DE GRUYTER Open Math. 2019; 17:1239–1248

Open Mathematics

Research Article

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On the two-term exponential sums and character sums of polynomials

https://doi.org/10.1515/math-2019-0107
Received June 24, 2019; accepted September 27, 2019

Abstract: The main aim of this paper is to use the analytic methods and the properties of the classical Gauss sums to research the computational problem of one kind hybrid power mean containing the character sums of polynomials and two-term exponential sums modulo p, an odd prime, and acquire several accurate asymptotic formulas for them.

Keywords: two-term exponential sums; character sums of polynomials; hybrid power mean; analytic method; asymptotic formula

MSC: 11L03; 11L40

1 Introduction

Let $q \geq 3$ be an integer and χ be a Dirichlet character modulo q. For any positive integers N and M with $M > N$, and rational coefficient polynomial $f(x)$ of x with degree n, the character sums of polynomials $\bmod\ q$ is defined by

$$S(\chi, f; q) = \sum_{a=N+1}^{N+M} \chi(f(a)).$$

It is well known that the upper bound estimate of $S(\chi, f; q)$ is a particularly vital classical problem in analytic number theory. Any substantial progress in this area will certainly play a valuable role in promoting the development of analytic number theory. For this reason, a great number of scholars have researched the estimate problem of $S(\chi, f; q)$, and obtained a series of meaningful results. For instance, Pólya and Vinogradov's ground breaking work (see [1]: Theorem 8.21 and Theorem 13.15) proved that for any non-principal character χ mod q, one has the estimate

$$\sum_{a=N+1}^{N+M} \chi(a) \ll q^{\frac{1}{2}} \ln q,$$

where the symbol $A \ll B$ denotes $|A| < cB$ for some constant c.

Suppose that $q = p$ is an odd prime. A. Weil's [2] proved a particularly significant conclusion: Let χ be a q-th character mod p, and polynomial $f(x)$ is not a perfect q-th power mod p, then the estimate

$$\left| \sum_{x=N+1}^{N+M} \chi\left(f(x)\right) \right| \ll p^{\frac{1}{2}} \ln p \tag{1.1}$$

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can be acquired, where the estimate $p^{\frac{1}{2}}$ in (1.1) is the best one. Actually, Zhang Wenpeng and Yi Yuan [3] gave a series of polynomials $f(x) = (x-r)^m(x-s)^n$ such that

$$\left|\sum_{a=1}^{q} \chi\left((a-r)^m(a-s)^n\right)\right| = \sqrt{q},$$

where $(r-s, q) = 1$, m, n and χ satisfy several special conditions.

The minor term $\ln p$ in (1.1) is difficult to improve, and it cannot even be improved to $\ln^{\lambda} p$, where $0 < \lambda < 1$ is any fixed real number.

A lot of results associated with character sums of polynomials can be found in various analytic number theory books, such as [4, 5], and several papers about character sums of polynomials can be found in [6-10]. We are not going to list them one by one.

On the other hand, for any integers m and n, the two-term exponential sums $G(m, n, r, s; q)$ is defined as

$$G(m, n, r, s; q) = \sum_{a=0}^{q} e\left(\frac{ma^r + na^s}{q}\right),$$

where $r > s \geq 1$ are integers, and $e(y) = e^{2\pi i y}$.

It is necessary to research two-term exponential sums. In fact, if $r = p$ is an odd prime, they are closely related to Fourier analysis on finite fields. Because of this, a lot of researchers have discussed the various properties of $G(m, n, r, s; q)$, and obtained a great number of meaningful results, see [11-23]. For instance, Zhang Wenpeng and Han Di [24] researched the sixth power mean of the two-term exponential sums, and obtained an exact computational formula.

Zhang Han and Zhang Wenpeng [25] proved a significant and precise formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2 & \text{if } 3 \mid p-1, \end{cases}$$

where p indicates an odd prime with $(n, p) = 1$.

