb{Z}$-graded Lie superalgebra.

Let $\widetilde{V}_i = \langle a\otimes v \mid a \in \mathcal{O}_i, v \in V\rangle$. Then $\widetilde{V} = \bigoplus_{i=0}^{\xi}\widetilde{V}_i$, where $\xi = \sum_{i=1}^{m}\pi_i + n$. Put $\widetilde{V}_i = 0$ for $i > \xi$. A direct verification shows that $\Omega_i \cdot \widetilde{V}_j \subseteq \widetilde{V}_{i+j}$. Hence $\widetilde{V}$ is a positively graded Ω-module. Since the P-expansion of $\mathfrak{gl}(m,n)$ and $\mathfrak{sl}(m,n))$ into W are ,respectively, W and $\overline{S}$, Proposition 3.2 shows the following corollary.

Corollary 3.3. *The following statements hold:*
(1) *If V is a $\mathfrak{gl}(m,n)$-module, then the mixed product $\widetilde{V}$ is a $\mathbb{Z}$-graded W-module.*
(2) *If V is an $\mathfrak{sl}(m,n)$-module, then the mixed product $\widetilde{V}$ is a $\mathbb{Z}$-graded $\overline{S}$-module and so is a $\mathbb{Z}$-graded S-module.*

If V is an irreducible $L(P)$-module with a highest weight λ, then denote $\widetilde{V}$ by $\widetilde{V}(\lambda)$. Furthermore, the weight vector associated with the highest weight λ is denoted by v_λ, where $v_\lambda \in \widetilde{V}(\lambda)$. Similar to [21, Theorem 1.2] one may obtain that $\tilde{V}(\lambda)$ has an unique irreducible submodule $U(L)(1\otimes V)$, where $U(L)$ is the universal enveloping algebra of L. We customarily denote the unique irreducible submodule by $\overline{V}$. Then $x^{(\alpha)}x^u\otimes v_\lambda \in \overline{V}$ for all $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{A}$, $u = (i_1,\ldots,i_r) \in \mathbb{B}$.

Suppose that $\pi = (\pi_1,\ldots,\pi_m)$, $E = (m+1,\ldots,s)$, then $\pi \in \mathbb{A}$, $E \in \mathbb{B}$. Since Lie superalgebra W_0 is isomorphic to $\mathfrak{gl}(m,n)$, the element $\sum_{i,j=1}^{s} a_{ij}E_{ij}$ of $L(P)$ can be identified as the element $\sum_{i,j=1}^{s} a_{ij}x_iD_j$ of W, where $a_{ij} \in \mathbb{F}$. Hence $x^{(\pi)}x^E\otimes V$ can be regarded as an $L(P)$-module. Similarly, by [21, Proposition 2.4], we can prove that V is an irreducible $L(P)$-module if and only if $x^{(\pi)}x^E\otimes V$ is an irreducible $L(P)$-module. The following proposition is the analogue of [21, Proposition 2.1].

Proposition 3.4. *Suppose that V is an irreducible $L(P)$-module, v_λ is the weight vector associated with the highest weight λ and $\overline{V}$ is the unique irreducible submodule of $\widetilde{V}(\lambda)$, where $L = \mathfrak{gl}(m,n)$ or $\mathfrak{sl}(m,n)$, then*
(i) *$\overline{V}$ contains $1 \otimes V$ as a submodule.*
(ii) *If $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$, then $\overline{V} = \widetilde{V}(\lambda)$, that is $\widetilde{V}(\lambda)$ is an irreducible Ω-module, where Ω is the P-expansion of $L(P)$ into W.*

Proof. Recall that $\Omega = W$ if $L = \mathfrak{gl}(m,n)$ and $\Omega = \overline{S}$ if $L = \mathfrak{sl}(m,n)$.
(i) Without loss of generality we suppose that $x^{(\alpha)}x^u \otimes v_\lambda \in \overline{V}$, then

$$1 \otimes v_\lambda = D_1^{\alpha_1} \cdots D_m^{\alpha_m} D_{i_1} \cdots D_{i_r} \cdot (x^{(\alpha)}x^u \otimes v_\lambda) \in \overline{V}.$$

