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Research Article

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Embeddings of harmonic mixed norm spaces on smoothly bounded domains in \mathbb{R}^n

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Abstract: The main result of this paper is the embedding

$$\mathcal{B}_\beta^{s,r}(\Omega) \hookrightarrow \mathcal{B}_{\beta+(n-1)\left(\frac{1}{s}-\frac{1}{s_1}\right)}^{s_1,r_1}(\Omega),$$

$0 < r \leq r_1 \leq \infty$, $0 < s \leq s_1 \leq \infty$, $\beta > -1$, of harmonic functions mixed norm spaces on a smoothly bounded domain $\Omega \subset \mathbb{R}^n$. We also extend a result on boundedness, in mixed norm, of a maximal function-type operator from the case of the unit disc and the unit ball to general domains in \mathbb{R}^n .

Keywords: mixed norm spaces, harmonic functions spaces, maximal functions**MSC:** 31B05, 42B25, 42B35

1 Introduction and preliminaries

The embedding theorems for harmonic or analytic function spaces with mixed norm have been studied extensively, especially in the case of the unit disc, where first results are due to Hardy and Littlewood [1, 2]. In the case of analytic functions such theorems were proved for general bounded strictly pseudoconvex domains in \mathbb{C}^n , see [3]. Mixed norm spaces of harmonic and analytic functions on the upper half plane were investigated in [4, 5], some of the methods we use here can be traced to these papers. For harmonic functions many authors considered embeddings of mixed norm spaces on \mathbb{B}^n or upper half-space \mathbb{H}^n , see for example [6] for \mathbb{B}^n , [7–9] for \mathbb{R}^n , or [10] for \mathbb{H}^n . However, it seems that the case of more general domains was not treated.

In this paper we prove an embedding theorem for mixed norm spaces of harmonic functions, Theorem 1 below, in the setting of bounded C^1 domains. This result generalizes Theorem 1.1 (iv) from [6]. In addition, we consider a maximal function-type operator $u \mapsto u^\times$ and prove its boundedness with respect to mixed norm in the class of quasi-nearly subharmonic functions u , see Theorem 2 below.

We note that the operator u^\times was discussed, in the case of the unit disc, in [11], and the corresponding result in $\Omega \subset \mathbb{R}^n$ is Theorem 2; see also a related result in [12] for weighted harmonic Bergman spaces on \mathbb{B}^n .

We denote the Lebesgue measure on \mathbb{R}^n by dV and the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ by $|E|$. The surface measure on $\partial\Omega$ is denoted by $d\sigma$. $B(a, r)$ denotes the usual Euclidean ball in \mathbb{R}^n , with center at $a \in \mathbb{R}^n$ and radius $r > 0$. We also use a standard convention: C denotes a constant which can actually change its value from one occurrence to the next one. Also, for positive quantities A and B , $A \asymp B$ means that $cA \leq B \leq CA$ for some constants $0 < c \leq C < \infty$.

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In this paper we work with a bounded domain $\Omega \subset \mathbb{R}^n$ with C^1 boundary. We fix a defining function ρ for Ω , which means $\rho \in C^1(\mathbb{R}^n)$, $\Omega = \{x \in \mathbb{R}^n : \rho(x) > 0\}$, $\partial\Omega = \{x \in \mathbb{R}^n : \rho(x) = 0\}$ and $\nabla\rho(\xi) \neq 0$ for all $\xi \in \partial\Omega$. We note that

$$\rho(x) \asymp \text{dist}(x, \partial\Omega) \quad \text{for } x \in \Omega.$$

By well known Tubular Neighborhood Theorem, there is a neighborhood U of $\partial\Omega$ and there is a C^1 -diffeomorphism $\chi : U \rightarrow \partial\Omega \times (-r_0, r_0)$ such that $\chi(\partial\Omega) = \partial\Omega \times \{0\}$, $\chi(U \cap \Omega) = \partial\Omega \times (0, r_0)$. We set $\varphi = \chi^{-1}$ and, for $-r_0 < t < r_0$, $\Gamma_t = \varphi(\partial\Omega \times \{t\})$. For a given measurable complex valued function f defined on $U \cap \Omega$ (or Ω), we define $\tilde{f} : \partial\Omega \times (0, r_0) \rightarrow \mathbb{C}$ by $\tilde{f}(\xi, t) = f(\varphi(\xi, t))$.