In this paper, we are going to consider the computational problem of the hybrid power mean containing character sums of polynomials and two-term exponential sums

$$H(r, s, t, \chi; p) = \sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} \chi\left(a^r + ma^s\right)\right|^2 \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^t + b}{p}\right)\right|^2. \tag{1.2}$$

Han Di [26] studied the asymptotic properties of the hybrid mean value involving the two-term exponential sums and polynomial character sums, and proved the following asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^k + na}{p}\right)\right|^2 \cdot \left|\sum_{a=1}^{p-1} \chi(ma + \overline{a})\right|^2 = \begin{cases} 2p^3 + O\left(|k|p^2\right) & \text{if } 2 \mid k, \\ 2p^3 + O\left(|k|p^{\frac{5}{2}}\right) & \text{if } 2 \nmid k, \end{cases}$$

where p is an odd prime, χ denotes any non-principal even Dirichlet character mod p, and $\overline{a}$ represents the multiplicative inverse of a mod p. That is, $a\overline{a} \equiv 1 \bmod p$.

If we take $k = -1$ in this theorem, one can deduce the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma + \overline{a}}{p}\right)\right|^2 \cdot \left|\sum_{b=1}^{p-1} \chi\left(mb + \overline{b}\right)\right|^2 = 2p^3 + O\left(p^2\right).$$

Du Xiaoying [27] researched a similar problem, and proved the following conclusion:
Let $p > 3$ be a prime with $(3, p-1) = 1$. Then for any non-principal even character χ mod p, one has the identity

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} \chi\left(ma^3 + a\right)\right|^2 \cdot \left|\sum_{b=1}^{p-1} e\left(\frac{mb^3 + b}{p}\right)\right|^2$$

$$= 2p\left(p^2 - p - 1\right) - p\left(2 + \left(\frac{3}{p}\right)\right)\sum_{u=1}^{p-1}\overline{\chi}(u)\sum_{a=1}^{p-1}\left(\frac{(a-1)(a^3-u^2)}{p}\right),$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol mod p.

According to this formula, Du Xiaoying [27] deduced the following asymptotic formula:

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2 = 2p^3 + O\left(p^2\right).$$

The main aim of this paper is to use the analytic methods and the properties of the classical Gauss sums to research the computational problem of (1.2) for special integers $r = 4$, $s = 1$ and $t = 3$ or 4. We will give several sharp asymptotic formulas for (1.2). That is, we will prove the following two main results:

Theorem 1. Let p be an odd prime with $p \equiv 1 \bmod 3$ and χ be any Dirichlet character mod p. If $\chi^3 \neq \chi_0$ and $\chi^4 \neq \chi_0$, then we acquire the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi^3\left(a^4+ma\right)\right|^2\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2 = 3p^3 + E(p),$$

where the error term $E(p)$ satisfies the estimate $\left|E(p)\right| \leq 18\cdot p^2$.

If $\chi^3 \neq \chi_0$ and $\chi^4 = \chi_0$, then we acquire the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi^3\left(a^4+ma\right)\right|^2\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2 = 2p^3 + E_1(p),$$

where the error term $E_1(p)$ satisfies the estimate $\left|E(p)\right| \leq 15\cdot p^2$.

Theorem 2. Let p be an odd prime with $p \equiv 1 \bmod 3$, χ be any Dirichlet character mod p. If $\chi^3 \neq \chi_0$ and $\chi^4 \neq \chi_0$, then we obtain the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi^3\left(a^4+ma\right)\right|^2\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^4+b}{p}\right)\right|^2 = 3p^3 + W(p),$$

where the error term $W(p)$ satisfies the estimate $\left|W(p)\right| \leq 27\cdot p^{\frac{5}{2}}$.

If $\chi^3 \neq \chi_0$ and $\chi^4 = \chi_0$, then we obtain the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi^3\left(a^4+ma\right)\right|^2\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^4+b}{p}\right)\right|^2 = 2p^3 + W_1(p),$$

where the error term $W_1(p)$ satisfies the estimate $\left|W_1(p)\right| \leq 23\cdot p^{\frac{5}{2}}$.

Some notes: Above all, if $3 \nmid (p-1)$ in Theorem 1, then for any non-principal character χ mod p, if $\chi^4 = \chi_0$, then we acquire the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right| = 1.$$

Therefore, in this case, the result is trivial. That is,

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|^2\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2 = p^2.$$

If $\chi^4 \neq \chi_0$, then we acquire the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right| = p.$$

In this case, we acquire the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|^2\cdot\left|\sum_{b=0}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2=p^3.$$

Secondly, if $3 \mid (p-1)$ and χ is not a third character mod p (that is, there is not any character χ_1 mod p such that $\chi=\chi_1^3$), then we obtain the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|=0.$$

Therefore, we did not discuss this special case.

