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A characterization of translational hulls of a strongly right type B semigroup

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Abstract: The aim of this paper is to study the translational hull of a strongly right type B semigroup. Our main result is to prove that the translational hull of a strongly right type B semigroup is itself a strongly right type B semigroup. As an application, we give a proof of a problem posted by Petrich on translational hulls of inverse semigroups in Petrich (Inverse Semigroups, Wiley, New York, 1984) to the cases of some strongly right type B semigroups.

Keywords: strongly rpp, right type B semigroups, translational hulls

MSC: 20M10, 06F05

1 Introduction

Recall from [1] that if S is a semigroup and $a, b \in S$, then $a\mathcal{L}^{\star}b$ $[a\mathcal{R}^{\star}b]$ if there is a semigroup T, containing S as a subsemigroup, such that $a\mathcal{L}b$ $[a\mathcal{R}b]$ in T. A semigroup S is called *rpp [lpp]* if each $\mathcal{L}^{\star}$ $[\mathcal{R}^{\star}]$class contains at least one idempotent. An rpp [lpp] semigroup in which the idempotents commute is *right [left] adequate*. A semigroup is said to be *adequate* if it is both right and left adequate. Recently, Guo, Shum and Guo have considered the so called *strongly rpp* semigroups (see, [2]). In fact, a strongly rpp semigroup S is an rpp semigroup in which for every $a \in S$, there exists a unique idempotent a^{00} which is $\mathcal{L}^{\star}$– related to a such that $a^{00}a = a$. As usual, we denote by $a^{\star}$ $[a^{+}]$ an idempotent $\mathcal{L}^{\star}$– $[\mathcal{R}^{\star}$–$]$ related to a; $E(S)$ denotes the set of idempotents of S. A right [left] adequate semigroup S is *right [left] ample*, if $ea = a(ea)^{\star}$ $[ae = (ae)^{+}a]$ for all $a \in S, e \in E(S)$. An *ample semigroup* is the one which is both right ample and left ample. Following [1], a right adequate semigroup S is *right type B*, if it satisfies the following conditions (B1) and (B2):

(B1) for all $e, f \in E(S^1)$, $a \in S$, $(efa)^{\star} = (ea)^{\star}(fa)^{\star}$;
(B2) if for all $a \in S, e \in E(S), e \leq a^{\star}$, then there is an element $f \in E(S^1)$ such that $e = (fa)^{\star}$, where " $\leq$ " is a natural partial order on $E(S)$ (i.e., $(\forall g, h \in E(S))\ g \leq h \Leftrightarrow g = gh = hg$).

A *left type B semigroup* is defined dually. A *type B semigroup* is the one which is both right type B and left type B (see, [1, 3]). In particular, we call a right type B which is strongly rpp a *strongly right type B semigroup*. Since it is known that for a right adequate semigroup S, each $\mathcal{L}^{\star}$–class of S contains exactly one idempotent (i.e., S is $\mathcal{L}^{\star}$*–unipotent*), we have $a^{00} = a^{\star}$ for all $a \in S$ if S is strongly right type B.

Following [4], a mapping λ from a semigroup S to itself is a *left translation* of S if $\lambda(ab) = (\lambda a)b$ for all $a, b \in S$. Similarly, a mapping ρ which maps a semigroup S to itself is called a *right translation* of S if $(ab)\rho =$

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$a(b\rho)$ for all $a, b \in S$. a left translation λ and a right translation ρ of a semigroup S are called *linked* if $a(\lambda b) = (a\rho)b$ for all $a, b \in S$, in which case the pair (λ, ρ) is called a *bitranslation* of S. We denote by $\Lambda(S)$ [resp., $I(S)$] the set of all left [resp., right] translations of S, and denote by $\Omega(S)$ the set of all bitranslations of S. It is easy to check that $\Omega(S)$ forms a subsemigroup of $\Lambda(S)\times I(S)$. We call $\Omega(S)$ *the translational hull* of S. The role played by the translational hull of a semigroup in the general theory of semigroups (and, especially, in the theory of semigroup extensions) is well known (see, [4–7]). There are a number of detailed researches on translational hulls of various classes of semigroups (see, [8–13]). It is well known that the translational hull of an inverse semigroup is itself an inverse semigroup [4]. In 1985, Fountain and Lawson [8] generalized the above result to the case of adequate semigroups. Recently, Guo and Shum [9] investigated the translational hull of an ample semigroup on the basis of [8]. Generally, the translational hull of a right (left) type B semigroup [resp., a right (left) ample semigroup] is not a semigroup of the same type.

In this paper, we shall prove that the translational hull of a strongly right type B semigroup is still a semigroup of the same type. As an application, we give a positive answer to a problem posted by Petrich (i.e., if a semigroup S is embeddable into an inverse semigroup, is $\Omega(S)$ also embeddable into an inverse semigroup (see, [4, V.3.11 Problems, p. 226])) to the case of a strongly right type B semigroup satisfying some condition.

2 Preliminaries

Throughout this paper, we shall use the notions and notations of [1, 4, 8, 14].

Lemma 2.1. [1] *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $a\mathcal{L}^*b$;
(2) *for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.*

Corollary 2.2. [1] *Let S be a semigroup and $e^2 = e, a \in S$. Then the following statements are true:*

(1) *$a\mathcal{L}^*e$ if and only if $ae = a$ and for all $x, y \in S^1$, $ax = ay$ implies $ex = ey$;*
(2) *$\mathcal{L}^*$ is a right congruence on S.*

Evidently, in an arbitrary semigroup, we have $\mathcal{L} \subseteq \mathcal{L}^*$. For $a, b \in Reg(S)$, we get $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$, where $Reg(S)$ denotes the set of regular elements of S.

Lemma 2.3. [1] *Let S be a right adequate semigroup and $a, b \in S$. Define $\mu_L = \{(a, b) \in S \times S \mid (ea)^* = (eb)^*$ for all $e \in E(S^1)\}$. Then μ_L is the largest idempotent-separating congruence on S contained in $\mathcal{L}^*$.*

As in [1], a right adequate semigroup S is called *right fundamental* if $\mu_L = 1_S$, where 1_S is the identity relation on S.