Let $h(\Omega) = \{u : \Omega \rightarrow \mathbb{C} \mid u \text{ is harmonic in } \Omega\}$. If $u_1, u_2 \in h(\Omega)$ and $u_1 = u_2$ on $U \cap \Omega$, then $u_1 = u_2$ on Ω . We set, by a slight abuse of notation, $\tilde{u} = (u|_{U \cap \Omega})^\sim$. By the above remark, if $\tilde{u}_1 = \tilde{u}_2$, then $u_1 = u_2$ for $u_1, u_2 \in h(\Omega)$.

Next we define certain spaces of functions on Ω and $\partial\Omega \times (0, r_0)$ which are a natural generalization of classical mixed norm spaces on the unit ball. For a Borel measurable function f on Ω or $\Omega \cap U$ we set

$$M_s(f, t) = \left\{ \int_{\partial\Omega} |\tilde{f}(\xi, t)|^s d\sigma(\xi) \right\}^{\frac{1}{s}}, \quad 0 < s < \infty, \quad 0 < t < r_0,$$

with the usual modification for $s = \infty$. Also for a Borel measurable function g on $\partial\Omega \times (0, r_0)$ we set

$$\tilde{M}_s(g, t) = \left\{ \int_{\partial\Omega} |g(\xi, t)|^s d\sigma(\xi) \right\}^{\frac{1}{s}}, \quad 0 < s < \infty, \quad 0 < t < r_0,$$

again with the usual modification for $s = \infty$. Now we have a mixed norm space

$$\mathcal{L}_\beta^{s,r} = \mathcal{L}_\beta^{s,r}(\partial\Omega \times (0, r_0)), \quad 0 < s, r \leq \infty, \quad \beta \in \mathbb{R},$$

as the space of Borel measurable function g on $\partial\Omega \times (0, r_0)$ such that the following (quasi) norm of g is finite

$$\|g\|_{\mathcal{L}_\beta^{s,r}} = \|t^\beta \tilde{M}_s(g, t)\|_{L^r((0, r_0), \frac{dt}{t})}.$$

The main object of study in this paper is the following space of harmonic functions

$$\mathcal{B}_\beta^{s,r}(\Omega) = \{u \in h(\Omega) : \tilde{u} \in \mathcal{L}_\beta^{s,r}(\partial\Omega \times (0, r_0))\},$$

with the following (quasi) norm

$$\|u\|_{\mathcal{B}_\beta^{s,r}(\Omega)} = \|\tilde{u}\|_{\mathcal{L}_\beta^{s,r}}.$$

Here $0 < s, r \leq \infty$ and $\beta > -1$. Note that these spaces are trivial for $\beta \leq -1$. Different choice of a defining function ρ and a different choice of tubular neighborhood map χ lead to different, but equivalent norms and the same mixed norm spaces.

For every point ξ on the boundary of Ω and $t > 0$ we define a "ball" $B_t^{\partial\Omega}(\xi)$ with center at point $\xi \in \partial\Omega$ and radius $t > 0$ by

$$B_t^{\partial\Omega}(\xi) = \{\eta \in \partial\Omega : |\xi - \eta| \leq t\}.$$

Note that the following area estimate is valid:

$$\sigma(B_t^{\partial\Omega}(\xi)) \asymp t^{n-1}, \quad 0 < t \leq \text{diam}(\partial\Omega). \quad (1.1)$$

We also consider a "cylinder" in Ω centered at $\varphi(\xi, t)$:

$$Q(\xi, t) = \left\{ z \in \Omega \cap U \mid \chi(z) \in B_t^{\partial\Omega}(\xi) \times \left[\frac{t}{2}, \frac{3t}{2} \right] \right\}, \quad \xi \in \partial\Omega, \quad 0 < t < \frac{2r_0}{3}.$$

We have the following two-sided volume estimate:

$$|Q(\xi, t)| \asymp t^n, \quad 0 < t \leq \text{diam}(\partial\Omega). \quad (1.2)$$

