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Research Article

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On some automorphic properties of Galois traces of class invariants from generalized Weber functions of level 5

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Abstract: After the significant work of Zagier on the traces of singular moduli, Jeon, Kang and Kim showed that the Galois traces of real-valued class invariants given in terms of the singular values of the classical Weber functions can be identified with the Fourier coefficients of weakly holomorphic modular forms of weight $3/2$ on the congruence subgroups of higher genus by using the Bruinier-Funke modular traces. Extending their work, we construct real-valued class invariants by using the singular values of the generalized Weber functions of level 5 and prove that their Galois traces are Fourier coefficients of a harmonic weak Maass form of weight $3/2$ by using Shimura's reciprocity law.

Keywords: modular forms, modular traces, Galois traces, class field theory

MSC 2010: Primary 11F37; Secondary 11F30, 11G15, 11R27, 11R37

1 Introduction

Let D be a negative integer with $D \equiv 0, 1 \pmod{4}$ so that D is an imaginary quadratic discriminant. More explicitly, if we let

$$\tau_D = \begin{cases} \frac{\sqrt{D}}{2} & \text{if } D \equiv 0 \pmod{4}, \\ \frac{-1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod{4}, \end{cases} \quad (1)$$

then the \mathbb{Z} -lattice $\mathcal{O}_D = [\tau_D, 1]$ becomes a quadratic order of discriminant $D = d_K \cdot t^2$ in the imaginary quadratic field $K = \mathbb{Q}(\tau_D)$ where d_K is a fundamental discriminant of K and a positive integer t is the conductor of \mathcal{O}_D .

Let \mathcal{Q}_D be the set of all positive definite integral binary quadratic forms of discriminant D , namely,

$$\mathcal{Q}_D = \{ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y] \mid a > 0, b^2 - 4ac = D\}.$$

The modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I_2\}$ acts on the set \mathcal{Q}_D from the right by the rule

$$\gamma = \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{bmatrix} : Q(x, y) = ax^2 + bxy + cy^2 \mapsto Q^\gamma(x, y) = Q(\gamma_1x + \gamma_2y, \gamma_3x + \gamma_4y), \quad (2)$$

where I_2 denotes the 2×2 identity matrix. Then the action induces an equivalence relation \sim on \mathcal{Q}_D as

$$Q_1 \sim Q_2 \quad \text{if and only if} \quad Q_1 = Q_2^\gamma \text{ for some } \gamma \in \Gamma(1).$$

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If we let $\mathcal{Q}_D^0 \subset \mathcal{Q}_D$ be the set of all primitive forms (i.e. $\gcd(a, b, c) = 1$), then the set of equivalence classes $\mathcal{Q}_D^0/\Gamma(1)$ becomes a finite abelian group under Dirichlet composition which is called the *form class group* of discriminant D and is denoted by $C(D)$.

For each quadratic form $Q = [a, b, c] = ax^2 + bxy + cy^2 \in \mathcal{Q}_D$, let τ_Q be the zero of $Q(x, 1) = 0$ in the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$, namely,

$$\tau_Q = \frac{-b + \sqrt{D}}{2a}. \quad (3)$$

The classical j -invariant on \mathbb{H} is a $\Gamma(1)$ -modular function defined by

$$j(\tau) = \frac{\left(1 + 240 \sum_{n=1}^{\infty} \sum_{m|n} m^3 q^n\right)^3}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}} = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$

where $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$. Letting H_D be the ring class field of order \mathcal{O}_D over K , we have

$$H_D = K(j(\tau_D))$$

by the theory of complex multiplication. Furthermore, we have the classical isomorphism

$$\operatorname{Gal}(H_D/K) \cong C(D) = \mathcal{Q}_D^0/\Gamma(1)$$

and the special values $j(\tau_Q)$ for all $Q \in \mathcal{Q}_D^0/\Gamma(1)$ become the Galois conjugates of $j(\tau_D)$ in H_D over K which are called the *singular moduli*.

Let $J(\tau) = j(\tau) - 744$ be the normalized Hauptmodul on the modular group $\Gamma(1)$. In [1], Zagier defined the modified Galois trace $\mathbf{t}_J(D)$ of index D as

$$\mathbf{t}_J(D) = \sum_{Q \in \mathcal{Q}_D^0/\Gamma(1)} \frac{J(\tau_Q)}{|\Gamma(1)_Q|},$$

where the sum allows the classes of imprimitive forms and $\Gamma(1)_Q$ is the stabilizer of Q . Furthermore, Kaneko [2] found another description for $\mathbf{t}_J(D)$ as

$$\mathbf{t}_J(D) = \sum_{\mathcal{O}_d \supset \mathcal{O}_D} \frac{2}{\omega_d} \cdot \sum_{\mathfrak{a} \in \operatorname{Cl}(\mathcal{O}_d)} J(\mathfrak{a}),$$

where the first sum runs over all imaginary quadratic orders $\mathcal{O}_d \supset \mathcal{O}_D$, $\operatorname{Cl}(\mathcal{O}_d)$ denotes the \mathcal{O}_d -ideal class group which is isomorphic to $\operatorname{Gal}(H_d/K)$ (see [3, §9]) and

$$\omega_d = \begin{cases} 6 & \text{if } d = -3, \\ 4 & \text{if } d = -4, \\ 2 & \text{otherwise,} \end{cases}$$

is the number of units in \mathcal{O}_d . Therefore, we can see that the modified trace of J is essentially a sum of usual Galois traces.

Zagier proved that the generating series

$$-q^{-1} + 2 + \sum_{D=1}^{\infty} \mathbf{t}_J(D) q^D = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + 7256q^8 + \cdots$$

is a weakly holomorphic modular form of weight $3/2$ for the Hecke subgroup $\Gamma_0(4)$. After Zagier's work, Bruinier and Funke [4] defined the modular traces of the CM values of modular functions for congruence subgroups of arbitrary genus and showed that modular traces of the values of an arbitrary modular function at Heegner points are Fourier coefficients of the holomorphic part of a harmonic weak Maass form of weight $3/2$.

On the other hand, it is well known that the value of every modular function at an imaginary quadratic number lies in a ray class field of an imaginary quadratic field. In particular, we call the value $f(\tau_D)$ of a modular function $f(\tau)$ at $\tau = \tau_D$ a *class invariant* if

$$K(f(\tau_D)) = K(j(\tau_D)),$$

following Weber [5]. We can easily see that the modular trace of the CM value of $J(\tau)$ at a Heegner point is naturally its Galois trace. However, it is not obvious to see whether the Galois trace of a given algebraic integer is a modular trace and hence a Fourier coefficient of a certain automorphic form. In [6], the authors paid attention to real-valued class invariants given in terms of the singular values of the classical Weber functions

$$f_\infty = \sqrt{3} \frac{\eta(3\tau)}{\eta(\tau)}, \quad f_0 = \frac{\eta(\frac{\tau}{3})}{\eta(\tau)}, \quad f_1 = \frac{\eta(\frac{\tau+1}{3})}{\eta(\tau)}, \quad f_2 = \frac{\eta(\frac{\tau+2}{3})}{\eta(\tau)},$$

where $\eta(\tau)$ is the classical Dedekind's eta-function. They proved that the *modified* Galois traces of those invariants can be identified with the Fourier coefficients of weakly holomorphic modular forms of weight $3/2$.

In this paper, we shall construct real-valued class invariants by using the generalized Weber functions of level 5 given by

$$\begin{aligned} g_\infty(\tau) &= 5^3 \cdot \left(\frac{\eta(5\tau)}{\eta(\tau)} \right)^6, & g_0(\tau) &= \left(\frac{\eta(\frac{\tau}{5})}{\eta(\tau)} \right)^6, & g_1(\tau) &= \left(\frac{\eta(\frac{\tau+96}{5})}{\eta(\tau)} \right)^6, \\ g_2(\tau) &= \left(\frac{\eta(\frac{\tau+72}{5})}{\eta(\tau)} \right)^6, & g_3(\tau) &= \left(\frac{\eta(\frac{\tau+48}{5})}{\eta(\tau)} \right)^6, & g_4(\tau) &= \left(\frac{\eta(\frac{\tau+24}{5})}{\eta(\tau)} \right)^6 \end{aligned}$$

(Theorems 4.3 and 4.5) by extending the argument of [6, §6]. Furthermore, we shall prove that their Galois traces are the Fourier coefficients of holomorphic parts of weight $3/2$ harmonic weak Maass forms (Theorem 6.4). To do this, we shall use the results on the Bruinier-Funke modular trace (Propositions 5.4 and 5.6) and Shimura's reciprocity law (Proposition 3.3).

2 Generalized Weber function of level 5

In this section, we shall briefly introduce some arithmetic properties of generalized Weber functions (See [5] or [7, §4] for details). Throughout this paper, we let N be a positive integer.

Let $\zeta_N = e^{2\pi i/N}$ be the primitive N -th root of unity and let \mathcal{F}_N be the field of modular functions on the principal congruence group $\Gamma(N) = \{\gamma \in \Gamma(1) \mid \gamma \equiv I_2 \pmod{N}\}$ whose Fourier coefficients lie in the N -th cyclotomic field $\mathbb{Q}(\zeta_N)$. Then, it is well known that \mathcal{F}_N is a Galois extension over $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$ with

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}.$$

The group $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ can be decomposed into

$$\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} = G_N \cdot \Gamma(N) = \Gamma(N) \cdot G_N, \quad (4)$$

where

$$G_N = \left\{ \sigma_u = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} \mid u \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

Each element $\sigma_u \in G_N$ acts on the function $f(\tau) \in \mathcal{F}_N$ by

$$\sigma_u \in G_N : f(\tau) = \sum_{n \geq n_0} c_n q^{n/N} \mapsto f^{\sigma_u}(\tau) = \sum_{n \geq n_0} c_n^{\sigma_u} q^{n/N},$$

where $c_n^{\sigma_u}$ denotes the image of $c_n \in \mathbb{Q}(\zeta_N)$ via the automorphism of $\mathbb{Q}(\zeta_N)$ defined by $\sigma_u : \zeta_N \mapsto \zeta_N^u$. Besides, the action of $\Gamma(N)$ is given by

$$\gamma \in \Gamma(N) : f(\tau) \mapsto f^\gamma(\tau) = f(\tilde{\gamma}\tau),$$

where $\tilde{\gamma} \in \Gamma(1)$ is a preimage of γ via the natural surjection $\Gamma(1) \rightarrow \Gamma(N)$ and $\tau \mapsto \tilde{\gamma}\tau$ is the fractional linear transformation with respect to $\tilde{\gamma}$.

Let $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ be the classical *Dedekind eta-function* and

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

be generators of $\Gamma(1)$. We have the following transformation formula of $\eta(\tau)$ under the composition of $\Gamma(1)$.