Lemma 2.4. [1] *Let S be a right adequate semigroup. Then the following statements are true:*

(1) *if S is a right ample semigroup, then S can be embedded into an inverse semigroup;*
(2) *if S is a right type B semigroup which is right fundamental, then S is right ample.*

Lemma 2.5. [15] *Let S be a right type B semigroup. Define a relation on S as follows: $(a, b) \in \sigma \Longleftrightarrow ea = eb$, for some $e \in E(S)$. Then σ is the least left cancellative monoid congruence on S.*

In this paper, we call a right type B semigroup S *proper* if $\sigma \bigcap \mathcal{L}^* = 1_S$. Naturally, one would ask whether right type B semigroups are special strongly rpp semigroups? Note that, not all right type B semigroups are strongly right type B. We now give an example of a right type B semigroup which is not strongly right type B.

Example 2.1. *(Due to Fountain [1]) Let* $\mathbb{N}$ *be the set of all non-negative integers and put* $I = \mathbb{N} \times \mathbb{N}$, $S = \mathbb{N} \cup I$. *Define a multiplication "* $\circ$ *" on S as follows:*

$$\begin{aligned} &m \circ n = m + n \\ &m \circ (h, k) = (m + h, k) \\ &(h, k) \circ m = (h, k + m) \\ &(h, k) \circ (m, n) = (h, k + m + n) \end{aligned}$$

It is readily verified that "$\circ$*" is associative, that the set of idempotents of S is* $\{0, (0, 0)\}$. *It is not difficult to check that the* $\mathcal{L}^*$*–classes of S are* $\mathbb{N}$ *and* I. *In fact, Fountain [2] proved that S is a right type B semigroup. However, S itself is not strongly rpp, for if otherwise, then*

$$(m, n)\mathcal{L}^*(0, 0) \text{ and } (0, 0) \circ (m, n) = (m, n).$$

But since $(0, 0) \circ (m, n) = (0, m+n)$, *we have* $(0, 0) \circ (m, n) \neq (m, n)$. *This contradiction shows S is not a strongly right type B semigroup.*

Next, we give an example of a proper strongly right type B semigroup.

Example 2.2. *Let* $\mathbb{N}$ *be the set of all non-negative integers and* $S = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \geq n\}$. *Define a multiplication "* $\bullet$ *" on S by*

$$(m, n) \bullet (p, q) = (m - n + t, q - p + t),$$

where $t = max\{n, p\}$. *Then, it is readily verified that* $(S, \bullet)$ *is a semigroup and* $E(S) = \{(m, m) \in \mathbb{N} \times \mathbb{N}\}$. *In addition, it is easy to check that S is right type B. On the other hand, since for all* $(m, n) \in S$, *there exists a unique idempotent* $(n, n) \in S$ *such that*

$$(m, n)\mathcal{L}^*(n, n) \text{ and } (n, n) \bullet (m, n) = (m, n).$$

We have that S is a strongly rpp semigroup with $(m, n)^{00} = (n, n)$.

Now, we show that S is also proper. For this purpose, let $(m, n), (p, q) \in S$ *be such that* $(m, n)[\mathcal{L}^* \cap \sigma](p, q)$. *Then* $m \geq n, p \geq q$, $(m, n)\mathcal{L}^*(p, q)$ *and* $(m, n)\sigma(p, q)$. *Hence,* $n = q$, *and there exists* $(k, k) \in E(S)$ *such that* $(k, k) \bullet (m, n) = (k, k) \bullet (p, q)$. *That is,* $(k, k) \bullet (m, n) = (k, k) \bullet (p, n)$. *Hence,* $(t,\ n - m + t) = (s,\ n - p + s)$, *where* $t = max\{k, m\}$ *and* $s = max\{k, p\}$. *Thus* $t = s$ *and* $n - m + t = n - p + s$, *this gives* $m = p$. *Therefore,* $(m, n) = (p, q)$. *That is, S is a proper strongly right type B semigroup.*

Corollary 2.6. *Let S be a proper right type B semigroup. Then S is right ample.*

Proof. Let $a \in S, e \in E(S)$. Then $a\mathcal{L}^*a^*$ and $ea\mathcal{L}^*(ea)^*$. Hence $a(ea)^*\mathcal{L}^*a^*(ea)^*$ and $ea = eaa^*\mathcal{L}^*(ea)^*a^*$ since $\mathcal{L}^*$ is a right congruence on S. Note that $E(S)$ is a semilattice. We have $a(ea)^*\mathcal{L}^*a^*(ea)^* = (ea)^*a^*\mathcal{L}^*ea$. On the other hand, it is easy to see that $(1, e) \in \sigma$ and $(1, (ea)^*) \in \sigma$ from Lemma 2.5. Hence $(a, ea) \in \sigma$ and $(a, a(ea)^*) \in \sigma$ since σ is a congruence on S from Lemma 2.5. Therefore, $ea[\mathcal{L}^* \bigcap \sigma]a(ea)^*$. But S is proper, we have $ea = a(ea)^*$. This gives that S is right ample. □

3 The translational hull of a strongly right type B semigroup

In this section, we first characterize the relation $\mathcal{L}^*$ on the translational hull of a strongly right type B semigroup and then we obtain the proof of our main result (i.e., if S is strongly right type B, then so is $\Omega(S)$).

To start with our study, we first define two mappings λ^{00} and ρ^{00} on a strongly rpp semigroup S as follows:

$$\lambda^{00}a = (\lambda a^{00})^{00}a, \quad a\rho^{00} = a(\lambda a^{00})^{00},$$

where $a \in S, (\lambda, \rho) \in \Omega(S)$. Obviously, λ^{00} and ρ^{00} map S into itself.