We define a metric on $\partial\Omega \times \mathbb{R}$ by

$$d_{\partial\Omega \times \mathbb{R}}((\xi_1, t_1), (\xi_2, t_2)) = \sqrt{|\xi_1 - \xi_2|^2 + |t_1 - t_2|^2},$$

for $(\xi_1, t_1), (\xi_2, t_2)$ in $\partial\Omega \times \mathbb{R}$. It is easy to see that $\chi : U_1 \rightarrow \partial\Omega \times [-r_1, r_1]$ and $\varphi : \partial\Omega \times [-r_1, r_1] \rightarrow U_1$ are Lipschitz continuous for any $r_1 \in (0, r_0)$, where $U_1 = \varphi(\partial\Omega \times [-r_1, r_1])$. In fact, these C^1 diffeomorphisms have continuous and bounded partial derivatives. Hence, without loss of generality, we can assume that χ and φ are Lipschitz continuous, i.e. there are constants $0 < l \leq L < \infty$ such that

$$l|z - w| \leq d_{\partial\Omega \times \mathbb{R}}(\chi(z), \chi(w)) \leq L|z - w|,$$

for all $z, w \in U$. Also, there are constants $0 < c \leq C < \infty$ such that for any measurable $E \subset U$ we have

$$c(d\sigma \times dt)(\chi(E)) \leq |E| \leq C(d\sigma \times dt)(\chi(E)).$$

Therefore, for any non-negative and measurable f on $\Omega \cap U$ we have:

$$\int_{U \cap \Omega} f dV \asymp \int_{\partial\Omega} \int_0^{r_0} \tilde{f} d\sigma dt. \quad (1.3)$$

This is, in view of (1.1), a generalization of (1.2).

Let $r_2 = \min(\frac{2r_0}{3}, \frac{r_0}{2L})$. Let us prove the following inclusions:

$$\overline{B}\left(\varphi(\xi, t), \frac{t}{2L}\right) \subset Q(\xi, t) \subset B\left(\varphi(\xi, t), \frac{2t}{l}\right), \quad \xi \in \partial\Omega, \quad 0 < t \leq r_2. \quad (1.4)$$

The first inclusion is equivalent to the following one:

$$\chi\left(\overline{B}\left(\varphi(\xi, t), \frac{t}{2L}\right)\right) \subset \chi(Q(\xi, t)) = B_t^{\partial\Omega}(\xi) \times \left[\frac{t}{2}, \frac{3t}{2}\right].$$

Now, for $z \in \overline{B}(\varphi(\xi, t), \frac{t}{2L})$ we have

$$d_{\partial\Omega \times \mathbb{R}}(\chi(z), (\xi, t)) = d_{\partial\Omega \times \mathbb{R}}(\chi(z), \chi(\varphi(\xi, t))) \leq L|z - \varphi(\xi, t)| \leq L \frac{t}{2L} = \frac{t}{2},$$

which proves a stronger inclusion:

$$\chi\left(\overline{B}\left(\varphi(\xi, t), \frac{t}{2L}\right)\right) \subset B_{t/2}^{\partial\Omega}(\xi) \times \left[\frac{t}{2}, \frac{3t}{2}\right].$$

Similarly one proves $Q(\xi, t) \subset B(\varphi(\xi, t), 2t/l)$.

Let us set

$$V = \varphi(\partial\Omega \times (0, r_2)) \subset \Omega \cap U. \quad (1.5)$$

Working within V has certain advantages: one can always consider $Q(\xi, t)$ when $\varphi(\xi, t) \in V$ and, within V , one can use inclusions (1.4).

The following lemma, due to Fefferman and Stein (see [13]), states that $|u|^p$ has subharmonic behavior for any $p > 0$.

Lemma 1. *Let $u \in h(\Omega)$ and let $B = B(z, r) \subset \Omega$. Then*

$$|u(z)|^p \leq \frac{C}{|B|} \int_B |u|^p dV,$$

where C is a constant which depends only on p and n .

