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Research Article

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F -biharmonic maps into general Riemannian manifolds

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Abstract: Let $\psi : (M, g) \rightarrow (N, h)$ be a map between Riemannian manifolds (M, g) and (N, h) . We introduce the notion of the F -bienergy functional

$$E_{F,2}(\psi) = \int_M F\left(\frac{|\tau(\psi)|^2}{2}\right) dV_g,$$

where $F : [0, \infty) \rightarrow [0, \infty)$ be C^3 function such that $F' > 0$ on $(0, \infty)$, $\tau(\psi)$ is the tension field of ψ . Critical points of $\tau_{F,2}$ are called F -biharmonic maps. In this paper, we prove a nonexistence result for F -biharmonic maps from a complete non-compact Riemannian manifold of dimension $m = \dim M \geq 3$ with infinite volume that admit an Euclidean type Sobolev inequality into general Riemannian manifold whose sectional curvature is bounded from above. Under these geometric assumptions we show that if the L^p -norm ($p > 1$) of the tension field is bounded and the m -energy of the maps is sufficiently small, then every F -biharmonic map must be harmonic. We also get a Liouville-type result under proper integral conditions which generalize the result of [Branding V., Luo Y., A nonexistence theorem for proper biharmonic maps into general Riemannian manifolds, 2018, arXiv: 1806.11441v2].

Keywords: Sobolev inequality, L^p -norm, harmonic

MSC: 58E20, 53C43

1 Introduction

In the past several decades harmonic map plays a central role in geometry and analysis. Harmonic maps between two Riemannian manifolds are critical points of the energy functional

$$E(\psi) = \frac{1}{2} \int_M |d\psi|^2 dV_g,$$

for smooth maps $\psi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds and for all compact domain $\Omega \subseteq M$. Here, dV_g denotes the volume element of g . Harmonic maps equation is simply the Euler-Lagrange equation of the energy function E which is given (cf. [2]) by $\tau(\psi) = 0$, where $\tau(\psi)$ is called the tension field of ψ . By extending notion of harmonic map, in 1983, Eells and Lemaire [3] proposed to consider the bienergy functional

$$E_2(\psi) = \frac{1}{2} \int_M |\tau(\psi)|^2 dV_g$$

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for smooth maps between two Riemannian manifolds. In 1986, Jiang [4, 5] studied the first and the second variational formulas of the bienergy functional and initiated the study of biharmonic maps. The Euler-Lagrange equation of $E_2(\psi)$ is given by

$$\tau_2(\psi) := -\Delta^\psi \tau(\psi) - \sum_{i=1}^m R^N(\tau(\psi), d\psi(e_i)) d\psi(e_i) = 0,$$

where $\Delta^\psi := \sum_{i=1}^m (\bar{\nabla}_{e_i} \bar{\nabla}_{e_i} - \bar{\nabla}_{\nabla_{e_i} e_i})$, and $\bar{\nabla}$ is the induced connection on the pullback ψ^*TN , ∇ is the Levi-Civita connection on (M, g) , and R^N is the Riemannian curvature tensor on N . Clearly, any harmonic map is always a biharmonic map.

Later, Ara [6], Han and Feng [7] introduced the F -bienergy functional

$$E_{F,2}(\psi) = \int_M F\left(\frac{|\tau(\psi)|^2}{2}\right) dV_g,$$

where $F : [0, \infty) \rightarrow [0, \infty)$ be C^3 function such that $F' > 0$ on $(0, \infty)$. The map ψ is called an F -biharmonic map if it is a critical point of that F -bienergy $E_{F,2}(\psi)$, which is a generalization of biharmonic maps (cf. [8–12]), p -biharmonic maps (cf. [13–16]) or exponentially biharmonic maps (cf. [7]). The Euler-Lagrange equation of $E_{F,2}(\psi)$ is

$$\tau_{F,2}(\psi) := -\Delta^\psi \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) - \sum_{i=1}^m R^N \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi), d\psi(e_i) \right) d\psi(e_i) = 0.$$

A map $\psi : (M, g) \rightarrow (N, h)$ is called a F -biharmonic map if $\tau_{F,2}(\psi) = 0$. Note that harmonic maps are always F -biharmonic by definition.