Lemma 3.1. *Let S be an rpp semigroup, $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$. Then the following statements are true:*

(1) $\rho_1 = \rho_2 \Longleftrightarrow (\forall e \in E(S))\ e\rho_1 = e\rho_2$;
(2) *if S is strongly rpp, then* $\lambda_1 = \lambda_2 \Longleftrightarrow (\forall e \in E(S))\ \lambda_1 e = \lambda_2 e$;
(3) *if $E(S)$ is a semilattice, then* $(\lambda_1, \rho_1) = (\lambda_2, \rho_2) \Longleftrightarrow \lambda_1 = \lambda_2$;
(4) *if S is strongly right type B, then*
$(\lambda_1, \rho_1) = (\lambda_2, \rho_2) \Longleftrightarrow \rho_1 = \rho_2 \Longleftrightarrow \lambda_1 = \lambda_2$;

Proof. (1) and (2) are trivial.

(3) Let $\lambda_1 = \lambda_2$. Then $\lambda_1 f = \lambda_2 f$ for all $f \in E(S)$. Hence, for all $e \in E(S)$, we have

$$\begin{aligned} \lambda_1 f = \lambda_2 f &\Rightarrow e(\lambda_1 f) = e(\lambda_2 f) \\ &\Rightarrow (e\rho_1)f = (e\rho_2)f \\ &\Rightarrow [(e\rho_1)f]^* = [(e\rho_2)f]^* \\ &\Rightarrow [(e\rho_1)^* f]^* = [(e\rho_2)^* f]^* \\ &\Rightarrow (e\rho_1)^* f = (e\rho_2)^* f \end{aligned}$$

since $E(S)$ is a semilattice. Choose an idempotent $(e\rho_1)^*$ of S to replace the element f of the above formula. We have $(e\rho_1)^* = (e\rho_2)^*(e\rho_1)^*$. Similarly, we can get $(e\rho_2)^* = (e\rho_1)^*(e\rho_2)^*$. Hence, $(e\rho_1)^* = (e\rho_2)^*$ since $E(S)$ is a semilattice. Thus, for all $e \in E(S)$, we have

$$\begin{aligned} e\rho_1 &= (e\rho_1)(e\rho_1)^* = e(\lambda_1(e\rho_1)^*) = e(\lambda_2(e\rho_1)^*) \\ &= (e\rho_2)(e\rho_1)^* = (e\rho_2)(e\rho_2)^* = e\rho_2. \end{aligned}$$

Therefore, by (1), $\rho_1 = \rho_2$. That is, $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$.

The converse is clear.

(4) By (3), we only need to prove that $\rho_1 = \rho_2$ implies that $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$. To see it, let $\rho_1 = \rho_2$. Then, by(1), for all $e \in E(S)$, we have

$$\lambda_1 e = (\lambda_1 e)^{00}\lambda_1 e = ((\lambda_1 e)^{00}\rho_1)e = ((\lambda_1 e)^{00}\rho_2)e = (\lambda_1 e)^{00}\lambda_2 e.$$

Similarly, $\lambda_2 e = (\lambda_2 e)^{00}\lambda_1 e$. Hence, $\lambda_1 e \mathcal{L} \lambda_2 e$, and so $\lambda_1 e \mathcal{L}^* \lambda_2 e$. Note that S is strongly right type B. We have $(\lambda_1 e)^{00} = (\lambda_2 e)^{00}$. Hence, $\lambda_1 e = \lambda_2 e$. By (2), $\lambda_1 = \lambda_2$. Therefore, $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$, as required. □

Proposition 3.2. *Let S be a strongly right type B semigroup, and $(\lambda, \rho) \in \Omega(S)$. Then the following statements are true:*

(1) $e\rho^{00} = \lambda^{00}e = (\lambda e)^{00} \in E(S)$, *for all* $e \in E(S)$;
(2) $(\lambda^{00}, \rho^{00}) \in E(\Omega(S))$;
(3) $(\lambda, \rho)\mathcal{L}^*(\lambda^{00}, \rho^{00})$;
(4) $e\rho^{00}\rho \mathcal{L} e\rho$, *for all* $e \in E(S)$;
(5) $(\lambda^{00}, \rho^{00})(\lambda, \rho) = (\lambda, \rho)$;
(6) $E(\Omega(S)) = \{(\lambda, \rho) \in \Omega(S) \mid \lambda E(S) \bigcup E(S)\rho \subseteq E(S)\}$.

Proof. (1) Let $e \in E(S)$. Then by the Definitions of λ^{00} and ρ^{00}, we have $e\rho^{00} = e(\lambda e)^{00} = (\lambda e)^{00}e = \lambda^{00}e$ since $E(S)$ is a semilattice. And $(\lambda^{00}e)^2 = (\lambda^{00}e)(\lambda^{00}e) = (\lambda e)^{00}e(\lambda e)^{00}e = (\lambda e)^{00}ee = (\lambda e)^{00}e = \lambda^{00}e \in E(S)$. Note that $\mathcal{L}^*$ is a right congruence. We have $\lambda^{00}e = (\lambda e)^{00}e\mathcal{L}^*(\lambda e)e = \lambda e\mathcal{L}^*(\lambda e)^{00}$. Again since S is $\mathcal{L}^*$–unipotent, we get that $\lambda^{00}e = (\lambda e)^{00}$. That is, $e\rho^{00} = \lambda^{00}e = (\lambda e)^{00} \in E(S)$.

(2) We first prove that $(\lambda^{00}, \rho^{00}) \in \Omega(S)$. To see it, let $a, b \in S$. Then by the Definitions of λ^{00} and ρ^{00} and (1), we have $(ab)\rho^{00} = (ab)(\lambda(ab)^{00})^{00} = (ab)(\lambda(ab)^{00}b^{00})^{00} = (ab)(\lambda b^{00}(ab)^{00})^{00} = ab(\lambda b^{00})^{00}(ab)^{00} = (ab)(ab)^{00}(\lambda b^{00})^{00} = ab(\lambda b^{00})^{00} = a(b\rho^{00})$. Hence, ρ^{00} is a right translation of S. Similarly, we can prove that λ^{00} is a left translation of S.