The above lemma combined with (1.2) and (1.4) gives the next result:

Lemma 2. Suppose $Q(\xi, t)$ is a cylinder in Ω , where $\xi \in \partial\Omega$, $0 < t \leq r_2$, and assume h is harmonic in Ω . Then for every $p > 0$ there is a constant $C > 0$ that depends only on p and n such that

$$|u(\varphi(\xi, t))|^p \leq \frac{C}{|Q(\xi, t)|} \int_{Q(\xi, t)} |u|^p dV.$$

Remark 1. In the above constructions one can use segment $[(1-\delta), (1+\delta)]$, where $0 < \delta < 1$ instead of $[\frac{t}{2}, 3\frac{t}{2}]$ (the case $\delta = \frac{1}{2}$). In particular, Lemma 2 is valid in this case, of course, the constant C depends on δ as well.

2 Main results

The main result of the paper is:

Theorem 1. For $0 < s \leq s_1 \leq \infty$ and $0 < r \leq r_1 \leq \infty$ we have a continuous embedding

$$\mathcal{B}_\beta^{s,r}(\Omega) \hookrightarrow \mathcal{B}_{\beta_1}^{s_1,r_1}(\Omega),$$

where $\beta_1 = \beta + (n-1)(\frac{1}{s} - \frac{1}{s_1})$.

The following lemma is a special case of Theorem 1, where $s = s_1$, $r_1 = \infty$:

Lemma 3. Suppose $0 < r \leq \infty$ and $\beta > -1$, then we have $\mathcal{B}_\beta^{s,r}(\Omega) \hookrightarrow \mathcal{B}_\beta^{s,\infty}(\Omega)$.

Proof. Let us fix $u \in \mathcal{B}_\beta^{s,r}(\Omega)$. We treat separately the cases $0 < s \leq r < \infty$ and $0 < r < s < \infty$.

Assume $0 < s \leq r < \infty$. For $0 < t < r_2$ we obtain, by Lemma 2 and (1.3), the following estimate:

$$|\tilde{u}(\xi, t)|^s \leq \frac{C}{|Q(\xi, t)|} \int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^s d\sigma(\eta) d\tau. \quad (2.1)$$

Integrating over $\xi \in \partial\Omega$ and applying Fubini's theorem we obtain

$$\int_{\partial\Omega} |\tilde{u}(\xi, t)|^s d\sigma(\xi) \leq \frac{C}{|Q(\xi, t)|} \int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{\partial\Omega} \int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^s d\sigma(\eta) d\sigma(\xi) d\tau. \quad (2.2)$$

For a fixed τ we have, again applying Fubini's theorem and (1.1):

$$\int_{\partial\Omega} \int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^s d\sigma(\eta) d\sigma(\xi) = \int_{\partial\Omega} |\tilde{u}(\eta, \tau)|^s \int_{\partial\Omega} \chi_{B_t^{\partial\Omega}(\xi)} d\sigma(\xi) d\sigma(\eta) \leq C\tau^{n-1} \int_{\partial\Omega} |\tilde{u}(\eta, \tau)|^s d\sigma(\eta).$$

We use the above inequality and (1.2) to estimate inner integrals in (2.2):

$$M_s^s(u, t) \leq \frac{C}{t^n} \int_{\frac{t}{2}}^{\frac{3t}{2}} \tau^{n-1} \int_{\partial\Omega} |\tilde{u}(\eta, \tau)|^s d\sigma(\eta) d\tau \leq C \int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^s(u, \tau) \frac{d\tau}{\tau},$$

note that we also used $\tau \asymp t$ for $\frac{t}{2} \leq \tau \leq \frac{3t}{2}$. Next we use Hölder's inequality with exponent $\frac{r}{s} \geq 1$ and get

$$M_s^s(u, \tau) \leq C \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^r(u, \tau) \frac{d\tau}{\tau} \right)^{\frac{s}{r}} \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} \frac{d\tau}{\tau} \right)^{1-\frac{s}{r}} = C \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^r(u, \tau) \frac{d\tau}{\tau} \right)^{\frac{s}{r}}.$$

Therefore we obtained

$$M_s(u, t) \leq C \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^r(u, \tau) \frac{d\tau}{\tau} \right)^{\frac{1}{r}}, \quad 0 < t < r_2. \quad (2.3)$$

Our next goal is to obtain the crucial estimate (2.3) also in the second case, i.e. for $0 < r \leq s < \infty$. Let us set $p = s/r \geq 1$. We fix $0 < t < r_2$ and, as in the first case, see (2.1), we obtain from Lemma 2 the following estimate:

$$|\tilde{u}(\xi, t)|^r \leq \frac{C}{|Q(\xi, t)|} \int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^r d\sigma(\eta) d\tau. \quad (2.4)$$