Proposition 2.1. *Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1)$. We may assume that*

$$c \geq 0 \quad \text{and} \quad d > 0 \text{ if } c = 0.$$

We write $c = 2^{\lambda(c)} \cdot c_0$ with c_0 odd and put $c_0 = \lambda(c) = 1$ if $c = 0$ for convenience. Then,

$$\eta(\gamma\tau) = \varepsilon(\gamma) \cdot \sqrt{c\tau + d} \cdot \eta(\tau) \quad \text{with } \operatorname{Re}(\sqrt{c\tau + d}) > 0,$$

where

$$\varepsilon(\gamma) = \left(\frac{a}{c_0} \right) \cdot \zeta_{24}^{ba+c(d(1-a^2)-a)+3(a-1)c_0+\frac{3}{2}\lambda(c)(a^2-1)}.$$

Here, $(\frac{a}{c_0})$ is the Legendre symbol.

In particular, we have

$$\eta \circ S(\tau) = \sqrt{-i\tau} \cdot \eta(\tau) \quad \text{and} \quad \eta \circ T(\tau) = \zeta_{24} \cdot \eta(\tau).$$

Proof. See [5, §38] and [8, §4]. □

The *generalized Weber functions* are defined by

$$v_{\infty, N}(\tau) = \sqrt{N} \cdot \frac{\eta(N\tau)}{\eta(\tau)} \quad \text{and} \quad v_{k, N}(\tau) = \frac{\eta(\frac{\tau+k}{N})}{\eta(\tau)}, \quad (\tau \in \mathbb{H}, k \in \mathbb{Z}).$$

Then, these functions have the following modular properties.

Proposition 2.2. *For a positive integer N and an integer k , we have*

- (i) $v_{\infty, N}$ and $v_{k, N}$ belong to \mathcal{F}_{24N} .
- (ii) Let $\{r_n\}_{n|N}$ be a set of integers indexed by the positive divisors of N . If $\gcd(N, 6) = 1$ and $k \equiv 0 \pmod{24}$, then we have

$$\sum_{n|N} (n-1)r_n \equiv 0 \pmod{24} \quad \text{if and only if} \quad \prod_{n|N} (v_{k, N})^{r_n} \in \mathcal{F}_N.$$

Proof. See [9, Theorem 3.2]. □

From now on, let us consider the case $N = 5$. For each $n \in \mathbb{Z}$, let k_n be an integer such that

$$k_n \equiv 0 \pmod{24} \quad \text{and} \quad k_n \equiv n \pmod{5}.$$

We then define

$$g_{\infty}(\tau) = v_{\infty, 5}(\tau)^6 = 5^3 \cdot \left(\frac{\eta(5\tau)}{\eta(\tau)} \right)^6, \quad g_n(\tau) = v_{k_n, 5}(\tau)^6 = \left(\frac{\eta(\frac{\tau+k_n}{5})}{\eta(\tau)} \right)^6.$$

Lemma 2.3. *Let n, n_1 and n_2 be integers. Then we have*

- (i) $g_{\infty}(\tau)$ and $g_n(\tau)$ belong to \mathcal{F}_5 .
- (ii) If $n_1 \equiv n_2 \pmod{5}$, then $g_{n_1}(\tau) = g_{n_2}(\tau)$.

Proof. (i) It is straightforward from Proposition 2.2 (ii) for $g_n(\tau)$, if we choose $r_1 = 1$, $r_5 = 6$. See [10, Theorem 1.64] for $g_\infty(\tau)$.

(ii) For each $i \in \{1, 2\}$, we may write $k_{n_i} = 5 \cdot K_i + v$ for some integers K_i and $v \in \{0, 1, 2, 3, 4\}$. Then we have

$$\eta\left(\frac{\tau + k_{n_i}}{5}\right) = \eta\left(\frac{\tau + v}{5} + K_i\right) = \zeta_{24}^{K_i} \cdot \eta\left(\frac{\tau + v}{5}\right)$$

by Proposition 2.1. Since $k_{n_1} - k_{n_2} = p \cdot (K_1 - K_2) \equiv 0 \pmod{24}$ and $\gcd(5, 24) = 1$, we get $K_1 \equiv K_2 \pmod{24}$ which implies that $\zeta_{24}^{K_1} = \zeta_{24}^{K_2}$. \square

By the above lemma, the indices of the generalized Weber functions of level 5 can be chosen from $\mathbb{Z}/5\mathbb{Z} \cup \{\infty\}$, namely,

$$\{g_\infty(\tau), g_0(\tau), g_1(\tau), g_2(\tau), g_3(\tau), g_4(\tau)\} \subset \mathcal{F}_5.$$

Remark 2.4. From the q -product of $\eta(\tau)$, one can easily see that

$$g_\infty(\tau) = 5^3 \cdot q \cdot \prod_{n=1}^{\infty} \left(\frac{1 - q^{5n}}{1 - q^n} \right)^6, \quad g_v(\tau) = \zeta_5^{-v} \cdot q^{-1/5} \cdot \prod_{n=1}^{\infty} \left(\frac{1 - \zeta_5^{vn} q^{n/5}}{1 - q^n} \right)^6$$

for $v \in \{0, 1, 2, 3, 4\}$.

By Proposition 2.1 and Remark 2.4, we obtain that

$$\begin{aligned} S &: (g_\infty, g_0, g_1, g_2, g_3, g_4) \mapsto (g_0, g_\infty, g_4, g_2, g_3, g_1), \\ T &: (g_\infty, g_0, g_1, g_2, g_3, g_4) \mapsto (g_\infty, g_1, g_2, g_3, g_4, g_0), \\ \sigma_u &: (g_\infty, g_0, g_1, g_2, g_3, g_4) \mapsto (g_\infty, g_0, g_u, g_{2u}, g_{3u}, g_{4u}), \end{aligned} \quad (5)$$

where $\sigma_u \in G_5$.

Further by using (4), (5) and the following lemma, we can compute explicitly the Galois actions on the generalized Weber functions of level 5.

Lemma 2.5. Let $N = p^r$ be a power of a rational prime number. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ so that either a or c is relatively prime to N . If $\gcd(c, N) = 1$, let $y \equiv (1 + a)c^{-1} \pmod{N}$. Otherwise, let $z \equiv (1 + c)a^{-1} \pmod{N}$. Then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{cases} T^y S T^c S T^{dy-b} & \text{if } \gcd(c, N) = 1, \\ S T^{-z} S T^{-a} S T^{bz-d} & \text{if } \gcd(a, N) = 1, \end{cases} \pmod{N}.$$

Proof. See [7, §5]. \square

Remark 2.6. We see that the Galois conjugates of $g_v(\tau)$ for $v \in \{0, 1, 2, 3, 4\} \cup \{\infty\}$ in \mathcal{F}_5 are given by

$$g_v^\sigma(\tau) = g_{v'}(\tau) \quad \text{for some } v' \in \{0, 1, 2, 3, 4\} \cup \{\infty\}$$

for any $\sigma \in \text{GL}_2(\mathbb{Z}/5\mathbb{Z})/\{\pm I_2\}$.

On the other hand, the generalized Weber functions of level 5 have the following algebraic relations with the j -invariant.

Lemma 2.7. $g_0(\tau), \dots, g_4(\tau)$, and $g_\infty(\tau)$ are the six distinct roots of

$$(X^2 + 10X + 5)^3 - j(\tau) \cdot X \in \mathbb{Z}[j(\tau)][X].$$

Proof. See [5, §72]. \square

3 The singular values of Weber functions

Let $D \equiv 0, 1 \pmod{4}$ be an imaginary quadratic discriminant. Then, the singular values of the generalized Weber functions of level 5 evaluated at τ_D lie in a finite abelian extension of an imaginary quadratic field by the theory of complex multiplication (See [11] and [3, §15]). In particular, there is a useful criterion for determining whether the values belonging to the ring class field H_D so that we can illustrate the Galois action of $C(D) \cong \text{Gal}(H_D/K)$ by Shimura's reciprocity law.

Let $F(X)$ denote the minimal polynomial of τ_D over \mathbb{Q} , namely,

$$F(X) = \begin{cases} X^2 - D/4 & \text{if } D \equiv 0 \pmod{4}, \\ X^2 + X + (1 - D)/4 & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Proposition 3.1. *Let n be a positive integer prime to 6 and k be an integer satisfying $k \equiv 0 \pmod{24}$ and $F(-k) \equiv 0 \pmod{n}$. If r is an even integer such that $r \cdot (n - 1) \equiv 0 \pmod{24}$, then we have*

$$\left(\frac{\eta\left(\frac{\tau_D + k}{n}\right)}{\eta(\tau_D)} \right)^r \in H_D.$$

Proof. See [12, Theorem 20]. □

From the above proposition, we obtain the following class invariants.

Lemma 3.2. *For an imaginary quadratic discriminant $D \equiv \square \pmod{100}$ with $\gcd(D, 5) = 1$, the values*

$$\begin{cases} g_2(\tau_D), g_3(\tau_D) & \text{if } D \equiv 0 \pmod{4}, D \equiv 1 \pmod{5} \\ g_1(\tau_D), g_4(\tau_D) & \text{if } D \equiv 0 \pmod{4}, D \equiv 4 \pmod{5} \\ g_0(\tau_D), g_1(\tau_D) & \text{if } D \equiv 1 \pmod{4}, D \equiv 1 \pmod{5} \\ g_2(\tau_D), g_4(\tau_D) & \text{if } D \equiv 1 \pmod{4}, D \equiv 4 \pmod{5} \end{cases}$$

are class invariants over $K = \mathbb{Q}(\tau_D)$.

Proof. Since $j(\tau) \in \mathbb{Q}(g_v(\tau))$ by Lemma 2.7, we have

$$H_D = K(j(\tau_D)) \subseteq K(g_v(\tau_D))$$

for each $v \in \{0, \dots, 4\} \cup \{\infty\}$. Conversely, if we put $n = 5$, $r = 6$, and $k = k_v$ for each $v \in \{0, \dots, 4\}$ in Proposition 3.1, then we can determine the values of v such that $g_v(\tau_D) \in H_D$. □

It is well known that the form class group $C(D) = \mathcal{Q}_D^0/\Gamma(1)$ is isomorphic to $\text{Gal}(H_D/K)$ (See [3, Theorem 3.9]). Let $Q = [a, b, c] \in \mathcal{Q}_D^0$ be a primitive quadratic form. For each prime integer p , we define the matrix $M_{Q,p} \in \text{GL}_2(\mathbb{Z}/p\mathbb{Z})$ as

(i) for $D \equiv 0 \pmod{4}$,

$$M_{Q,p} = \begin{cases} \begin{bmatrix} a & b/2 \\ 0 & 1 \end{bmatrix} & \text{if } p \nmid a, \\ \begin{bmatrix} -b/2 & -c \\ 1 & 0 \end{bmatrix} & \text{if } p \mid a \text{ and } p \nmid c, \\ \begin{bmatrix} -a - b/2 & -c - b/2 \\ 1 & -1 \end{bmatrix} & \text{if } p \mid a \text{ and } p \mid c, \end{cases} \quad (6)$$

(ii) for $D \equiv 1 \pmod{4}$,

$$M_{Q,p} = \begin{cases} \begin{bmatrix} a(b-1)/2 & \\ 0 & 1 \end{bmatrix} & \text{if } p \nmid a, \\ \begin{bmatrix} -(b+1)/2 & -c \\ 1 & 0 \end{bmatrix} & \text{if } p \mid a \text{ and } p \nmid c, \\ \begin{bmatrix} -a - (b+1)/2 & -c + (1-b)/2 \\ 1 & -1 \end{bmatrix} & \text{if } p \mid a \text{ and } p \mid c. \end{cases} \quad (7)$$

Note that for a given $N \geq 2$, we can obtain a unique matrix M_Q in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ satisfying $M_Q \equiv M_{Q,p} \pmod{p^r}$ for all primes p with $p^r \parallel N$ by Chinese remainder theorem.