Third, our methods can also be applied to the general hybrid power mean $H(3,1,k,\chi;p)$ for all integers $k \geq 3$. Actually, if $3 \mid k$, then the asymptotic formula for $H(3,1,k,\chi;p)$ is the same as in Theorem 1. If $3 \nmid k$, then the asymptotic formula for $H(3,1,k,\chi;p)$ is the same as in Theorem 2.

Finally, it is worthwhile to improve the error term in Theorem 2.

2 Some lemmas

In this part, firstly, we introduce several simple properties related to classical Gauss sums mod q, which is defined as

$$\tau(\chi)=\sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right),\ \text{ where } e(y)=e^{2\pi iy}.$$

If χ is a primitive character mod q, then one has the identities

$$\sum_{a=1}^{q}\chi(a)e\left(\frac{na}{q}\right)=\overline{\chi}(n)\tau(\chi)\ \text{and}\ \ |\tau(\chi)|=\sqrt{q}.$$

The other properties of $\tau(\chi)$ can also be found in a great number of analytic number theory text books, such as [1] or [4] and [5], here we will not repeat them.

Lemma 1. Let p be an odd prime with $p \equiv 1 \bmod 3$ and m be any integer with $(m,p)=1$. If χ is not a third character mod p, then we acquire

$$\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|^2=0;$$

If χ is a third character mod p and $\chi^4 \neq \chi_0$, then we acquire the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|^2=3p+\overline{\lambda}(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\lambda\left(b^4a^3-1\right)\overline{\lambda}(b-1)+\lambda(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\lambda}\left(b^4a^3-1\right)\lambda(b-1).$$

If χ is a third character mod p and $\chi^4=\chi_0$, then we acquire the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|^2=2p+1+\overline{\lambda}(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\lambda\left(b^4a^3-1\right)\overline{\lambda}(b-1)$$
$$+\lambda(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\lambda}\left(b^4a^3-1\right)\lambda(b-1),$$

where λ is a third-order character mod p. That is, $\lambda \neq \chi_0$ and $\lambda^3 = \chi_0$.

Proof. If χ is not a third character mod p, then there exists an integer r such that $r^3 \equiv 1 \bmod p$ and $\chi(r) \neq 1$. According to the properties of the reduced residue system mod p, we obtain

$$\sum_{a=1}^{p-1} \chi\left(a^4 + ma\right) = \sum_{a=1}^{p-1} \chi\left((ra)^4 + mra\right) = \chi(r) \sum_{a=1}^{p-1} \chi\left(r^3 a^4 + ma\right) = \chi(r) \sum_{a=1}^{p-1} \chi\left(a^4 + ma\right)$$

or

$$(1 - \chi(r)) \cdot \sum_{a=1}^{p-1} \chi\left(a^4 + ma\right) = 0. \tag{2.1}$$

Since $\chi(r) \neq 1$, applying with (2.1) we obtain the identity

$$\left|\sum_{a=1}^{p-1} \chi\left(a^4 + ma\right)\right|^2 = 0. \tag{2.2}$$

If χ is a third character mod$p$, for any integer m with $(m, p) = 1$, we obtain the identity

$$\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) = 1 + \sum_{a=1}^{p-1} \left(1 + \lambda(a) + \overline{\lambda}(a)\right) e\left(\frac{ma}{p}\right) = \lambda(m)\tau\left(\overline{\lambda}\right) + \overline{\lambda}(m)\tau(\lambda),$$

where λ is any third-order character mod p.