This gives, using (1.2):

$$|\tilde{u}(\xi, t)|^s \leq \left(\frac{C}{t^n} \right)^p \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^r d\sigma(\eta) d\tau \right)^p.$$

Now we integrate with respect to $d\sigma(\xi)$ and obtain:

$$M_s^r(u, t) \leq \left(\frac{C}{t^n} \right)^p \int_{\partial\Omega} \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^r d\sigma(\eta) d\tau \right)^p d\sigma(\xi),$$

which gives

$$M_s^r(u, t) \leq \frac{C}{t^n} \left(\int_{\partial\Omega} \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^r d\sigma(\eta) d\tau \right)^p d\sigma(\xi) \right)^{\frac{1}{p}}.$$

Now we use Minkowski's integral inequality with exponent $p = s/r$ and obtain

$$M_s^r(u, t) \leq \frac{C}{t^n} \int_{\frac{t}{2}}^{\frac{3t}{2}} \left(\int_{\partial\Omega} \left(\int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^r d\sigma(\eta) \right)^p d\sigma(\xi) \right)^{\frac{1}{p}} d\tau.$$

We set

$$\varphi_\tau(\xi) = \int_{B_t^{\partial\Omega}(\xi)} |\tilde{u}(\eta, \tau)|^r d\sigma(\eta), \quad (2.5)$$

and write the above estimate as

$$M_s^r(u, t) \leq \frac{C}{t^n} \int_{\frac{t}{2}}^{\frac{3t}{2}} \left(\int_{\partial\Omega} \varphi_\tau^p(\xi) d\sigma(\xi) \right)^{\frac{1}{p}} d\tau = \frac{C}{t^n} \int_{\frac{t}{2}}^{\frac{3t}{2}} \|\varphi_\tau\|_{L^p(\partial\Omega, d\sigma(\xi))} d\tau. \quad (2.6)$$

Next we want to estimate the $L^p(\partial\Omega, d\sigma)$ norm of φ_τ , where $t/2 \leq \tau \leq 3t/2$, to that end we define a function $\theta : \partial\Omega \times \partial\Omega \rightarrow \mathbb{R}$ by

$$\theta(\xi, \eta) = \begin{cases} 1, & |\xi - \eta| \leq t \\ 0, & |\xi - \eta| > t \end{cases},$$

clearly $\theta(\xi, \eta) = \theta(\eta, \xi)$ and

$$\varphi_\tau(\xi) = \int_{\partial\Omega} \theta(\xi, \eta) |\tilde{u}(\eta, \tau)|^r d\sigma(\eta).$$

We will use a duality argument: let us fix $\psi \in L^q(\partial\Omega, d\sigma(\xi))$, $\|\psi\|_q \leq 1$, where $1/p + 1/q = 1$. Then we have

$$\begin{aligned}
\left| \int_{\partial\Omega} \varphi_\tau(\xi) \psi(\xi) d\sigma(\xi) \right| &= \left| \int_{\partial\Omega} \int_{\partial\Omega} |\tilde{u}(\eta, \tau)|^r \theta(\xi, \eta) |\psi(\xi)| d\sigma(\eta) d\sigma(\xi) \right| \\
&\leq \int_{\partial\Omega} \int_{\partial\Omega} |\tilde{u}(\eta, \tau)|^r \theta^{\frac{1}{p}}(\xi, \eta) \theta^{\frac{1}{q}}(\xi, \eta) |\psi(\xi)| d\sigma(\xi) d\sigma(\eta) \\
&\leq AB,
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\int_{\partial\Omega} \int_{\partial\Omega} (\tilde{u}(\eta, \tau))^{pr} \theta(\xi, \eta) d\sigma(\xi) d\sigma(\eta) \right)^{\frac{1}{p}} \leq t^{\frac{n-1}{p}} \left(\int_{\partial\Omega} |\tilde{u}(\eta, \tau)|^s d\sigma(\eta) \right)^{r/s}, \\
B &= \left(\int_{\partial\Omega} \int_{\partial\Omega} |\psi(\xi)|^q \theta(\xi, \eta) d\sigma(\xi) d\sigma(\eta) \right)^{\frac{1}{q}} \leq t^{\frac{n-1}{q}} \|\psi\|_q \leq t^{\frac{n-1}{q}}.
\end{aligned}$$

