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Coexistence for a kind of stochastic three-species competitive models

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Abstract: The coexistence of species sustains the ecological balance in nature. This paper focuses on sufficient conditions for the coexistence of a three-species stochastic competitive model, where the model has non-linear diffusion parts. Three values λ_{3z}, λ_{3x} and λ_{3y} are introduced and calculated from the coefficients, which can be considered as threshold values. Moreover, convergence in distribution of the positive solution of the model is also addressed. A few numerical simulations are carried out to illustrate the theoretical results.

Keywords: three-species competitive model, coexistence, environmental noise, non-linear diffusion, ergodicity

MSC: 60H10; 60J70; 92D25; 93D15

1 Introduction

The coexistence of species plays a vital role in protecting ecological balance, and the situation is not desirable when any species extinct. Therefore, this paper aims to discuss what the conditions for coexistence is. In term of deterministic model, Lotka-Volterra model, which was applied to study species interactions, was originally proposed by A. Lotka and V. Volterra [1–3]. Later Lotka-Volterra model is widely applied in the field of chemistry. Literature [4], based on the reversible Lotka-Volterra model, investigated the dynamic behavior of oscillatory reaction. In economy, Lotka-Volterra model is also a very useful tool to analyse enterprise competition [5]. Nowadays, Lotka-Volterra model plays a vital role in engineering field.

The classical three-dimensional competitive Lotka-Volterra model can be written as

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)(r_1 - a_{11}x(t) - a_{12}y(t) - a_{13}z(t)), \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)(r_2 - a_{21}x(t) - a_{22}y(t) - a_{23}z(t)), \\ \frac{\mathrm{d}z(t)}{\mathrm{d}t} = z(t)(r_3 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)), \end{cases} \tag{1.1}$$

where $x(t)$, $y(t)$ and $z(t)$ denote the densities of three species at time t. For $i, j = 1, 2, 3$, the parameters r_i, a_{ij} are all positive; r_i stand for intrinsic growth rates; a_{ii} describe as intra-specific competition rates; $a_{ij}(i \neq j)$ represent the inter-specific competition rates. For deterministic Lotka-Volterra model, quite a few scholars contribute to investigate coexistence of species (see [6–12] and references therein).

However, as the development of stochastic analysis, deterministic model may not be suitable for reality sometimes. As a matter of fact, population systems are usually subject to environmental noise. Therefore, to

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describe more realistically population systems, stochastic population models have attracted considerable attention [13–17]. For the sake of generality, suppose that all coefficients of model(1.1) are perturbed by Brownian motions, and the model becomes

$$\begin{cases} \mathrm{d}x(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y(t) - a_{13}z(t))\mathrm{d}t \\ \qquad +(\Delta_1 x(t) + \sigma_{11}x^2(t))\mathrm{d}B_1(t) + \sigma_{12}x(t)y(t)\mathrm{d}B_2(t) + \sigma_{13}x(t)z(t)\mathrm{d}B_3(t), \\ \mathrm{d}y(t) = y(t)(r_2 - a_{21}x(t) - a_{22}y(t) - a_{23}z(t))\mathrm{d}t \\ \qquad +\sigma_{21}x(t)y(t)\mathrm{d}B_2(t) + (\Delta_2 y(t) + \sigma_{22}y^2(t))\mathrm{d}B_4(t) + \sigma_{23}y(t)z(t)\mathrm{d}B_5(t), \\ \mathrm{d}z(t) = z(t)(r_3 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t))\mathrm{d}t \\ \qquad +\sigma_{31}x(t)z(t)\mathrm{d}B_3(t) + \sigma_{22}y(t)z(t)\mathrm{d}B_5(t) + (\Delta_3 z(t) + \sigma_{33}z^2(t))\mathrm{d}B_6(t), \end{cases} \tag{1.2}$$

where B_i $(i = 1, 2, 3, 4, 5, 6)$ are mutually independent Brownian motions. Although there may be an appropriate Lyapunov function to provide the condition for coexistence, it is really hard to be found in practice. Motivated this, some skillful technique should be introduced to solve this problem. To reduce unnecessary computations due to notational complexity and to make our ideas more understandable but still preserve important properties, we assume that the lowest-power terms are not affected by environment noise for simplicity, which means $\Delta_1 = \Delta_2 = \Delta_3 = 0$. Thus, throughout the rest of the paper, the following model will be considered

$$\begin{cases} \mathrm{d}x(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y(t) - a_{13}z(t))\mathrm{d}t \\ \qquad +\sigma_{11}x^2(t)\mathrm{d}B_1(t) + \sigma_{12}x(t)y(t)\mathrm{d}B_2(t) + \sigma_{13}x(t)z(t)\mathrm{d}B_3(t), \\ \mathrm{d}y(t) = y(t)(r_2 - a_{21}x(t) - a_{22}y(t) - a_{23}z(t))\mathrm{d}t \\ \qquad +\sigma_{21}x(t)y(t)\mathrm{d}B_2(t) + \sigma_{22}y^2(t)\mathrm{d}B_4(t) + \sigma_{23}y(t)z(t)\mathrm{d}B_5(t), \\ \mathrm{d}z(t) = z(t)(r_3 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t))\mathrm{d}t \\ \qquad +\sigma_{31}x(t)z(t)\mathrm{d}B_3(t) + \sigma_{22}y(t)z(t)\mathrm{d}B_5(t) + \sigma_{33}z^2(t)\mathrm{d}B_6(t). \end{cases} \tag{1.3}$$

To proceed, The rest of the paper is as follows. In Section 2, basic concept and three essential constants λ_{3z}, λ_{3x} and λ_{3y} are provided. Sections 3 are devoted to proving the stochastic coexistence. Then a few numerical simulations are given to manifest our results in Section 4. The last but not least, brief discussions are made to conclude the paper.

2 Basic concept and essential constants

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. In model (1.3), B_i $(i = 1, 2, 3, 4, 5, 6)$ are $\mathcal{F}_t$-adapted, mutually independent Brownian motions. Suppose that a_{ij} $(i, j = 1, 2, 3)$ are positive constants. Set $\sigma_{ii} \neq 0$ $(i = 1, 2, 3)$ in order to make the diffusion be non-degenerate. For convenience, denote $w = (x, y, z)$, $w_0 = (x_0, y_0, z_0)$, and $w(t) = (x(t), y(t), z(t))$. Denote $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, $\mathbb{R}^{2,o}_+ = \{(x, y) : x > 0, y > 0\}$ and $\mathbb{R}^{3,o}_+ = \{(x, y, z) : x > 0, y > 0, z > 0\}$. Let $w_{w_0}(t) = (x_{w_0(t)}, y_{w_0}(t), z_{w_0}(t))$ be the solution to (1.3) with initial value w_0. Previous study [15] proved that $w_{w_0}(t)$ remains in $\mathbb{R}^{3,o}_+$ with probability one if $w_0 \in \mathbb{R}^{3,o}_+$. The definition of coexistence is given as following.

Definition 2.1. *For model (1.3), stochastic coexistence takes place in three species if for any $\varepsilon > 0$, there exists an $M = M(\varepsilon) > 1$ satisfying that*

$$\liminf_{t\to\infty} \mathbb{P}\left\{M^{-1} \leq x(t), y(t), z(t) \leq M\right\} \geq 1 - \varepsilon.$$

Next, a few useful properties of solutions are presented, whose proofs can be found in [15, 19, 20].

Proposition 2.1. *The following assertions hold:*
(i) For all $z_0 \in \mathbb{R}^3_+ \backslash \{(0, 0, 0)\}$, there exists an $M_0 > 0$ satisfying that

$$\limsup_{t\to\infty} \mathbb{E}V(x_{w_0(t)}, y_{w_0}(t), z_{w_0}(t)) \leq M_0,$$

where $V(x,y,z) = (x+y+z) + (x+y+z)^{-1}$.
(ii) For any $\varepsilon > 0$, $H > 1$, $T > 0$, *there exists an* $\overline{H} = \overline{H}(\varepsilon, H, T) > 1$ *satisfying that*

$$\mathbb{P}\{\overline{H}^{-1} \le x_{w_0}(t) \le \overline{H}\ \forall t \in [0,T]\} \le 1-\varepsilon \ \ if\ w_0 \in [H^{-1}, H] \times [0,H] \times [0,H],$$

$$\mathbb{P}\{\overline{H}^{-1} \le y_{w_0}(t) \le \overline{H}\ \forall t \in [0,T]\} \le 1-\varepsilon \ \ if\ w_0 \in [0,H] \times [H^{-1}, H] \times [0,H]$$

and that

$$\mathbb{P}\{\overline{H}^{-1} \le z_{w_0}(t) \le \overline{H}\ \forall t \in [0,T]\} \le 1-\varepsilon \ \ if\ w_0 \in [0,H] \times [0,H] \times [H^{-1}, H].$$

(iii) For any $p \in (0,3)$, *there exists an* $M_p > 0$ *satisfying that*

$$\mathbb{E}\int_0^t |w_{w_0}(s)|^p \mathrm{d}s \le M_p(t + |w_0|)\ \forall z_0 \in \mathbb{R}_+^2,\ t \ge 0,$$

where $|(x,y,z)| = \sqrt{x^2+y^2+z^2}$.