Because V is an irreducible $L(P)$-module, $1 \otimes V$ is an irreducible Ω_0-module. Then $1 \otimes V = U(\Omega_0)(1 \otimes v_\lambda) \subseteq \overline{V}$, where $U(\Omega_0)$ is the universal enveloping algebra of Ω_0.
(ii) Since module V is non-trivial, $\widetilde{V}(\lambda)_\xi$ is non-trivial. Let $x^{(\pi)}x^E \otimes V'$ be a proper Ω_0-submodule of $\widetilde{V}(\lambda)_\xi = x^{(\pi)}x^E \otimes V$. Then $1 \otimes V'$ is a proper Ω_0-submodule of $1 \otimes V$. Hence V' is a proper $L(P)$-submodule of V, which contradicts that V is irreducible. Since $\widetilde{V}(\lambda)_\xi$ is an irreducible Ω_0-module, we have

$$U(\Omega_0)(x^{(\pi)}x^E \otimes v_\lambda) = x^{(\pi)}x^E \otimes V \subseteq \overline{V}.$$

Let $\alpha \in \mathbb{A}$, $u \in \mathbb{B}$. Assuming that $(\beta_1, \ldots, \beta_m) = \pi - \alpha$ and $w = (j_1, \ldots, j_k) \in \mathbb{B}$ such that $\{w\} = \{E\} \backslash \{u\}$, then

$$x^{(\alpha)}x^u \otimes v_\lambda = D_1^{\beta_1} \cdots D_m^{\beta_m} D_{j_1} \cdots D_{j_k} \cdot (x^{(\pi)}x^E \otimes v_\lambda) \in \overline{V}.$$

Hence $\overline{V} = \widetilde{V}(\lambda)$. □

4 Irreducibility of module $\widetilde{V}(\lambda)$ over W and S

Let V be an irreducible $\mathfrak{sl}(m,n)$-module with the highest weight λ. Proposition 3.2 shows that $\widetilde{V}(\lambda)$ is an $\overline{S}$-module. Clearly, it is also an S-module. Assuming that $\overline{V}$ is irreducible as an S-module and if $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$, then it follows that $\overline{V} = \widetilde{V}(\lambda)$ from the similar methods used in Proposition 3.4 (ii). Therefore, $\widetilde{V}(\lambda)$ is an irreducible S-module.

We know that the standard Cartan subalgebra H of S is $\langle h_i \mid i = 1, 2, \ldots, s-1 \rangle$, where

$$h_i = E_{ii} - (-1)^{\tau(i)+\tau(i+1)} E_{i+1 i+1}, \quad i = 1, 2, \ldots, s-1.$$

Let Λ_i be the linear function on $\langle E_{11}, \ldots, E_{ss} \rangle$ such that $\Lambda_i(E_{jj}) = \delta_{ij}$, where $i, j = 1, 2, \ldots, s$. Set

$$\lambda_i = \sum_{j=1}^{i} \Lambda_j, \quad i = 1, 2, \ldots, m-1,$$
$$\lambda_m = \sum_{j=m+1}^{s} \Lambda_j,$$
$$\lambda_i = -\sum_{j=1}^{m} \Lambda_j + \sum_{j=m+1}^{i} \Lambda_j, \quad i = m+1, \ldots, s.$$

Then λ_i, $i = 1, 2, \ldots, s-1$, is a fundamental weight of $\mathfrak{sl}(m,n)$ and $\lambda_i(h_j) = \delta_{ij}$. We know that $\Lambda_i - \Lambda_j$ is a positive root of $\mathfrak{sl}(m,n)$ and the corresponding vectors of the positive root are E_{ij}, where $1 \le i < j \le s$.