Combining the above estimates we obtain

$$\left| \int_{\partial\Omega} \varphi_\tau(\xi) \psi(\xi) d\sigma(\xi) \right| \leq t^{n-1} M_s^r(u, \tau), \quad \|\psi\|_q \leq 1,$$

and, by duality, this gives $\|\varphi_\tau\|_{L^p(\partial\Omega, d\sigma(\xi))} \leq t^{n-1} M_s^r(u, \tau)$. Using (2.6) and remembering that $t \asymp \tau$ for $t/2 \leq \tau \leq 3t/2$ we finally obtain

$$M_s^r(u, t) \leq C \int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^r(u, \tau) \frac{d\tau}{\tau},$$

which means we proved (2.3) also in the case $0 < r \leq s$. Thus, again using $\tau \asymp t$, in both cases we have:

$$t^\beta M_s^r(u, t) \leq C \int_{\frac{t}{2}}^{\frac{3t}{2}} \tau^\beta M_s^r(u, \tau) \frac{d\tau}{\tau} \leq C \|u\|_{\mathcal{B}_\beta^{s,r}(\Omega)}^r, \quad 0 < t < r_2,$$

and consequently $\|u\|_{\mathcal{B}_\beta^{s,\infty}(\Omega)} \leq C \|u\|_{\mathcal{B}_\beta^{s,r}(\Omega)}$. □

In order to proceed from this special case of Theorem 1 to the full scope of Theorem 1 we need to investigate a class of quasi-nearly subharmonic functions. A key result in this direction is Theorem 2 below.

Let, for $K \geq 1$, $QNS_K(W)$ denote the class of nonnegative, locally bounded Borel measurable functions u on a domain $W \subset \mathbb{R}^n$ satisfying

$$u(x) \leq \frac{K}{|B(x, r)|} \int_{B(x, r)} u dV, \quad B(x, r) \subset W.$$

Functions in the class $QNS(W) = \bigcup_{K \geq 1} QNS_K(W)$ are called *quasi-nearly subharmonic* functions. We need the next result, which generalizes Lemma 1.

Theorem A [14, 15] *Let $0 < p < \infty$. If $u \in QNS(W)$, then $u^p \in QNS(W)$. More precisely, if $u \in QNS_K(W)$, then $u^p \in QNS_{K_1}(W)$, where K_1 depends only on K , n and p .*

Let

$$u^\times(\varphi(\xi, t)) = \sup_{\frac{3t}{4} \leq \tau \leq t} u(\varphi(\xi, \tau)), \quad (\xi, t) \in \partial\Omega \times (0, r_0),$$

u^\times is a function defined on $\Omega \cap U$.

Using Remark 1 and estimates (1.1) and (1.2) one easily proves that we have:

$$\sup_{\frac{3}{4}t \leq \tau \leq t} |\tilde{u}(\xi, \tau)| \leq \frac{K_1}{|Q(\xi, t)|} \int_{Q(\xi, t)} u dV, \quad u \in QNS_K(\Omega), \quad \xi \in \partial\Omega, \quad 0 < t \leq r_2,$$

where K_1 depends only on K, n and Lipschitz constants L, l of χ, φ . This means that for $u \in QNS_K(\Omega)$ we have:

$$u^\times(\varphi(\xi, t)) \leq \frac{K_1}{|Q(\xi, t)|} \int_{Q(\xi, t)} u dV, \quad \xi \in \partial\Omega, \quad 0 < t \leq r_2. \quad (2.7)$$

As already noted, this version of maximal operator was studied in [11, 12].

The space $L_\beta^{p,q}(\Omega \cap U)$ consists of all measurable functions $f : \Omega \cap U \rightarrow \mathbb{C}$ such that

$$\|f\|_{L_\beta^{p,q}} = \left(\int_0^{r_0} \left(\int_{\partial\Omega} |\tilde{f}(\xi, t)|^p d\sigma(\xi) \right)^{\frac{q}{p}} t^{\beta q - 1} \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

In other words, $\|f\|_{L_\beta^{p,q}} = \|\tilde{f}\|_{L_\beta^{p,q}}$.