- When $F(t) = e^t$, we have exponential bienergy functional $E_{e,2}(\psi) = \int_M e^{\frac{|\tau(\psi)|^2}{2}} dV_g$.
- When $F(t) = (2t)^{\frac{p}{2}}$, we have p -bienergy functional $E_{p,2}(\psi) = \int_M |\tau(\psi)|^p dV_g$, where $p \geq 2$. The Euler-Lagrange equation of $E_{p,2}$ is

$$\tau_{p,2}(\psi) := -\Delta^\psi \left(p |\tau(\psi)|^{p-2} \tau(\psi) \right) - \sum_{i=1}^m R^N \left(d\psi(e_i), p |\tau(\psi)|^{p-2} \tau(\psi) \right) d\psi(e_i) = 0.$$

A map $\psi : (M, g) \rightarrow (N, h)$ is called a p -biharmonic map if $\tau_{p,2}(\psi) = 0$.

B. Y. Chen [17] raised so called B. Y. Chen's conjecture and later, [18] and [19] raised the generalized B. Y. Chen's conjecture.

Conjecture 1.1. *Every biharmonic submanifold of Euclidean space must be harmonic (minimal).*

Here, if $\psi : (M, g) \rightarrow (N, h)$ is a biharmonic isometric immersion, then M is called a biharmonic submanifold in N .

Conjecture 1.2. *Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be harmonic (minimal).*

Indeed, biharmonic maps are not harmonic maps (the counter examples of [20]). It is interesting to find some conditions to solve the biharmonic map equation which solutions reduce to harmonic map. About this results, there exists a large number of celebrated results (for instance, see [21–26]). For harmonic maps, it is well known that: If a domain manifold (M, g) is complete and has non-negative Ricci curvature and the sectional curvature of a target manifold (N, h) is non-positive, then every finite harmonic map is a constant map [27]. See [28] and [29] for recent works on harmonic map. If M is noncompact, the maximum principle is no longer application. In this case we can use the integration by parts argument, by choosing proper test functions. Based on this idea, Baird, Fardoun and Ouakkas ([30]) showed that: If a non-compact manifold (M, g) is

complete and has non-negative Ricci curvature and (N, h) has non-positive sectional curvature, then every bienergy finite biharmonic map of (M, g) into (N, h) is harmonic. It is natural to ask whether we can abandon the curvature restriction on the domain manifold and weaken the integrability condition on the bienergy. In this direction, Nakauchi et al. [8] showed that every biharmonic map of a complete Riemannian manifold into a Riemannian manifold of non-positive curvature whose bienergy and energy are finite must be harmonic. Later, Maeta [31] obtained that biharmonic maps from a complete Riemannian manifold into a non-positive curved manifold with finite p -bienergy $\int_M |\tau(\psi)|^p dV_g < \infty$ ($p \geq 2$) and energy are harmonic. In [14], Han and Zhang investigated harmonicity of p -biharmonic maps, as the cases of F -biharmonic maps. Moreover, in [15] Han introduced the notion of p -biharmonic submanifold and proved that p -biharmonic submanifold (M, g) in a Riemannian manifold (N, h) with non-positive sectional curvature which satisfies certain condition must be minimal. Furthermore, Luo in [32, 33] respectively generalized these results.

Recently, Branding and Luo [26] proved a nonexistence result for proper biharmonic maps from complete non-compact Riemannian manifolds, by assuming that the sectional curvature of the Riemannian manifold have an upper bound and give a more natural integrability condition, which generalized Branding's. It is natural to ask whether we can generalize the results on the F -biharmonic maps. In [7], Han and Feng proved that F -biharmonic map from a compact orientable Riemannian manifold into a Riemannian manifold with non-positive sectional curvature are harmonic. There have been extensive studies in this area (for instance, [34, 35]). In the following, we aim to establish another nonexistence result for F -biharmonic maps that does not require any assumption on the curvature of the target manifold, instead we will demand that the tension field is small in a certain L^p -norm and satisfies m -energy finiteness. To state our theorem, we first give a definition.

