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Research Article

Hongwei Liang* and Zhongping Wan

Characterizations of Benson proper efficiency of set-valued optimization in real linear spaces

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Abstract: A new class of generalized convex set-valued maps termed relatively solid generalized cone-subconvexlike maps is introduced in real linear spaces not equipped with any topology. This class is a generalization of generalized cone-subconvexlike maps and relatively solid cone-subconvexlike maps. Necessary and sufficient conditions for Benson proper efficiency of set-valued optimization problem are established by means of scalarization, Lagrange multipliers, saddle points and duality. The results generalize and improve some corresponding ones in the literature. Some examples are afforded to illustrate our results.

Keywords: set-valued maps, relative algebraic interior, vector closure, Benson proper efficiency, generalized cone subconvexlikeness

MSC: 90C26, 90C30, 90C46

1 Introduction

It is well known that the notion of (weak) efficiency plays a very critical role in vector optimization problem. Since the range of the set of (weakly) efficient solutions is often too large, various concepts of proper efficiency have been proposed in the literature [1–4]. By virtue of weak or proper efficiency, some scholars have involved many research areas of set-valued optimization problems [5–12]. Meanwhile, in the past few years, the authors [13–18] introduced generalized convexity of set-valued maps to characterize weak efficiency and proper efficiency. The usual framework of the above papers is that of topological linear spaces or locally convex spaces. As we know, the linear spaces are wider than the topological spaces or locally convex spaces, how to extend the concepts of weak or proper efficiency and generalized convexity from the topological spaces or locally convex spaces to the linear spaces is a meaningful topic.

Through the concept of algebraic interior, Li [19] defined the notions of weakly efficient solutions of vector optimization and cone subconvexlike set-valued maps in real linear spaces. Some optimality conditions had been obtained. To avoid the algebraic interior is empty, in real linear spaces, by virtue of relative algebraic interior and vector closure, Adán and Novo [20–22] also defined weakly efficient solutions, some kinds of properly efficient solutions of vector optimization problems and generalized convexlikeness of vector-valued maps in real linear spaces. Scalarization, multiplier rules, and saddle-point theorems had been also obtained. Adán and Novo [23] established Lagrangian type duality theorems and saddle-points theorems for weak and

***Corresponding Author: Hongwei Liang:** School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China, E-mail: 2015102010029@whu.edu.cn
Zhongping Wan: School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China, E-mail: zpwan-whu@126.com

Benson proper efficient solutions of vector optimization problems with generalized convexity introduced in [22].

Under the assumption of generalized cone subconvexlikeness refering to relative algebraic interior, Hernǽndez et al. [24] and Zhou et al. [25] obtained Fritz-John and Kuhn-Tucker optimality conditions for weakly efficient solutions of set-valued optimization problem in linear spaces. Some approximate solution concepts had been generalized to the algebraic setting and their characterizations had been derived via linear scalarizations (see [26, 27]), Lagrangian optimality conditions and saddle point theorems (see [28]).

It is well-known that many maps are not generalized convex set-valued maps introduced in [19, 21, 22, 24, 25], the reason is that the conditions for these generalized convex maps are too strong. Hence, in this work, we introduce a more general generalized convex set-valued maps and extend scalarization theorems and Lagrange multiplier theorems under this generalized convexity hypotheses in real linear spaces. Meanwhile, to our knowledge, up to now there are few works focus on Benson proper saddle points and duality of set-valued optimization problems in linear spaces. In this paper, we further investigate Benson proper efficiency of the set-valued optimization problem in terms of saddle points theorems and duality theorems in linear spaces. Moreover, the duality theorems is important because of the several applications to, for instance, population dynamics as dynamics vaccination. Results in this line had been derived in [14, 22–24, 29].

The remaining of this article is organized as follows. Section 2 provide some basic definitions we need in this paper. In Section 3, a new class generalized cone subconvexlike of set-valued maps is introduced in real linear spaces. We state the major results for the characterizations of Benson proper efficiency of the set-valued optimization problem in Sections 4-7.