On the other hand, by (1), we have

$$\begin{aligned} a(\lambda^{00}b) &= a(\lambda^{00}b^{00})b = aa^{00}(\lambda^{00}b^{00})b \\ &= aa^{00}(b^{00}\rho^{00})b = a(a^{00}b^{00})\rho^{00}b \\ &= a(b^{00}a^{00})\rho^{00}b = ab^{00}(a^{00}\rho^{00})b \\ &= a(a^{00}\rho^{00})b^{00}b = (a\rho^{00})b, \end{aligned}$$

this yields that $(\lambda^{00}, \rho^{00}) \in \Omega(S)$.

Now, we prove that $(\lambda^{00}, \rho^{00}) \in E(\Omega(S))$. To see it, let $e \in E(S)$. Then by (1), $e(\rho^{00})^2 = (e\rho^{00})\rho^{00} = (e(e\rho^{00}))\rho^{00} = (e\rho^{00}e)\rho^{00} = e\rho^{00}e\rho^{00} = e\rho^{00}$. By Lemma 3.1(1), $(\rho^{00})^2 = \rho^{00}$. Thus, by Lemma 3.1(4), $(\lambda^{00}, \rho^{00})^2 = (\lambda^{00}, \rho^{00})(\lambda^{00}, \rho^{00}) = ((\lambda^{00})^2, (\rho^{00})^2) = (\lambda^{00}, \rho^{00}) \in E(\Omega(S))$, as required.

(3) Let $e \in E(S)$. Then by (1), we have $\lambda\lambda^{00}e = \lambda(e\rho^{00}) = \lambda(ee\rho^{00}) = (\lambda e)(e\rho^{00}) = (\lambda e)(\lambda^{00}e) = (\lambda e)(\lambda e)^{00} = \lambda e$. Hence, by Lemma 3.1(2), $\lambda\lambda^{00} = \lambda$. Therefore, by Lemma 3.1(4), $(\lambda, \rho)(\lambda^{00}, \rho^{00}) = (\lambda\lambda^{00}, \rho\rho^{00}) = (\lambda, \rho)$.

On the other hand, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in [\Omega(S)]^1$, and

$$(\lambda, \rho)(\lambda_1, \rho_1) = (\lambda, \rho)(\lambda_2, \rho_2).$$

Then, by Lemma 3.1(4), $\rho\rho_1 = \rho\rho_2$, and so $e\rho\rho_1 = e\rho\rho_2$ for all $e \in E(S)$. That is, $e\rho[(e\rho)^{00}\rho_1] = e\rho[(e\rho)^{00}\rho_2]$. Hence, by Corollary 2.2, $(e\rho)^{00}(e\rho)^{00}\rho_1 = (e\rho)^{00}(e\rho)^{00}\rho_2$. That is,

$$(e\rho)^{00}\rho_1 = (e\rho)^{00}\rho_2 \quad \textit{for all } e \in E(S).$$

Choose an idempotent $(\lambda e)^{00}$ of S to replace the element e of the above formula. We have

$$((\lambda e)^{00}\rho)^{00}\rho_1 = ((\lambda e)^{00}\rho)^{00}\rho_2.$$

Note that $((\lambda e)^{00}\rho)e = (\lambda e)^{00}(\lambda e) = \lambda e$. We have $((\lambda e)^{00}\rho)^{00}e = (\lambda e)^{00}$. Thus,

$$\begin{aligned} e\rho^{00}\rho_1 &= (e\rho^{00})\rho_1 = (\lambda^{00}e)\rho_1 = (\lambda e)^{00}\rho_1 \\ &= [((\lambda e)^{00}\rho)^{00}e]\rho_1 = [e((\lambda e)^{00}\rho)^{00}]\rho_1 \\ &= e((\lambda e)^{00}\rho)^{00}\rho_1 = e((\lambda e)^{00}\rho)^{00}\rho_2 \\ &= [((\lambda e)^{00}\rho)^{00}e]\rho_2 = (\lambda e)^{00}\rho_2 \\ &= e\rho^{00}\rho_2, \end{aligned}$$

which implies that $\rho^{00}\rho_1 = \rho^{00}\rho_2$. By Lemma 3.1(4),

$$(\lambda^{00}\lambda_1, \rho^{00}\rho_1) = (\lambda^{00}\lambda_2, \rho^{00}\rho_2).$$

That is,

$$(\lambda^{00}, \rho^{00})(\lambda_1, \rho_1) = (\lambda^{00}, \rho^{00})(\lambda_2, \rho_2).$$

This together with the fact $(\lambda, \rho) = (\lambda, \rho)(\lambda^{00}, \rho^{00})$, yields that $(\lambda, \rho)\mathcal{L}^\star(\lambda^{00}, \rho^{00})$.

(4) Obviously, for any $a \in S$, $a\mathcal{L}^\star a^{00}$ implies that $a\rho\mathcal{L}^\star a^{00}\rho$, where ρ is a right translation of S. Hence, for all $e \in E(S)$, we have

$$(e\rho)^{00}\rho^{00}\mathcal{L}^\star(e\rho)\rho^{00} = e\rho\rho^{00} = e\rho\mathcal{L}^\star(e\rho)^{00}$$

since $\rho\rho^{00} = \rho$ ($i.e$ $(\lambda, \rho) = (\lambda, \rho)(\lambda^{00}, \rho^{00})$). Thus, $(e\rho)^{00} = (e\rho)^{00}\rho^{00}$ since S is $\mathcal{L}^\star$–unipotent. Note that $e\rho^{00}e\rho = e\rho^{00}\rho$. We have

$$\begin{aligned} e\rho &= (e\rho)^{00}(e\rho) = ((e\rho)^{00}\rho^{00})(e\rho) = (e(e\rho)^{00}\rho^{00})\rho \\ &= (e(e\rho)^{00})\rho^{00}\rho = ((e\rho)^{00}e)\rho^{00}\rho \\ &= (e\rho)^{00}e\rho^{00}\rho \end{aligned}$$

Therefore, $e\rho^{00}\rho\mathcal{L}e\rho$ for all $e \in E(S)$, as required.