To further study, the equations on the boundary need to be considered. On the x-axis, one has

$$\mathrm{d}\varphi(t) = \varphi(t)(r_1 - a_{11}\varphi(t))\mathrm{d}t + \sigma_{11}\varphi^2(t)\mathrm{d}B_1(t). \tag{2.1}$$

There exists a unique stationary distribution $\pi_1^\star$ in $(0,\infty)$ with density

$$f_1^\star(u) = \frac{c_1^\star}{u^4}\exp\left(\frac{2a_{11}}{\sigma_{11}^2}\frac{1}{u} - \frac{r_1}{\sigma_{11}^2}\frac{1}{u^2}\right),\ u > 0,$$

where $c_1^\star = \left(\int_0^\infty x^2 \exp\left(\frac{2a_{11}}{\sigma_{11}^2}x - \frac{r_1}{\sigma_{11}^2}x^2\right)\mathrm{d}x\right)^{-1}$. Owing to the ergodicity, for any measurable function $h(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}$ satisfying that $\int_0^\infty |h(u)|f_1^\star(u)\mathrm{d}u < \infty$, the following equation holds,

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T h(\varphi_{x_0}(t))\mathrm{d}t = \int_0^\infty h(u)f_1^\star(u)\mathrm{d}u\right\} = 1,\ \forall x_0 > 0, \tag{2.2}$$

where $\varphi_{x_0}(t)$ is the solution to (2.1) starting at x_0. Specially, for any $p \in (-\infty, 3)$,

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi_{x_0}^p(t)\mathrm{d}t = Q_{px} := \int_0^\infty u^p f_1^\star(u)\mathrm{d}u < \infty\right\} = 1,\ \forall x_0 > 0. \tag{2.3}$$

Similarly, on the y-axis, one has

$$\mathrm{d}\psi(t) = \psi(t)(r_2 - a_{22}\psi(t))\mathrm{d}t + \sigma_{22}\psi^2(t)\mathrm{d}B_4(t), \tag{2.4}$$

which in $(0,\infty)$ has a unique stationary distribution $\pi_2^\star$ with density

$$f_2^\star(v) = \frac{c_2^\star}{v^4}\exp\left(\frac{2a_{22}}{\sigma_{22}^2}\frac{1}{v} - \frac{r_2}{\sigma_{22}^2}\frac{1}{v^2}\right),\ v > 0,$$

$$c_2^\star = \left(\int_0^\infty x^2 \exp\left(\frac{2a_{22}}{\sigma_{22}^2}x - \frac{r_2}{\sigma_{22}^2}x^2\right)\mathrm{d}x\right)^{-1}.$$

By the ergodicity, for $p \in (-\infty, 3)$, one gets

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \psi_{y_0}^p(t)\mathrm{d}t = Q_{py} := \int_0^\infty v^p f_2^\star(v)\mathrm{d}v < \infty\right\} = 1,\ \forall x_0 > 0, \tag{2.5}$$

where $\psi_{y_0}(t)$ is the solution to (2.4) starting at y_0.
On the z-axis, one has

$$\mathrm{d}\rho(t) = \rho(t)(r_3 - a_{33}\rho(t))\mathrm{d}t + \sigma_{33}\rho^2(t)\mathrm{d}B_6(t), \tag{2.6}$$

which in $(0, \infty)$ has a unique stationary distribution $\pi_3^\star$ with density

$$f_3^\star(\phi) = \frac{c_3^\star}{\phi^4}\exp\left(\frac{2a_{33}}{\sigma_{33}^2}\frac{1}{\phi} - \frac{r_3}{\sigma_{33}^2}\frac{1}{\phi^2}\right), \ \phi > 0$$

$$c_3^\star = \left(\int_0^\infty x^2 \exp\left(\frac{2a_{33}}{\sigma_{33}^2}x - \frac{r_3}{\sigma_{33}^2}x^2\right)\mathrm{d}x\right)^{-1}.$$

In view of the ergodicity, for $p \in (-\infty, 3)$, one has

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \rho_{z_0}^p(t)\mathrm{d}t = Q_{pz} := \int_0^\infty \phi^p f_3^\star(\phi)\mathrm{d}\phi < \infty\right\} = 1, \ \forall z_0 > 0, \tag{2.7}$$

where $\rho_{z_0}(t)$ is the solution to (2.6) starting at z_0. As to the x-y plane, Literature [18] has considered following model

$$\begin{cases} \mathrm{d}X(t) = X(t)(r_1 - a_{11}X(t) - a_{12}Y(t))\mathrm{d}t + \sigma_{11}X^2(t)\mathrm{d}B_1(t) + \sigma_{12}X(t)Y(t)\mathrm{d}B_2(t), \\ \mathrm{d}Y(t) = Y(t)(r_2 - a_{21}X(t) - a_{22}Y(t))\mathrm{d}t + \sigma_{21}X(t)Y(t)\mathrm{d}B_2(t) + \sigma_{22}Y^2(t)\mathrm{d}B_4(t), \end{cases} \tag{2.8}$$

which can be seen as a case on the x-y plane for (1.3). Its result provides that (2.8) has a unique invariant measure $\mu_1^\star$ in $\mathbb{R}_+^{2,o}$ if

$$\lambda_{1z} := \int_0^\infty (r_2 - a_{21}x - \frac{\sigma_{21}^2}{2}x^2)f_1^\star(x)\mathrm{d}x = r_2 - a_{21}Q_{1x} - \frac{\sigma_{21}^2}{2}Q_{2x} > 0$$

and

$$\lambda_{2z} := \int_0^\infty (r_1 - a_{12}y - \frac{\sigma_{12}^2}{2}y^2)f_2^\star(y)\mathrm{d}y = r_1 - a_{12}Q_{1y} - \frac{\sigma_{12}^2}{2}Q_{2y} > 0.$$

Moreover, for any $\mu_1^\star$-integrable function $F(x, y) : \mathbb{R}_+^{2,o} \to \mathbb{R}$, one has

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t F(X_\gamma(s), Y_\gamma(s))\mathrm{d}s = \int_{\mathbb{R}_+^{2,o}} F(x, y)\mu_1^\star(\mathrm{d}x, \mathrm{d}y) \ a.s. \ \forall\gamma \in \mathbb{R}_+^{2,o}. \tag{2.9}$$

Let $f_1(x, y)$ be the density of probability measure $\mu_1^\star$ in $\mathbb{R}_+^{2,o}$. Then $f_1(x, y)$ has following properties,

$$\int_0^\infty\int_0^\infty f_1(x, y)dxdy = 1,$$

$$\int_0^\infty f_1(x, y)dy = f_1^\star(x) \text{ and } \int_0^\infty f_1(x, y)dx = f_2^\star(y),$$

where $f_1^\star(x)$ and $f_2^\star(y)$ are density of $\pi_1^\star$ and $\pi_2^\star$ respectively. Define that

$$\begin{aligned}\lambda_{3z} :=& r_3 - \int_0^\infty (a_{31}x + \frac{\sigma_{31}^2}{2}x^2)f_1^\star(x)\mathrm{d}x - \int_0^\infty (a_{32}y + \frac{\sigma_{32}^2}{2}y^2)f_2^\star(y)dy \\ =& r_3 - a_{31}Q_{1x} - \frac{\sigma_{31}^2}{2}Q_{2x} - a_{32}Q_{1y} - \frac{\sigma_{32}^2}{2}Q_{2y}.\end{aligned} \tag{2.10}$$

Similarly, on y-z plane, one gets

$$\begin{cases} \mathrm{d}Y(t) = Y(t)(r_2 - a_{22}Y(t) - a_{23}Z(t))\mathrm{d}t + \sigma_{22}Y^2(t)\mathrm{d}B_4(t) + \sigma_{23}Y(t)Z(t)\mathrm{d}B_5(t), \\ \mathrm{d}Z(t) = Z(t)(r_3 - a_{32}Y(t) - a_{33}Z(t))\mathrm{d}t + \sigma_{32}Y(t)Z(t)\mathrm{d}B_5(t) + \sigma_{33}Z^2(t)\mathrm{d}B_6(t). \end{cases} \tag{2.11}$$

Using the method of [18], if

$$\lambda_{1x} := \int_0^\infty (r_3 - a_{32}y - \frac{\sigma_{32}^2}{2}y^2)f_2^\star(y)\mathrm{d}y = r_3 - a_{32}Q_{1y} - \frac{\sigma_{32}^2}{2}Q_{2y} > 0$$

and

$$\lambda_{2x} := \int_0^\infty (r_2 - a_{23}z - \frac{\sigma_{23}^2}{2}z^2)f_3^\star(z)\mathrm{d}\phi = r_2 - a_{23}Q_{1z} - \frac{\sigma_{23}^2}{2}Q_{2z} > 0,$$

then (2.11) has a unique invariant measure $\mu_2^\star$ in $\mathbb{R}_+^{2,o}$ Moreover, for any $\mu_2^\star$-integrable function $F(y,z) : \mathbb{R}_+^{2,o} \to \mathbb{R}$, one has

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t F(Y_\gamma(s), Z_\gamma(s))\mathrm{d}s = \int_{\mathbb{R}_+^{2,o}} F(y,z)\mu_2^\star(\mathrm{d}y, \mathrm{d}z) \ a.s. \ \forall \gamma \in \mathbb{R}_+^{2,o}.$$

Let $f_2(y,z)$ be the density of probability measure $\mu_1^\star$ in $\mathbb{R}_+^{2,o}$, and then $f_2(y,z)$ has following properties,

$$\int_0^\infty \int_0^\infty f_2(y,z)\mathrm{d}y\mathrm{d}z = 1,$$

$$\int_0^\infty f_2(y,z)\mathrm{d}z = f_2^\star(y) \text{ and } \int_0^\infty f_2(y,z)\mathrm{d}y = f_3^\star(z),$$

where $f_2^\star(y)$ and $f_3^\star(z)$ are density of $\pi_2^\star$ and $\pi_3^\star$ respectively. Define that

$$\begin{aligned} \lambda_{3x} :=& r_1 - \int_0^\infty (a_{12}y + \frac{\sigma_{12}^2}{2}y^2)f_2^\star(y)\mathrm{d}y - \int_0^\infty (a_{13}z + \frac{\sigma_{13}^2}{2}z^2)f_3^\star(z)dz \\ =& r_1 - a_{12}Q_{1y} - \frac{\sigma_{12}^2}{2}Q_{2y} - a_{13}Q_{1z} - \frac{\sigma_{13}^2}{2}Q_{2z}. \end{aligned} \tag{2.12}$$