Then, Shimura's reciprocity law tells us that

Proposition 3.3. For $f \in \mathcal{F}_N$ and $Q \in C(D) \cong \text{Gal}(H_D/K)$, we have

$$f(\tau_D)^{Q^{-1}} = f^{M_Q}(\tau_Q),$$

where Q^{-1} denotes the inverse of Q in $C(D)$.

Proof. See [13, §6]. □

Remark 3.4. (i) The principal form

$$\begin{cases} [1, 0, -D/4] & \text{if } D \equiv 0 \pmod{4}, \\ [1, 1, (1-D)/4] & \text{if } D \equiv 1 \pmod{4} \end{cases}$$

represents the identity class in $C(D)$ ([3, Theorem 3.9]).

(ii) The form class group $C(D)$ is usually represented by *reduced* quadratic forms $Q = [a, b, c] \in \mathcal{Q}_D^0$ characterized by the condition

$$(-a < b \leq a < c \text{ or } 0 \leq b \leq a = c) \quad \text{and} \quad b^2 - 4ac = D$$

([3, Theorem 2.8]). One can easily derive that if the class of Q is not the identity, then

$$2 \leq a \leq \sqrt{-D/3}.$$

(iii) Let h_D be the class number of an imaginary quadratic discriminant D . Then, it is well known that $h_D = 1$ if and only if

$$D = -3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163$$

([3, Theorem 7.30]).

We observe that the pair of class invariants appearing in Lemma 3.2 are not necessarily real numbers. However, it is guaranteed that their sums or products are real numbers for arbitrary discriminants D by the following lemma.

Lemma 3.5. We have

$$\begin{cases} \overline{\mathfrak{g}_v(\tau_D)} = \mathfrak{g}_{-v}(\tau_D) & \text{if } D \equiv 0 \pmod{4}, \\ \overline{\mathfrak{g}_v(\tau_D)} = \mathfrak{g}_{1-v}(\tau_D) & \text{if } D \equiv 1 \pmod{4}, \end{cases}$$

for each $v \in \{0, 1, 2, 3, 4\}$. Here, the indices $-v$ and $1-v$ are integers in a complete set of residues $\{0, 1, 2, 3, 4\}$ modulo 5 such that $v + (-v) \equiv 0 \pmod{5}$ and $v + (1-v) \equiv 1 \pmod{5}$.

Proof. Let $B = 0$ if $D \equiv 0 \pmod{4}$ and $B = -1$ if $D \equiv 1 \pmod{4}$ so that

$$\tau_D = \frac{B + \sqrt{D}}{2} \quad \text{and} \quad q_D = e^{2\pi i \tau_D} = e^{B\pi i} \cdot r_D, \quad \text{where} \quad r_D = |q_D| = e^{-\pi\sqrt{-D}}.$$

By Remark 2.4, for $v \in \{0, 1, 2, 3, 4\}$, we have

$$\begin{aligned} \zeta_5^{-v} \cdot q_D^{-1/5} \cdot \prod_{n=1}^{\infty} \left(\frac{1 - \zeta_5^{vn} q_D^{n/5}}{1 - q_D^n} \right)^6 &= \zeta_5^{-v} \cdot e^{-B\pi i/5} \cdot r_D^{-1/5} \cdot \prod_{n=1}^{\infty} \left(\frac{1 - \zeta_5^{vn} \cdot e^{nB\pi i/5} \cdot r_D^{n/5}}{1 - e^{nB\pi i} \cdot r_D^n} \right)^6 \\ &= e^{(-2v-B)\pi i/5} \cdot r_D^{-1/5} \cdot \prod_{n=1}^{\infty} \left(\frac{1 - e^{n(2v+B)\pi i/5} \cdot r_D^{n/5}}{1 - e^{nB\pi i} \cdot r_D^n} \right)^6. \end{aligned}$$

One can see that the complex numbers appearing in the above product are of the form

$$e^{(-2v-B)\pi i/5} \quad \text{and} \quad e^{n(2v+B)\pi i/5} \quad \text{for all } n \geq 1.$$

Then, we find that only $v' \in \{0, 1, 2, 3, 4\}$ with $B + v + v' \equiv 0 \pmod{5}$ satisfy

$$e^{(-2v-B)\pi i/5} \cdot e^{(-2v'-B)\pi i/5} = 1, \quad e^{n(2v+B)\pi i/5} \cdot e^{n(2v'+B)\pi i/5} = 1 \quad \text{for all } n \geq 1.$$

This completes the proof. \square

4 Real valued class invariants from the generalized Weber functions of level 5

In this section, we construct a real valued class invariants from the generalized Weber functions of level 5 by using Shimura's reciprocity law and the lemmas on the absolute values of Galois conjugates. We shall assume that $D \equiv \square \pmod{100}$ and $\gcd(D, 5) = 1$, i.e. 5 splits completely in $K = \mathbb{Q}(\tau_D)$.

We start with the basic inequalities.

Lemma 4.1. *We have*

- (i) $1 + X < e^X$, for all $X > 0$.
- (ii) If $0 < X \leq 1/11$, then

$$\frac{1}{1-X} \leq 1 + 1.1X.$$

Proof. The proofs of (i) and (ii) are straightforward by basic calculus. \square

Lemma 4.2. *Let $x + yi \in \mathbb{H}$ and $r = e^{-2\pi y}$.*

- (i) *If $0 < r < 1/11$, then $|\mathfrak{g}_{\infty}(x + yi)| < 5^3 \cdot r \cdot e^{\frac{6r^5}{1-r^5} + \frac{6.6r}{1-r}}$.*
- (ii) *If $0 < r < 1/11$, then $|\mathfrak{g}_v(x + yi)| < r^{-1/5} \cdot e^{\frac{6r^{1/5}}{1-r^{1/5}} + \frac{6.6r}{1-r}}$ for all $v \in \{0, 1, 2, 3, 4\}$.*
- (iii) *If $0 < r^{1/5} < 1/11$, then $|\mathfrak{g}_v(x + yi)| > r^{-1/5} \cdot e^{-\frac{6.6r^{1/5}}{1-r^{1/5}} - \frac{6.6r}{1-r}}$ for all $v \in \{0, 1, 2, 3, 4\}$.*

Proof. (i) We deduce that

$$\begin{aligned} |\mathfrak{g}_{\infty}(x + yi)| &\leq 5^3 \cdot r \cdot \prod_{n=1}^{\infty} \left(\frac{1 + r^{5n}}{1 - r^n} \right)^6 \quad \text{by Remark 2.4} \\ &< 5^3 \cdot r \cdot \prod_{n=1}^{\infty} \left(1 + r^{5n} \right)^6 (1 + 1.1 \cdot r^n)^6 \quad \text{by Lemma 4.1 (ii)} \\ &< 5^3 \cdot r \cdot \prod_{n=1}^{\infty} \left(e^{r^{5n}} \right)^6 \left(e^{1.1r^n} \right)^6 \quad \text{by Lemma 4.1 (i)} \end{aligned}$$

$$\begin{aligned}
&= 5^3 \cdot r \cdot e^{6 \cdot \sum_{n=1}^{\infty} r^{5n} + 6 \cdot \sum_{n=1}^{\infty} r^n} \\
&= 5^3 \cdot r \cdot e^{\frac{6r^5}{1-r^5} + \frac{6 \cdot 6r}{1-r}}.
\end{aligned}$$

(ii) The proof is similar to the proof of (i).

(iii) We establish that

$$\begin{aligned}
|g_v(x + yi)| &\geq r^{-1/5} \cdot \prod_{n=1}^{\infty} \left(\frac{1 - r^{n/5}}{1 + r^n} \right)^6 \quad \text{by Remark 2.4} \\
&> r^{-1/5} \cdot \prod_{n=1}^{\infty} (1 - r^{n/5})^6 \cdot (1 - r^n)^6 \quad \text{since } \frac{1}{1+X} > 1 - X \text{ for all } X > 0 \\
&> r^{-1/5} \cdot \prod_{n=1}^{\infty} (e^{-1.1 \cdot r^{n/5}})^6 (e^{-1.1 \cdot r^n})^6 \quad \text{by Lemma 4.1 (i),(ii)} \\
&= r^{-1/5} \cdot e^{-6.6 \cdot \sum_{n=1}^{\infty} r^{n/5} - 6.6 \cdot \sum_{n=1}^{\infty} r^n} \\
&= r^{-1/5} \cdot e^{-\frac{6.6r^{1/5}}{1-r^{1/5}} - \frac{6.6r}{1-r}}.
\end{aligned}$$

□

Extending the arguments in [6, §6], we achieve the following theorems.

Theorem 4.3. *For an imaginary quadratic discriminant $D \leq -31$, we assume that $D \equiv \square \pmod{100}$ and $\gcd(D, 5) = 1$. Then the singular values*

$$g_{\text{prod}}(\tau_D) = \begin{cases} g_2(\tau_D) \cdot g_3(\tau_D) & \text{if } D \equiv 0 \pmod{4}, D \equiv 1 \pmod{5}, \\ g_1(\tau_D) \cdot g_4(\tau_D) & \text{if } D \equiv 0 \pmod{4}, D \equiv 4 \pmod{5}, \\ g_0(\tau_D) \cdot g_1(\tau_D) & \text{if } D \equiv 1 \pmod{4}, D \equiv 1 \pmod{5}, \\ g_2(\tau_D) \cdot g_4(\tau_D) & \text{if } D \equiv 1 \pmod{4}, D \equiv 4 \pmod{5} \end{cases}$$

are real-valued class invariants over $K = \mathbb{Q}(\tau_D)$.

Proof. We may assume that $h_D \geq 2$ so that $D \leq -24$ by Remark 3.4 (iii). Let $Q = [a, b, c] \in \mathcal{Q}_D^0$ be a non-principal reduced form. By Proposition 3.3 and Remark 2.6, we have

$$(g_{\text{prod}}(\tau_D))^{Q^{-1}} = g_{v_1}(\tau_Q) \cdot g_{v_2}(\tau_Q)$$

for some $v_1, v_2 \in \{0, \dots, 4\} \cup \{\infty\}$. Further by the above definition of g_{prod} and Lemma 3.5, we see that

$$|g_{\text{prod}}(\tau_D)| = |g_v(\tau_D)|^2 \quad \text{for some } v \in \{0, 1, 2\}.$$

Therefore, it suffices to show that

$$|g_v(\tau_D)| > |g_{v'}(\tau_Q)|$$

for all $v' \in \{0, \dots, 4\} \cup \{\infty\}$.