According to the properties of Gauss sums and reduced residue system, we obtain

$$\begin{aligned}
\left|\sum_{a=1}^{p-1} \chi\left(a^4 + ma\right)\right|^2 &= \frac{1}{p} \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi\left(a\overline{b}\right) \sum_{c=0}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{dc^3\left(ba^3 - 1\right) + md\,(b-1)}{p}\right) \\
&= \sum_{\substack{a=1 \\ ba^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \chi\left(a\overline{b}\right) \sum_{d=0}^{p-1} e\left(\frac{md\,(b-1)}{p}\right) - \sum_{\substack{a=1 \\ ba^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \chi\left(a\overline{b}\right) \\
&\quad + \frac{1}{p} \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi\left(a\overline{b}\right) \sum_{d=1}^{p-1} \left(\lambda\left(d(ba^3 - 1)\right)\right) \tau\left(\overline{\lambda}\right) + \overline{\lambda}\left(d(ba^3 - 1\right) \tau(\lambda)\right) e\left(\frac{md\,(b-1)}{p}\right) \\
&= p \sum_{\substack{a=1 \\ a^3 \equiv 1 \bmod p}}^{p-1} \chi(a) - \sum_{a=1}^{p-1} \chi^4(a) + \frac{\tau(\lambda)\tau\left(\overline{\lambda}\right)}{p} \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi\left(a\overline{b}\right) \lambda\left(ba^3 - 1\right) \overline{\lambda}\left(m(b-1)\right) \\
&\quad + \frac{\tau(\lambda)\tau\left(\overline{\lambda}\right)}{p} \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi\left(a\overline{b}\right) \overline{\lambda}\left(ba^3 - 1\right) \lambda\left(m(b-1)\right) \\
&= 3p + \overline{\lambda}(m) \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a) \lambda\left(b^4 a^3 - 1\right) \overline{\lambda}(b-1) + \lambda(m) \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a) \overline{\lambda}\left(b^4 a^3 - 1\right) \lambda(b-1).
\end{aligned} \tag{2.3}$$

If $\chi^4 \neq \chi_0$, we can use the identity $\tau(\lambda)\tau\left(\overline{\lambda}\right) = p$.

If $\chi^4 = \chi_0$, then we get

$$\left|\sum_{a=1}^{p-1} \chi\left(a^4 + ma\right)\right|^2 = 2p + 1 + \overline{\lambda}(m) \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a) \lambda\left(b^4 a^3 - 1\right) \overline{\lambda}(b-1) + \lambda(m) \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a) \overline{\lambda}\left(b^4 a^3 - 1\right) \lambda(b-1). \tag{2.4}$$

Now, Lemma 1 follows from (2.2), (2.3) and (2.4).

Lemma 2. Let p be an odd prime, then we obtain the identity

$$\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = \begin{cases} p(p-2) & \text{if } 3 \mid (p-1), \\ p^2 & \text{if } 3 \nmid (p-1). \end{cases}$$

Proof. For any positive integer $q > 1$, applying with the trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n \end{cases}$$

and the properties of the reduced residue system mod p, we get

$$\begin{aligned}\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 &= \sum_{m=0}^{p-1}\left(1+\sum_{a=1}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 \\ &= p + \sum_{a=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) + \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{mb^3\left(a^3-1\right)+b(a-1)}{p}\right) \\ &= p + p\sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{b(a-1)}{p}\right) = \begin{cases} p(p-2) & \text{if } 3 \mid (p-1), \\ p^2 & \text{if } 3 \nmid (p-1). \end{cases}\end{aligned}$$

This proves Lemma 2.

Lemma 3. Let p be an odd prime, then we obtain the identity

$$\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 = \begin{cases} p(p-3) & \text{if } 4 \mid (p-1), \\ p(p-1) & \text{if } 4 \nmid (p-1). \end{cases}$$

Proof. It is not difficult to prove above formula by the same method of proving Lemma 2, so we omit details of the proof.