Analogously, on x-z plane, one has

$$\begin{cases} \mathrm{d}X(t) = X(t)(r_1 - a_{11}X(t) - a_{13}Z(t))\mathrm{d}t + \sigma_{11}X^2(t)\mathrm{d}B_1(t) + \sigma_{13}X(t)Z(t)\mathrm{d}B_3(t), \\ \mathrm{d}Z(t) = Z(t)(r_3 - a_{31}X(t) - a_{33}Z(t))\mathrm{d}t + \sigma_{31}X(t)Z(t)\mathrm{d}B_3(t) + \sigma_{33}Z^2(t)\mathrm{d}B_6(t). \end{cases} \tag{2.13}$$

If

$$\lambda_{1y} := \int_0^\infty (r_3 - a_{31}x - \frac{\sigma_{31}^2}{2}x^2)f_2^\star(x)\mathrm{d}x = r_3 - a_{31}Q_{1x} - \frac{\sigma_{31}^2}{2}Q_{2x} > 0$$

and

$$\lambda_{2y} := \int_0^\infty (r_1 - a_{13}z - \frac{\sigma_{13}^2}{2}z^2)f_3^\star(z)\mathrm{d}z = r_1 - a_{13}Q_{1z} - \frac{\sigma_{13}^2}{2}Q_{2z} > 0,$$

then (2.13) has a unique invariant measure $\mu_3^\star$ in $\mathbb{R}_+^{2,o}$ Moreover, for any $\mu_3^\star$-integrable function $F(x,z)$: $\mathbb{R}_+^{2,o} \to \mathbb{R}$, one has

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t F(X_\gamma(s), Z_\gamma(s))\mathrm{d}s = \int_{\mathbb{R}_+^{2,o}} F(x,z)\mu_3^\star(\mathrm{d}x, \mathrm{d}z) \ a.s. \ \forall \gamma \in \mathbb{R}_+^{2,o}.$$

Let $f_3(x,z)$ be the density of probability measure $\mu_3^\star$ in $\mathbb{R}_+^{2,o}$, and then $f_3(x,z)$ has following properties,

$$\int_0^\infty\int_0^\infty f_3(x,z)dxdz = 1,$$

$$\int_0^\infty f_3(x,z)dz = f_1^\star(x) \text{ and } \int_0^\infty f_3(x,z)dy = f_3^\star(z),$$

where $f_1^\star(x)$ and $f_3^\star(z)$ are density of $\pi_1^\star$ and $\pi_3^\star$ respectively. Define that

$$\begin{aligned}\lambda_{3y} :=& r_2 - \int_0^\infty (a_{21}x + \frac{\sigma_{21}^2}{2}x^2)f_1^\star(x)\mathrm{d}x - \int_0^\infty (a_{23}z + \frac{\sigma_{23}^2}{2}z^2)f_3^\star(z)dz \\ =& r_2 - a_{21}Q_{1x} - \frac{\sigma_{21}^2}{2}Q_{2x} - a_{23}Q_{1z} - \frac{\sigma_{23}^2}{2}Q_{2z}. \end{aligned} \tag{2.14}$$

The reason of definition of λ_{3z}, λ_{3x} and λ_{3y} is as following. To determine whether $z_{w_0}(t)$ converges to 0 or not, it is required to consider the Lyapunov exponent of $z_{w_0}(t)$. The Itô's formula manifests that

$$\begin{aligned}\frac{\ln z_{w_0}(T)}{T} =& \frac{\ln z_0}{T} + \frac{1}{T}\int_0^T \left(r_3 - a_{31}x_{w_0}(t) - \frac{\sigma_{31}^2}{2}x_{w_0}^2(t) - a_{32}y_{w_0}(t) - \frac{\sigma_{32}^2}{2}y_{w_0}^2(t) - a_{33}z_{w_0}(t) - \frac{\sigma_{33}^2}{2}z_{w_0}^2(t)\right)\mathrm{d}t \\ &+ \frac{1}{T}\int_0^T (\sigma_{31}x_{w_0}(t)\mathrm{d}B_3(t) + \sigma_{32}y_{w_0}(t)\mathrm{d}B_5(t) + \sigma_{33}z_{w_0}(t)\mathrm{d}B_6(t)).\end{aligned} \tag{2.15}$$

When T is large enough, the first and the third terms on the right-side of (2.15) are small. Clearly, if $z_{w_0}(t)$ is small in $[0,T]$, the solution $x_{w_0}(t)$ is close to $X_{(x_0,y_0)}(t)$ and $y_{w_0}(t)$ is close to $Y_{(x_0,y_0)}(t)$, where $X_{(x_0,y_0)}(t)$ and $Y_{(x_0,y_0)}(t)$ are solutions to (2.8) starting at (x_0,y_0). Employing the ergodicity (2.9), $\frac{\ln z_{w_0}(T)}{T}$ is close to λ_{3z}. The definitions of λ_{3x} and λ_{3y} are also gotten in the similar way.

3 Main result

This section focuses on providing the conditions for stochastic coexistence in model (1.3).

Theorem 3.1. *If λ_{3z}, λ_{3x} and λ_{3y} are all positive, the three species coexist. Moreover, in $R_+^{3,o}$, the solution process $w(t)$ has a unique invariant measure $m^\star$ satisfying that*
(i) the transition probability $\mathbb{P}(t, w_0, \cdot)$ of $z(t)$ converges to $m^\star$, $\forall w_0 \in \mathbb{R}_+^{3,o}$;
(ii) for any $m^\star$-integrable function $J(w) : \mathbb{R}_+^{3,o} \to \mathbb{R}$, one has

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t J(w_{w_0}(s))\mathrm{d}s = \int_{\mathbb{R}_+^{3,o}} J(w)m^\star(\mathrm{d}w)\ a.s.\ \forall w_0 \in \mathbb{R}_+^{3,o}.$$

To prove Theorem 3.1, some arguments need to be introduced.

The following formula is the well-known exponential martingale inequality. It asserts that for any $a, b > 0$,

$$\mathbb{P}\left\{\int_0^t g(s)\mathrm{d}B(s) - \frac{a}{2}\int_0^t g^2(s)\mathrm{d}s > b\ \ \forall t \geq 0\right\} \leq \mathrm{e}^{-ab}, \tag{3.1}$$

if $B(s)$ is a $\mathcal{F}_t$-adapted Brownian motion while $g(t)$ is a real-valued $\mathcal{F}_t$-adapted process and $\int_0^t g^2(s)\mathrm{d}s < \infty$, $\forall t \geq 0$ almost surely. Ordinarily, the inequality holds in finite time, but (3.1) holds since

$$\mathbb{P}\left\{\int_0^t g(s)\mathrm{d}B(s) - \frac{a}{2}\int_0^t g^2(s)\mathrm{d}s > b \ \forall t \geq 0\right\} = \lim_{T\to\infty} \mathbb{P}\left\{\int_0^t g(s)\mathrm{d}B(s) - \frac{a}{2}\int_0^t g^2(s)\mathrm{d}s > b \ \forall t \in [0, T]\right\}.$$

Let any $T > 1$, $p \in (1, 1.5)$, and $\frac{1}{p} + \frac{1}{q} = 1$. For $A \in \mathcal{F}$, $\mathbf{I}_A$ stands for the indicator function of A. Owing to part (iii) of Proposition 2.1 and Holder's inequality, it is estimated that

$$\begin{aligned}
\mathbb{E}\mathbf{I}_A|\ln z_{w_0} - \ln z_0| &\leq \mathbb{E}\int_0^T \mathbf{I}_A \left| r_3 + a_{31}x_{w_0}(t) + \frac{\sigma_{31}^2}{2}x_{w_0}^2(t) + a_{32}y_{w_0}(t) + \frac{\sigma_{32}^2}{2}y_{w_0}^2(t) + a_{33}z_{w_0}(t) + \frac{\sigma_{33}^2}{2}z_{w_0}^2(t)\right| \mathrm{d}t \\
&\quad + \mathbb{E}\mathbf{I}_A \left| \int_0^T (\sigma_{31}x_{w_0}(t)\mathrm{d}B_3(t) + \sigma_{32}y_{w_0}(t)\mathrm{d}B_5(t)) + \sigma_{33}z_{w_0}(t)\mathrm{d}B_6(t)) \right| \\
&\leq \theta_1 \left[\mathbb{P}(A)T\right]^{\frac{1}{q}} \left[\mathbb{E}\int_0^T (1 + x_{w_0}^{2p}(t) + y_{w_0}^{2p}(t) + z_{w_0}^{2p}(t))\mathrm{d}t\right]^{\frac{1}{p}} \\
&\quad + \sqrt{\mathbb{P}(A)} \cdot (|\sigma_{31}| \vee |\sigma_{32}| \vee |\sigma_{33}|) \sqrt{\mathbb{E}\int_0^T \left[x_{w_0}^2(t) + y_{w_0}^2(t) + z_{w_0}^2(t)\right] \mathrm{d}t} \\
&\leq \theta_2 T[P(A)]^{\frac{1}{q}}(1 + |w_0|)^{\frac{1}{p}},
\end{aligned} \tag{3.2}$$

for some constants θ_1, θ_2 independent of z_0, T and A. In particular, when $A = \Omega$,

$$\mathbb{E}\left|\frac{\ln z_{w_0}(T) - \ln z_0}{T}\right| \leq \theta_2(1 + |w_0|)^{\frac{1}{p}}, \tag{3.3}$$

and consequently,

$$\mathbb{P}\left\{\left|\frac{\ln y_{z_0}(T) - \ln y_0}{T}\right| \geq \frac{\theta_3(1 + |w_0|)^{\frac{1}{p}}}{\varepsilon}\right\} \leq \varepsilon. \tag{3.4}$$

To satisfy the condition that $z_{w_0}(t)$ is small in $[0, T]$, it requires to introduce Lemma 3.1. Let $\tau_{w_0}^{\sigma}$ be the stopping time that $\tau_{w_0}^{\sigma} = \inf\{t \geq 0 : z_{w_0}(t) \geq \sigma\}$.