If V is a finite-dimensional irreducible $\mathfrak{sl}(m,n)$-module, then $\lambda = \sum_{i=1}^{s-1} c_i\lambda_i$, where $c_i \in \mathbb{F}$. Let $\lambda|_{-m} = \sum_{i=1}^{m-1} c_i\lambda_i$ and $\lambda|_{+m} = \sum_{i=m+1}^{s-1} c_i\lambda_i$. Then $\lambda = \lambda|_{-m} + c_m\lambda_m + \lambda|_{+m}$. Also, assuming that ρ and $\widetilde{\rho}$ are representations corresponding with the modules V and $\widetilde{V}(\lambda)$, respectively.

Lemma 4.1. *Assuming that $\lambda|_{-m} \neq 0$. If $\lambda \neq \lambda_i$, $i = 1, 2, \ldots, m-1$, then $\widetilde{V}(\lambda)$ is an irreducible S-module.*

Proof. Since $\lambda|_{-m} \neq 0$ and $\lambda \neq \lambda_i$, $i = 1, 2, \ldots, m-1$, it was observed that λ must be one of the two cases: (1) $\lambda|_{-m} = \lambda_i$ and $c_m\lambda_m + \lambda|_{+m} \neq 0$, (2) $\lambda|_{-m} \neq \lambda_i$, $i = 1, 2, \ldots, m-1$.
(1) If $\lambda|_{-m} = \lambda_i$ and $c_m\lambda_m + \lambda|_{+m} \neq 0$, where $1 \leq i \leq m-1$, then $c_i \neq 0$ for $m \leq i \leq s-1$. Let c_k be the first non-zero element of $\{c_m, c_{m+1}, \ldots, c_{s-1}\}$. If $k > m$, then by virtue of formulae (1), (2), (8) and (9) and

$$\rho(E_{ii} - E_{k+1k+1})v_\lambda = \rho(\sum_{j=i}^{m} h_j - \sum_{j=m+1}^{k} h_j)v_\lambda = (1 - c_k)v_\lambda,$$

where v_λ is a weight vector associated with the highest weight λ,

$$\begin{aligned}
\widetilde{\rho}(D_{ik+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda) &= \widetilde{\rho}(x^{(\pi-\varepsilon_i)}x^E D_{k+1} - (-1)^{n+k-m}x^{(\pi)}x^{E-\langle k+1\rangle}D_i)(1 \otimes v_\lambda)\\
&= \rho_1(\sum_{j=1}^{m} D_j(x^{(\pi-\varepsilon_i)}x^E) \otimes E_{jk+1}\\
&\quad -(-1)^{n-1}\sum_{j=m+1}^{s} D_j(x^{(\pi-\varepsilon_i)}x^E) \otimes E_{jk+1})(1 \otimes v_\lambda)\\
&\quad -(-1)^{n+k-m}\rho_1(\sum_{j=1}^{m} D_j(x^{(\pi)}x^{E-\langle k+1\rangle}) \otimes E_{ji}\\
&\quad +(-1)^{n-1}\sum_{j=m+1}^{s} D_j(x^{(\pi)}x^{E-\langle k+1\rangle} \otimes E_{ji}))(1 \otimes v_\lambda)\\
&= -(-1)^{n+k-m}x^{(\pi-\varepsilon_i)}x^{E-\langle k+1\rangle} \otimes (1 - c_k)v_\lambda\\
&\quad + \sum_{j=k+2}^{s}(-1)^{n+j-m}x^{(\pi-\varepsilon_i)}x^{E-\langle j\rangle} \otimes \rho(E_{jk+1})v_\lambda\\
&\quad -(-1)^{n+k-m}\sum_{j=i+1}^{m} x^{(\pi-\varepsilon_j)}x^{E-\langle k+1\rangle} \otimes \rho(E_{ji})v_\lambda\\
&\quad + \sum_{j=m+1}^{k}(-1)^{k+j}x^{(\pi-\varepsilon_i)}x^{E-\langle j\rangle-\langle k+1\rangle} \otimes \rho(E_{ji})v_\lambda\\
&\quad + \sum_{j=k+2}^{s}(-1)^{j+k}x^{(\pi-\varepsilon_i)}x^{E-\langle k+1\rangle-\langle j\rangle} \otimes \rho(E_{ji})v_\lambda. \qquad (10)
\end{aligned}$$