Lemma 4. Let p be an odd prime with $p \equiv 1 \bmod 3$, then for any third-order character λ mod p, we acquire

$$\sum_{m=1}^{p-1}\lambda(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = -\tau^2\left(\overline{\lambda}\right).$$

Proof. Applying with the properties of Gauss sums, we acquire

$$\begin{aligned}\sum_{m=1}^{p-1}\lambda(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 &= \sum_{m=1}^{p-1}\lambda(m) + 2\sum_{a=1}^{p-1}\sum_{m=1}^{p-1}\lambda(m)e\left(\frac{ma^3+a}{p}\right) \\ &\quad + \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\lambda(m)e\left(\frac{mb^3\left(a^3-1\right)+b(a-1)}{p}\right) \\ &= 2\tau(\lambda)\sum_{a=1}^{p-1} e\left(\frac{a}{p}\right) + \tau(\lambda)\sum_{a=1}^{p-1}\overline{\lambda}\left(a^3-1\right)\sum_{b=1}^{p-1} e\left(\frac{b(a-1)}{p}\right) \\ &= -2\tau(\lambda) - \tau(\lambda)\sum_{a=1}^{p-1}\overline{\lambda}\left(a^3-1\right) = -\tau(\lambda)\sum_{a=0}^{p}\overline{\lambda}\left(a^3-1\right). \end{aligned} \tag{2.5}$$

It is obvious that

$$\begin{aligned}\sum_{a=0}^{p}\lambda\left(a^3-1\right) &= 2 + \sum_{a=1}^{p-1}\left(1+\lambda(a)+\overline{\lambda}(a)\right)\lambda(a-1) \\ &= \sum_{a=0}^{p-1}\lambda(a-1) + \sum_{a=0}^{p-1}\overline{\lambda}(a-1) + \sum_{a=1}^{p-1}\lambda(a)\lambda(a-1)\end{aligned}$$

$$= \frac{1}{\tau\left(\overline{\lambda}\right)} \sum_{b=1}^{p-1} \overline{\lambda}(b) \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{b(a-1)}{p}\right) = \frac{\tau(\lambda)}{\tau\left(\overline{\lambda}\right)} \sum_{b=1}^{p-1} \overline{\lambda}^2(b) e\left(\frac{-b}{p}\right)$$
$$= \frac{\tau^2(\lambda)}{\tau\left(\overline{\lambda}\right)} = \frac{\tau^3(\lambda)}{\tau(\lambda)\tau\left(\overline{\lambda}\right)} = \frac{\tau^3(\lambda)}{p}. \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\sum_{m=1}^{p-1} \lambda(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) \right|^2 = -\tau^2\left(\overline{\lambda}\right).$$

This proves Lemma 4.

Lemma 5. Let p be an odd prime with $p \equiv 1 \bmod 3$, then for any third-order character $\lambda \bmod p$, we can acquire the estimate

$$\left| \sum_{m=1}^{p-1} \lambda(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 \right| \leq p^{\frac{3}{2}} + p.$$

Proof. Note that $\tau(\lambda)\tau\left(\overline{\lambda}\right) = p$, and $\lambda^4 = \lambda$, from the method of proving Lemma 4, we obain

$$\sum_{m=1}^{p-1} \lambda(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right) \right|^2 = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} \lambda(m) e\left(\frac{mb^4\left(a^4-1\right)+b(a-1)}{p}\right)$$
$$+ \sum_{m=1}^{p-1} \lambda(m) + \sum_{a=1}^{p-1} \sum_{m=1}^{p-1} \lambda(m) e\left(\frac{ma^4+a}{p}\right) + \sum_{a=1}^{p-1} \sum_{m=1}^{p-1} \lambda(m) e\left(\frac{-ma^4-a}{p}\right)$$
$$= 2\tau(\lambda)\tau\left(\overline{\lambda}\right) + \tau(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}\left(a^4-1\right) \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{b(a-1)}{p}\right)$$
$$= 2p + p \sum_{a=1}^{p-1} \overline{\lambda}\left(a^4-1\right) \lambda(a-1) = 2p + p \sum_{a=2}^{p-1} \overline{\lambda}\left(a^3+a^2+a+1\right)$$
$$= p\left(1-\lambda(2)\right) + p \sum_{a=0}^{p-1} \overline{\lambda}\left(a^3+a^2+a+1\right). \tag{2.7}$$