2 Preliminaries

Throughout this paper, we always assume that X, Y and Z are three real linear spaces. Let 0 denote the zero element of every space. Let $K \subset Y$ be nonempty. We denote by

$$\mathrm{cone}(K) := \{\lambda k | k \in K,\ \lambda \geq 0\},$$

$$\mathrm{aff}(K) := \{\sum_{i=1}^{n} \lambda_i k_i : \forall i \in \{1, \dots, n\},\ k_i \in K,\ \sum_{i=1}^{n} \lambda_i = 1\},$$

the generated cone, affine hull of K, respectively. K is called a cone if $\lambda K \subseteq K$ for any $\lambda \geq 0$. A cone K is convex if $K + K \subseteq K$. A cone K is called pointed if $K \cap (-K) = \{0\}$. A cone $K \subseteq Y$ is called nontrivial if $\{0\} \neq K \neq Y$. Let $Y^\star$ and $Z^\star$ stand for the algebraic dual spaces of Y and Z, respectively. From now on, let C and D be nontrivial pointed convex cones in Y and Z, respectively. The algebraic dual cone C^+ and strictly algebraic dual cone C^{+i} of C are defined as

$$C^+ := \{y^\star \in Y^\star \mid y^\star(y) \geq 0,\ \forall y \in C\}$$

and

$$C^{+i} := \{y^\star \in Y^\star \mid y^\star(y) > 0,\ \forall y \in C \setminus \{0\}\}.$$

The meaning of D^+ is similar to that of C^+.

In the sequel, Y is assumed to be ordered through the following quasi order:

$$y_1, y_2 \in Y,\ y_1 \leq_C y_2 \Longleftrightarrow y_2 - y_1 \in C.$$

Definition 2.1. [30] *Let K be a nonempty subset in Y. The algebraic interior of K is the set*

$$\mathrm{cor}(K) := \{k \in K \mid \forall v \in Y, \exists \lambda' > 0, \forall \lambda \in [0, \lambda'],\ \text{s.t. } k + \lambda v \in K\}.$$

We say that K is algebraic open, if $\mathrm{cor}(K) = K$.

Definition 2.2. [31] *Let K be a nonempty subset in Y. The relative algebraic interior of K is the set*

$$\mathrm{icr}(K) := \{k \in K \mid \forall v \in \mathrm{aff}(K) - k, \exists \lambda' > 0, \forall \lambda \in [0, \lambda'], \text{ s.t. } k + \lambda v \in K\}.$$

We say that K is solid (relatively solid) if $\mathrm{cor}(K) \neq \emptyset$ ($\mathrm{icr}(K) \neq \emptyset$). Clearly, $\mathrm{cor}(K) \subseteq \mathrm{icr}(K)$, if $\mathrm{cor}K \neq \emptyset$, then $\mathrm{cor}(K) = \mathrm{icr}(K)$ i.e., if K is solid, then K is relatively solid.

Remark 2.1. *Let C be a nontrivial pointed convex cones in Y. Then, $\forall \varphi \in C^{+i}$, $\varphi(y) > 0, \ \forall y \in \mathrm{icr}(C)$.*

Proof. Since C be a nontrivial pointed cone in Y, by Remark 2.3 of [25], $0 \notin \mathrm{icr}(C)$. Hence, $\mathrm{icr}(C) \subset C \setminus \{0\}$. It follows from the Definition of C^{+i} that the conclusion holds.

Definition 2.3. [21] *Let K be a nonempty subset in Y. The vector closure of K is the set*

$$\begin{aligned}\mathrm{vcl}(K) :=& \{k \in Y \mid \exists v \in Y, \forall \lambda' > 0, \exists \lambda \in [0, \lambda'], \text{ s.t. } k + \lambda v \in K\}\\ =& \{k \in Y \mid \exists v \in Y, \ \exists \lambda_n \to 0^+, \text{ s.t. } k + \lambda_n v \in K, \ \forall n \in \mathbb{N}\}.\end{aligned}$$

The set K is called vectorially closed (v-closed) if $K = \mathrm{vcl}(K)$.