(5) We first prove that $e\rho^{00}\rho = e\rho$ for all $e \in E(S)$.

$$\begin{aligned}(e\rho^{00}\rho)(e\rho)^{00} &= [(e\rho^{00})\rho](e\rho)^{00} = [(\lambda^{00}e)\rho](e\rho)^{00}\\ &= (\lambda^{00}e)(\lambda(e\rho)^{00}) = (\lambda^{00}e)[(\lambda(e\rho)^{00})^{00}](\lambda(e\rho)^{00})\\ &= (\lambda^{00}e)[(\lambda(e\rho)^{00})^{00}(e\rho)^{00}](\lambda(e\rho)^{00})\\ &= \lambda^{00}[e(\lambda(e\rho)^{00})^{00}(e\rho)^{00}](\lambda(e\rho)^{00})\\ &= \lambda^{00}[(e\rho)^{00}e(\lambda(e\rho)^{00})^{00}](\lambda(e\rho)^{00})\\ &= \lambda^{00}(e\rho)^{00}e(\lambda(e\rho)^{00})^{00}\lambda(e\rho)^{00}\\ &= \lambda^{00}(e\rho)^{00}e\lambda(e\rho)^{00} = e\lambda^{00}(e\rho)^{00}\lambda(e\rho)^{00}\\ &= e(\lambda(e\rho)^{00})^{00}\lambda(e\rho)^{00} = e(\lambda(e\rho)^{00})\\ &= (e\rho)(e\rho)^{00} = e\rho.\end{aligned}$$

On the other hand, by (4), we have $e\rho^{00}\rho\mathcal{L}^*e\rho$ and $e\rho = (e\rho)(e\rho)^{00}$. Thus, by Lemma 2.1, we have $e\rho^{00}\rho = e\rho^{00}\rho(e\rho)^{00} = e\rho$. By Lemma 3.1(1), $\rho = \rho^{00}\rho$. Therefore, by Lemma 3.1(4), $(\lambda^{00}, \rho^{00})(\lambda, \rho) = (\lambda, \rho)$.

(6) Let $(\lambda, \rho) \in \Omega(S)$ be such that $\lambda E(S) \bigcup E(S)\rho \subseteq E(S)$. Then $e\rho \in E(S)$ for all $e \in E(S)$. Hence, $e\rho^2 = (ee\rho)\rho = (e\rho)e\rho = e\rho$, and so $\rho^2 = \rho$. By Lemma 3.1(4), $(\lambda, \rho)^2 = (\lambda, \rho) \in E(\Omega(S))$.

Conversely, if $(\lambda, \rho) \in E(\Omega(S))$, then $(\lambda^{00}, \rho^{00}) = (\lambda^{00}, \rho^{00})(\lambda, \rho)$ since $(\lambda^{00}, \rho^{00})\mathcal{L}^*(\Omega(S))(\lambda, \rho)$. On the other hand, by (5), $(\lambda, \rho) = (\lambda^{00}, \rho^{00})(\lambda, \rho)$. Thus $(\lambda, \rho) = (\lambda^{00}, \rho^{00})$, and so $\lambda = \lambda^{00}, \rho = \rho^{00}$, this gives $\lambda E(S) \bigcup E(S)\rho \subseteq E(S)$. This completes the proof. □

Theorem 3.3. *Let S be a strongly right type B semigroup. Then so is $\Omega(S)$.*

Proof. By Proposition 3.2 (2) and (3), $\Omega(S)$ is an rpp semigroup.

Now, we prove that $\Omega(S)$ is right adequate. To see it, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in E(\Omega(S))$. Then, by Proposition 3.2 (6), $e\rho_1, e\rho_2 \in E(S)$ for all $e \in E(S)$. Hence, $e\rho_1\rho_2 = e\rho_1 e\rho_2 = e\rho_2 e\rho_1 = e\rho_2\rho_1$, which implies that $\rho_1\rho_2 = \rho_2\rho_1$. By Lemma 3.1(4),

$$(\lambda_1\lambda_2, \rho_1\rho_2) = (\lambda_2\lambda_1, \rho_2\rho_1).$$

That is,

$$(\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_2, \rho_2)(\lambda_1, \rho_1).$$

Therefore, $E(\Omega(S))$ is a semilattice. That is, $\Omega(S)$ is a right adequate semigroup. This together with Proposition 3.2(2), (3) and (5), yields that $\Omega(S)$ is a strongly rpp semigroup with semilattice of idempotents $E(\Omega(S))$.

Next, we only need to prove that $\Omega(S)$ satisfies Conditions (B1) and (B2). To see it, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in E[(\Omega(S))^1]$, $(\lambda, \rho) \in \Omega(S)$, and $e \in E(S)$. Then

$$\begin{aligned}& e(\rho_1\rho)^{00}(\rho_2\rho)^{00} = [ee(\rho_1\rho)^{00}](\rho_2\rho)^{00} = e(\rho_1\rho)^{00}e(\rho_2\rho)^{00}\\ &= (\lambda_1\lambda)^{00}e(\lambda_2\lambda)^{00}e = (\lambda_1\lambda e)^{00}(\lambda_2\lambda e)^{00} && \text{(by Proposition 3.2(1))}\\ &= (\lambda_1(\lambda e)^{00}(\lambda e))^{00}(\lambda_2(\lambda e)^{00}(\lambda e))^{00} && \text{(since S is strongly rpp)}\\ &= (\lambda_1(\lambda e)^{00}\lambda_2(\lambda e)^{00}(\lambda e))^{00} && \text{(since S satisfies Condition (B1))}\\ &= (\lambda_1((\lambda e)^{00}\lambda_2(\lambda e)^{00})(\lambda e))^{00}\\ &= (\lambda_1(\lambda_2(\lambda e)^{00}(\lambda e)^{00})(\lambda e))^{00}\\ &= (\lambda_1\lambda_2\lambda e)^{00}\\ &= (\lambda_1\lambda_2\lambda)^{00}e && \text{(by Proposition 3.2(1))}\\ &= e(\rho_1\rho_2\rho)^{00} && \text{(by Proposition 3.2(1))}\end{aligned}$$