As in the proof of Lemma 3.5, let

$$q_D = e^{2\pi i \tau_D}, \quad q_Q = e^{2\pi i \tau_Q} \quad \text{and} \quad r_D = |q_D| = e^{-\pi \sqrt{-D}}, \quad r_Q = |q_Q| = e^{-\pi \sqrt{-D}/a}.$$

One can immediately see that for $D \leq -24$,

$$r_D^{1/5} = e^{-\pi \sqrt{-D}/5} < 1/11 \quad \text{and} \quad r_D^{1/2} \leq r_Q = r_D^{1/a} \leq e^{-\pi \sqrt{3}} < 1/11$$

since $2 \leq a \leq \sqrt{-D/3}$ from Remark 3.4 (ii). Then, we get

$$|g_{\infty}(\tau_Q)| < 5^3 \cdot r_Q \cdot e^{\frac{6r_Q^5}{1-r_Q^5} + \frac{6 \cdot 6r_Q}{1-r_Q}} \quad \text{by Lemma 4.2 (i)}$$

$$\leq 5^3 \cdot e^{-\pi\sqrt{3}} \cdot e^{\frac{6e^{-5\pi\sqrt{3}}}{1-e^{-5\pi\sqrt{3}}} + \frac{6.6e^{-\pi\sqrt{3}}}{1-e^{-\pi\sqrt{3}}}} \quad \text{because } r_Q \leq e^{-\pi\sqrt{3}} \quad (8)$$

$$\approx 0.55747$$

and

$$|g_{v'}(\tau_Q)| < r_Q^{-1/5} \cdot e^{\frac{6r_Q^{1/5}}{1-r_Q^{1/5}} + \frac{6.6r_Q}{1-r_Q}} \quad \text{by Lemma 4.2 (ii)}$$

$$\leq r_D^{-1/10} \cdot e^{\frac{6e^{-\pi\sqrt{3}/5}}{1-e^{-\pi\sqrt{3}/5}} + \frac{6.6e^{-\pi\sqrt{3}}}{1-e^{-\pi\sqrt{3}}}} \quad \text{because } r_D^{1/2} \leq r_Q \leq e^{-\pi\sqrt{3}} \quad (9)$$

$$\approx 21.66520 \cdot r_D^{-1/10}$$

for $v' \in \{0, 1, 2, 3, 4\}$. Further we have

$$|g_v(\tau_D)| > r_D^{-1/5} \cdot e^{-\frac{6.6r_D^{1/5}}{1-r_D^{1/5}} - \frac{6.6r_D}{1-r_D}} \quad \text{by Lemma 4.2 (iii).}$$

Then we deduce that

$$\left| \frac{g_\infty(\tau_Q)}{g_v(\tau_D)} \right| \leq 0.55747 \cdot r_D^{1/5} \cdot e^{\frac{6.6r_D^{1/5}}{1-r_D^{1/5}} + \frac{6.6r_D}{1-r_D}}$$

$$= 0.55747 \cdot e^{-\pi\sqrt{-D}/5} \cdot e^{\frac{6.6e^{-\pi\sqrt{-D}/5}}{1-e^{-\pi\sqrt{-D}/5}} + \frac{6.6e^{-\pi\sqrt{-D}}}{1-e^{-\pi\sqrt{-D}}}} \quad \text{since } r_D = e^{-\pi\sqrt{-D}}$$

$$\leq 0.55747 \cdot e^{-\pi\sqrt{24}/5} \cdot e^{\frac{6.6e^{-\pi\sqrt{24}/5}}{1-e^{-\pi\sqrt{24}/5}} + \frac{6.6e^{-\pi\sqrt{24}}}{1-e^{-\pi\sqrt{24}}}} \approx 0.03530 < 1 \quad \text{for } D \leq -24$$

and

$$\left| \frac{g_{v'}(\tau_Q)}{g_v(\tau_D)} \right| \leq 21.66520 \cdot r_D^{-1/10} \cdot r_D^{1/5} \cdot e^{\frac{6.6r_D^{1/5}}{1-r_D^{1/5}} + \frac{6.6r_D}{1-r_D}}$$

$$= 21.66520 \cdot r_D^{1/10} \cdot e^{\frac{6.6r_D^{1/5}}{1-r_D^{1/5}} + \frac{6.6r_D}{1-r_D}}$$

$$= 21.66520 \cdot e^{-\pi\sqrt{-D}/10} \cdot e^{\frac{6.6e^{-\pi\sqrt{-D}/5}}{1-e^{-\pi\sqrt{-D}/5}} + \frac{6.6e^{-\pi\sqrt{-D}}}{1-e^{-\pi\sqrt{-D}}}} \quad \text{since } r_D = e^{-\pi\sqrt{-D}}$$

$$\leq 21.66520 \cdot e^{-\pi\sqrt{99}/10} \cdot e^{\frac{6.6e^{-\pi\sqrt{99}/5}}{1-e^{-\pi\sqrt{99}/5}} + \frac{6.6e^{-\pi\sqrt{99}}}{1-e^{-\pi\sqrt{99}}}} \approx 0.96330 < 1 \quad \text{for } D \leq -99$$

for all $v' \in \{0, 1, 2, 3, 4\}$. Hence we obtain the assertion for $D \leq -99$.

For the remaining finite cases where $-96 \leq D \leq -31$, we observe that

$$|g_{v'}(\tau_Q)| \leq r_Q^{-1/5} \cdot e^{\frac{6r_Q^{1/5}}{1-r_Q^{1/5}} + \frac{6.6r_Q}{1-r_Q}} = r_D^{-1/5a} \cdot e^{\frac{6r_D^{1/5a}}{1-r_D^{1/5a}} + \frac{6.6r_D^{1/a}}{1-r_D^{1/a}}} \quad \text{because } r_Q = r_D^{1/a}$$

for $v' \in \{0, 1, 2, 3, 4\}$ and $a \geq 2$. We then deduce that

$$\left| \frac{g_{v'}(\tau_Q)}{g_v(\tau_D)} \right| < r_D^{1/5-1/5a} \cdot e^{\frac{6r_D^{1/5a}}{1-r_D^{1/5a}} + \frac{6.6r_D^{1/a}}{1-r_D^{1/a}} + \frac{6.6r_D^{1/5}}{1-r_D^{1/5}} + \frac{6.6r_D}{1-r_D}}$$

$$= e^{-\pi\sqrt{-D} \cdot (1/5-1/5a)} \cdot e^{\frac{6e^{-\pi\sqrt{-D}/5a}}{1-e^{-\pi\sqrt{-D}/5a}} + \frac{6.6e^{-\pi\sqrt{-D}/a}}{1-e^{-\pi\sqrt{-D}/a}} + \frac{6.6e^{-\pi\sqrt{-D}/5}}{1-e^{-\pi\sqrt{-D}/5}} + \frac{6.6e^{-\pi\sqrt{-D}}}{1-e^{-\pi\sqrt{-D}}}}. \quad (10)$$

By using the algorithm for counting reduced forms (see [14, Algorithm 5.3.5]), we can make the list of the actual values of a for each D (see Table 1 below). Evaluating (10) at those values, we attain the assertion for $-96 \leq D \leq -31$.

Therefore, we conclude that the only reduced form in \mathcal{Q}_D^0 that fixes $g_{\text{prod}}(\tau_D)$ is the principal form, which represents the identity in the group $C(D) \cong \text{Gal}(H_D/K)$.

This completes the proof of our theorem by Galois theory. \square

Table 1: The coefficients of x^2 of non-principal reduced form $Q = [a, b, c] \in C(D)$ for $D \equiv \square \pmod{100}$ relatively prime to 5 for $D > -200$.

D	a	D	a	D	a	D	a
-24	2	-71	2, 3, 4	-116	2, 3, 5	-159	2, 3, 4, 5, 6
-31	2	-76	4	-119	2, 3, 4, 5, 6	-164	2, 3, 5, 6
-36	2	-79	2, 4	-124	5	-171	5, 7
-39	2, 3	-84	2, 3, 5	-131	3, 5	-176	3, 4, 5
-44	3	-91	5	-136	2, 5	-179	3, 5
-51	3	-96	3, 4, 5	-139	5	-184	2, 5
-56	2, 3	-99	5	-144	4, 5	-191	2, 3, 4, 5, 6
-59	3	-104	2, 3, 5	-151	2, 4, 5	-196	2, 5
-64	4	-111	2, 3, 4, 5	-156	3, 5	-199	2, 4, 5, 7

Remark 4.4. In fact, we can see that Theorem 4.3 is still valid for $D = -24$. We have

$$C(-24) = \{Q_0 = [1, 0, 6], Q_1 = [2, 0, 3]\}$$

so that the corresponding CM points are given by

$$\tau_{Q_0} = \tau_D = \frac{\sqrt{-24}}{2} = \sqrt{-6} \quad \text{and} \quad \tau_{Q_1} = \frac{\sqrt{-24}}{4} = \frac{\sqrt{-6}}{2}.$$

Since

$$\mathfrak{g}_{\text{prod}}(\tau_D)^{Q_1^{-1}} = (\mathfrak{g}_2(\tau_D) \cdot \mathfrak{g}_3(\tau_D))^{Q_1^{-1}} = \mathfrak{g}_1(\tau_{Q_1}) \cdot \mathfrak{g}_4(\tau_{Q_1})$$

by Proposition 3.3, it is enough to show that

$$\left| \mathfrak{g}_2(\sqrt{-6}) \right| > \left| \mathfrak{g}_1(\sqrt{-6}/2) \right|.$$

From Lemma 4.2 (iii), a lower bound of $\left| \mathfrak{g}_2(\sqrt{-6}) \right|$ is given by

$$\left| \mathfrak{g}_2(\sqrt{-6}) \right| \geq e^{2\pi\sqrt{6}/5} \cdot e^{-\frac{6.6e^{-2\pi\sqrt{6}/5}}{1-e^{-2\pi\sqrt{6}/5}} - \frac{6.6e^{-2\pi\sqrt{6}}}{1-e^{-2\pi\sqrt{6}}}} > 15.79269.$$

On the other hand, for an upper bound of $\left| \mathfrak{g}_1(\sqrt{-6}/2) \right|$, we observe that

$$\prod_{n=1}^{\infty} \frac{(1 - \zeta_5^n q^{n/5})^6}{(1 - q^n)^6} = \prod_{t=0}^{\infty} \prod_{s=1}^4 (1 - \zeta_5^{5t+s} q^{(5t+s)/5})^6 = \prod_{t=0}^{\infty} \prod_{s=1}^4 (1 - \zeta_5^s q^{t+s/5})^6$$

since

$$1 - \zeta_5^n q^{n/5} = 1 - \zeta_5^{5n_0} q^{5n_0/5} = 1 - q^{n_0} \quad \text{for } n = 5n_0.$$