Lemma 3.1. *For any $\varepsilon > 0$, $\sigma > 0$, $T > 1$, there is a $\delta = \delta(T, \varepsilon, \sigma) > 0$ satisfying that*

$$\mathbb{P}\{\tau_{w_0}^{\sigma} \geq T\} \geq 1 - \varepsilon, \ \ \forall w_0 \in (0, \infty) \times (0, \infty) \times (0, \delta].$$

Proof. By (3.1), $\mathbb{P}(\Omega_1^{w_0}) \geq 1 - \varepsilon$, where

$$\begin{aligned}
\Omega_1^{w_0} = \Bigg\{ &\int_0^t [\sigma_{31}x_{w_0}(s)\mathrm{d}B_3(s) + \sigma_{32}y_{w_0}(s)\mathrm{d}B_5(s) + \sigma_{33}z_{w_0}(s)\mathrm{d}B_6(s)] \\
&-\frac{1}{2}\int_0^t [\sigma_{31}^2x_{w_0}^2(s) + \sigma_{32}^2y_{w_0}^2(s) + \sigma_{33}^2z_{w_0}^2(s)]\mathrm{d}s < \ln\frac{1}{\varepsilon}, \ \forall t \geq 0\Bigg\}.
\end{aligned}$$

In view of (2.15), when $\omega \in \Omega_1^{w_0}$, it is seen that

$$\begin{aligned}
\ln z_{w_0}(t) &< \ln z_0 + \ln\frac{1}{\varepsilon} + \int_0^t r_3\mathrm{d}s - \int_0^t [a_{31}x_{w_0}(s) + a_{32}y_{w_0}(s) + a_{33}z_{w_0}(s)]\mathrm{d}s \\
&< \ln z_0 + \ln\frac{1}{\varepsilon} + r_3 t.
\end{aligned}$$

Letting $\delta = \sigma\varepsilon e^{-r_3 T}$, if $z_0 \leq \delta$, then $z_{w_0}(t) < \sigma$ for all $t < T$ and $\omega \in \Omega_1^{w_0}$. The proof is complete.

Lemma 3.2. *For any $\varepsilon, \beta > 0$ and $H, T > 1$, there exists $\sigma > 0$ satisfying that for all $w_0 \in [H^{-1}, H] \times [H^{-1}, H] \times [0, \sigma]$,*

$$\mathbb{P}\{|X_{(x_0,y_0)}(t) - x_{w_0}(t)| < \beta \text{ and } |Y_{(x_0,y_0)}(t) - y_{w_0}(t)| < \beta, \ \forall 0 \le t \le T \wedge \tau^{\sigma}_{z_0}\} \ge 1 - \varepsilon.$$

Proof. In view of part (ii) of Proposition 2.1, there is an $\overline{H}$ sufficiently large such that

$$\mathbb{P}\left\{\max\{X_{(x_0,y_0)}(t), x_{w_0}(t), Y_{(x_0,y_0)}(t), y_{w_0}(t)\} \le \overline{H}, \forall t \le T\right\} > 1 - \varepsilon,$$
$$\forall w_0 \in [H^{-1}, H] \times [H^{-1}, H] \times (0, \sigma].$$

Let $\xi_{w_0} := \tau^{\sigma}_{w_0} \wedge \inf\left\{t : \max\{X_{(x_0,y_0)}(t), x_{w_0}(t), Y_{(x_0,y_0)}(t), y_{w_0}(t)\} \le \overline{H}\right\}$. The Itô's formula manifests that

$$\begin{aligned}
|X_{(x_0,y_0)}(s) - x_{w_0}(s)| \le & \int_0^s |X_{(x_0,y_0)}(u) - x_{w_0}(u)| \left[r_1 + a_{11}(X_{(x_0,y_0)}(u) + x_{w_0}(u))\right] \mathrm{d}u \\
& + a_{12} \int_0^s \left|X_{(x_0,y_0)}(u)Y_{(x_0,y_0)}(u) - x_{w_0}(s)y_{w_0}(u)\right| \mathrm{d}u + a_{13} \int_0^s |x_{w_0}(u)z_{w_0}(u)| \mathrm{d}u \\
& + |\sigma_{11}| \left| \int_0^s \left[X_{(x_0,y_0)}(u) - x_{w_0}(u)\right] \left[X_{(x_0,y_0)}(u) + x_{w_0}(u)\right] \mathrm{d}B_1(u) \right| \\
& + |\sigma_{12}| \left| \int_0^s \left[X_{(x_0,y_0)}(u)Y_{(x_0,y_0)}(u) - x_{w_0}(u)y_{w_0}(u)\right] \mathrm{d}B_2(u) \right| \\
& + |\sigma_{13}| \left| \int_0^s x_{w_0}(u)z_{w_0}(u) \mathrm{d}B_3(u) \right|.
\end{aligned}$$

By the elementary inequality $(\Sigma_{i=1}^n a_i)^2 \le 2^n \Sigma_{i=1}^n a_i^2$ leads to

$$\begin{aligned}
& \mathbb{E} \sup_{s \le t} \ (X_{(x_0,y_0)}(s \wedge \xi_{w_0}) - x_{w_0}(s \wedge \xi_{w_0}))^2 \\
& \le 64\mathbb{E} \left\{ \int_0^{t \wedge \xi_{w_0}} |X_{(x_0,y_0)}(u) - x_{w_0}(u)| \left[r_1 + a_{11}(X_{(x_0,y_0)}(u) + x_{w_0}(u))\right] \mathrm{d}u \right\}^2 \\
& + 64a_{12}^2 \mathbb{E} \int_0^{t \wedge \xi_{w_0}} [X_{(x_0,y_0)}(u)Y_{(x_0,y_0)}(u) - x_{w_0}(s)y_{w_0}(u)]^2 \mathrm{d}u + 64a_{13} \mathbb{E} \int_0^{r \wedge \xi_{w_0}} x^2_{w_0}(u)z^2_{w_0}(u) \mathrm{d}u \\
& + 64\sigma_{11}^2 \mathbb{E} \sup_{s \le t} \left| \int_0^{s \wedge \xi_{w_0}} \left[X_{(x_0,y_0)}(u) - x_{w_0}(u)\right] \left[X_{(x_0,y_0)}(u) + x_{w_0}(u)\right] \mathrm{d}B_1(u) \right|^2 \\
& + 64\sigma_{12}^2 \mathbb{E} \sup_{s \le t} \left| \int_0^{s \wedge \xi_{w_0}} \left[X_{(x_0,y_0)}(u)Y_{(x_0,y_0)}(u) - x_{w_0}(u)y_{w_0}(u)\right] \mathrm{d}B_2(u) \right|^2 \\
& + 64\sigma_{13}^2 \mathbb{E} \sup_{s \le t} \left| \int_0^{s \wedge \xi_{w_0}} x_{w_0}(u)z_{w_0}(u) \mathrm{d}B_3(u) \right|^2.
\end{aligned} \tag{3.5}$$

For $t \in [0, T]$, a few estimates are made in the following,

$$\mathbb{E} \left\{ \int_0^{t \wedge \xi_{w_0}} |X_{(x_0,y_0)}(u) - x_{w_0}(u)| \left[r_1 + a_{11}(X_{(x_0,y_0)}(u) + x_{w_0}(u))\right] \mathrm{d}u \right\}^2$$

$$\leq (r_1 + 2a_{11}\overline{H})^2 T\mathbb{E} \int_0^{t\wedge\xi_{w_0}} \left[X_{(x_0,y_0)}(u) - x_{w_0}(u)\right]^2 \mathrm{d}s, \tag{3.6}$$

$$\mathbb{E} \int_0^{t\wedge\xi_{w_0}} [X_{(x_0,y_0)}(u)Y_{(x_0,y_0)}(u) - x_{w_0}(s)y_{w_0}(u)]^2 \mathrm{d}u \leq 4\overline{H}^4 T, \tag{3.7}$$

$$\mathbb{E} \int_0^{r\wedge\xi_{w_0}} x_{w_0}^2(u)z_{w_0}^2(u)\mathrm{d}u \leq \overline{H}^2 \sigma^2 T. \tag{3.8}$$

It follows from the Burkholder-Davis-Gundy inequality that

$$\begin{aligned}
&\mathbb{E}\sup_{s\leq t} \left| \int_0^{s\wedge\xi_{w_0}} \left[X_{(x_0,y_0)}(u) - x_{w_0}(u)\right] \left[X_{(x_0,y_0)}(u) + x_{w_0}(u)\right] \mathrm{d}B_1(u) \right|^2 \\
&\leq \mathbb{E} \int_0^{t\wedge\xi_{w_0}} \left[X_{(x_0,y_0)}(u) - x_{w_0}(u)\right]^2 \left[X_{(x_0,y_0)}(u) + x_{w_0}(u)\right]^2 \mathrm{d}u, \\
&\leq 4\overline{H}^2 \mathbb{E} \int_0^{t\wedge\xi_{w_0}} \left[X_{(x_0,y_0)}(u) - x_{w_0}(u)\right]^2 \mathrm{d}u,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\mathbb{E}\sup_{s\leq t} \left| \int_0^{s\wedge\xi_{w_0}} \left[X_{(x_0,y_0)}(u)Y_{(x_0,y_0)}(u) - x_{w_0}(u)y_{w_0}(u)\right] \mathrm{d}B_2(u) \right|^2 \\
&\leq \mathbb{E} \int_0^{t\wedge\xi_{w_0}} \left[X_{(x_0,y_0)}(u)Y_{(x_0,y_0)}(u) - x_{w_0}(u)y_{w_0}(u)\right]^2 \mathrm{d}u, \\
&\leq 4\overline{H}^4 T,
\end{aligned} \tag{3.10}$$

$$\mathbb{E}\sup_{s\leq t} \left| \int_0^{s\wedge\xi_{w_0}} x_{w_0}(u)z_{w_0}(u)\mathrm{d}B_3(u) \right|^2 \leq 4\overline{H}^2 \sigma^2 T. \tag{3.11}$$