Remark 2.2. *Shi [31] (p17) introduced a concept that equivalent to Definition 2.3 called algebraic closure and denoted by K^c.*

Lemma 2.1. [31] *Let $K \subset Y$ is convex. Then K^c is convex, i.e.,* $\mathrm{vcl}(K)$ *is convex.*

Lemma 2.2. [31] (Strong Separation Theorem) *Let Y be a linear space and $S, T \subset Y$ be two nonempty sets with $S - T$ is a convex set in Y. If* $\mathrm{icr}(S - T) \neq \emptyset$ *and* $0 \notin (S - T)^c$, *then there exists $y^\star \in Y^\star \setminus \{0\}$ such that* $\sup_{t \in T} y^\star(t) < \inf_{s \in S} y^\star(s)$.

Lemma 2.3. [20] *Let K_1 and K_2 be two nontrivial convex cones in Y_1 and Y_2, respectively. If* $\mathrm{icr}(K_1) \neq \emptyset$ *and* $\mathrm{icr}(K_2) \neq \emptyset$, *then* $\mathrm{icr}(K_1 \times K_2) = \mathrm{icr}(K_1) \times \mathrm{icr}(K_2)$.

The following lemma was stated in [23, Propositions 5(iii) and 6(i)].

Lemma 2.4. [21] *Let $K \subset Y$ and $C \subset Y$ be a nontrivial convex cone. Then,*
(i)$\mathrm{vcl}(\mathrm{cone}(K) + C) = \mathrm{vcl}(\mathrm{cone}(K + C))$.
(ii) If C is relatively solid, then $\mathrm{vcl}(K + C) = \mathrm{vcl}(K + \mathrm{icr}(C))$.

Lemma 2.5. [23] *Let $a \in Y$ and C be a relatively solid convex cone of Y. Then* $\{a\} + \mathrm{icr}(C) = \mathrm{icr}(\{a\} + \mathrm{icr}(C))$.

Lemma 2.6. [22] *If $C \subset Y$ is a convex cone, then C is relatively solid if and only if C^+ is relatively solid.*

The following cone separation theorem is due to Adán and Novo [24, Theorem 2.2]. By Lemma 2.6, the assumption on the relative solidness of K supposed in [24, Theorem 2.2] can be removed.

Lemma 2.7. [22] (Separation Theorem) *Let M, K be two v-closed convex cones in Y. Let M be relatively solid and K^+ be solid. If $M \cap K = \{0\}$, then there exists a linear functional $l \in Y^\star \setminus \{0\}$ such that $\forall k \in K, m \in M$,*

$$l(k) \geq 0 \geq l(m),$$

and furthermore, $\forall k \in K \setminus \{0\}, l(k) > 0$.

Lemma 2.8. [24] *Let K be a relatively solid convex subset of Y and $\varphi : Y \to Z$ be a linear map. Then* $\varphi(\mathrm{icr}(K)) = \mathrm{icr}(\varphi(K))$.

Lemma 2.9. [24] *Let K be a relatively solid convex subset of Y. Then* $\mathrm{icr}(K) \subset \mathrm{icr}(\mathrm{cone}(K))$.

Definition 2.4. [24] *Let $B \subset Y$, $y \in B$ be called a Benson proper minimal point of B (denoted by $y \in \mathrm{PMin}(B, C)$) if* $\mathrm{vcl}(\mathrm{cone}(B - y + C)) \cap (-C) = \{0\}$.

By Definition 2.4, we can get the following Definition.

Definition 2.5. *Let $B \subset Y$, $y \in B$ be called a Benson proper maximal point of B (denoted by $y \in \mathrm{PMax}(B, C)$) if* $\mathrm{vcl}(\mathrm{cone}(B - y - C)) \cap C = \{0\}$.

3 Generalized cone subconvexlike set-valued maps

In this section, we introduce some classes of generalized convex set-valued maps and discuss their relationships. From now on, suppose that A is a nonempty subset of X and $F : A \rightrightarrows Y$ is a set-valued map on A. Let $F(A) = \bigcup_{x \in A} F(x)$.