For each $1 \leq s \leq 4$, we then obtain

$$\begin{aligned} \left| 1 - \zeta_5^s q^{t+s/5} \right|^2 &= \left(1 - r_{Q_1}^{t+s/5} \cos \frac{2s\pi}{5} \right)^2 + \left(r_{Q_1}^{t+s/5} \sin \frac{2s\pi}{5} \right)^2 \quad \text{since } q_{Q_1} = r_{Q_1} = e^{-\pi\sqrt{6}} \in \mathbb{R} \\ &= 1 - 2r_{Q_1}^{t+s/5} \cos \frac{2s\pi}{5} + r_{Q_1}^{2(t+s/5)}. \end{aligned}$$

Then, it is routine to check that

$$\prod_{s=1}^4 \left(1 - 2r_{Q_1}^{t+s/5} \cos \frac{2s\pi}{5} + r_{Q_1}^{2(t+s/5)} \right)^3 = \prod_{s=1}^4 \left(1 - 2e^{-\pi\sqrt{6} \cdot (t+s/5)} \cos \frac{2s\pi}{5} + e^{-2\pi\sqrt{6} \cdot (t+s/5)} \right)^3$$

is a monotone increasing function for $t \geq 1$ and has the limit 1 when $t \rightarrow \infty$. Moreover, its value at $t = 0$ is less than 1. Hence we get

$$\left| g_1 \left(\sqrt{-6}/2 \right) \right| < r_{Q_1}^{-1/5} = e^{\pi\sqrt{6}/5} \leq 4.66021 < 15.79269 < \left| g_2 \left(\sqrt{-6} \right) \right|.$$

Note that the minimal polynomial of $g_{\text{prod}}(\tau_D)$ is given by

$$\left(X - g_2(\sqrt{-6}) \cdot g_3(\sqrt{-6}) \right) \left(X - g_1(\sqrt{-6}/2) \cdot g_4(\sqrt{-6}/2) \right) = X^2 - 750X + 15625.$$

Theorem 4.5. For an imaginary quadratic discriminant D , we assume that $D \equiv \square \pmod{100}$ and $\gcd(D, 5) = 1$. Then the singular values defined by

$$g_{\text{sum}}(\tau_D) = \begin{cases} g_2(\tau_D) + g_3(\tau_D) & \text{if } D \equiv 0 \pmod{4}, D \equiv 1 \pmod{5}, D \leq -44, \\ g_1(\tau_D) + g_4(\tau_D) & \text{if } D \equiv 0 \pmod{4}, D \equiv 4 \pmod{5}, D \leq -56, \\ g_0(\tau_D) + g_1(\tau_D) & \text{if } D \equiv 1 \pmod{4}, D \equiv 1 \pmod{5}, D \leq -59, \\ g_2(\tau_D) + g_4(\tau_D) & \text{if } D \equiv 1 \pmod{4}, D \equiv 4 \pmod{5}, D \leq -71, \end{cases}$$

are real-valued class invariants over $K = \mathbb{Q}(\tau_D)$.

Proof. We prove the case $D \equiv 0 \pmod{4}$, $D \equiv 1 \pmod{5}$. The proofs for the other cases can be done similarly.

If $h_D = 1$, there is nothing to prove. Therefore, we may assume that $h_D \geq 2$. Let $Q = [a, b, c] \in \mathcal{O}_D^0$ be a non-principal reduced form so that

$$2 \leq a \leq \sqrt{-D/3}$$

by Remark 3.4 (ii). From the definition of $g_{\text{sum}}(\tau_D)$ and Lemma 3.5, we have

$$g_{\text{sum}}(\tau_D) = g_2(\tau_D) + g_3(\tau_D) = 2 \cdot \text{Re}(g_2(\tau_D)).$$

Further by Remark 2.6 and Proposition 3.3, we see that

$$g_{\text{sum}}(\tau_D)^{Q^{-1}} = g_{v_1}(\tau_Q) + g_{v_2}(\tau_Q) \quad \text{for some } v_1, v_2 \in \{0, 1, 2, 3, 4\} \cup \{\infty\}. \quad (11)$$

Hence, it is enough to prove that

$$|\text{Re}(g_2(\tau_D))| > |g_v(\tau_Q)|$$

for all $v \in \{0, 1, 2, 3, 4\} \cup \{\infty\}$.

We estimate a lower bound of $|\text{Re}(g_2(\tau_D))|$. Let us set

$$q_D = e^{2\pi i \tau_D} \quad \text{and} \quad r_D = |q_D| = e^{-\pi\sqrt{-D}}.$$

In fact, $q_D = r_D$ for $D \equiv 0 \pmod{4}$. By Remark 2.4, we then have

$$g_2(\tau_D) = \zeta_5^{-2} \cdot r_D^{-1/5} \cdot \prod_{n=1}^{\infty} \left(\frac{1 - \zeta_5^{2n} r_D^{n/5}}{1 - r_D^n} \right)^6.$$

We put

$$z_D = \zeta_5^{-2} \cdot r_D^{-1/5} = \left(\cos \frac{4\pi}{5} - i \sin \frac{4\pi}{5} \right) \cdot r_D^{-1/5} \quad \text{and} \quad w_D = \prod_{n=1}^{\infty} \left(\frac{1 - \zeta_5^{2n} \cdot r_D^{n/5}}{1 - r_D^n} \right)^6$$

so that $g_2(\tau_D) = z_D \cdot w_D$. Furthermore, let

$$\mathfrak{L}(D) = e^{-\frac{6.6e^{-\pi\sqrt{-D}/5}}{1-e^{-\pi\sqrt{-D}/5}} - \frac{6.6e^{-\pi\sqrt{-D}}}{1-e^{-\pi\sqrt{-D}}}} \quad \text{and} \quad \mathfrak{U}(D) = e^{\frac{6e^{-\pi\sqrt{-D}/5}}{1-e^{-\pi\sqrt{-D}/5}} + \frac{6.6e^{-\pi\sqrt{-D}}}{1-e^{-\pi\sqrt{-D}}}}.$$

One can check that $\mathfrak{L}(D)$ and $\mathfrak{U}(D)$ are decreasing and increasing functions for $D \leq -44$, respectively. By substituting $e^{-\pi\sqrt{-D}}$ for r in Lemma 4.2 (ii) and (iii), we see that

$$\mathfrak{L}(D) < |w_D| < \mathfrak{U}(D) \quad \text{for } D \leq -44.$$

Now we compute $|\operatorname{Re}(w_D)|$ and $|\operatorname{Im}(w_D)|$ by estimating the argument of w_D . For $-\pi \leq \theta$, $\theta_n \leq \pi$, we let $w_D = |w_D| \cdot e^{i\theta}$ and θ_n be the argument of

$$\frac{1 - \zeta_5^{2n} \cdot r_D^{n/5}}{1 - r_D^n} = \frac{1 - e^{4n\pi i/5} \cdot r_D^{n/5}}{1 - r_D^n}$$

for each n . Since

$$1 - r_D^{n/5} \cdot e^{4n\pi i/5} = \left(1 - r_D^{n/5} \cos \frac{4n\pi}{5}\right) - i r_D^{n/5} \sin \frac{4n\pi}{5},$$

we get

$$|\tan \theta_n| = \left| \frac{r_D^{n/5} \sin \frac{4n\pi}{5}}{1 - r_D^{n/5} \cos \frac{4n\pi}{5}} \right| \leq \frac{r_D^{n/5}}{1 - r_D^{n/5}} \leq r_D^{n/5} \cdot (1 + 1.1 \cdot r_D^{n/5}) \quad \text{by Lemma 4.1 (ii).}$$

Define

$$\Theta(D) = \frac{6e^{-\pi\sqrt{-D}/5}}{1 - e^{-\pi\sqrt{-D}/5}} + \frac{6.6e^{-2\pi\sqrt{-D}/5}}{1 - e^{-2\pi\sqrt{-D}/5}}$$

which is an increasing function for $D \leq -44$. Then, by using the fact that $x \leq \tan x$ for $0 < x < \pi/2$, we obtain that

$$\begin{aligned} |\theta| &= \left| 6 \cdot \sum_{n=1}^{\infty} \theta_n \right| \leq 6 \cdot \sum_{n=1}^{\infty} |\tan \theta_n| \leq 6 \cdot \sum_{n=1}^{\infty} \left(r_D^{n/5} + 1.1 r_D^{2n/5} \right) \\ &\leq \frac{6r_D^{1/5}}{1 - r_D^{1/5}} + \frac{6.6r_D^{2/5}}{1 - r_D^{2/5}} = \Theta(D) \quad \text{because } r_D = e^{-\pi\sqrt{-D}}. \end{aligned}$$

Thus, by using that $\sin x \leq x$ for $x > 0$, we get

$$|\operatorname{Im}(w_D)| = |w_D| \cdot |\sin \theta| \leq |w_D| \cdot |\theta| < \mathfrak{L}(D) \cdot \Theta(D). \quad (12)$$

We then arrive at

$$|\operatorname{Re}(w_D)| = \sqrt{|w_D|^2 - |\operatorname{Im}(w_D)|^2} \geq \sqrt{\mathfrak{L}(D)^2 - \mathfrak{L}(D)^2 \cdot \Theta(D)^2}. \quad (13)$$

Note that

$$\mathfrak{L}(D)^2 - \mathfrak{L}(D)^2 \cdot \Theta(D)^2 > 0 \quad \text{for } D \leq -44.$$

Therefore, we achieve by (12), (13) that

$$\begin{aligned} |\operatorname{Re}(\mathfrak{g}_2(\tau_D))| &= |\operatorname{Re}(z_D) \cdot \operatorname{Re}(w_D) - \operatorname{Im}(z_D) \cdot \operatorname{Im}(w_D)| \\ &\geq ||\operatorname{Re}(z_D)| \cdot |\operatorname{Re}(w_D)| - |\operatorname{Im}(z_D)| \cdot |\operatorname{Im}(w_D)|| \\ &= \left| |\operatorname{Re}(w_D)| \cdot \left| \cos \frac{4\pi}{5} \right| - |\operatorname{Im}(w_D)| \cdot \left| \sin \frac{4\pi}{5} \right| \right| \cdot r_D^{-1/5} \\ &> \left(\sqrt{\mathfrak{L}(D)^2 - \mathfrak{L}(D)^2 \cdot \Theta(D)^2} \cdot \cos \frac{\pi}{5} - \mathfrak{L}(D) \cdot \Theta(D) \cdot \sin \frac{\pi}{5} \right) \cdot e^{\pi\sqrt{-D}/5}. \end{aligned} \quad (14)$$

On the other hand, from (8), we have

$$|\mathfrak{g}_\infty(\tau_Q)| < 0.55747$$

for any reduced form $Q \in \mathcal{D}_D^0$. By evaluating $\mathfrak{L}(D)$, $\mathfrak{L}(D)$ and $\Theta(D)$ at $D = -44$, we obtain from (14) that

$$|\operatorname{Re}(\mathfrak{g}_2(\tau_D))| > 0.66224 \cdot e^{\pi\sqrt{44}/5} \approx 42.76270 > |\mathfrak{g}_\infty(\tau_Q)| \quad \text{for } D \leq -44.$$