By Lemma 3.1(1), $(\rho_1\rho)^{00}(\rho_2\rho)^{00} = (\rho_1\rho_2\rho)^{00}$. On the other hand, we have

$$\begin{aligned}[(\lambda_1, \rho_1)(\lambda_2, \rho_2)(\lambda, \rho)]^* &= [(\lambda_1, \rho_1)(\lambda_2, \rho_2)(\lambda, \rho)]^{00}\\ &= (\lambda_1\lambda_2\lambda,\ \rho_1\rho_2\rho)^{00}\\ &= ((\lambda_1\lambda_2\lambda)^{00},\ (\rho_1\rho_2\rho)^{00})\end{aligned}$$

and

$$\begin{aligned}[(\lambda_1,\rho_1)(\lambda,\rho)]^\star[(\lambda_2,\rho_2)(\lambda,\rho)]^\star &= [(\lambda_1,\rho_1)(\lambda,\rho)]^{00}[(\lambda_2,\rho_2)(\lambda,\rho)]^{00}\\ &= (\lambda_1\lambda,\ \rho_1\rho)^{00}(\lambda_2\lambda,\ \rho_2\rho)^{00}\\ &= ((\lambda_1\lambda)^{00},\ (\rho_1\rho)^{00})((\lambda_2\lambda)^{00},\ (\rho_2\rho)^{00})\\ &= ((\lambda_1\lambda)^{00}(\lambda_2\lambda)^{00},\ (\rho_1\rho)^{00}(\rho_2\rho)^{00}).\end{aligned}$$

By Lemma 3.1(4), $[(\lambda_1,\rho_1)(\lambda_2,\rho_2)(\lambda,\rho)]^\star = [(\lambda_1,\rho_1)(\lambda,\rho)]^\star[(\lambda_2,\rho_2)(\lambda,\rho)]^\star$. Therefore, $\Omega(S)$ satisfies Condition (B1).

Let $(\lambda_1,\rho_1) \in E(\Omega(S))$, $(\lambda,\rho) \in \Omega(S)$ be such that $(\lambda_1,\rho_1) \leq (\lambda,\rho)^\star$. Then $(\lambda_1,\rho_1) \leq (\lambda,\rho)^{00} = (\lambda^{00},\rho^{00})$ since $\Omega(S)$ is $\mathcal{L}^\star$–unipotent. Hence $(\lambda_1,\rho_1) = (\lambda_1,\rho_1)(\lambda^{00},\rho^{00}) = (\lambda_1\lambda^{00},\rho_1\rho^{00})$, and so $\rho_1 = \rho_1\rho^{00}$. By Lemma 3.1(1), $e\rho_1 = e\rho_1\rho^{00} = e\rho_1 e\rho^{00} = e\rho^{00}e\rho_1$ for all $e \in E(S)$. Hence, $e\rho \leq e\rho^{00} = (\lambda e)^{00} = (\lambda e)^\star$. Again since S satisfies Condition (B2), we have

$$e\rho_1 = [f(\lambda e)]^\star = [f(\lambda e)]^{00} \quad \text{for some } f \in E(S^1).$$

That is,

$$e\rho_1 = [f(\lambda e)]^{00} = [\lambda_f(\lambda e)]^{00} = (\lambda_f\lambda e)^{00} = (\lambda_f\lambda)^{00}e = e(\rho_f\rho)^{00},$$

where λ_f (ρ_f) is the inner left (right) translation on S^1 determined by $f \in E(S^1)$. By Lemma 3.1(1), $\rho_1 = (\rho_f\rho)^{00}$. Hence, by Lemma 3.1(4),

$$\begin{aligned}[(\lambda_f,\rho_f)(\lambda,\rho)]^\star &= [(\lambda_f,\rho_f)(\lambda,\rho)]^{00}\\ &= (\lambda_f\lambda,\ \rho_f\rho)^{00}\\ &= ((\lambda_f\lambda)^{00},\ (\rho_f\rho)^{00})\\ &= (\lambda_1,\rho_1),\end{aligned}$$

where $(\lambda_f,\rho_f) \in E[(\Omega(S))^1]$. That is, $\Omega(S)$ satisfies Condition (B2).

Summing up the above arguments, we conclude $\Omega(S)$ is a strongly right type B semigroup. □

Corollary 3.4. *Let S be a strongly right type B semigroup. Then the following statements are true:*

(1) *for all* $e \in E(S)$, $(\lambda_1,\rho_1),(\lambda_2,\rho_2) \in \Omega(S)$, $(\lambda_1,\rho_1)\mathcal{L}^\star(\Omega(S))(\lambda_2,\rho_2)$ *if and only if* $\lambda_1 e\mathcal{L}^\star(S)\lambda_2 e$;
(2) *for all* $(\lambda_1,\rho_1),(\lambda_2,\rho_2) \in \Omega(S)$, $(\lambda_1,\rho_1)\mu_L^{\Omega(S)}(\lambda_2,\rho_2)$ *if and only if for all* $e \in E(S)$, $\lambda_1 e\mu_L^S\lambda_2 e$;