Definition 3.1. [24] *Let $\mathrm{icr}(C) \neq \emptyset$. The set-valued map $F : A \rightrightarrows Y$ is said to be C-subconvexlike on A iff $F(A) + \mathrm{icr}(C)$ is a convex set in Y.*

Definition 3.2. [24] *Let $\mathrm{icr}(C) \neq \emptyset$. The set-valued map $F : A \rightrightarrows Y$ is said to be relatively solid C-subconvexlike on A iff F is C-subconvexlike on A and $F(A) + \mathrm{icr}(C)$ is relatively solid.*

Definition 3.3. [25] *Let $\mathrm{icr}(C) \neq \emptyset$. The set-valued map $F : A \rightrightarrows Y$ is said to be generalized C-subconvexlike on A iff $\mathrm{cone}(F(A)) + \mathrm{icr}(C)$ is a convex set in Y.*

Lemma 3.1. [25] *If the set-valued map $F : A \rightrightarrows Y$ is C-subconvexlike on A, then F is generalized C-subconvexlike on A.*

By Definition 3.2 and Definition 3.3, we give the following definition.

Definition 3.4. *Let $\mathrm{icr}(C) \neq \emptyset$. The set-valued map $F : A \rightrightarrows Y$ is said to be relatively solid generalized C-subconvexlike on A, if F is generalized C-subconvexlike on A and $\mathrm{cone}(F(A)) + \mathrm{icr}(C)$ is relatively solid.*

Proposition 3.1. *If the set-valued map $F : A \rightrightarrows Y$ is relatively solid C-subconvexlike on A, then F is relatively solid generalized C-subconvexlike on A.*

Proof. From Lemma 3.1, F is generalized C-subconvexlike on A is valid. We need to prove the following expression holds:

$$\mathrm{icr}[\mathrm{cone}(F(A)) + \mathrm{icr}(C)] \neq \emptyset. \tag{3.1}$$

Since $F(A) + \mathrm{icr}(C)$ is convex and $\mathrm{icr}(F(A) + \mathrm{icr}(C)) \neq \emptyset$. It follows from Lemma 2.9 that $\mathrm{cone}(F(A) + \mathrm{icr}(C))$ is convex and $\mathrm{icr}[\mathrm{cone}(F(A) + \mathrm{icr}(C))] \neq \emptyset$. Due to Proposition 4(i) of [21], one obtains

$$\mathrm{icr}[\mathrm{vcl}(\mathrm{cone}(F(A) + \mathrm{icr}(C)))] \neq \emptyset.$$

As $\mathrm{cone}(F(A) + \mathrm{icr}(C)) = \mathrm{cone}(F(A) + C)$, then,

$$\mathrm{icr}[\mathrm{vcl}(\mathrm{cone}(F(A) + C))] \neq \emptyset. \tag{3.2}$$

On the other hand, again, by Proposition 4(i) of [21], we get

$$\mathrm{icr}[\mathrm{cone}(F(A)) + \mathrm{icr}(C)] = \mathrm{icr}[\mathrm{vcl}(\mathrm{cone}(F(A)) + \mathrm{icr}(C))]. \tag{3.3}$$

By Lemma 2.4, we obtain

$$\mathrm{vcl}(\mathrm{cone}(F(A)) + \mathrm{icr}(C)) = \mathrm{vcl}(\mathrm{cone}(F(A)) + C)) = \mathrm{vcl}(\mathrm{cone}(F(A) + C)). \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\mathrm{icr}[\mathrm{cone}(F(A)) + \mathrm{icr}(C)] = \mathrm{icr}[\mathrm{vcl}(\mathrm{cone}(F(A) + C))],$$

which together with (3.2), (3.1) holds.

Remark 3.1. *The example below illustrates relatively solid generalized C-subconvexlike set-valued maps not necessarily imply relatively solid C-subconvexlike. Thus, the relatively solid generalized C-subconvexlike maps is more extensive than relatively solid C-subconvexlikeness. So, in this paper, it makes more sense for us to study relevant issues in virtue of relatively solid generalized C-subconvexlikeness.*

Example 3.1. *Let $X = Y = \mathbb{R}^2$, $A = \{(2,0),(1,1)\}$ and $C = \{(y_1, 0) \mid y_1 \geq 0\}$. Consider the set-valued map $F : A \rightrightarrows Y$ defined by*

$$F(2,0) = \{y_1, y_2) \mid \frac{3}{2} \leq y_2 \leq -y_1 + 3,\ y_1 \geq 0\},$$

$$F(1,1) = \{y_1, y_2) \mid 0 \leq y_2 \leq -y_1 + 3,\ y_1 \geq \frac{3}{2}\}.$$

Obviously, $\mathrm{cone}(F(A)) + \mathrm{icr}(C)$ is a convex set in Y and $\mathrm{icr}(\mathrm{cone}(F(A)) + \mathrm{icr}(C)) = \{y_1, y_2) \mid y_1 > 0,\ y_2 > 0\} \neq \emptyset$, i.e., F is relatively solid generalized C-subconvexlike on A. However, $F(A) + \mathrm{icr}(C)$ is not a convex set in Y, so F is not C-subconvexlike on A. By definition 3.3, F is not relatively solid C-subconvexlike on A.