Applying (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11) to (3.5), one has

$$\begin{aligned}
\mathbb{E}\sup_{s\leq t}(X_{(x_0,y_0)}(s\wedge\xi_{w_0}) - x_{w_0}(s\wedge\xi_{w_0}))^2 &\leq \overline{\theta}\left(\sigma^2 + \mathbb{E}\int_0^{t\wedge\xi_{w_0}} (X_{(x_0,y_0)}(u) - x_{w_0}(u))^2 \mathrm{d}u\right) \\
&\leq \overline{\theta}\left(\sigma^2 + \int_0^t \mathbb{E}\sup_{s\leq u}(X_{(x_0,y_0)}(s\wedge\xi_{w_0}) - x_{w_0}(s\wedge\xi_{w_0}))^2 \mathrm{d}u\right) \quad \forall t \in [0, T],
\end{aligned}$$

where $\overline{\theta} = \overline{\theta}(\overline{H}, T) > 0$. It then follows from Granwall's inequality that

$$\mathbb{E}\sup_{s\leq T}(X_{(x_0,y_0)}(s\wedge\xi_{w_0}) - x_{w_0}(s\wedge\xi_{w_0}))^2 \leq \overline{\theta}\sigma^2 \mathrm{e}^{\overline{\theta}T}.$$

Consequently, $\mathbb{P}\left\{\sup_{s\le T}(X_{(x_0,y_0)}(s\wedge\xi_{w_0})-x_{w_0}(s\wedge\xi_{w_0}))^2\ge\beta^2\right\}\le\frac{\overline{\theta}\sigma^2e^{\overline{\theta}T}}{\beta^2}<\frac{\varepsilon}{4}$ when σ is sufficiently small $\left(\sigma^2<\frac{\varepsilon\beta^2}{4\overline{\theta}}e^{-\overline{\theta}T}\right)$. The similar way yields that

$$\mathbb{P}\left\{\sup_{s\le T}(Y_{(x_0,y_0)}(s\wedge\xi_{w_0})-y_{w_0}(s\wedge\xi_{w_0}))^2\ge\beta^2\right\}<\frac{\varepsilon}{4}.$$

Since

$$\begin{aligned}\mathbb{P}\left\{s\wedge\xi_{w_0}=s\wedge\tau^{\sigma}_{w_0},\ \forall s\in[0,T]\right\}&\ge\mathbb{P}\left\{\sup_{s\le T}\{\max\left\{x_{w_0}(s),y_{w_0}(s),X_{(x_0,y_0)}(s),Y_{(x_0,y_0)}(s)\right\}\le\overline{H}\right\}\\&\ge1-\frac{\varepsilon}{2},\end{aligned}$$

it follows that

$$\begin{aligned}\mathbb{P}\{|X_{(x_0,y_0)}(s)\ &-x_{w_0}(s)|<\beta\text{and }\ |Y_{(x_0,y_0)}(s)-y_{w_0}(s)|<\beta,\ \forall 0\le t\le T\wedge\tau^{\sigma}_{w_0}\}\\&\ge\mathbb{P}\left\{s\wedge\xi_{w_0}=s\wedge\tau^{\sigma}_{w_0}\ \forall s\in[0,T]\right\}-\mathbb{P}\left\{\sup_{r\le t}(X_{(x_0,y_0)}(r\wedge\xi_{w_0})-x_{w_0}(r\wedge\xi_{w_0}))^2\ge\beta^2\right\}\\&\quad-\mathbb{P}\left\{\sup_{r\le t}(X_{(x_0,y_0)}(r\wedge\xi_{w_0})-x_{w_0}(r\wedge\xi_{w_0}))^2\ge\beta^2\right\}\\&\ge1-\varepsilon.\end{aligned}$$

The proof is complete.

Remark 3.1. Lemma 3.2 aims at describing that $x_{w_0}(t)$ is close to $X_{(x_0,y_0)}(t)$ and $y_{w_0}(t)$ is close to $Y_{(x_0,y_0)}(t)$ in certain conditions.

Lemma 3.3. *For any $\varepsilon>0$, there exists an $\overline{M}>0$ satisfying that*

$$\mathbb{P}\left\{\left|\int_0^T\sigma_{31}x_{w_0}(t)\mathrm{d}B_3(t)+\sigma_{32}y_{w_0}(t)\mathrm{d}B_5(t)+\sigma_{33}z_{w_0}(t)\mathrm{d}B_6(t)\right|\le\frac{\overline{M}}{\varepsilon}\sqrt{T|w_0|}\right\}\ge1-\varepsilon.$$

Proof. Since

$$\mathbb{E}\left|\int_0^T\sigma_{31}x_{w_0}(t)\mathrm{d}B_3(t)+\sigma_{32}y_{w_0}(t)\mathrm{d}B_5(t)+\sigma_{33}z_{w_0}(t)\mathrm{d}B_6(t)\right|^2=\mathbb{E}\int_0^T\left|\sigma_{31}^2x_{z_0}^2(t)+\sigma_{32}^2y_{w_0}^2(t)+\sigma_{33}^2z_{w_0}^2(t)\right|\mathrm{d}t.$$

By (iii) of Proposition 2.1 and Chebyshev inequality, the desired result can be obtained.

Two propositions should be introduced to make proof of Theorem 3.1 clear.

Proposition 3.1. *Suppose λ_{1z} λ_{2z}* and *$\lambda_{3z}>0$. For any $\varepsilon>0$, $H>1$, there exist $T=T(\varepsilon,H)$ and $\delta_0=\delta_0(\varepsilon,H)$ satisfying that $\mathbb{P}(\overline{\Omega}^{w_0})>1-4\varepsilon$, where*

$$\overline{\Omega}^{w_0}=\left\{\frac{\lambda_{3z}}{5}\le\ln z_{w_0}(t)-\ln z_0\right\},\ \forall w_0\in[H^{-1},H]\times[H^{-1},H]\times[0,\delta_0].$$

Proof. In view of (2.10), for sufficiently small β, one has

$$\iint_{\mathbb{R}^{2,o}_+}\left(r_3-a_{31}(u+\beta)-\frac{\sigma_{31}^2}{2}(u+\beta)^2-a_{32}(v+\beta)-\frac{\sigma_{32}^2}{2}(v+\beta)^2\right)f_1(u,v)\mathrm{d}u\mathrm{d}v\ge\frac{4\lambda_{3z}}{5}.$$

Let $\overline{M}$ be as in Lemma 3.3. Using (2.9), there is $T=T(\varepsilon,T)>\frac{25H\overline{M}^2}{\varepsilon^2\lambda_{3z}^2}$ such that

$$\mathbb{P}(\Omega^H)\ge1-\varepsilon,$$

where

$$\Omega^H = \left\{ \frac{1}{T}\int_0^\infty \Big(r_3 - a_{31}(X_{(H,H)}(t)+\beta) - \frac{\sigma_{31}^2}{2}(X_{(H,H)}(t)+\beta)^2 - a_{31}(Y_{(H,H)}(t)+\beta) - \frac{\sigma_{31}^2}{2}(Y_{(H,H)}(t)+\beta)^2\Big)\,\mathrm{d}t \geq \frac{3\lambda_{3z}}{5} \right\}.$$

In view of the uniqueness of solution, for all $(x_0, y_0) \in [H^{-1}, H] \times [H^{-1}, H]$, one has $X_{(x_0,y_0)}(t) \leq X_{(H,H)}(t)$ and $Y_{(x_0,y_0)}(t) \leq Y_{(H,H)}(t)$. Therefore, $\mathbb{P}(\Omega_2^{w_0}) \geq 1-\varepsilon$ where

$$\Omega_2^{w_0} = \left\{ \frac{1}{T}\int_0^T \Big(r_3 - a_{31}(X_{(x_0,y_0)}(t)+\beta) - \frac{\sigma_{31}^2}{2}(X_{(x_0,y_0)}(t)+\beta)^2 - a_{31}(Y_{(x_0,y_0)}(t)+\beta) - \frac{\sigma_{31}^2}{2}(Y_{(x_0,y_0)}(t)+\beta)^2\Big)\,\mathrm{d}t \geq \frac{3\lambda_{3z}}{5} \right\}.$$

Owing to Lemma 3.2, there is a $\sigma = \sigma(\varepsilon, H) > 0$ satisfying that

$$a_{33}\sigma + \frac{\sigma_{33}^2}{2}\sigma^2 < \frac{\lambda_{3z}}{5}$$

and $\mathbb{P}(\Omega_3^{w_0}) \geq 1-\varepsilon$, where

$$\Omega_3^{w_0} = \left\{ |X_{(x_0,y_0)}(t) - x_{w_0}(t)| \leq \alpha \text{ and } |Y_{(x_0,y_0)}(t) - y_{w_0}(t)| \leq \alpha,\ \forall 0 \leq t \leq T \wedge \tau_{w_0}^\sigma \right\}.$$

By virtue of Lemma 3.1, there exists $\delta_0 = \delta_0(\varepsilon, H)$ satisfying that $\mathbb{P}(\Omega_4^{w_0}) \geq 1-\varepsilon$ for all $w_0 \in [H^{-1}, H]\times[H^{-1}, H]\times(0, \delta_0]$, where

$$\Omega_4^{w_0} = \{\tau_{w_0}^\sigma \geq T\}.$$

Since $T > \frac{25\overline{M}^2 H}{\varepsilon^2\lambda_{3z}^2}$, it is derived from Lemma 3.3 that $\mathbb{P}(\Omega_5^{w_0}) \geq 1-\varepsilon$, where

$$\Omega_5^{w_0} = \mathbb{P}\left\{ \left| \int_0^T \sigma_{31}x_{w_0}(t)\mathrm{d}B_3(t) + \sigma_{32}y_{w_0}(t)\mathrm{d}B_5(t) + \sigma_{33}z_{w_0}(t)\mathrm{d}B_6(t) \right| \leq \frac{\lambda_{3z}}{5}T \right\}.$$