On the other hand, for any integer b with $(b,p)=1$, note that the identity

$$\sum_{a=0}^{p-1} e\left(\frac{ba^2}{p}\right) = 1 + \sum_{a=1}^{p-1} \left(1+\chi_2(a)\right) e\left(\frac{ba}{p}\right) = \chi_2(b)\tau(\chi_2),$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre symbol mod p. We also deduce the identity

$$\sum_{a=0}^{p-1} \overline{\lambda}\left(a^3+a^2+a+1\right) = \sum_{a=0}^{p-1} \overline{\lambda}\left((a+1)^3 - 2(a+1)^2 + 2(a+1)\right)$$
$$= \sum_{a=1}^{p-1} \overline{\lambda}\left(2\overline{a}^2 - 2\overline{a} + 1\right) = \sum_{a=0}^{p-1} \overline{\lambda}\left(2a^2-2a+1\right) - 1$$
$$= \lambda(2) \sum_{a=0}^{p-1} \overline{\lambda}\left((2a-1)^2+1\right) - 1 = \lambda(2) \sum_{a=0}^{p-1} \overline{\lambda}\left(a^2+1\right) - 1$$
$$= \frac{\lambda(2)}{\tau(\lambda)} \sum_{b=1}^{p-1} \lambda(b) \sum_{a=0}^{p-1} e\left(\frac{ba^2+b}{p}\right) - 1$$

$$= \frac{\lambda(2)\tau(\chi_2)}{\tau(\lambda)} \sum_{b=1}^{p-1} \lambda(b)\chi_2(b) e\left(\frac{b}{p}\right) - 1 = \frac{\lambda(2)\tau(\chi_2)\tau(\lambda\chi_2)}{\tau(\lambda)} - 1. \tag{2.8}$$

Now Lemma 5 follows from (2.7), (2.8) and the identity $|\tau(\chi)| = \sqrt{p}$.

Lemma 6. Let p be an odd prime with $p \equiv 1 \bmod 3$, then for any non-principal third character χ mod p, we can obtain the estimate

$$\left|\sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \lambda\left(b^4a^3 - 1\right) \overline{\lambda}(b-1)\right| \le 6p,$$

where λ is any third-order character mod p.

Proof. Since $3 \mid (p-1)$, there exists an integer $1 < r < p-1$ such that $r^3 \equiv 1 \bmod p$. For any integer m with $(m, p) = 1$, note that $\lambda(m) + \lambda(mr) + \lambda\left(r^2m\right) = \lambda(m)\left(1 + \lambda(r) + \lambda(r^2)\right) = 0$, and combining with Lemma 1, we know that for any third character χ mod p with $\chi \neq \chi_0$, we obtain the identity

$$\begin{aligned}
&\left|\sum_{a=1}^{p-1} \chi\left(a^4 + ma\right)\right|^2 + \left|\sum_{a=1}^{p-1} \chi\left(a^4 + mra\right)\right|^2 + \left|\sum_{a=1}^{p-1} \chi\left(a^4 + mr^2a\right)\right|^2 \\
&= 9p + \left(\overline{\lambda}(m) + \overline{\lambda}(rm) + \overline{\lambda}\left(r^2m\right)\right) \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \lambda\left(b^4a^3 - 1\right) \overline{\lambda}(b-1) \\
&\quad + \left(\lambda(m) + \lambda(rm) + \lambda\left(r^2m\right)\right) \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \overline{\lambda}\left(b^4a^3 - 1\right) \lambda(b-1) \\
&= 9p.
\end{aligned} \tag{2.9}$$

In summary, for $i = 0, 1, 2$, according to (2.9) we acquire the estimate

$$\left|\sum_{a=1}^{p-1} \chi\left(a^4 + mr^ia\right)\right|^2 \le 9p. \tag{2.10}$$