Consider the following two systems:

System 1. There exists $x_0 \in A$ such that $F(x_0) \cap (-\mathrm{icr}(C)) \neq \emptyset$.

System 2. There exists $\varphi \in C^+ \setminus \{0\}$ such that $\varphi(y) \geq 0$, $\forall y \in F(x)$.

Lemma 3.2. [25] *Let C be pointed convex cone with nonempty relative algebraic interior in Y.*

(i) Suppose that F is relatively solid generalized C-subconvexlike on A. If System 1 has no solutions, then System 2 has a solution.

(ii) If $\varphi \in C^{+i}$ is a solution of System 2, then System 1 has no solutions.

4 Scalarization

Now, we consider the following unconstrainted vector optimization problem with set-valued maps:

$$\text{(UVP)}\ \ \mathrm{Min}\ F(x) \text{s.t.}\ \ x \in A$$

and its scalar problem

$$\text{(UVP)}_\varphi \quad \mathrm{Min}(\varphi \circ F)(x) \quad \text{s.t. } x \in A,$$

where $\varphi \in Y^* \setminus \{0_{Y^*}\}$.

Definition 4.1. *A point $\bar{x} \in A$ is called Benson proper efficient solution of* (UVP) *if $F(\bar{x}) \cap \mathrm{PMin}(F(A), C) \neq \emptyset$. A pair $(\bar{x}, \bar{y})$ is called a Benson proper efficient element of* (UVP) *if $\bar{y} \in F(\bar{x}) \cap \mathrm{PMin}(F(A), C)$.*

Definition 4.2. *$\bar{x} \in A$ is called a minimal solution of* (UVP)_φ *if there exists $\bar{y} \in F(\bar{x})$ such that*

$$\varphi(\bar{y}) \leq \varphi(y),\ \ \forall y \in F(x).$$

The pair $(\bar{x}, \bar{y})$ is called a minimizer element of (UVP)_φ.

Theorem 4.1. *Let $\varphi \in C^{+i}$. If $(\bar{x}, \bar{y})$ is a minimizer element of* (UVP)$_\varphi$*, then $(\bar{x}, \bar{y})$ is a Benson proper efficient element of* (UVP).

Proof. Let $c \in \mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y} + C)) \cap (-C)$. Then, $c \in \mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y} + C))$. By Definition 2.3, there exist $v \in Y$ and a sequence $\lambda_n \to 0^+$, such that $c + \lambda_n v \in \mathrm{cone}(F(A) - \bar{y} + C)$ for all $n \in \mathbb{N}$. Therefore, there exist sequences $\{\mu_n\} \subset \mathbb{R}_+$, $\{c_n\} \subset C$ and $\{y_n\} \subset F(A)$ such that

$$c + \lambda_n v = \mu_n(y_n + c_n - \bar{y}).$$

Hence,

$$\varphi(c) + \lambda_n \varphi(v) = \mu_n(\varphi(y_n) + \varphi(c_n) - \varphi(\bar{y})). \tag{4.1}$$

As $(\bar{x}, \bar{y})$ is a minimizer element of (UVP)$_\varphi$ and $\varphi \in C^{+i}$, we have $\varphi(y_n) \geq \varphi(\bar{y})$ and $\varphi(c_n) \geq 0$ for all $n \in \mathbb{N}$. Therefore, the right-hand side of (4.1) is nonnegative, and so

$$\varphi(c) + \lambda_n \varphi(v) \geq 0. \tag{4.2}$$

Taking $\lambda_n \to 0^+$ in (4.2), we have

$$\varphi(c) \geq 0.$$

On the other hand, since $c \in -C$, we obtain

$$\varphi(c) \leq 0.$$

Thus,

$$\varphi(c) = 0,$$

then $c = 0$ and

$$\mathrm{vcl}(\mathrm{cone}(F(S) - \bar{y} + C)) \cap (-C) = 0,$$

which together with $\bar{y} \in F(\bar{x})$, it results

$$\bar{y} \in F(\bar{x}) \cap \mathrm{PMin}(F(A), C).$$

Hence, $(\bar{x}, \bar{y})$ is a Benson proper efficient element of (UVP).