The following theorem is a result on boundedness of $u \mapsto u^\times$ in the class of quasi-nearly subharmonic functions. It will be used in the proof of our main result, Theorem 1

Theorem 2. Let $0 < s, r \leq \infty$ and $\beta > -1$. A function $u \in QNS_K(\Omega \cap U)$ belongs to $L_\beta^{s,r}(\Omega \cap U)$ if and only if u^\times belongs to $L_\beta^{s,r}(V)$. Moreover we have

$$\|u^\times\|_{L_\beta^{s,r}(V)} \leq C \|u\|_{L_\beta^{s,r}(\Omega \cap U)},$$

where C depends on K and Ω but is independent of u .

Proof. Since u is locally bounded, we only have to prove the implication $u \in L_\beta^{s,r} \Rightarrow u^\times \in L_\beta^{s,r}$. Assume that $0 < s < r < \infty$. Since u^s is, by Theorem A, a QNS function we have, using (2.7)

$$(u^\times(\varphi(\xi, t)))^s \leq \frac{C}{|Q(\xi, t)|} \int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} u^s(\varphi(\eta, \tau)) d\sigma(\eta) d\tau. \quad (2.8)$$

Integration over $\xi \in \partial\Omega$ gives:

$$\int_{\partial\Omega} (u^\times(\varphi(\xi, t)))^s d\sigma(\xi) \leq \frac{C}{|Q(\xi, t)|} \int_{\partial\Omega} \int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} u^s(\varphi(\eta, \tau)) d\sigma(\eta) d\tau d\sigma(\xi).$$

Arguing as in the proof of Lemma 3 we obtain

$$M_s^s(u^\times, t) \leq \frac{C}{t^n} \int_{\frac{t}{2}}^{\frac{3t}{2}} \tau^{n-1} \int_{\partial\Omega} |u(\varphi(\eta, \tau))|^s d\sigma(\eta) d\tau \leq C \int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^s(u, \tau) \frac{d\tau}{\tau}.$$

Then we use Hölder's inequality with exponent $\frac{r}{s}$ and obtain

$$M_s(u^\times, t) \leq C \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^r(u, \tau) \frac{d\tau}{\tau} \right)^{\frac{1}{r}}.$$

If $r < s < \infty$, we have as in (2.8)

$$|u^\times(\varphi(\xi, t))|^r \leq \frac{C}{|Q(\xi, t)|} \int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} |u(\varphi(\eta, \tau))|^r d\sigma(\eta) d\tau,$$

which gives

$$M_s^r(u^\times, t) \leq \frac{C}{t^n} \left(\int_{\partial\Omega} \left(\int_{\frac{t}{2}}^{\frac{3t}{2}} \int_{B_t^{\partial\Omega}(\xi)} u^r(\varphi(\eta, \tau)) d\sigma(\eta) d\tau \right)^{\frac{s}{r}} d\sigma(\xi) \right)^{\frac{r}{s}}.$$

Arguing as in Lemma 3 we get

$$M_s^r(u^\times, t) \leq C \int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^r(u, \tau) \frac{d\tau}{\tau}, \quad 0 < t < r_2.$$

Multiplying by t^β and integrating over $0 < t < r_2$ gives

$$\begin{aligned} \|u^\times\|_{L_\beta^{s,r}}^r &= \int_0^{r_2} t^{\beta r} M_s^r(u^\times, t) \frac{dt}{t} \leq C \int_0^{r_2} \left(t^{\beta r} \int_{\frac{t}{2}}^{\frac{3t}{2}} M_s^r(u, \tau) \frac{d\tau}{\tau} \right) \frac{dt}{t} \\ &\leq C \int_0^{r_2} t^{\beta r} \int_0^{r_0} \chi_{[\frac{t}{2}, \frac{3t}{2}]}(\tau) M_s^r(u, \tau) \frac{d\tau}{\tau} \frac{dt}{t} \leq C \int_0^{r_0} \tau^{\beta r} M_s^r(u, \tau) \frac{d\tau}{\tau} \\ &= C \|u\|_{L_\beta^{s,r}}^r. \end{aligned}$$

□

Theorem 3. Let $0 < s < s_1 \leq \infty$, $0 < r \leq \infty$ and $\beta > -1$. If a function u belongs to $QNS_K(\Omega \cap U) \cap L_\beta^{s,r}$, then it belongs to $L_{\beta_1}^{s_1,r}(\Omega \cap U)$, where $\beta_1 = \beta + (n-1)(\frac{1}{s} - \frac{1}{s_1})$, and we have $\|u\|_{L_{\beta_1}^{s_1,r}} \leq C \|u\|_{L_\beta^{s,r}}$, where C is a constant independent of u .