Furthermore, by (9), we have

$$|\mathfrak{g}_v(\tau_Q)| \leq 21.66520 \cdot r_D^{-1/10} = 21.66520 \cdot e^{\pi\sqrt{-D}/10}$$

for $v \in \{0, 1, 2, 3, 4\}$. Then, by specializing $\mathfrak{L}(D)$, $\mathfrak{L}(D)$ and $\Theta(D)$ at $D = -124$, we get from (14) that

$$|\operatorname{Re}(\mathfrak{g}_2(\tau_D))| > 0.80087 \cdot e^{\pi\sqrt{-D}/5} \quad \text{for } D \leq -124.$$

Hence, we achieve that for $D \leq -124$,

$$\left| \frac{g_v(\tau_Q)}{\operatorname{Re}(g_2(\tau_D))} \right| < \frac{21.66520}{0.80087} \cdot e^{-\pi\sqrt{124}/10} \approx 0.81827 < 1.$$

The finite remaining cases are given by

$$D = -104, -84, -64, -44.$$

We see that for $v \in \{0, 1, 2, 3, 4\}$,

$$\begin{aligned} |g_v(\tau_Q)| &\leq r_Q^{-1/5} \cdot e^{\frac{6r_Q^{1/5}}{1-r_Q^{1/5}} + \frac{7.2r_Q}{1-r_Q}} \quad \text{by Lemma 4.2 (ii)} \\ &= r_D^{-1/5a} \cdot e^{\frac{6r_D^{1/5a}}{1-r_D^{1/5a}} + \frac{7.2r_D^{1/a}}{1-r_D^{1/a}}} \quad \text{because } r_Q = r_D^{1/a} \\ &= e^{\pi\sqrt{-D}/5a} \cdot e^{\frac{6e^{-\pi\sqrt{-D}/5a}}{1-e^{-\pi\sqrt{-D}/5a}} + \frac{7.2e^{-\pi\sqrt{-D}/a}}{1-e^{-\pi\sqrt{-D}/a}}}. \end{aligned}$$

By evaluating (14) at $D = -44$ and the last formula at the actual values of a (see Table 1) of non-principal reduced forms $Q = [a, b, c] \in \mathcal{Q}_D^0$, we again achieve that

$$\left| \frac{g_v(\tau_Q)}{\operatorname{Re}(g_2(\tau_D))} \right| < 1$$

for the remaining cases.

Hence, we conclude from (11) that

$$|g_{\text{sum}}(\tau_D)| = 2 \cdot \operatorname{Re}(g_2(\tau_D)) > |g_{v_1}(\tau_D)| + |g_{v_2}(\tau_D)| \geq |g_{\text{sum}}(\tau_D)|^{Q^{-1}}$$

for any reduced forms Q representing non-identity classes in $C(D) \cong \operatorname{Gal}(H_D/K)$. This completes the proof by Galois theory. \square

Remark 4.6. (i) The finite exceptional cases of D with $h_D \geq 2$ in the above theorem are given by $D = -24, -31, -36, -39, -51$. Using Proposition 3.3, we can directly compute the minimal polynomials of

$$g_{\text{sum}}(\tau_D) = \begin{cases} g_2(\tau_D) + g_3(\tau_D) & \text{if } D = -24, \\ g_1(\tau_D) + g_4(\tau_D) & \text{if } D = -36, \\ g_0(\tau_D) + g_1(\tau_D) & \text{if } D = -39, \\ g_2(\tau_D) + g_4(\tau_D) & \text{if } D = -31, -51 \end{cases}$$

over \mathbb{Q} , namely,

$$\begin{cases} X^2 + 56X + 392 & \text{if } D = -24, h_{-24} = 2, \\ X^3 + 57X^2 + 991X + 6383 & \text{if } D = -31, h_{-31} = 3, \\ X^2 - 16X + 16 & \text{if } D = -36, h_{-36} = 2, \\ X^4 - 29X^3 - 2321X^2 - 37041X - 187867 & \text{if } D = -39, h_{-39} = 4, \\ X^2 + 68X + 68 & \text{if } D = -51, h_{-51} = 2, \end{cases}$$

which are irreducible over \mathbb{Q} . Thus, we can establish Theorem 4.3 again.

(ii) In fact, for $D = -24, -36, -39, -51$, we can apply the same argument as in Remark 4.4. However, we shall not repeat the same computations.

Example 4.7. Let $D = -96$ and $K = \mathbb{Q}(\sqrt{-6})$. Then we have

$$C(-96) = \{Q_0 = [1, 0, 24], Q_1 = [3, 0, 8], Q_2 = [4, 4, 7], Q_3 = [5, 2, 5]\}$$

with

$$\tau_{-96} = \tau_{Q_0} = 2\sqrt{-6}, \tau_{Q_1} = \frac{2\sqrt{-6}}{3}, \tau_{Q_2} = \frac{-1 + \sqrt{-16}}{2}, \tau_{Q_3} = \frac{-2 + 2\sqrt{-6}}{5}.$$

By Proposition 3.3, the class polynomials of

$$\mathfrak{g}_{\text{prod}}(\tau_{-96}) = \mathfrak{g}_1(\tau_{-96}) \cdot \mathfrak{g}_4(\tau_{-96}) \quad \text{and} \quad \mathfrak{g}_{\text{sum}}(\tau_{-96}) = \mathfrak{g}_1(\tau_{-96}) + \mathfrak{g}_4(\tau_{-96})$$

are given by

$$\begin{aligned} \min(\mathfrak{g}_{\text{prod}}(\tau_{-96}), K) &= (X - \mathfrak{g}_1(\tau_{Q_0}) \cdot \mathfrak{g}_4(\tau_{Q_0})) \cdot (X - \mathfrak{g}_2(\tau_{Q_1}) \cdot \mathfrak{g}_3(\tau_{Q_1})) \\ &\quad \cdot (X - \mathfrak{g}_2(\tau_{Q_2}) \cdot \mathfrak{g}_4(\tau_{Q_2})) \cdot (X - \mathfrak{g}_0(\tau_{Q_3}) \cdot \mathfrak{g}_{\infty}(\tau_{Q_3})) \\ &= X^4 - 221000X^3 + 60281250X^2 - 3453125000X + 244140625 \end{aligned}$$

and

$$\begin{aligned} \min(\mathfrak{g}_{\text{sum}}(\tau_{-96}), K) &= (X - (\mathfrak{g}_1(\tau_{Q_0}) + \mathfrak{g}_4(\tau_{Q_0}))) \cdot (X - (\mathfrak{g}_2(\tau_{Q_1}) + \mathfrak{g}_3(\tau_{Q_1}))) \\ &\quad \cdot (X - (\mathfrak{g}_2(\tau_{Q_2}) + \mathfrak{g}_4(\tau_{Q_2}))) \cdot (X - (\mathfrak{g}_0(\tau_{Q_3}) + \mathfrak{g}_{\infty}(\tau_{Q_3}))) \\ &= X^4 - 236X^3 - 11712X^2 - 125528X + 20164, \end{aligned}$$

respectively.

5 Modular trace of a weakly holomorphic modular function

Throughout this section, we shall assume that an imaginary quadratic discriminant $D = d_K \cdot t^2$ is congruent to a square modulo $4N^2$ and relatively prime to N .

For each positive integer N , let

$$\Gamma = \Gamma_0^0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1) \mid b \equiv c \equiv 0 \pmod{N} \right\}$$

which is a congruence subgroup of level N . We denote

$$\mathcal{Q}_{D,(N)} = \{[a, b, c] \in \mathcal{Q}_D \mid a \equiv c \equiv 0 \pmod{N}\}.$$

Then, the elements of $\mathcal{Q}_{D,(N)}$ can be written as $Q = [Na, b, Nc]$. From (2), one can check that Γ acts on $\mathcal{Q}_{D,(N)}$ and the action preserves the value of $b \pmod{2N^2}$. Thus we obtain the following decomposition

$$\mathcal{Q}_{D,(N)}/\Gamma = \bigcup_{\beta \in \mathbb{Z}/2N^2\mathbb{Z}} \mathcal{Q}_{D,(N),\beta}/\Gamma,$$

where $\mathcal{Q}_{D,(N),\beta} = \{[Na, b, Nc] \in \mathcal{Q}_D \mid b \equiv \beta \pmod{2N^2}\}$ for each $\beta \in \mathbb{Z}/2N^2\mathbb{Z}$.

Remark 5.1. (i) The values β with $\mathcal{Q}_{D,(N),\beta} \neq \emptyset$ can be determined by the congruence equation $\beta^2 \equiv D \pmod{4N^2}$. If N has ℓ distinct prime divisors, then the number of such β is equal to 2^ℓ by Chinese remainder theorem.

(ii) Let \mathcal{O}_d be a quadratic order containing \mathcal{O}_D in $K = \mathbb{Q}(\tau_D)$. Then we can write $d = d_K \cdot (t/t')^2$ for some positive divisor t' of t . By assigning

$$Q = [Na, b, Nc] \text{ (with } \gcd(Na, b, Nc) = t') \longmapsto \tilde{Q} = \frac{1}{t'}[Na, b, Nc]$$

for each $t' \mid t$, we then obtain the decomposition

$$\mathcal{Q}_{D,(N),\beta} = \bigcup_{d \mid D} t' \cdot \mathcal{Q}_{d,(N),\beta t'^{-1}}.$$

Moreover, we can easily see that $\tau_Q = \tau_{\tilde{Q}}$ and $\Gamma_Q = \Gamma_{\tilde{Q}}$.

From now on, we assume that $\mathcal{Q}_{D,(N),\beta} \neq \emptyset$ for some suitable $\beta \in \mathbb{Z}/2N^2\mathbb{Z}$. Let $\mathcal{Q}_{D,(N),\beta}^0 \subset \mathcal{Q}_{D,(N),\beta}$ be the subset of primitive forms. Then, we have the following lemma.

Lemma 5.2. *We have a canonical bijection between $\mathcal{Q}_D/\Gamma(1)$ (resp. $\mathcal{Q}_D^0/\Gamma(1)$) and $\mathcal{Q}_{D,(N),\beta}/\Gamma$ (resp. $\mathcal{Q}_{D,(N),\beta}^0/\Gamma$).*

Proof. See [15, Proposition in §I.1] and [6, Lemma 5.1]. □

Let f be a modular function on Γ . We define the Zagier-type trace $\mathbf{t}_f^{(\beta)}(D)$ of index D as

$$\mathbf{t}_f^{(\beta)}(D) = \sum_{Q \in \mathcal{Q}_{D,(N),\beta}/\Gamma} \frac{1}{|\Gamma_Q|} f(\tau_Q),$$

where the weights of the summands are determined by the following lemma.