Applying the formulae (8), (9) and (10) and

$$\rho(E_{ii+1}E_{i+1i})v_\lambda = \rho(E_{ii} - E_{i+1i+1})v_\lambda = v_\lambda,$$

we get

$$\begin{aligned}
\widetilde{\rho}(D_{ii+1}(x^{(2\varepsilon_i)}x_{i+1}x_{k+1}))\widetilde{\rho}(D_{ik+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda) &= (-1)^n x^{(\pi)}x^E \otimes v_\lambda - (-1)^n(1 - c_k)x^{(\pi)}x^E \otimes \rho(E_{ii} - E_{i+1i+1})v_\lambda\\
&= -(-1)^n c_k x^{(\pi)}x^E \otimes v_\lambda.
\end{aligned}$$

By Proposition 3.4 (i), we know that $1 \otimes v_\lambda \in \overline{V}$. Since $c_k \neq 0$, it shows $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$. Hence $\widetilde{V}(\lambda)$ is an irreducible S-module.
(2) If $\lambda|_{-m} \neq \lambda_i$, $i = 1, 2, \ldots, m-1$, then

$$\widetilde{\rho}(D_{i+1i}(x^{(2\varepsilon_i+2\varepsilon_{i+1})}))\widetilde{\rho}(D_{ii+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda) = (-c_i^2 + c_i)x^{(\pi)}x^E \otimes v_\lambda.$$

If $c_i \neq 0$ or 1, then $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$ and $\widetilde{V}(\lambda)$ is an irreducible S-module. So it was assumed that $c_k = 0$ or 1, where $k = 1, 2, \ldots, m-1$. Since $\lambda|_{-m} \neq 0$ and $\lambda|_{-m} \neq \lambda_i$, there exist at least two $k \in \{1, 2, \ldots, m-1\}$ such that $c_k = 1$. Without loss of generality we assumed that c_i and c_j, $i < j$, are the first and second coefficients which are equal to 1. Then

$$\widetilde{\rho}(D_{ij+1}(x^{(2\varepsilon_i+2\varepsilon_{j+1})}))\widetilde{\rho}(D_{ij+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda) = ((-c_i + c_j)^2 + (c_i + c_j))x^{(\pi)}x^E \otimes v_\lambda = 2x^{(\pi)}x^E \otimes v_\lambda.$$

Therefore, $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$ and $\widetilde{V}(\lambda)$ is an irreducible S-module. □

Lemma 4.2. *Suppose that n is odd. If $c_m \neq -1$ or 0, then $\widetilde{V}(\lambda)$ is an irreducible S-module.*

Proof. If $c_{m+1} \neq c_m + 1$, then $c_m(1 + c_m - c_{m+1}) \neq 0$. A direct computation shows that

$$\begin{aligned}\widetilde{\rho}(D_{mm+1}(x^{(2\varepsilon_m)}x_{m+1}x_{m+2}))\ \widetilde{\rho}(D_{mm+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda)\\ = c_m(1 - (-1)^n(c_m - c_{m+1}))x^{(\pi)}x^E \otimes v_\lambda = c_m(1 + c_m - c_{m+1})x^{(\pi)}x^E \otimes v_\lambda.\end{aligned}$$

Then $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$ and $\widetilde{V}(\lambda)$ is an irreducible S-module. If $c_{m+1} = c_m + 1$, then

$$\begin{aligned}\widetilde{\rho}(D_{mm+2}(x^{(2\varepsilon_m)}x_{m+1}x_{m+2}))\ \widetilde{\rho}(D_{mm+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda)\\ = (-(c_m - c_{m+1}) - (-1)^n c_{m+1} + (-1)^n c_m(c_m - c_{m+1}))x^{(\pi)}x^E \otimes v_\lambda\\ = (2 + 2c_m)x^{(\pi)}x^E \otimes v_\lambda.\end{aligned}$$