Remark 4.1. *The proof of Theorem 4.1 is different from Theorem 4.1 of [22] and Theorem 5.3 of [24].*

Theorem 4.2. *Let C be v-closed and C^+ be solid. Let $\bar{x} \in A$ and $\bar{y} \in F(\bar{x})$. Suppose that the following conditions hold:*

(i)$(\bar{x}, \bar{y})$ is a Benson proper efficient element of (UVP)*;*

(ii) $F - \bar{y}$ is relatively solid generalized C-subconvexlike on A.

Then, there exists $\varphi \in C^{+i}$ such that $(\bar{x}, \bar{y})$ is a minimizer element of (UVP)$_\varphi$.

Proof. By condition (i). Then,

$$\mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y} + C)) \cap (-C) = \{0\}. \tag{4.3}$$

It follows from condition (ii) that $\mathrm{cone}(F(A) - \bar{y}) + \mathrm{icr(C)}$ is a relatively solid, convex set in Y. Then, according to Proposition 3(iii)-(iv) and Proposition 4(i) in [21], we conclude that $\mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y}) + \mathrm{icr(C)})$ is a relatively solid, v-closed, convex set in Y. Furthermore, it follows from (i) and (ii) of Lemma 2.4 that

$$\mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y}) + \mathrm{icr(C)}) = \mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y}) + C) = \mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y} + C)).$$

Hence, by Lemma 2.5, there exists a linear functional $\varphi \in C^{+i}$ such that

$$\varphi(y) \geq 0, \ \forall y \in \mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y} + C)).$$

Clearly, $F(A) - \bar{y} + C \subset \mathrm{vcl}(\mathrm{cone}(F(A) - \bar{y} + C))$, then

$$\varphi(y) - \varphi(\bar{y}) + \varphi(c) \geq 0, \ \forall y \in F(A), \ \ \forall c \in C. \tag{4.4}$$

Taking $c = 0$ in (4.4), this yields

$$\varphi(y) \geq \varphi(\bar{y}), \ \forall y \in f(A).$$

Thus, $(\bar{x}, \bar{y})$ is a minimizer element of $(\mathrm{UVP})_\varphi$.

Remark 4.2. *Theorem 4.2 extends Theorem 4.2 of [22] and Theorem 5.4 of [24].*

Corollary 4.1. *Let C be v-closed and C^+ be solid, $\bar{x} \in A$ and $\bar{y} \in F(\bar{x})$. Let $F - \bar{y}$ be relatively solid generalized C-subconvexlike on A. Then $(\bar{x}, \bar{y})$ is a Benson proper efficient element of* (UVP) *if and only if $(\bar{x}, \bar{y})$ is a minimizer element of* $(\mathrm{UVP})_\varphi$ *for $\varphi \in C^{+i}$.*

5 Lagrange multiplier rules

Let $F : A \rightrightarrows Y$ and $G : A \rightrightarrows Z$ be two set-valued maps on A. We suppose that $\mathrm{icr}(C) \times \mathrm{icr}(D) \neq \emptyset$.

Now, we consider the following constrainted vector optimization problem with set-valued maps:

$$(\mathrm{CVP}) \ \mathrm{Min}\ F(x) \ \ \text{s.t. } G(x) \cap (-D) \neq \emptyset, \ x \in A.$$

Denote the feasible solution set of (CVP) by

$$S = \{x \in A \mid G(x) \cap (-D) \neq \emptyset\}$$

and write $F(S) = \bigcup_{x \in S} F(x)$.