Proof. Let $u \in QNS_K \cap L_\beta^{s,r}$. Then, by Theorem A, $u^s \in QNS_{K_1}$, and it easily follows that:

$$M_\infty(u, t) \leq \frac{C}{t^{\frac{n-1}{s}}} \sup_{\frac{t}{2} \leq \tau \leq \frac{3t}{2}} M_s(u, \tau), \quad \frac{3t}{2} < r_0.$$

Therefore, we obtain an estimate:

$$M_{s_1}^{s_1}(u, t) = \int_{\partial\Omega} u^{s_1-s}(\varphi(\xi, t)) u^s(\varphi(\xi, t)) d\sigma(\xi) \leq M_\infty^{s_1-s}(u, t) M_s^s(u, t).$$

Then

$$M_{s_1}(u, t) \leq \frac{C}{t^{\frac{n-1}{s_1} - \frac{s_1-s}{s_1}}} \sup_{\frac{t}{2} \leq \tau \leq \frac{3t}{2}} M_s(u, \tau) = \frac{C}{t^{(n-1)(\frac{1}{s} - \frac{1}{s_1})}} \sup_{\frac{t}{2} \leq \tau \leq \frac{3t}{2}} M_s(u, \tau). \quad (2.9)$$

Since $[\frac{t}{2}, \frac{3t}{2}] \subset \bigcup_{j=1}^4 \Delta_j$, where $\Delta_j = [(\frac{3}{4})^j \frac{t}{2}, (\frac{3}{4})^{j-1} \frac{3t}{2}]$ we have

$$\sup_{\frac{t}{2} \leq \tau \leq \frac{3t}{2}} M_s(u, \tau) \leq \sum_{j=1}^4 \sup_{\tau \in \Delta_j} M_s(u, \tau) \leq \sum_{j=1}^4 M_s \left(u^\times, \frac{3^{j-1}}{4^{j-1}} \frac{3t}{2} \right). \quad (2.10)$$

Therefore, using (2.9) and (2.10), we obtain

$$M_{s_1}(u, t) t^{\beta+(n-1)(\frac{1}{s} - \frac{1}{s_1})} \leq C \sum_{j=1}^4 t^\beta M_s \left(u^\times, \frac{3^{j-1}}{4^{j-1}} \frac{3t}{2} \right). \quad (2.11)$$

Now the result follows from the previous theorem. □

We finish this paper with a proof of Theorem 1.

Proof of Theorem 1: Let $u \in \mathcal{B}_\beta^{s,r}(\Omega)$. Then $|u|$ is subharmonic and therefore in $QNS_1(\Omega)$. Now $|u| \in L_\beta^{s,r}$, because of $u \in \mathcal{B}_\beta^{s,r}(\Omega)$. Hence, by Theorem 3, $|u| \in L_{\beta_1}^{s_1,r}$. Since u is harmonic, this means $u \in \mathcal{B}_{\beta_1}^{s_1,r}(\Omega)$. Lemma 3 gives us $u \in \mathcal{B}_{\beta_1}^{s_1,\infty}(\Omega)$ and hence $u \in \mathcal{B}_{\beta_1}^{s_1,r_1}(\Omega)$. Therefore $\mathcal{B}_\beta^{s,r}(\Omega) \subset \mathcal{B}_{\beta_1}^{s_1,r_1}(\Omega)$. The continuity of the embedding $\mathcal{B}_\beta^{s,r}(\Omega) \hookrightarrow \mathcal{B}_{\beta_1}^{s_1,r_1}(\Omega)$ follows from the estimates given in Theorem 3 and Lemma 3, or from the Closed Graph Theorem. \square

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