Lemma 5.3. *For each $Q \in \mathcal{Q}_{D,(N),\beta}$, we have*

$$|\Gamma_Q| = \begin{cases} 2 & \text{if } D = -4 \cdot t^2 \text{ and } Q \text{ is } \Gamma(1)\text{-equivalent to } [t, 0, t], \\ 3 & \text{if } D = -3 \cdot t^2 \text{ and } Q \text{ is } \Gamma(1)\text{-equivalent to } [t, t, t], \\ 1 & \text{otherwise.} \end{cases}$$

In particular, if Q is primitive, then t should be 1.

Proof. It is a straightforward consequence from the fact that for each $Q \in \mathcal{Q}_D$,

$$|\Gamma(1)_Q| = \begin{cases} 2 & \text{if } Q \text{ is } \Gamma(1)\text{-equivalent to } [t, 0, t], \\ 3 & \text{if } Q \text{ is } \Gamma(1)\text{-equivalent to } [t, t, t], \\ 1 & \text{otherwise.} \end{cases}$$

□

Now we briefly introduce the Bruinier-Funke modular trace of modular functions on Γ (see [4] for general statements). Let

$$V(\mathbb{Q}) = \left\{ X = \begin{bmatrix} b & a \\ c & -b \end{bmatrix} \mid a, b, c \in \mathbb{Q} \right\}$$

be the vector space of dimension 3 over \mathbb{Q} consisting of trace zero 2×2 matrices. It becomes a quadratic space of signature $(1, 2)$ with the quadratic form $q(X) = \det(X)$ and the associated bilinear form $(X, Y) = -\text{tr}(XY)$ for $X, Y \in V(\mathbb{Q})$. One can see that the group $\text{SL}_2(\mathbb{Q})$ acts on V by conjugation $\gamma.X = \gamma X \gamma^{-1}$ for $X \in V(\mathbb{Q})$ and $\gamma \in \text{SL}_2(\mathbb{Q})$.

Let \mathcal{D} be the space of positive lines in $V(\mathbb{R}) = V(\mathbb{Q}) \otimes \mathbb{R}$, namely,

$$\mathcal{D} = \{ z \subset V(\mathbb{R}) \mid \dim(z) = 1, q|_z > 0 \}.$$

We can identify \mathcal{D} with \mathbb{H} by assigning $\tau = x + yi \in \mathbb{H}$ to the line spanned by

$$X_\tau = \frac{1}{y} \begin{bmatrix} -\frac{1}{2}(\tau + \bar{\tau}) & \tau \bar{\tau} \\ -1 & \frac{1}{2}(\tau + \bar{\tau}) \end{bmatrix}.$$

By direct computation, one can easily check that $q(X_\tau) = 1$ and $\gamma.X_\tau = X(\gamma\tau)$ for $\gamma \in \text{SL}_2(\mathbb{R})$. Then, the CM points in \mathbb{H} can be viewed as positive lines $\mathbb{R}X$ with the vectors $X \in V(\mathbb{Q})$ of positive norms.

Let L be an even \mathbb{Z} -lattice of $V(\mathbb{Q})$ defined by

$$L = \left\{ \begin{bmatrix} Nb & c \\ a & -Nb \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Then, the level of L is $4N^2$ and the dual lattice is given by

$$L^\sharp = \left\{ \begin{bmatrix} Nb & c \\ a & -Nb \end{bmatrix} \mid a, c \in \mathbb{Z} \text{ and } b \in \frac{1}{2N^2}\mathbb{Z} \right\}. \quad (15)$$

We then see that Γ acts on L by conjugation and acts trivially on the discriminant group L^\sharp/L . Furthermore, the group L^\sharp/L is isomorphic to a cyclic group $\mathbb{Z}/2N^2\mathbb{Z}$. Therefore, each coset can be written in the form

$$L + h = \left\{ X = \begin{bmatrix} Nb + h/2N & c \\ -a & -Nb - h/2N \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\} \quad (16)$$

for $h \in \{0, 1, \dots, 2N^2 - 1\}$.

Meanwhile, by using the fact that the stabilizer of each $X \in V(\mathbb{R})$ in $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SO}(2)$ is compact, we get that $\Gamma_X = (\mathrm{SL}_2(\mathbb{R}))_X \cap \Gamma$ is finite. Besides, if we let m be a positive rational number and h be a representative in $L^\sharp/L \cong \mathbb{Z}/2N^2\mathbb{Z}$, the group Γ acts on the set

$$L_{h,m} = \{X \in L + h \mid q(X) = m\}$$

with the finite number of orbits. Then, the *modular trace* of a weakly holomorphic modular function f on Γ with respect to the lattice L for positive index m is defined by

$$\mathrm{MT}_f^L(h, m) = \sum_{X \in \Gamma_N \backslash L_{h,m}} \frac{1}{|\Gamma_X|} f(\tau_X),$$

where τ_X is a CM point corresponding to the vector

$$\begin{cases} (1/\sqrt{m}) \cdot X & \text{if } a > 0, \\ (1/\sqrt{m}) \cdot (-X) & \text{if } a < 0. \end{cases}$$

The modular traces for zero or negative index are described by using a regularized integral or an infinite geodesic in \mathbb{H} (See [4, Definition 4.3]). Their explicit computations are given in [4, Proposition 4.7 and Remark 4.9]. We then have the following analytic property of modular traces.

Proposition 5.4. *Let f be a weakly holomorphic modular function on Γ . Then the series*

$$\sum_{n \gg -\infty} \mathrm{MT}_f^L(h, n) q^n$$

is the holomorphic part of a harmonic weak Maass form of weight $3/2$ on $\Gamma(4N^2)$.

Proof. See [4, Theorem 4.5]. □

Remark 5.5. If $h = 0$, then the above series is the holomorphic part of a harmonic weak Maass form of weight $3/2$ on a bigger group $\Gamma_0(4N^2)$ (See [4, §§3-4]).

Furthermore, the modular traces with respect to the lattice L can be related to the Zagier-type traces of modular functions.

Proposition 5.6. *We have*

$$\mathrm{MT}_f^L(\beta, -D/4N^2) = \mathfrak{t}_f^{(\beta)}(D) + \mathfrak{t}_f^{(-\beta)}(D).$$

Proof. See [6, Lemma 2.3]. □

6 Modular property of Galois traces of class invariants

Let us assume that $N = 5$ and D is an imaginary quadratic discriminant such that $D \equiv \square \pmod{100}$ and $\gcd(D, 5) = 1$. In this section, we shall identify the Galois traces of real-valued class invariants defined in

Theorems 4.3 and 4.5 with the Fourier coefficients of harmonic weak Maass forms of weight $3/2$ by using the Bruinier-Funke modular traces and Shimura's reciprocity law. For $N = 5$, we recall that $\Gamma = \Gamma_0^0(5)$ and

$$L = \left\{ \begin{bmatrix} 5b & c \\ a & -5b \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Before we go further, we need some lemmas.

Lemma 6.1. g_0 and g_∞ are Γ -modular functions.

Proof. By the definition of $\Gamma = \Gamma_0^0(5)$, the only nontrivial transformation is given by the matrices $\gamma \equiv \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \pmod{5}$ in Γ . By Lemma 2.5, we have the decomposition

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \equiv ST^{-3}ST^{-2}ST^{-3} \pmod{5}.$$

Using (5), we deduce that

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} : \begin{array}{l} g_0 \xrightarrow{S} g_\infty \xrightarrow{T^{-3}} g_\infty \xrightarrow{S} g_0 \xrightarrow{T^{-2}} g_3 \xrightarrow{S} g_3 \xrightarrow{T^{-3}} g_0, \\ g_\infty \xrightarrow{S} g_0 \xrightarrow{T^{-3}} g_2 \xrightarrow{S} g_2 \xrightarrow{T^{-2}} g_0 \xrightarrow{S} g_\infty \xrightarrow{T^{-3}} g_\infty. \end{array}$$

This completes the proof. \square

For a given discriminant D , we choose $\beta \in \mathbb{Z}/50\mathbb{Z}$ satisfying $\beta^2 \equiv D \pmod{50}$ so that $\mathcal{Q}_{D,(5),\beta}$ is nonempty.

Lemma 6.2. Let $Q = [5a, b, 5c] \in \mathcal{Q}_{D,(5),\beta}^0$. Then we have

$$\begin{cases} (g_{\text{prod}}(\tau_D))^{Q^{-1}} = g_0(\tau_Q) \cdot g_\infty(\tau_Q), \\ (g_{\text{sum}}(\tau_D))^{Q^{-1}} = g_0(\tau_Q) + g_\infty(\tau_Q). \end{cases}$$

Proof. Since $D = b^2 - 100ac$ and $\gcd(D, 5) = 1$, we have $D \equiv b^2 \pmod{5}$ and $\gcd(b, 5) = 1$. From (6) and (7), the corresponding matrix $M_Q \in \text{GL}_2(\mathbb{Z}/5\mathbb{Z})/\{\pm I_2\}$ is given by

$$\begin{cases} \begin{bmatrix} -5a - b/2 & -5c - b/2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -b/2 & -b/2 \\ 1 & -1 \end{bmatrix} & \text{if } D \equiv 0 \pmod{4}, \\ \begin{bmatrix} -5a - (b+1)/2 & -5c + (1-b)/2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -(b+1)/2 & (1-b)/2 \\ 1 & -1 \end{bmatrix} & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

By Lemma 2.5, we obtain

$$\begin{cases} \begin{bmatrix} -b/2 & -b/2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \cdot T^{b(1+2b)}ST^{b-1}ST^{b-1} & \text{if } D \equiv 0 \pmod{4}, \\ \begin{bmatrix} -(b+1)/2 & (1-b)/2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \cdot T^{b(3+2b)}ST^{b-1}ST^{b-1} & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Since the computations for other cases are similar, we suppose that $D \equiv 0 \pmod{4}$ and $D \equiv 1 \pmod{5}$. Then, we get

$$b = 1 : \begin{cases} g_2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} g_2 \xrightarrow{T^3} g_0 \xrightarrow{S} g_\infty \xrightarrow{T^1} g_\infty \xrightarrow{S} g_0 \xrightarrow{T^0} g_0, \\ g_3 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} g_3 \xrightarrow{T^3} g_1 \xrightarrow{S} g_4 \xrightarrow{T^1} g_0 \xrightarrow{S} g_\infty \xrightarrow{T^0} g_\infty, \end{cases}$$

$$b = 4 : \begin{cases} \mathfrak{g}_2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}} \mathfrak{g}_3 \xrightarrow{T^1} \mathfrak{g}_4 \xrightarrow{S} \mathfrak{g}_1 \xrightarrow{T^4} \mathfrak{g}_0 \xrightarrow{S} \mathfrak{g}_\infty \xrightarrow{T^3} \mathfrak{g}_\infty, \\ \mathfrak{g}_3 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}} \mathfrak{g}_2 \xrightarrow{T^1} \mathfrak{g}_3 \xrightarrow{S} \mathfrak{g}_3 \xrightarrow{T^4} \mathfrak{g}_2 \xrightarrow{S} \mathfrak{g}_2 \xrightarrow{T^3} \mathfrak{g}_0. \end{cases}$$

This completes the proof by the definitions of $\mathfrak{g}_{\text{prod}}$ and $\mathfrak{g}_{\text{sum}}$. \square

Following Kaneko's description, the *modified* Galois traces of $\mathfrak{g}_{\text{prod}}$ and $\mathfrak{g}_{\text{sum}}$ of index D are given by

$$\begin{cases} \text{GT}_{\mathfrak{g}_{\text{prod}}}(D) = \sum_{\mathcal{O}_d \supset \mathcal{O}_D} \frac{2}{\omega_d} \cdot \text{Tr}_{H_d/K}(\mathfrak{g}_{\text{prod}}(\tau_d)), \\ \text{GT}_{\mathfrak{g}_{\text{sum}}}(D) = \sum_{\mathcal{O}_d \supset \mathcal{O}_D} \frac{2}{\omega_d} \cdot \text{Tr}_{H_d/K}(\mathfrak{g}_{\text{prod}}(\tau_d)), \end{cases}$$

where Tr is the usual Galois trace.