Since $c_m \neq -1$, we have $2 + 2c_m \neq 0$. Therefore, $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$ and $\widetilde{V}(\lambda)$ is irreducible. □

Lemma 4.3. *Suppose that n is odd and $c_m \neq 0$. If $\lambda \neq -\lambda_m$, then $\widetilde{V}(\lambda)$ is an irreducible S-module.*

Proof. If $c_m \neq -1$, then Lemma 4.2 shows that $\widetilde{V}(\lambda)$ is irreducible. If $c_m = -1$ and $\lambda|_{-m} \neq 0$, then Lemma 4.1 shows that $\widetilde{V}(\lambda)$ is irreducible. Suppose that $c_m = -1$, $\lambda|_{-m} = 0$ and $\lambda|_{+m} \neq 0$. Let c_k be the first non-zero element of $\{c_{m+1}, c_{m+2}, \ldots, c_{s-1}\}$. Then

$$\widetilde{\rho}(D_{mk}(x^{(2\varepsilon_m)}x_kx_{k+1}))\widetilde{\rho}(D_{mk+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda) = (c_m - (-1)^n(c_m - c_k)c_m)x^{(\pi)}x^E \otimes v_\lambda = c_kx^{(\pi)}x^E \otimes v_\lambda.$$

Hence $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$. Consequently, the module $\widetilde{V}(\lambda)$ is an irreducible S-module. □

Lemma 4.4. *Suppose that $\lambda|_{-m} = 0$, $c_m \neq 0$ and $\lambda|_{+m} \neq 0$. If $\widetilde{V}(\lambda)$ is a reducible S-module, then there exists $k \in \{m + 1, m + 2, \ldots, s - 2\}$ such that $\lambda = c_k\lambda_k - (c_k + 1)\lambda_{k+1}$.*

Proof. Since $\lambda|_{+m} \neq 0$, the elements $c_{m+1}, \ldots, c_{s-1}$ of $\mathbb{F}$ are not all zero. So we may suppose that c_k is the first non-zero element of $\{c_{m+1}, c_{m+2}, \ldots, c_{s-1}\}$. Assuming that there exists a $c_{k+i} \neq 0$, where $i > 1$. A direct computation shows that

$$\widetilde{\rho}(D_{k+ik+i+1}(x_kx_{k+1}x_{k+i}x_{k+i+1}))\widetilde{\rho}(D_{kk+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda) = (-1)^n c_kc_{k+i}x^{(\pi)}x^E \otimes v_\lambda.$$

As $c_kc_{k+i} \neq 0$, we have $x^{(\pi)}x^E \otimes v_\lambda \in \overline{V}$. But this conclusion confutes that $\widetilde{V}(\lambda)$ is a reducible S-module. As $c_{k+i} = 0$, $i > 1$ and $\lambda = c_k\lambda_k + c_{k+1}\lambda_{k+1}$. Then

$$\begin{aligned}\widetilde{\rho}(D_{k-1k+2}(x_{k-1}x_kx_{k+1}x_{k+2}))\ \widetilde{\rho}(D_{kk+1}(x^{(\pi)}x^E))(1 \otimes v_\lambda)\\ = \begin{cases}(-1)^n c_k(1 + c_k + c_{k+1})x^{(\pi)}x^E \otimes v_\lambda, & m + 2 \leq k \leq s - 2;\\ -(-1)^n c_k(1 + c_k + c_{k+1})x^{(\pi)}x^E \otimes v_\lambda, & k = m + 1.\end{cases}\end{aligned}$$

Since $\widetilde{V}(\lambda)$ is a reducible S-module, we have $1 + c_k + c_{k+1} = 0$. Therefore, $c_{k+1} = -(c_k + 1)$ and $\lambda = c_k\lambda_k - (c_k + 1)\lambda_{k+1}$. □