Definition 1.1. [36, 37] An m -dimensional ($m \geq 3$) complete non-compact Riemannian manifold M of infinite volume admits an Euclidean type Sobolev inequality

$$\left(\int_M |u|^{\frac{2m}{m-2}} dV_g \right)^{\frac{m-2}{m}} \leq C_{Sob}^M \int_M |\nabla u|^2 dV_g, \quad (1.1)$$

for all $u \in W^{1,2}(M)$ with compact support, where C_{Sob}^M is a positive constant that depends on the geometry of M .

Remark 1.1. Such an inequality holds in \mathbb{R}^m and is well-known as Gagliardo-Nirenberg inequality in this case.

Remark 1.2. Suppose that (M, g) is a complete non-compact Riemannian manifold of dimension $m = \dim M$ with nonnegative Ricci curvature. Suppose that for some point $x \in M$,

$$\lim_{R \rightarrow \infty} \frac{\text{Vol}_g(B_R(x))}{\omega_m R^m} > 0$$

holds, then (1.1) holds true (cf. [38]). Here, ω_m denotes the volume of the unit ball in \mathbb{R}^m . For a discussion of this kind of problem, we suggest the reader refer to [36].

Motivated by these aspects, we actually can prove the following result:

Theorem 1.1. Suppose that (M, g) is a complete, connected non-compact Riemannian manifold of $m = \dim M \geq 3$ with infinite volume that admits an Euclidean type Sobolev inequality of the form (1.1). Moreover, suppose that (N, h) is another Riemannian manifold whose sectional curvature satisfy $K^N \leq K$, where K is a positive constant. Let $\psi : (M, g) \rightarrow (N, h)$ be a smooth F -biharmonic map. If

$$\int_M \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^p dV_g < \infty$$

and

$$\int_M |d\psi|^m dV_g < \epsilon,$$

for $p > 1$ and $\epsilon > 0$ (depending on p , K , and the geometry of M) sufficiently small, then ψ must be harmonic.

Remark 1.3. For a better understand of Theorem 1.1, the readers could consult the papers for examples of p -harmonic maps (cf. [8]), biharmonic maps (cf. [1]) and [39] for a more general result.

Furthermore, we can get the following Liouville-type result.

Corollary 1.1. Suppose that (M, g) is a complete, connected non-compact Riemannian manifold of $m = \dim M \geq 3$ with nonnegative Ricci curvature that admits an Euclidean type Sobolev inequality of the form (1.1). Moreover, suppose that (N, h) is another Riemannian manifold whose sectional curvature satisfy $K^N \leq K$, where K is a positive constant. Let $\psi : (M, g) \rightarrow (N, h)$ be a smooth F -biharmonic map. If

$$\int_M \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^p dV_g < \infty$$

and

$$\int_M |d\psi|^m dV_g < \epsilon,$$

for $p > 1$ and $\epsilon > 0$ (depending on p , K , and the geometry of M) sufficiently small, then ψ is a constant map.

Remark 1.4. Note that due to a classical result of Yau [40], a complete noncompact Riemannian manifold with nonnegative Ricci curvature has infinite volume.

2 Lemmas

In order to prove our theorems, we need the following lemmas.

For a fixed point $x_0 \in M$ and for every $R > 0$, let us consider the following cut-off function $\eta(x)$ on M :

$$\begin{cases} 0 \leq \eta(x) \leq 1, x \in M; \\ \eta(x) = 0, x \in M \setminus B_{2R}(x_0); \\ \eta(x) = 1, x \in B_R(x_0); \\ |\nabla \eta(x)| \leq \frac{C}{R}, x \in M, \end{cases}$$

where $B_R(x_0) = \{x \in M : d(x, x_0) < R\}$, C is a positive constant and d is the distance on (M, g) .