Definition 5.1. *A point $\bar{x}$ is called Benson proper efficient solution of* (CVP) *if $\bar{x} \in S$ and $F(\bar{x}) \cap \mathrm{PMin}(F(S), C) = \emptyset$. A pair $(\bar{x}, \bar{y})$ is called a Benson proper efficient element of* (CVP) *if $\bar{y} \in F(\bar{x}) \cap \mathrm{PMin}(F(S), C)$.*

We denote by $L(Z, Y)$ the set of all linear operators from Z to Y and by $L^+(Z, Y)$ the set of positive linear operators defined as

$$L^+(Z, Y) = \{T \in L(Z, Y) \mid T(D) \subset C\}.$$

Denote by (F, G) the set-valued map from A to $Y \times Z$ defined by $(F, G)(x) = F(x) \times G(x)$. where $Y \times Z$ is an linear space with nontrivial pointed convex cone $C \times D$.

By Definition 3.4, $(F, G)(x)$ is relatively solid generalized $C \times D$-subconvexlike on A if $\mathrm{cone}((F, G)(A)) + \mathrm{icr}(C \times D)$ is a relatively solid convex set in $Y \times Z$.

The Lagrangian set-valued map of (CVP) is defined by

$$L(x, T) = F(x) + T(G(x)), \ \forall (x, T) \in A \times L^+(Z, Y).$$

Consider the following unconstrained set-valued optimization problem:

$$(\mathrm{UVP})_{\mathrm{T}} \ \mathrm{Min}\ L(x, T) \ \ \text{s.t. } (x, T) \in A \times L^+(Z, Y).$$

Proposition 5.1. *Let $(F, G) : A \rightrightarrows Y \times Z$ be generalized $C \times D$-subconvexlike on A.*

(i)If $\varphi \in C^+$, then $(\varphi \circ F, G)$ is generalized $\mathbb{R}_+ \times D$-subconvexlike on A.

(ii) If $T \in L^+(Z, Y)$, then $F + T \circ G$ is generalized K-subconvexlike on A.

Furthermore, if (F, G) is relatively solid, then $(\varphi \circ F, G)$ is relatively solid in part (i) and $F + T \circ G$ is relatively solid in part (ii).

Proof. (i) Denote by i the identity map in Y. A linear map $(\varphi, i) : Y \times Z \to \mathbb{R} \times Z$ be defined by $(\varphi, i)(y, z) = (\varphi(y), z)$. Because $\mathrm{cone}((F, G)(A)) + \mathrm{icr}(C \times D)$ is a convex set in $Y \times Z$. By the linearity of φ, the image set $(\varphi, i)(\mathrm{cone}((F, G)(A)) + \mathrm{icr}(C \times D))$ is convex. Because of Lemma 2.3 and Lemma 2.8, then,

$$\begin{aligned}(\varphi, i)(\mathrm{cone}((F, G)(A)) + \mathrm{icr}(C \times D)) &= \mathrm{cone}((\varphi \circ F, G)(A)) + \varphi(\mathrm{icr}(C)) \times \mathrm{icr}(D)\\ &= \mathrm{cone}((\varphi \circ F, G)(A)) + \mathrm{icr}(\varphi(C)) \times \mathrm{icr}(D)\\ &= \mathrm{cone}((\varphi \circ F, G)(A)) + \mathrm{icr}(\mathbb{R}_+) \times \mathrm{icr}(D)\\ &= \mathrm{cone}((\varphi \circ F, G)(A)) + \mathrm{icr}(\mathbb{R}_+ \times D).\end{aligned}$$

As a consequence, $\mathrm{cone}((\varphi \circ F, G)(A)) + \mathrm{icr}(\mathbb{R}_+ \times D)$ is convex, i.e., $(\varphi \circ F, G)$ is generalized $\mathbb{R}_+ \times D$-subconvexlike on A.

(ii) Define $(i, T) : Y \times Z \to Y$ by $(i, T)(y, z) = y + T(z)$. Similar to (i), $(i, T)(\mathrm{cone}((F, G)(A)) + \mathrm{icr}(C \times D))$ is convex. Moreover,

$$\begin{aligned}(i, T)(\mathrm{cone}((F, G)(A))) + \mathrm{icr}(C \times D)) &= (i, T)\mathrm{cone}((F, G)(A)) + (i, T)(\mathrm{icr}(C) \times \mathrm{icr}(D))\\ &= \mathrm{cone}(F(A) + T(G(A))) + \mathrm{icr}(C) + T(\mathrm{icr}(D))\\ &= \mathrm{cone}(F(A) + T(G(A))) + \mathrm{icr}(C).\end{aligned}$$

Therefore, $F + T \circ G$ is generalized C-subconvexlike on A.