Proof. (1) By Theorem 3.3, $\Omega(S)$ is a strongly right type B semigroup. Let $(\lambda_1,\rho_1),(\lambda_2,\rho_2) \in \Omega(S)$ be such that $(\lambda_1,\rho_1)\mathcal{L}^\star(\Omega(S))(\lambda_2,\rho_2)$. Then, by Proposition 3.2(3), $(\lambda_1^{00},\rho_1^{00})\mathcal{L}^\star(\Omega(S))(\lambda_2^{00},\rho_2^{00})$. Hence, $(\lambda_1^{00},\rho_1^{00}) = (\lambda_2^{00},\rho_2^{00})$ since $\Omega(S)$ is $\mathcal{L}^\star$–unipotent. Thus $\lambda_1^{00} = \lambda_2^{00}$. By Lemma 3.1(2), $\lambda_1^{00}e = \lambda_2^{00}e$ for all $e \in E(S)$). Hence,

$$\lambda_1 e\mathcal{L}^\star(S)(\lambda_1 e)^{00} = \lambda_1^{00}e = \lambda_2^{00}e = (\lambda_2 e)^{00}\mathcal{L}^\star(S)\lambda_2 e.$$

Conversely, if for all $e \in E(S)$, $\lambda_1 e\mathcal{L}^\star(S)\lambda_2 e$, then $\lambda_1^{00}e = \lambda_2^{00}e$ since S is both $\mathcal{L}^\star$–unipotent and strongly rpp . Hence, by Lemma 3.1(2), $\lambda_1^{00} = \lambda_2^{00}$. By Lemma 3.1(4), $(\lambda_1^{00},\rho_1^{00}) = (\lambda_2^{00},\rho_2^{00})$. Thus, by Proposition 3.2(3), $(\lambda_1,\rho_1)\mathcal{L}^\star(\Omega(S))(\lambda_2,\rho_2)$.

(2) Suppose that $(\lambda_1,\rho_1),(\lambda_2,\rho_2) \in \Omega(S)$ and $(\lambda_1,\rho_1)\mu_L^{\Omega(S)}(\lambda_2,\rho_2)$. Then for all $f \in E(S)$,

$$(\lambda_f,\rho_f)(\lambda_1,\rho_1)\mu_L^{\Omega(S)}(\lambda_f,\rho_f)(\lambda_2,\rho_2),$$

where $(\lambda_f,\rho_f) \in E(\Omega(S))$. Hence,

$$(\lambda_f,\rho_f)(\lambda_1,\rho_1)\mathcal{L}^\star(\Omega(S))(\lambda_f,\rho_f)(\lambda_2,\rho_2),$$

That is,

$$(\lambda_f\lambda_1,\ \lambda_f\rho_1)\mathcal{L}^\star(\Omega(S))\,(\lambda_f\lambda_2,\ \lambda_f\rho_2).$$

By (1), $\lambda_f\lambda_1 e\mathcal{L}^\star(S)\lambda_f\lambda_2 e$ for all $e \in E(S)$. That is, $f\lambda_1 e\mathcal{L}^\star(S)f\lambda_2 e$. Thus, $\lambda_1 e\mu_L^S\lambda_2 e$.

Conversely, if for all $e \in E(S)$, $\lambda_1 e\mu_L^S\lambda_2 e$, then $\lambda_1 e\mathcal{L}^\star(S)\lambda_2 e$. Note S is a strongly right type B semigroup. We have $(\lambda_1 e)^{00} = (\lambda_2 e)^{00}$. Hence, for all $(\lambda,\rho) \in \Omega(S)$, we have

$$[\lambda(\lambda_1 e)^{00}]\lambda_1 e \, \mu_L^S \, [\lambda(\lambda_2 e)^{00}]\lambda_2 e.$$

That is, $\lambda\lambda_1 e \, \mu_L^S \lambda\lambda_2 e$. Hence, $\lambda\lambda_1 e \, \mathcal{L}^*(S)\lambda\lambda_2 e$. By (1), we have

$$(\lambda\lambda_1, \ \rho\rho_1)\mathcal{L}^*(\Omega(S))(\lambda\lambda_2, \ \rho\rho_2).$$

That is,

$$(\lambda, \rho)(\lambda_1, \rho_1)\mathcal{L}^*(\Omega(S))(\lambda, \rho)(\lambda_2, \rho_2).$$

Choose any idempotent (λ', ρ') of $\Omega(S)$ to replace the element (λ, ρ) of the above formula. We have $(\lambda_1, \rho_1)\mu_L^{\Omega(S)}(\lambda_2, \rho_2)$, as required. □

4 Some special cases

In this section, we shall consider the translational hulls of some special strongly right type B semigroups.

Proposition 4.1. *Let S be a strongly right type B semigroup. Then for all $e \in E(S)$, $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$,*
$(\lambda_1, \rho_1)\sigma_{\Omega(S)}(\lambda_2, \rho_2) \Longleftrightarrow e\rho_1\sigma_S e\rho_2 \Longleftrightarrow \lambda_1 e\sigma_S\lambda_2 e$.

Proof. We first prove that $(\lambda_1, \rho_1)\sigma_{\Omega(S)}(\lambda_2, \rho_2)$ implies $e\rho_1\sigma_S e\rho_2$ and $\lambda_1 e\sigma_S\lambda_2 e$ for all $e \in E(S)$. To see it, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ such that $(\lambda_1, \rho_1)\sigma_{\Omega(S)} (\lambda_2, \rho_2)$. Then, by the definition of σ, there exists $(\lambda, \rho) \in E(\Omega(S))$ such that

$$(\lambda, \rho)(\lambda_1, \rho_1) = (\lambda, \rho)(\lambda_2, \rho_2),$$

that is, $(\lambda\lambda_1, \rho\rho_1) = (\lambda\lambda_2, \rho\rho_2)$. Hence, $\lambda\lambda_1 = \lambda\lambda_2$ and $\rho\rho_1 = \rho\rho_2$. By Lemma 3.1 (1) and (2), $\lambda\lambda_1 e = \lambda\lambda_2 e$ and $e\rho\rho_1 = e\rho\rho_2$ for all $e \in E(S)$. Thus,

$$\lambda(\lambda_1 e)^{00}\lambda_1 e = \lambda(\lambda_2 e)^{00}\lambda_2 e \ \ and \ \ e\rho e\rho_1 = e\rho e\rho_2.$$