Lemma 2.1. Let $\psi : (M, g) \rightarrow (N, h)$ be a smooth F -biharmonic map and assume the sectional curvature of N satisfy $K^N \leq K$, where K is a positive constant. Let δ be a positive constant, then the following inequality holds:

(1) If $1 < p < 2$, we get

$$\begin{aligned} & \left(1 - \frac{p-1}{2} \right) \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g \\ & \leq - (p-2) \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g \\ & \quad + K \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} |d\psi|^2 dV_g \\ & \quad + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} dV_g. \end{aligned}$$

(2) If $p \geq 2$, we get

$$\begin{aligned} & \frac{1}{2} \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g \\ & \leq K \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p}{2}} |d\psi|^2 dV_g + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p}{2}} dV_g. \end{aligned}$$

Proof. Multiplying the F -biharmonic map equation by a test function of the term:

$$\eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi),$$

where $p > 1$ and $\delta > 0$, then we have

$$\begin{aligned} & \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \left\langle \Delta \psi \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle \\ & = -\eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \sum_{i=1}^m R^N \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi), d\psi(e_i), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi), d\psi(e_i) \right). \end{aligned} \quad (2.1)$$

Integrating over M and using integration by parts we get

$$\begin{aligned} & \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \left\langle \Delta \psi \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle dV_g \\ & = -2 \int_M \eta \nabla \eta \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \left\langle \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle dV_g \\ & \quad - (p-2) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \left\langle \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle \right|^2 dV_g \\ & \quad - \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 dV_g \\ & \leq -2 \int_M \eta \nabla \eta \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \left\langle \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle dV_g \\ & \quad - (p-2) \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \left\langle \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle \right|^2 dV_g \\ & \quad - \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g. \end{aligned} \quad (2.2)$$

We divide the arguments into two cases:

– Case 1: When $1 < p < 2$, we obtain from (2.2) that

$$\begin{aligned} & \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \left\langle \Delta \psi \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle dV_g \\ & \leq \left(\frac{p-1}{2} - 1 \right) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g \\ & \quad + \frac{2}{p-1} \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p}{2}} |\nabla \eta|^2 dV_g \end{aligned}$$

$$\begin{aligned}
& - (p-2) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \left\langle \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle \right|^2 dV_g \\
& \leq \left(\frac{p-1}{2} - 1 \right) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g \\
& \quad + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p}{2}} dV_g \\
& \quad - (p-2) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g,
\end{aligned} \tag{2.3}$$

where the inequality follows from

$$\left| \left\langle \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right\rangle \right|^2 \leq \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2.$$

Using (2.1) and (2.3) we deduce that

$$\begin{aligned}
& \left(1 - \frac{p-1}{2} \right) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g \\
& \leq \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \sum_{i=1}^m R^N \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi), d\psi(e_i), F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi), d\psi(e_i) \right) dV_g \\
& \quad + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p}{2}} dV_g \\
& \quad - (p-2) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g \\
& \leq K \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 |d\psi|^2 dV_g \\
& \quad + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p}{2}} dV_g \\
& \quad - (p-2) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g \\
& \leq K \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-2}{2}} |d\psi|^2 dV_g + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p}{2}} dV_g \\
& \quad - (p-2) \int_M \eta^2 \left([F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right) \right|^2 \left| F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi) \right|^2 dV_g,
\end{aligned}$$

which proves the first claim.

– Case 2: When $p \geq 2$, from (2.2) we get

$$\begin{aligned}
 & \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-2}{2}} \left\langle \Delta \psi \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right), F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right\rangle dV_g \\
 & \leq -2 \int_M \eta \nabla \eta \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-2}{2}} \left\langle \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right), F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right\rangle dV_g \\
 & \quad - \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g \quad (2.4) \\
 & \leq -\frac{1}{2} \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g \\
 & \quad + 2 \int_M \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} |\nabla \eta|^2 dV_g.
 \end{aligned}$$

By a straightforward computation we can rewrite (2.4) as

$$\begin{aligned}
 & \frac{1}{2} \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g \\
 & \leq \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-2}{2}} \sum_{i=1}^m R^N \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi), d\psi(e_i), F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi), d\psi(e_i) \right) dV_g \\
 & \quad + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} dV_g \\
 & \leq K \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-2}{2}} \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 |d\psi|^2 dV_g \\
 & \quad + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} dV_g \\
 & \leq K \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} |d\psi|^2 dV_g + \frac{C}{R^2} \int_{B_{2R}(x_0)} \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} dV_g.
 \end{aligned}$$

This completes the proof of Lemma 2.1. \square

Next, we deal with the term $K \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} |d\psi|^2 dV_g$.