Theorem 4.5. *Let V be a finite-dimensional irreducible $\mathfrak{sl}(m, n)$-module with a non-zero highest weight λ. Suppose that n is odd. If $\lambda \neq -\lambda_m$ or $(-1)^{\tau(i)}\lambda_i$, $i = 1, \ldots, m - 1, m + 2, \ldots, s - 1$, then $\widetilde{V}(\lambda)$ is an irreducible S-module.*

Proof. Assume that $\widetilde{V}(\lambda)$ is a reducible S-module. It suffices to prove $\lambda = -\lambda_m$ or $(-1)^{\tau(i)}\lambda_i$, where $i = 1, \dots, m-1, m+2, \dots, s-1$.

If $\lambda|_{-m} \neq 0$, by Lemma 4.1, then $\lambda = \lambda_i = (-1)^{\tau(i)}\lambda_i$, where $i \in \{1, \dots, m-1\}$.

Suppose that $\lambda|_{-m} = 0$. If $c_m \neq 0$, by Lemma 4.3, then $\lambda = -\lambda_m$.

Suppose that $\lambda|_{-m} = 0$ and $c_m = 0$. Then $\lambda|_{-m} \neq 0$. Lemma 4.4 shows that $\lambda = c_i\lambda_i - (c_i+1)\lambda_{i+1}$, where $i \in \{m+1, \dots, s-2\}$. A direct computation shows that

$$\widetilde{\rho}(D_{mi+1}(x^{(2\varepsilon_m)}x_{i+1}x_{i+2}))\widetilde{\rho}(D_{mi+2}(x^{(\pi)}x^E))(1 \otimes v_\lambda) = -2c_i x^{(\pi)}x^E \otimes v_\lambda.$$

Since $\widetilde{V}(\lambda)$ is a reducible S-module, we have $x^{(\pi)}x^E \otimes v_\lambda \notin \overline{V}$. Therefore, $c_i = 0$ and $\lambda = -\lambda_{i+1} = (-1)^{\tau(i+1)}\lambda_{i+1}$, where $i \in \{m+1, \dots, s-2\}$, that is $\lambda = (-1)^{\tau(i)}\lambda_i$, where $i \in \{m+1, \dots, s-2\}$. The proof is completed. □

The W-module $\widetilde{V}(\lambda)$ will be discussed in the following theorem.

Theorem 4.6. *Let V be a finite-dimensional irreducible $\mathfrak{gl}(m,n)$-module with the non-zero highest weight λ. Suppose that n is odd. If $\lambda \neq -\lambda_m$, $(-1)^{\tau(i)}\lambda_i$, $i = 1, \dots, m-1, m+2, \dots, s-1$, then $\widetilde{V}(\lambda)$ is an irreducible W-module.*

Proof. We may suppose that $\lambda = \sum_{j=1}^{s} c_j\Lambda_j$, where $c_j \in \mathbb{F}$, and ρ is the representation corresponding to the module V. Assume that $\widetilde{V}(\lambda)$ is a reducible W-module. It suffices to prove $\lambda = -\lambda_m$ or $(-1)^{\tau(i)}\lambda_i$, where $i = 1, \dots, m-1, m+2, \dots, s-1$. Clearly, $\widetilde{V}(\lambda)$ is also a reducible S-module. By Theorem 4.5, we know that $\lambda|_H$, the restriction of λ to the Cartan subalgebra H of $\mathfrak{sl}(m,n)$, is equal to $-\lambda_m$ or $(-1)^{\tau(i)}\lambda_i$, where $i = 1, \dots, m-1, m+2, \dots, s-1$.