Thus, for $w_0 \in [H^{-1}, H]\times[H^{-1}, H]\times(0, \delta_0]$ and $\omega \in \bigcap_{i=2}^{5}\Omega_i^{w_0}$, it follows that

$$\begin{aligned}
\ln z_{w_0}(T) - \ln z_0 &\geq \int_0^T \Big(r_3 - a_{31}x_{w_0}(t) - \frac{\sigma_{31}^2}{2}x_{w_0}(t) - a_{32}y_{w_0}(t) - \frac{\sigma_{32}^2}{2}y_{w_0}(t)\Big)\,\mathrm{d}t \\
&\quad - \int_0^T \Big(a_{33}z_{w_0}(t) - \frac{\sigma_{33}^2}{2}z_{w_0}(t)\Big)\,\mathrm{d}t \\
&\quad - \left| \int_0^T \sigma_{31}x_{w_0}(t)\mathrm{d}B_3(t) + \sigma_{32}y_{w_0}(t)\mathrm{d}B_5(t) + \sigma_{33}z_{w_0}(t)\mathrm{d}B_6(t) \right| \\
&\geq \int_0^T \Big(a_2 - c_2(u_{z_0}(t)+\alpha) - \frac{\beta_2^2}{2}(u_{z_0}(t)+\alpha)^2\Big)\,\mathrm{d}t \\
&\geq \frac{3\lambda_{3z}}{5}T - \frac{\lambda_{3z}}{5}T - \frac{\lambda_{3z}}{5}T \\
&\geq \frac{\lambda_{3z}}{5}T
\end{aligned}$$

and $\mathbb{P}\{\frac{\lambda_{3z}}{5}T \leq \ln z_{w_0}(T) - \ln z_0\} \geq \mathbb{P}\left(\bigcap_{i=2}^{5}\Omega_i^{w_0}\right) \geq 1-4\varepsilon$. The proof is complete.

Proposition 3.2. *If λ_{1z}, λ_{2z} and $\lambda_{3z} > 0$, then for any $\Delta > 0$, there exist $T = T(\Delta)$ and $\delta_2 = \delta_2(\Delta)$ satisfying that*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{P}\left\{z_{w_0}(kT) \le \delta_2\right\} \le \Delta, \ \forall w_0 \in \mathbb{R}_+^{3,o}.$$

Proof. Let $\varepsilon = \varepsilon(\Delta) \in (0,1)$ and $H = H(\Delta) > 1$ be chosen later. Put $\Lambda = \frac{\theta_2}{\varepsilon}(1+3H)^{\frac{1}{p}}$ where θ_2 is as in (3.2). In view of (3.4) and Proposition 3.1, there exist $\delta_0 \in (0,1)$ and $T > 1$ such that for all $w_0 \in [H^{-1}, H] \times [H^{-1}, H] \times (0, \delta_0]$, $\mathbb{P}(\Omega_0^{z_0}) > 1 - 5\varepsilon$ where

$$\Omega_0^{z_0} = \left\{\frac{\lambda_{3z}}{5} T \le \ln z_{w_0}(T) - \ln z_0 \le \Lambda T\right\}.$$

Let $L_1 = \Lambda T + \ln H - \ln \delta_0$. By using $|\ln H - \ln z_{w_0}(T)| \le |\ln H - \ln z_0| + |\ln z_{w_0}(T) - \ln z_0|$, if $w_0 \in [H^{-1}, H] \times [H^{-1}, H] \times (0, \delta_0]$, one can be derived from (3.4) that

$$\mathbb{P}\{|\ln H - \ln z_{w_0}(T)| \ge L_1\} \le \mathbb{P}\{|\ln z_{w_0}(T) - \ln z_0| \ge \Lambda T\} \le 5\varepsilon. \tag{3.12}$$

Let δ_1, δ_2 satisfy $L_1 = \ln H - \ln \delta_1$, $L_2 := L_1 + \Lambda T = \ln H - \ln \delta_2$. It's clear that $\delta_2 < \delta_1 < \delta_0$. Define $U(z) = (\ln H - \ln z) \vee L_1$. Obviously,

$$U(z') - U(z'') \le |\ln z' - \ln z''|. \tag{3.13}$$

Now, $\frac{1}{T}[\mathbb{E}U(z_{w_0}(T) - U(z_0))]$ needs to be estimated for different w_0. First, it follows form (3.2) and (3.13) that for any $w_0 \in \mathbb{R}_+^{3,o}$ and $\omega \in \Omega_0^{w_0,c} = \Omega \backslash \Omega_0^{w_0}$,

$$\begin{aligned}\frac{1}{T}\mathbb{E}\mathbf{I}_{\Omega_0^{w_0,c}}\left[U(z_{w_0}(T)) - U(z_0)\right] &\le \frac{1}{T}\mathbb{E}\mathbf{I}_{\Omega_0^{w_0,c}}|\ln z_{w_0}(T) - \ln z_0| \\ &\le \theta_2(1+3H)^{\frac{1}{p}}[\mathbb{P}(\Omega_0^{w_0,c})]^{\frac{1}{q}} \\ &\le \theta_2(1+3H)^{\frac{1}{p}}(5\varepsilon)^{\frac{1}{q}}.\end{aligned} \tag{3.14}$$

If $w_0 \in D_1 := [H^{-1}, H] \times [H^{-1}, H] \times (0, \delta_2]$, then $\ln H - \ln z_0 \ge \ln H - \ln \delta_2 \ge L_1 + \Lambda T$. It follows that in $\Omega_0^{\omega_0}$,

$$\ln H - \ln z_{w_0}(T) \ge \ln H - \ln z_0 - \Lambda T \ge L_2 - \Lambda T \ge L_1,$$

$$\ln H - \ln z_{w_0}(T) \le \ln H - \ln z_0 - \frac{\lambda_{3z}}{5} T.$$

Consequently, one gets

$$\begin{aligned}\frac{1}{T}\mathbb{E}\mathbf{I}_{\Omega_0^{w_0}}\left[U(z_{w_0}(T)) - U(z_0)\right] &= \frac{1}{T}\mathbb{P}(\Omega_0^{w_0})\left[\ln H - U(z_{w_0}(T)) - \ln H + U(z_0)\right] \\ &\le -\frac{\lambda_{3z}}{5}(1-\varepsilon).\end{aligned} \tag{3.15}$$

One can be derived from (3.14) and (3.15) that for all $w_0 \in D_1$,

$$\begin{aligned}\frac{1}{T}\mathbb{E}\left[U(z_{w_0}(T)) - U(z_0)\right] &\le \frac{1}{T}\mathbb{E}\left(\mathbf{I}_{\Omega_0^{w_0}} + \mathbf{I}_{\Omega_0^{w_0,c}}\right)\left[U(z_{w_0}(T)) - U(z_0)\right] \\ &\le -\frac{\lambda_{3z}}{5}(1+\varepsilon) + \theta_2(1+3H)^{\frac{1}{p}}(5\varepsilon)^{\frac{1}{q}}.\end{aligned} \tag{3.16}$$

If $w_0 \in D_2 := [H^{-1}, H] \times [H^{-1}, H] \times (\delta_2, \delta_1]$, then $\ln H - \ln z_0 \ge \ln H - \ln z_0 = L_1$. In $\Omega_0^{w_0}$,

$$\ln H - \ln z_{w_0}(T) \le \ln H - \ln z_0 - \frac{\lambda_{3z}}{5} T \le \ln H - \ln z_0 = U(z_0).$$

Therefore, $U(y_{w_0}(T)) - U(w_0) \le 0$ in $\Omega_0^{w_0}$. As a result of (3.14),

$$\frac{1}{T}\mathbb{E}\left[U(z_{w_0}(T)) - U(z_0)\right] \le \theta_2(1+3H)^{\frac{1}{p}}(5\varepsilon)^{\frac{1}{q}}, \ \forall w_0 \in D_2. \tag{3.17}$$

If $w_0 \in D_3 := [H^{-1}, H]\times[H^{-1}, H]\times(\delta_1, \delta_0]$, then $\ln H - \ln z_0 \le \ln H - \ln \delta_1 = L_1$. And in $\Omega_0^{w_0}$, $\ln H - \ln z_{w_0}(T) \le \ln H - \ln z_0 \le L_1$, which means that $U(z_{w_0}(T)) = U(z_0) = L_1$. Therefore,

$$\frac{1}{T}\mathbb{E}\left[U(z_{w_0}(T)) - U(z_0)\right] \le \theta_2(1+3H)^{\frac{1}{p}}(5\varepsilon)^{\frac{1}{q}}, \ \forall w_0 \in D_3. \tag{3.18}$$

If $w_0 \in D_4 := [H^{-1}, H]\times[H^{-1}, H]\times(\delta_0, H]$, it leads to that $\ln H - \ln z_0 \le \ln H - \ln \delta_0 = L_1 - \Lambda T$. In the same way of obtaining (3.18), one gets

$$\frac{1}{T}\mathbb{E}\left[U(z_{w_0}(T)) - U(z_0)\right] \le \theta_2(1+3H)^{\frac{1}{p}}(5\varepsilon)^{\frac{1}{q}}, \ \forall w_0 \in D_4. \tag{3.19}$$

For any initial value $w_0 \in \mathbb{R}_+^{3,o}$, the Markov property of $w(t)$ leads to

$$\frac{1}{T}\mathbb{E}[U(z_{w_0}(kT+T)) - U(z_{w_0}(kT))] \le \int\limits_{\mathbb{R}_+^{2,o}} \mathbb{P}\left\{z_{w_0}(kT) \in \mathrm{d}w\right\}\left[\frac{1}{T}\mathbb{E}U(z_w(T)) - U(z_0)\right].$$