On the other hand, from Lemma 1 we also obtain

$$\begin{aligned}
&\left|\sum_{a=1}^{p-1} \chi\left(a^4 + ma\right)\right|^4 + \left|\sum_{a=1}^{p-1} \chi\left(a^4 + mra\right)\right|^4 + \left|\sum_{a=1}^{p-1} \chi\left(a^4 + mr^2a\right)\right|^4 \\
&= 27p^2 + 6p\left(\overline{\lambda}(m) + \overline{\lambda}(rm) + \overline{\lambda}\left(r^2m\right)\right) \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \lambda\left(b^4a^3 - 1\right) \overline{\lambda}(b-1) \\
&\quad +6p\left(\lambda(m) + \lambda(rm) + \lambda\left(r^2m\right)\right) \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \overline{\lambda}\left(b^4a^3 - 1\right) \lambda(b-1) \\
&\quad + \left(\lambda(m) + \lambda(rm) + \lambda\left(r^2m\right)\right) \left(\sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \lambda\left(b^4a^3 - 1\right) \overline{\lambda}(b-1)\right)^2 \\
&\quad + \left(\overline{\lambda}(m) + \overline{\lambda}(rm) + \overline{\lambda}\left(r^2m\right)\right) \left(\sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \overline{\lambda}\left(b^4a^3 - 1\right) \lambda(b-1)\right)^2 \\
&\quad +6 \cdot \left|\sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \lambda\left(b^4a^3 - 1\right) \overline{\lambda}(b-1)\right|^2 \\
&= 27p^2 + 6 \cdot \left|\sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \lambda\left(b^4a^3 - 1\right) \overline{\lambda}(b-1)\right|^2.
\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) we may immediately deduce the estimate

$$\left|\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\lambda\left(b^4a^3-1\right)\overline{\lambda}(b-1)\right| \le 6p.$$

This proves Lemma 6.

3 Proofs of the theorems

In this section, we are going to prove our main theorem. Let p be an odd prime with $p \equiv 1 \bmod 3$. For any non-principal third character χ mod p with $\chi^4 \neq \chi_0$, from Lemma 1, Lemma 2 and Lemma 4 we acquire

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|^2 \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1}\overline{\lambda}(m)\left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^2 \left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\lambda\left(b^4a^3-1\right)\overline{\lambda}(b-1)\right) \\
&+\sum_{m=1}^{p-1}\lambda(m)\left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^2 \left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\lambda}\left(b^4a^3-1\right)\lambda(b-1)\right) \\
&+3p\sum_{m=1}^{p-1}\left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^2 \\
&= 3p^2(p-2)-\tau^2\left(\overline{\lambda}\right)\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\lambda\left(b^4a^3-1\right)\overline{\lambda}(b-1)\right) \\
&-\tau^2(\lambda)\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\lambda}\left(b^4a^3-1\right)\lambda(b-1)\right). \qquad (3.1)
\end{aligned}$$

Note that $|\tau(\lambda)|^2 = p$. From (3.1) and Lemma 6 we may instantly deduce the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|^2 \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^2 = 3p^3+E(p), \qquad (3.2)$$

where the error term $E(p)$ satisfies the estimate $\left|E(p)\right| \le 18p^2$.

If $\chi \neq \chi_0$ is a third character mod p with $\chi^4 = \chi_0$, then from Lemma 1 and the method of proving (3.2), we can also deduce the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^4+ma\right)\right|^2 \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^2 = 2p^3+E_1(p), \qquad (3.3)$$

where the error term $E_1(p)$ satisfies the estimate $\left|E_1(p)\right| \le 15p^2$.

It is obviously that Theorem 1 follows from (3.2) and (3.3).

Applying Lemma 1, Lemma 3, Lemma 5, Lemma 6 and the method of proving Theorem 1 we can easily deduce the conclusion of Theorem 2. That is,

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi^3\left(a^4+ma\right)\right|^2 \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^4+b}{p}\right)\right|^2 = \begin{cases} 3p^3+W(p) & \text{if } \chi^4 \neq \chi_0, \\ 2p^3+W_1(p) & \text{if } \chi_4 = \chi_0, \end{cases}$$

where the error terms $W(p)$ and $W_1(p)$ satisfy the estimates $\left|W(p)\right| \le 27 \cdot p^{\frac{3}{2}}$ and $\left|W_1(p)\right| \le 23 \cdot p^{\frac{3}{2}}$.

This completes the proofs of our all results.

4 Conclusion

The main results of this paper are two theorems. We obtained two sharp asymptotic formulas for the hybrid power mean involving character sums of polynomials and two-term exponential sums. In addition to this, we also acquired the upper bound estimation of the error terms. These results profoundly reveal the law of the value distribution of the character sums of polynomials and two-term exponential sums, and it can also be used for reference in the research of similar problems.

Acknowledgment: The authors would like to thank the referees for their very helpful and detailed comments, which have significantly improved the presentation of this paper. This work is supported by the N. S. F. (11771351) of P. R. China.

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