Lemma 2.2. Assume that (M, g) satisfies the assumptions of Theorem 1.1, then the following inequality holds

$$\begin{aligned}
 & \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} |d\psi|^2 dV_g \\
 & \leq 2C^2 \cdot C_{Sob}^M \left(\int_M |d\psi|^m dV_g \right)^{\frac{2}{m}} \cdot \frac{1}{R^2} \int_{B_{2R}(x_0)} \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} dV_g + \frac{p^2 \cdot C_{Sob}^M}{2} \left(\int_M |d\psi|^m dV_g \right)^{\frac{2}{m}} \\
 & \quad \times \int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g.
 \end{aligned}$$

Proof. Set $f = \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) + \delta \right)^{\frac{p}{4}}$, then we get

$$\int_M \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} |d\psi|^2 dV_g = \int_M \eta^2 |d\psi|^2 f^2 dV_g.$$

By using Hölder's inequality, we have

$$\int_M \eta^2 |d\psi|^2 f^2 dV_g \leq \left(\int_M (\eta f)^{\frac{2m}{m-2}} dV_g \right)^{\frac{m-2}{m}} \left(\int_M |d\psi|^m dV_g \right)^{\frac{2}{m}}. \quad (2.5)$$

Applying (1.1) to $u = \eta f$ we get

$$\left(\int_M (\eta f)^{\frac{m-2}{m}} dV_g \right)^{\frac{2m}{m-2}} \leq C_{Sob}^M \int_M |d(\eta f)|^2 dV_g. \quad (2.6)$$

Substituting (2.6) into (2.5) yields

$$\int_M \eta^2 |d\psi|^2 f^2 dV_g \leq 2C_{Sob}^M \left(\int_M |d\psi|^m dV_g \right)^{\frac{2}{m}} \left(\int_M |d\eta|^2 f^2 dV_g + \int_M |\eta|^2 |df|^2 dV_g \right).$$

Using $f = \left([F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi)]^2 + \delta \right)^{\frac{p}{4}}$ and

$$\begin{aligned} |df|^2 &= \frac{p^2}{4} \left([F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \left\langle \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right), F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right\rangle \right|^2 \\ &\leq \frac{p^2}{4} \left([F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2. \end{aligned}$$

Thus, we obtain the Lemma 2.2. \square

In the following we will make use of the following result due to Gaffney [41].

Lemma 2.3. *Let (M, g) be a complete Riemannian manifold. If a C^1 1-form ω satisfies that $\int_M |\omega| dV_g < \infty$ and $\int_M |\delta\omega| dV_g < \infty$, or equivalently, a C^1 vector field X defined by $\omega(Y) = \langle X, Y \rangle$ ($\forall Y \in TM$), satisfies that $\int_M |X| dV_g < \infty$ and $\int_M |\operatorname{div} X| dV_g < \infty$, then*

$$\int_M \delta\omega dV_g = \int_M \operatorname{div} X dV_g = 0.$$

3 Proof of the main result

In this section we will give a proof of Theorem 1.1.