Putting $D := [H^{-1}, H]\times[H^{-1}, H]\times[H^{-1}, H] = \bigcap_{i=1}^{4} D_i$, and using (3.2), (3.16), (3.17), (3.18) and (3.19), one gets

$$\begin{aligned}\frac{1}{T}\mathbb{E}[U(z_{w_0}(kT+T)) - U(z_{w_0}(kT))] \le & \mathbb{P}\left\{w_{w_0}(kT) \in D_1\right\}\left[-\frac{\lambda_{3z}}{5}(1-\varepsilon+\varepsilon_1)\right] + \varepsilon_1\mathbb{P}\left\{w_{w_0}(kT) \in D\setminus D_1\right\} \\ & + \mathbb{E}\mathbf{I}_{\{w_{w_0}(kT)\notin D\}}\frac{1}{T}\mathbb{E}\left|\ln(z_{w_{w_0}(kT)}(T)) - \ln(z_{w_0}(kT))\right| \\ \le & -\frac{\lambda_{3z}}{5}(1-\varepsilon)\mathbb{P}\left\{w_{w_0}(kT) \in D_1\right\} + \varepsilon_1 \\ & + \theta_2\left(1+\mathbb{E}|w_{w_0}(kT)|\right)^{\frac{1}{p}}\left(P\left\{w_{w_0}(kT) \notin D\right\}\right)^{\frac{1}{q}},\end{aligned} \tag{3.20}$$

where θ_2 is as in (3.2) and $\varepsilon_1 = \theta_2(1+3H)^{\frac{1}{p}}(5\varepsilon)^{\frac{1}{q}}$. It follows from Markov's inequality and (i) of Proposition 2.1 that

$$\begin{aligned}\limsup_{k\to\infty}\mathbb{P}\{w_{w_0}(kT) \notin D\} & \le \limsup_{k\to\infty}\mathbb{P}\{[x_{w_0}(kT)\vee y_{w_0}(kT)\vee z_{w_0}(kT)] > H\} \\ & \le \limsup_{k\to\infty}\mathbb{P}\{V(w_{w_0}(kT)) > H\} \\ & \le \limsup_{k\to\infty}\frac{\mathbb{E}V(w_{w_0}(kT))}{H} \\ & \le \frac{M_0}{H}\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\limsup_{k\to\infty}\left(1+\mathbb{E}|w_{w_0}(kT)|\right)^{\frac{1}{p}}\left(\mathbb{P}\left\{w_{w_0}(kT) \notin D\right\}\right)^{\frac{1}{q}} & \le \left(\frac{M_0}{H}\right)^{\frac{1}{q}}\limsup_{k\to\infty}[1+3\mathbb{E}V(w_{w_0}(kT)]^{\frac{1}{p}} \\ & \le \left(\frac{1+3M_0}{H}\right)^{\frac{1}{q}}(1+3M_0)^{\frac{1}{p}} \\ & \le \frac{1+3M_0}{H^{\frac{1}{q}}}.\end{aligned} \tag{3.22}$$

Obviously,

$$\liminf_{n\to\infty}\frac{1}{nT}\sum_{k=0}^{n-1}\mathbb{E}\left[U(z_{w_0}(kT+T)) - U(z_{w_0}(kT))\right] = \liminf_{n\to\infty}\frac{1}{nT}\mathbb{E}\left[U(z_{w_0}(nT) - U(z_0))\right] \ge 0. \tag{3.23}$$

It follows from (3.20), (3.21) and (3.23) that

$$0 \le \limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\left\{w_{w_0}(kT) \in D_1\right\}\left[-\frac{\lambda_{3z}}{5}(1-\varepsilon)\right] + \varepsilon_1 + \frac{\theta_2(1+3M_0)}{H^{\frac{1}{q}}}$$

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\left\{w_{w_0}(kT)\in D_1\right\}\le\frac{5}{\lambda_{3z}(1-\varepsilon)}\left[\varepsilon_1+\frac{\theta_2(1+3M_0)}{H^{\frac{1}{q}}}\right],\tag{3.24}$$

where $\varepsilon_1=\theta_2(1+3H)^{\frac{1}{p}}(5\varepsilon)^{\frac{1}{q}}$. It should be noted that $\mathbb{P}\{z_{w_0}(kT)\le\delta_2\}\le\mathbb{P}\{w_{w_0}(kT)\in D_1\}+\mathbb{P}\{w_{w_0}(kT)\notin D\}$. As a result of (3.21) and (3.24), one has

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\{z_{w_0}(kT)\le\delta_2\}\le\frac{5}{\lambda_{3z}(1-\varepsilon)}\left[\varepsilon_1+\frac{\theta_2(1+3M_0)}{H^{\frac{1}{q}}}\right]+\frac{M_0}{H}.$$

Consequently, there are sufficiently large $H=H(\Delta)$ and sufficiently small $\varepsilon=\varepsilon(\Delta)$ satisfying the desired result. The proof is complete.

Now, it is time to prove Theorem 3.1.

Proof of Theorem 3.1. Set any $\varepsilon>0$. Owing to symmetry of (1.3), similar to Proposition 3.2, it can be proved that if $\lambda_{1x},\ \lambda_{2x}$ and $\lambda_{3x}>0$, then there are $T'>1$ and $\delta_2'>0$ satisfying that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\left\{x_{w_0}(kT')\le\delta_2'\right\}\le\Delta,\ \forall w_0\in\mathbb{R}_+^{3,o}.$$

And if $\lambda_{1y},\ \lambda_{2y}$ and $\lambda_{3y}>0$, then there are $T''>1$ and $\delta_2''>0$ satisfying that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\left\{y_{w_0}(kT'')\le\delta_2''\right\}\le\Delta,\ \forall w_0\in\mathbb{R}_+^{3,o}.$$

In fact, if $\lambda_{3z},\ \lambda_{3x}$ and $\lambda_{3y}>0$, then $\lambda_{1z},\ \lambda_{2z},\ \lambda_{1x},\ \lambda_{2x},\ \lambda_{1y}$ and λ_{2y} will be positive, and $\lambda_{3z},\ \lambda_{3x}$ and λ_{3y} have been defined in (2.10), (2.12) and (2.14) respectively. Furthermore, by choosing sufficiently large T and sufficiently small δ_2, one has

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\left\{|x_{w_0}(kT)|\wedge|y_{w_0}(kT)|\wedge|z_{w_0}(kT)|\le\delta_2\right\}\le 3\Delta,\ \forall w_0\in\mathbb{R}_+^{3,o}.$$

Combining this with (i) of Proposition 2.1 implies that there exists a compact set $G\subset\mathbb{R}_+^{3,o}$ satisfying that

$$\liminf\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\{w_{w_0}(kT)\in G\}\ge 1-4\Delta,\ w_0\in\mathbb{R}_+^{3,o}.$$

Owing to (ii) of Proposition 2.1, there is an $M>1$ satisfying that

$$\mathbb{P}\{M^{-1}\le x_w(t),\ y_w(t),\ z_w(t)\le M\}\ge 1-\Delta,\ \forall t\le T,\ w\in G.$$

By virtue of Markov property, one has

$$\mathbb{P}\{M^{-1}\le x_{w_0}(kT+t),\ y_{w_0}(kT+t),\ z_{w_0}(kT+t)\le M\}\ge\mathbb{P}\{w_{w_0}(kT)\in G\}\mathbb{P}\{M^{-1}\le x_w(t),\ y_w(t),\ z_w(t)\le M\},$$
$$w\in G,\ w_0\in\mathbb{R}_+^{3,o}$$

As a result, for any $w_0\in\mathbb{R}_+^{3,o}$,

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{nT}\int_0^{nT}\mathbb{P}\{M^{-1}x_{w_0}(t),\ y_{w_0}(t),\ z_{w_0}(t)\le M\}\mathrm{d}t\ge&(1-\Delta)\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\{w_{w_0}(kT)\in G\}\\ \ge&(1-\Delta)(1-4\Delta)\\ \ge&1-5\Delta.\end{aligned}$$

It means that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{P}\{M^{-1}\le x_{w_0}(s),\ y_{w_0}(s),\ z_{w_0}(s)\le M\}\mathrm{d}s\ge 1-5\Delta,$$

and there exist an invariant probability measure consequently. Since the diffusion is non-degeneracy, the rest of the results are yield (see [21]). The proof is complete.

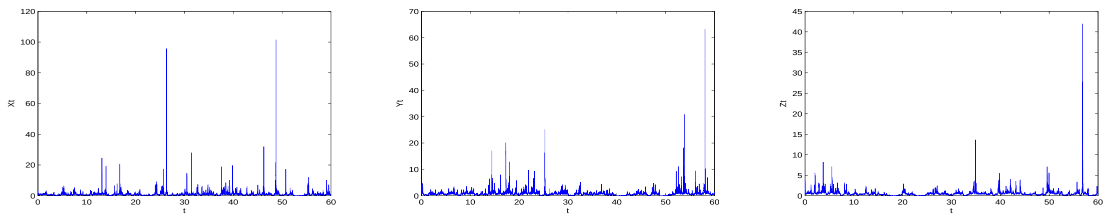

Figure 1: Sample paths of $x_{w_0}(t)$, $y_{w_0}(t)$ and $z_{w_0}(t)$ in Example 5.1 with $w_0 = (3, 3, 3)$.