Note that $\lambda(\lambda_1 e)^{00}, \lambda(\lambda_2 e)^{00}, e\rho \in E(S)$. We have $e\rho e\rho_1 = e\rho e\rho_2$ implies that $e\rho_1\sigma_S e\rho_2$. On the other hand,

$$\begin{aligned}
&\lambda(\lambda_1 e)^{00}\lambda_1 e = \lambda(\lambda_2 e)^{00}\lambda_2 e \\
&\Rightarrow e \cdot \lambda(\lambda_1 e)^{00} \cdot \lambda_1 e = e \cdot \lambda(\lambda_2 e)^{00} \cdot \lambda_2 e \\
&\Rightarrow \lambda(\lambda_1 e)^{00} \cdot e \cdot \lambda_1 e = \lambda(\lambda_2 e)^{00} \cdot e \cdot \lambda_2 e \\
&\Rightarrow \lambda((\lambda_1 e)^{00} \cdot e) \cdot \lambda_1 e = \lambda((\lambda_2 e)^{00} \cdot e) \cdot \lambda_2 e \\
&\Rightarrow \lambda e(\lambda_1 e)^{00}\lambda_1 e = \lambda e(\lambda_2 e)^{00}\lambda_2 e \\
&\Rightarrow (\lambda e)(\lambda_1 e) = (\lambda e)(\lambda_2 e),
\end{aligned}$$

where $\lambda e \in E(S)$. Thus, $\lambda_1 e\sigma_S\lambda_2 e$.

Now, we prove that for all $e \in E(S)$,

$$e\rho_1\sigma_S e\rho_2 \Rightarrow (\lambda_1, \rho_1)\sigma_{\Omega(S)}(\lambda_2, \rho_2).$$

To see it, let $e\rho_1\sigma_S e\rho_2$. Then there exists $f \in E(S)$ such that $f(e\rho_1) = f(e\rho_2)$. That is, $(fe)\rho_1 = (fe)\rho_2$. Hence, $(ef)\rho_1 = (ef)\rho_2$ since $E(S)$ is a semilattice. That is, $e\rho_f\rho_1 = e\rho_f\rho_2$. By Lemma 3.1(1), $\rho_f\rho_1 = \rho_f\rho_2$. Thus, by Lemma 3.1(4), we have $(\lambda_f\lambda_1, \rho_f\rho_1) = (\lambda_f\lambda_2, \rho_f\rho_2)$. That is,

$$(\lambda_f, \rho_f)(\lambda_1, \rho_1) = (\lambda_f, \rho_f)(\lambda_2, \rho_2).$$

Note that $(\lambda_f, \rho_f) \in E(\Omega(S))$. We have $(\lambda_1, \rho_1)\sigma_{\Omega(S)}(\lambda_2, \rho_2)$.

Similarly, we can prove that for all $e \in E(S)$,

$$\lambda_1 e\sigma_S\lambda_2 e \Rightarrow (\lambda_1, \rho_1)\sigma_{\Omega(S)}(\lambda_2, \rho_2).$$

This completes the proof. □

Theorem 4.2. *Let S be a proper strongly right type B semigroup. Then so is $\Omega(S)$.*

Proof. By Theorem 3.3, $\Omega(S)$ is a strongly right type B semigroup. It only remains to show that $\Omega(S)$ is proper. To see it, let $(\lambda_1,\rho_1),(\lambda_2,\rho_2)\in\Omega(S)$ be such that $(\lambda_1,\rho_1)[{\mathcal{L}^*}_{\Omega(S)}\bigcap\sigma_{\Omega(S)}](\lambda_2,\rho_2)$. Then, by Corollary 3.4(1) and Proposition 4.1, we have $\lambda_1 e\mathcal{L}^*(S)\lambda_2 e$ and $\lambda_1 e\sigma_S\lambda_2 e$ for all $e\in E(S)$. Hence, $\lambda_1 e[{\mathcal{L}^*}_S\bigcap\sigma_S]\lambda_2 e$. Again, since S is proper, we have $\lambda_1 e=\lambda_2 e$ for all $e\in E(S)$. By Lemma 3.1(2), $\lambda_1=\lambda_2$. Therefore, by Lemma 3.1(4), $(\lambda_1,\rho_1)=(\lambda_2,\rho_2)$. This completes the proof.

□

Corollary 4.3. *Let S be a strongly right type B semigroup which is right fundamental. Then so is $\Omega(S)$.*

Proof. By Theorem 3.3, $\Omega(S)$ is strongly right type B. Let $(\lambda_1,\rho_1),(\lambda_2,\rho_2)\in\Omega(S)$ be such that $(\lambda_1,\rho_1)\mu_L^{\Omega(S)}(\lambda_2,\rho_2)$. Then, by Corollary 3.4(2), $\lambda_1 e\mu_L^S\lambda_2 e$ for all $e\in E(S)$. Again, since S is right fundamental, we have $\lambda_1 e=\lambda_2 e$. By Lemma 3.1(2), $\lambda_1=\lambda_2$. Hence, by Lemma 3.1(4), $(\lambda_1,\rho_1)=(\lambda_2,\rho_2)$. That is, $\mu_L^{\Omega(S)}=1_{\Omega(S)}$. This completes the proof.

□

As applications of Theorem 4.2 and Corollary 4.3, we shall give a positive answer to a problem posted by Petrich (i.e., if a semigroup S is embeddable into an inverse semigroup, is $\Omega(S)$ also embeddable into an inverse semigroup (see, [4, V. 3.11 Problems, p. 226])) to the cases of some strongly right type B semigroups. The answer is given in the following Corollaries.

Corollary 4.4. *Let S be a proper strongly right type B semigroup. Then $\Omega(S)$ is embeddable into an inverse semigroup.*

Proof. It follows from Corollary 2.6, Lemma 2.4 and Theorem 4.2. □

Corollary 4.5. *Let S be a strongly right type B semigroup which is right fundamental. Then $\Omega(S)$ is embeddable into an inverse semigroup.*

Proof. It follows from Lemma 2.4 and Corollary 4.3. □

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