If $\lambda|_H = -\lambda_m$, then $\lambda|_H(h_i) = -\lambda_m(h_i)$, $i = 1, \dots, s-1$. Hence

$$\lambda = c\sum_{j=1}^{m}\Lambda_j + (1+c)\sum_{j=m+1}^{s}\Lambda_j,$$

where $c \in \mathbb{F}$. A direct computation shows that

$$\widetilde{\rho}(x^{(2\varepsilon_1)}D_1)\widetilde{\rho}(x^{(\pi)}x^E D_1)(1 \otimes v_\lambda) = (c - c^2)x^{(\pi)}x^E \otimes v_\lambda.$$

Since $\widetilde{V}(\lambda)$ is reducible, we have $c - c^2 = 0$. It follows $c = 0$ or 1. By virtue of

$$\widetilde{\rho}(x^{(2\varepsilon_1)}D_1)\widetilde{\rho}(x^{(\pi)}x^E D_{m+1})(1 \otimes v_\lambda) = -(-1)^{n-1}(1+c)cx^{(\pi)}x^E \otimes v_\lambda$$

and $\widetilde{V}(\lambda)$ is reducible, we have $(1+c)c = 0$. Then $c = 0$ and $\lambda = \sum_{j=m+1}^{s}\Lambda_j = -\lambda_m$.

If $\lambda|_H = (-1)^{\tau(i)}\lambda_i$, where $i = 1, \dots, m-1$, then $\lambda|_H = \lambda_i$. Since $\lambda(h_j) = \lambda_i(h_j)$, $j = 1, 2, \dots, s-1$, we have

$$\lambda = c\sum_{j=1}^{i}\Lambda_j + (c-1)\sum_{j=i+1}^{m}\Lambda_j + (c-1)\sum_{j=m+1}^{s}\Lambda_j,$$

where $c \in \mathbb{F}$. It follows that $c = 0$ or 1 from

$$\widetilde{\rho}(x^{(2\varepsilon_1)}D_1)\widetilde{\rho}(x^{(\pi)}x^E D_m)(1 \otimes v_\lambda) = (c - c^2)x^{(\pi)}x^E \otimes v_\lambda$$

and $\widetilde{V}(\lambda)$ is reducible. But the equation

$$\widetilde{\rho}(x^{(2\varepsilon_m)}D_m)\widetilde{\rho}(x^{(\pi)}x^E D_m)(1 \otimes v_\lambda) = (-(c-1)^2 + (c-1))x^{(\pi)}x^E \otimes v_\lambda,$$

shows that $c \neq 0$. Hence $c = 1$ and $\lambda = \lambda_i = (-1)^{\tau(i)}\lambda_i$, where $i = 1, \dots, m-1$.

If $\lambda|_H = (-1)^{\tau(i)}\lambda_i$, where $i = m+2, \dots, s-1$, then $\lambda|_H = -\lambda_i$. Since $\lambda(h_j) = -\lambda_i(h_j)$, $j = 1, 2, \dots, s-1$, we have

$$\lambda = c\sum_{j=1}^{m}\Lambda_j - c\sum_{j=m+1}^{i}\Lambda_j + (1-c)\sum_{j=i+1}^{s}\Lambda_j,$$

where $c \in \mathbb{F}$. It follows that $c = 0$ or 1 from

$$\widetilde{\rho}(x^{(2\varepsilon_1)}D_1)\widetilde{\rho}(x^{(\pi)}x^E D_1)(1 \otimes v_\lambda) = (c - c^2)x^{(\pi)}x^E \otimes v_\lambda$$

and $\widetilde{V}(\lambda)$ is reducible. Furthermore, the equation

$$\widetilde{\rho}(x_1 x_{j+1} D_1)\widetilde{\rho}(x^{(\pi)}x^E D_{j+1})(1 \otimes v_\lambda) = (-1)^n(1 - c)^2 x^{(\pi)}x^E \otimes v_\lambda,$$

shows that $c = 1$. Hence

$$\lambda = \sum_{j=1}^{m} \Lambda_j - \sum_{j=m+1}^{i} \Lambda_j = -\lambda_i = (-1)^{\tau(i)}\lambda_i,$$

where $i = m + 2, \ldots, s - 1$.

In conclusion, the proof is completed. □

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