Lemma 6.3. *We have*

$$\begin{cases} \text{GT}_{\mathfrak{g}_{\text{prod}}}(D) = \mathbf{t}_{\mathfrak{g}_0 \cdot \mathfrak{g}_\infty}^{(\beta)}(D) = \mathbf{t}_{\mathfrak{g}_0 \cdot \mathfrak{g}_\infty}^{(-\beta)}(D), \\ \text{GT}_{\mathfrak{g}_{\text{sum}}}(D) = \mathbf{t}_{\mathfrak{g}_0 + \mathfrak{g}_\infty}^{(\beta)}(D) = \mathbf{t}_{\mathfrak{g}_0 + \mathfrak{g}_\infty}^{(-\beta)}(D). \end{cases}$$

Proof. Since $\mathfrak{g}_0 \cdot \mathfrak{g}_\infty$ is Γ -modular function by Lemma 6.1, we deduce that

$$\begin{aligned} \mathbf{t}_{\mathfrak{g}_0 \cdot \mathfrak{g}_\infty}^{(\beta)}(D) &= \sum_{Q \in \mathcal{Q}_{D, (5), \beta} / \Gamma} \frac{1}{|\Gamma_Q|} \cdot \mathfrak{g}_0(\tau_Q) \cdot \mathfrak{g}_\infty(\tau_Q) \quad \text{by definition} \\ &= \sum_{d|D} \left(\sum_{\tilde{Q} \in \mathcal{Q}_{d, (5), \beta t'^{-1}}^0 / \Gamma} \frac{1}{|\Gamma_{\tilde{Q}}|} \cdot \mathfrak{g}_0(\tau_{\tilde{Q}}) \cdot \mathfrak{g}_\infty(\tau_{\tilde{Q}}) \right) \quad \text{by Remark 5.1 (ii)} \\ &= \sum_{d|D} \frac{1}{|\Gamma_{\tilde{Q}}|} \cdot \left(\sum_{\tilde{Q} \in \mathcal{Q}_{d, (5), \beta t'^{-1}}^0 / \Gamma} \mathfrak{g}_{\text{prod}}(\tau_d)^{\tilde{Q}^{-1}} \right) \quad \text{by Lemmas 5.3 and 6.2} \\ &= \sum_{d|D} \frac{1}{|\Gamma_{\tilde{Q}}|} \cdot \text{Tr}_{H_d/K}(\mathfrak{g}_{\text{prod}}(\tau_d)) \quad \text{by Lemma 5.2} \\ &= \sum_{d|D} \frac{2}{\omega_d} \cdot \text{Tr}_{H_d/K}(\mathfrak{g}_{\text{prod}}(\tau_d)) \quad \text{by Lemma 5.3} \\ &= \text{GT}_{\mathfrak{g}_{\text{prod}}}(D), \end{aligned}$$

where $d = d_K \cdot t'^2$ runs over all discriminants of orders $\mathcal{O}_d \supset \mathcal{O}_D$ in $K = \mathbb{Q}(\tau_D)$. Similarly, we have

$$\mathbf{t}_{\mathfrak{g}_0 + \mathfrak{g}_\infty}^{(\beta)}(D) = \text{GT}_{\mathfrak{g}_{\text{sum}}}(D).$$

Since GT is independent of the choice of β , we obtain the equalities on the right side. \square

By combining the above lemmas, we deduce the following theorem.

Theorem 6.4. *Let D be an imaginary quadratic discriminant congruent to a square modulo 100 and relatively prime to 5. Let $\beta \in \mathbb{Z}/50\mathbb{Z}$ such that $\beta^2 \equiv D \pmod{50}$. Then we have*

$$\text{GT}_{\mathfrak{g}_{\text{prod}}}(D) = \frac{1}{2} \cdot \text{MT}_{\mathfrak{g}_0 \cdot \mathfrak{g}_\infty}^L(\beta, -D/100) \quad \text{and} \quad \text{GT}_{\mathfrak{g}_{\text{sum}}}(D) = \frac{1}{2} \cdot \text{MT}_{\mathfrak{g}_0 + \mathfrak{g}_\infty}^L(\beta, -D/100).$$

Moreover, there are finite principal parts $\mathcal{A}(\tau) = \sum_{m \leq 0} a(m)q^m$ and $\mathcal{B}(\tau) = \sum_{m \leq 0} b(m)q^m$ such that each of

$$\mathcal{A}(\tau) + \sum_{\substack{D \equiv \square \pmod{100} \\ \gcd(D, 5) = 1}} \text{GT}_{\mathfrak{g}_{\text{prod}}}(D) q^{-D/100}$$

and

$$\mathcal{B}(\tau) + \sum_{\substack{D \equiv \square (100) \\ \gcd(D, 5) = 1}} \text{GT}_{\mathfrak{g}_{\text{sum}}}(D) q^{-D/100}$$

is the holomorphic part of a harmonic weak Maass form of weight $3/2$ on $\Gamma(100)$.

Proof. The first assertion directly comes from Lemmas 5.6 and 6.3. Precisely, if $D \equiv \beta^2 \pmod{50}$ for some $\beta \in \mathbb{Z}/50\mathbb{Z}$, we deduce that

$$\begin{cases} \text{MT}_{\mathfrak{g}_0 \cdot \mathfrak{g}_{\infty}}^L(\beta, -D/100) = \mathfrak{t}_{\mathfrak{g}_0 \cdot \mathfrak{g}_{\infty}}^{(\beta)}(D) + \mathfrak{t}_{\mathfrak{g}_0 \cdot \mathfrak{g}_{\infty}}^{(-\beta)}(D) = 2 \cdot \text{GT}_{\mathfrak{g}_{\text{prod}}}(D), \\ \text{MT}_{\mathfrak{g}_0 + \mathfrak{g}_{\infty}}^L(\beta, -D/100) = \mathfrak{t}_{\mathfrak{g}_0 + \mathfrak{g}_{\infty}}^{(\beta)}(D) + \mathfrak{t}_{\mathfrak{g}_0 + \mathfrak{g}_{\infty}}^{(-\beta)}(D) = 2 \cdot \text{GT}_{\mathfrak{g}_{\text{sum}}}(D). \end{cases}$$

For the second assertion, let $h \in \{0, 1, \dots, 49\}$ with $\gcd(h, 5) = 1$. Then, a vector $X \in L + h$ is of the form

$$X = \begin{bmatrix} 5b + h/10 & c \\ -a & -5b - h/10 \end{bmatrix} \in L + h$$

from (16). If it has a positive norm $-D/100 \in \mathbb{Q}$, then the corresponding point τ_X is a root of a positive definite form

$$Q = \begin{cases} [5a, 50b + h, 5c] & \text{if } a > 0, \\ [-5a, -50b - h, -5c] & \text{if } a < 0, \end{cases}$$

whose discriminant is given by $(50b + h)^2 - 100ac \equiv h^2 \pmod{100}$. This implies that if $\gcd(h, 5) = 1$, the generating series of $\text{MT}_{\mathfrak{g}_0 \cdot \mathfrak{g}_{\infty}}^L(h, -D/100)$ and $\text{MT}_{\mathfrak{g}_0 + \mathfrak{g}_{\infty}}^L(h, -D/100)$ only allow the terms $q^{-D/100}$ with $D \equiv \square \pmod{100}$ and $\gcd(D, 5) = 1$. This completes the proof by Proposition 5.4. \square

Example 6.5. Let $D = -96$ and $K = \mathbb{Q}(\sqrt{-6})$. If we choose $\beta = 2$, then we have

$$\mathcal{Q}_{-96, (5), 2} / \Gamma = \left\{ \begin{array}{l} Q_0 = [25, 102, 105], \quad Q_1 = [20, -48, 30], \quad Q_2 = [20, 52, 35], \\ Q_3 = [10, 52, 70], \quad Q_4 = [15, -48, 40], \quad Q_5 = [5, 52, 140] \end{array} \right\}$$

with

$$\begin{aligned} \tau_{Q_0} &= \frac{-102 + \sqrt{-96}}{50}, \quad \tau_{Q_1} = \frac{48 + \sqrt{-96}}{40}, \quad \tau_{Q_2} = \frac{-52 + \sqrt{-96}}{40}, \\ \tau_{Q_3} &= \frac{-52 + \sqrt{-96}}{20}, \quad \tau_{Q_4} = \frac{48 + \sqrt{-96}}{30}, \quad \tau_{Q_5} = \frac{-52 + \sqrt{-96}}{10}. \end{aligned}$$

We obtain that

$$\begin{cases} \mathfrak{t}_{\mathfrak{g}_0 \cdot \mathfrak{g}_{\infty}}^{(2)}(-96) = \sum_{k=0}^5 (\mathfrak{g}_0(\tau_{Q_k}) \cdot \mathfrak{g}_{\infty}(\tau_{Q_k})) = 221750, \\ \mathfrak{t}_{\mathfrak{g}_0 + \mathfrak{g}_{\infty}}^{(2)}(-96) = \sum_{k=0}^5 (\mathfrak{g}_0(\tau_{Q_k}) + \mathfrak{g}_{\infty}(\tau_{Q_k})) = 180. \end{cases}$$

On the other hand, we have

$$\begin{aligned} \text{GT}_{\mathfrak{g}_{\text{prod}}}(D) &= \text{Tr}_{H_{-24}/K}(\mathfrak{g}_{\text{prod}}(\tau_{-24})) + \text{Tr}_{H_{-96}/K}(\mathfrak{g}_{\text{prod}}(\tau_{-96})) \\ &= 221000 + 750 = 221750 \quad \text{by Remark 4.4 and Example 4.7} \end{aligned}$$

and

$$\begin{aligned} \text{GT}_{\mathfrak{g}_{\text{sum}}}(D) &= \text{Tr}_{H_{-24}/K}(\mathfrak{g}_{\text{sum}}(\tau_{-24})) + \text{Tr}_{H_{-96}/K}(\mathfrak{g}_{\text{sum}}(\tau_{-96})) \\ &= -56 + 236 = 180 \quad \text{by Remark 4.6 (i) and Example 4.7.} \end{aligned}$$

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