Proof. We divide the arguments into two cases:

- Case 1: When $1 < p < 2$, from Lemma 2.1 and Lemma 2.2, we can choose ϵ sufficiently small such that $\frac{Kp^2 \cdot C_{Sob}^M}{2} \epsilon^{\frac{2}{m}} \leq \frac{p-1}{4}$, a straightforward computation shows that

$$\begin{aligned} &\frac{p-1}{4} \int_M \eta^2 \left([F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi)]^2 + \delta \right)^{\frac{p-4}{2}} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g \\ &\leq \frac{C}{R^2} \int_{B_{2R}(x_0)} \left([F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi)]^2 + \delta \right)^{\frac{p}{2}} dV_g, \end{aligned} \quad (3.1)$$

where C is a constant depending on p , K and the geometry of M . Now, let $M_1 = \{x \in M \mid F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)(x) = 0\}$, and $M_2 = M \setminus M_1$. If M_2 is an empty set, this completes the proof. Hence, we assume that M_2 is nonempty and we will get a contradiction below. Since ψ is smooth, M_2 is an open set, we use (3.1) to have

$$\begin{aligned} \frac{p-1}{4} \int_{M_2} \eta^2 \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p-4}{2}} + \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g \\ \leq \frac{C}{R^2} \int_{B_{2R}(x_0)} \left(\left[F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right]^2 + \delta \right)^{\frac{p}{2}} dV_g. \end{aligned}$$

Letting $\delta \rightarrow 0$ we have

$$\begin{aligned} \frac{p-1}{4} \int_{M_2} \eta^2 \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^{p-2} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 \leq \frac{C}{R^2} \int_{B_{2R}(x_0)} \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^p dV_g \\ \leq \frac{C}{R^2} \int_M \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^p dV_g. \end{aligned}$$

Letting $R \rightarrow \infty$ we get

$$\frac{p-1}{4} \int_{M_2} \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^{p-2} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 dV_g = 0.$$

– Case 2: When $p \geq 2$, by a similar discussion we can prove that

$$\frac{p-1}{4} \int_{M_2} \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^{p-2} \left| \bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) \right|^2 dV_g = 0.$$

Therefore we conclude that $\bar{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) = 0$ everywhere in M_2 . It follows that M_2 is an open and closed nonempty set, hence, $M_2 = M$ and $|F'(\frac{|\tau(\psi)|^2}{2})\tau(\psi)| \equiv c$ for some constant $c \neq 0$. Thus $\text{Vol}(M) < \infty$ by $\int_M c^p dV_g < \infty$.

Next, making using of Gaffney's result (Lemma 2.3). Define a 1-form ω on M as follows

$$\omega(X) = \left\langle d\psi(X), F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right\rangle, \forall X \in TM.$$

Then we have

$$\begin{aligned} \int_M |\omega| dV_g &= \int_M \left[\sum_{i=1}^m |\omega(e_i)|^2 \right]^{\frac{1}{2}} dV_g \\ &\leq \int_M \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right| |d\psi| dV_g \\ &\leq c(\text{Vol}M)^{1-\frac{1}{m}} \left(\int_M |d\psi|^m dV_g \right)^{\frac{1}{m}} < \infty. \end{aligned}$$

Now, we compute $-\delta\omega = \sum_{i=1}^m (\nabla_{e_i}\omega)(e_i)$ to have

$$\begin{aligned} -\delta\omega &= \sum_{i=1}^m \nabla_{e_i} (\omega(e_i)) - \omega(\nabla_{e_i} e_i) \\ &= \sum_{i=1}^m \left[\left\langle \tilde{\nabla}_{e_i} d\psi(e_i), F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right\rangle - \left\langle d\psi(\nabla_{e_i} e_i), F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right\rangle \right] \\ &= \sum_{i=1}^m \left\langle \tilde{\nabla}_{e_i} d\psi(e_i) - d\psi(\nabla_{e_i} e_i), F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right\rangle \\ &= \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2, \end{aligned}$$

where in obtaining the second equality we have used $\tilde{\nabla} \left(F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right) = 0$. Therefore,

$$\int_M |\delta\omega| dV_g = c^2 \cdot \text{Vol}(M) < \infty.$$

Now by Gaffney's theorem and the above equality, we have

$$0 = \int_M (-\delta\omega) dV_g = \int_M \left| F' \left(\frac{|\tau(\psi)|^2}{2} \right) \tau(\psi) \right|^2 dV_g = c^2 \cdot \text{Vol}(M),$$

which implies that $c = 0$, a contradiction. Hence, we have $M_1 = M$ and ψ is a harmonic map. This completes the proof of Theorem 1.1. \square

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