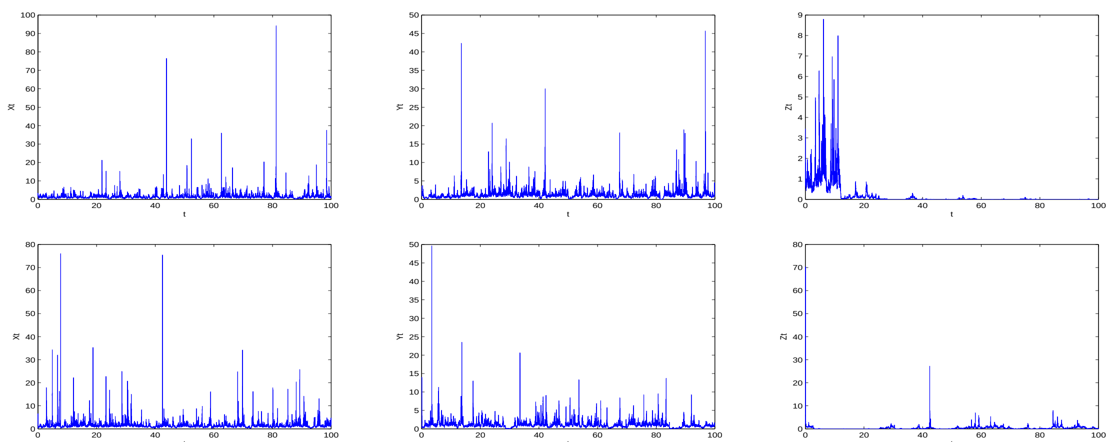

Figure 2: Sample paths of $x_{w_0}(t)$, $y_{w_0}(t)$ and $z_{w_0}(t)$ of Example 5.2 with $w_0 = (3, 3, 3)$ in two trials.

4 Examples and complexity discussion

In this section, we consider model (1.3),

$$\begin{cases} \mathrm{d}x(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y(t) - a_{13}z(t))\mathrm{d}t + \sigma_{11}x^2(t)\mathrm{d}B_1(t) + \sigma_{12}x(t)y(t)\mathrm{d}B_2(t) + \sigma_{13}x(t)z(t)\mathrm{d}B_3(t), \\ \mathrm{d}y(t) = y(t)(r_2 - a_{21}x(t) - a_{22}y(t) - a_{23}z(t))\mathrm{d}t + \sigma_{21}x(t)y(t)\mathrm{d}B_2(t) + \sigma_{22}y^2(t)\mathrm{d}B_4(t) + \sigma_{23}y(t)z(t)\mathrm{d}B_5(t), \\ \mathrm{d}z(t) = z(t)(r_3 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t))\mathrm{d}t + \sigma_{31}x(t)z(t)\mathrm{d}B_3(t) + \sigma_{22}y(t)z(t)\mathrm{d}B_5(t) + \sigma_{33}z^2(t)\mathrm{d}B_6(t). \end{cases}$$

According to Theorems 3.1, λ_{3z}, λ_{3x} and λ_{3y} govern the coexistence of three species; λ_{3z}, λ_{3x} and $\lambda_{3y} > 0$ lead to coexistence of three species; Note that λ_{3z}, λ_{3x} and λ_{3y} are defined in (2.10), (2.12) and (2.14) respectively. Examples 4.1 and the simulations illustrate theoretical results that three species coexist.

Example 4.1. Consider (1.3) with parameters $r_1 = 6$; $r_2 = 6$; $r_3 = 6$; $a_{11} = 1$; $a_{12} = 1$; $a_{13} = 2$; $a_{21} = 2$; $a_{22} = 2$; $a_{23} = 1$; $a_{31} = 1$; $a_{32} = 3$; $a_{33} = 3$; $\sigma_{11} = 2$; $\sigma_{12} = 1$; $\sigma_{13} = 0.5$; $\sigma_{21} = 1$; $\sigma_{22} = 2$; $\sigma_{23} = 1$; $\sigma_{31} = 0.5$; $\sigma_{32} = 1$; $\sigma_{33} = 2$. Direct calculation demonstrates that $\lambda_{3z} = 0.1258$, $\lambda_{3x} = 1.7079$, $\lambda_{3y} = 0.5557$. As a result of Theorem 3.1, the three species coexist. Sample paths of $x_{w_0}(t)$, $y_{w_0}(t)$ and $z_{w_0}(t)$ with $w_0 = (3, 3, 3)$ are illustrated in Fig. 1.

Remark 4.1. When λ_{3z}, λ_{3x} and λ_{3y} are not all positive, the properties of the solution will become very complicated. Three species may die out or coexist; see Example 4.2.

Example 4.2. Consider (1.3) with parameters $r_1 = 6$; $r_2 = 5$; $r_3 = 4$; $a_{11} = 2$; $a_{12} = 1$; $a_{13} = 1$; $a_{21} = 1$; $a_{22} = 2$; $a_{23} = 1$; $a_{31} = 1$; $a_{32} = 1$; $a_{33} = 2$; $\sigma_{11} = 1$; $\sigma_{12} = 0.5$; $\sigma_{13} = 0.5$; $\sigma_{21} = 0.5$; $\sigma_{22} = 1$; $\sigma_{23} = 0.8$; $\sigma_{31} = 0.5$; $\sigma_{32} = 0.8$; $\sigma_{33} = 1$. Direct calculation demonstrates that $\lambda_{3z} = -1.1692$, $\lambda_{3x} = 2.2482$, $\lambda_{3y} = 0.4026$. The species $z(t)$ may die out or three species coexist. Sample paths of $x_{w_0}(t)$, $y_{w_0}(t)$ and $z_{w_0}(t)$ with $w_0 = (3, 3, 3)$ are illustrated in Fig. 2.

5 Conclusions

This paper focuses on providing the conditions for a three-species stochastic competitive model. The obtaining results show that three values λ_{3z}, λ_{3x} and λ_{3y} calculated by the coefficients can be served as threshold values. For model (1.3), three species will coexist and all positive solutions converge to a unique invariant probability measure when λ_{3z}, λ_{3x} and $\lambda_{3y} > 0$. Besides, compared with the system in [18], the results in this paper is in agreement with Theorem 2.1 in [18] by chosen that $r_3 = a_{31} = a_{32} = a_{33} = \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$.

It is noteworthy that in Section 3, $\Delta_i = 0$, $i = 1, 2, 3$. However, for model (1.2), that is $\Delta_i \neq 0$, $i = 1, 2, 3$, equation on the x-axis, y-axis and z-axis are

$$\mathrm{d}\varphi(t) = \varphi(t)(r_1 - a_{11}\varphi(t))\mathrm{d}t + (\Delta_1\varphi(t) + \sigma_{11}\varphi^2(t))\mathrm{d}B_1(t), \tag{5.1}$$

$$\mathrm{d}\psi(t) = \psi(t)(r_2 - a_{22}\psi(t))\mathrm{d}t + (\Delta_2\psi(t) + \sigma_{22}\psi^2(t))\mathrm{d}B_4(t) \tag{5.2}$$

and

$$\mathrm{d}\rho(t) = \rho(t)(r_3 - a_{33}\rho(t))\mathrm{d}t + (\Delta_3\rho(t) + \sigma_{33}\rho^2(t))\mathrm{d}B_6(t), \tag{5.3}$$

respectively. Employing Theorem 3.1 of [22], one can be obtained that if $r_i - \frac{\Delta_i^2}{2} < 0$ (i=1,2,3), then $\mathbb{P}\left\{\lim\limits_{t\to\infty}\varphi(t) = 0 \text{ or } \lim\limits_{t\to\infty}\psi(t) = 0 \text{ or } \lim\limits_{t\to\infty}\rho(t) = 0\right\} = 1$ for all positive solutions $\varphi(t)$, $\psi(t)$ and $\rho(t)$. Then, by using arguments similar to the proof of Proposition 4.1 in [18], it shows that $\mathbb{P}\left\{\lim\limits_{t\to\infty}x(t) = 0 \text{ or } \lim\limits_{t\to\infty}y(t) = 0 \text{ or } \lim\limits_{t\to\infty}z(t) = 0\right\} = 1$. In case $r_i - \frac{\Delta_i^2}{2} > 0$ $(i = 1, 2, 3)$, (5.1), (5.2) and (5.3) respectively have unique invariant probability measure whose density $\widetilde{f}_1^\star$, $\widetilde{f}_2^\star$ and $\widetilde{f}_3^\star$ can be solved from the Fokker-Planck equation. Define

$$\widetilde{\lambda}_{3z} = r_3 - \int_0^\infty (a_{31}x + \frac{\sigma_{31}^2}{2}x^2)\widetilde{f}_1^\star(x)\mathrm{d}x - \int_0^\infty (a_{32}z + \frac{\sigma_{32}^2}{2}z^2)\widetilde{f}_2^\star(y)dy,$$

$$\widetilde{\lambda}_{3x} = r_1 - \int_0^\infty (a_{12}x + \frac{\sigma_{12}^2}{2}x^2)\widetilde{f}_2^\star(y)\mathrm{d}y - \int_0^\infty (a_{13}z + \frac{\sigma_{13}^2}{2}z^2)\widetilde{f}_3^\star(z)dz$$

and

$$\widetilde{\lambda}_{3y} = r_2 - \int_0^\infty (a_{21}x + \frac{\sigma_{21}^2}{2}x^2)\widetilde{f}_1^\star(x)\mathrm{d}x - \int_0^\infty (a_{23}z + \frac{\sigma_{23}^2}{2}z^2)\widetilde{f}_3^\star(z)dz.$$

If $r_i - \frac{\Delta_i^2}{2} > 0$ $(i = 1, 2, 3)$. Using the argument in Section 3 with modifications, one can be also gotten that if $\widetilde{\lambda}_{3z}$, $\widetilde{\lambda}_{3z}$ and $\widetilde{\lambda}_{3z} > 0$, then model (1.2) has an invariant probability measure in $\mathbb{R}_+^{3,o}$. It implies that the results stated in Theorem 3.1 hold for (1.2) with λ_{3z}, λ_{3z}, λ_{3z} replaced by $\widetilde{\lambda}_{3z}$, $\widetilde{\lambda}_{3x}$, $\widetilde{\lambda}_{3y}$. Furthermore, the results recover the main findings that Theorem 3.5 and 4.3 in [17].

Though this paper has verified that three species coexist when λ_{3z}, λ_{3z} and $\lambda_{3z} > 0$ for model (1.3), three species may also coexist in the case that λ_{3z}, λ_{3z} and λ_{3z} are not all positive. It therefore still required novel techniques to deal with this problem.

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