Finally, since $\mathrm{icr}[\mathrm{cone}((F, G)(A)) + \mathrm{icr}(C \times D)] \neq \emptyset$, then,

$$\begin{aligned}\emptyset &\neq (\varphi, i)[\mathrm{icr}(\mathrm{cone}((F, G)(A) + \mathrm{icr}(C \times D))]\\ &= \mathrm{icr}[\mathrm{cone}((\varphi \circ F, G)(A)) + \varphi(\mathrm{icr}(C)) \times \mathrm{icr}(D)]\\ &= \mathrm{icr}[\mathrm{cone}((\varphi \circ F, G)(A)) + \mathrm{icr}(\varphi(C)) \times \mathrm{icr}(D)]\\ &= \mathrm{icr}[\mathrm{cone}((\varphi \circ F, G)(A)) + \mathrm{icr}(\mathbb{R}_+) \times \mathrm{icr}(D)]\\ &= \mathrm{icr}[\mathrm{cone}((\varphi \circ F, G)(A)) + \mathrm{icr}(\mathbb{R}_+ \times D)].\end{aligned}$$

Thus, $(\varphi \circ F, G)$ is relatively solid. By the proof of above, it is easy to cheak that $F + T \circ G$ is relatively solid.

Definition 5.2. *We say that the optimization problem* (CVP) *satisfies the generalized Slater constraint qualification if there exists $x \in S$ such that $G(x) \cap -\mathrm{icr}(D) \neq \emptyset$.*

Theorem 5.1. *Let C is v-closed and C^+ is solid. Let $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$. Assume that the following conditions hold:*

(i)$(\bar{x}, \bar{y})$ is a Benson proper efficient element of (CVP) *and $F - \bar{y}$ is relatively solid generalized C-subconvexlike on S;*

(ii)$(F - \bar{y}, G)$ is relatively solid generalized $C \times D$-subconvexlike on A;

(iii)(CVP) *satisfies the generalized Slater constraint condition;*

(iv)$\mathrm{icr}(D) \subset \mathrm{icr}[\mathrm{cone}(G(A)) + \mathrm{icr}(D)]$.

Then, there exists $T \in L^+(Z, Y)$ such that $0 \in T(G(\bar{x}) \cap -D)$ and $(\bar{x}, \bar{y})$ is a Benson proper efficient element of the unconstrained problem $(\mathrm{UVP})_{\mathrm{T}}$.

Proof. By assumption (i), we can apply Theorem 4.2 to problem (CVP), then there exists $\varphi \in C^{+i}$ such that

$$\varphi(y) \geq \varphi(\bar{y}), \ \forall y \in f(S). \tag{5.1}$$

Define set-valued map $I : A \rightrightarrows \mathbb{R} \times Z$ by

$$I(x) := (\varphi \circ (F - \bar{y}), G)(x) = [\varphi(F(x) - \bar{y})] \times G(x) = \varphi(F(x)) \times G(x) - (\varphi(\bar{y}), 0).$$

By assumption (ii) and Proposition 5.1(ii), we deduce that $I(x)$ is relatively solid generalized $\mathbb{R}_+ \times D$-subconvexlike on A. Together with (5.1), it is easy to check that

$$I(A) \cap -\mathrm{icr}(\mathbb{R}_+ \times D) \neq \emptyset. \tag{5.2}$$

By Lemma 3.2, there exists $(r, \psi) \in \mathbb{R}_+ \times D^+ \setminus \{(0,0)\}$ such that

$$r(\varphi(F(x) - \bar{y})) + \psi(G(x)) \geq 0, \ \forall x \in A, \tag{5.3}$$

and

$$(r, \psi)(y_1, y_2) > 0, \ \forall (y_1, y_2) \in \operatorname{icr}[\operatorname{cone}((\varphi \circ (F - \bar{y}), G)(A)) + \operatorname{icr}(\mathbb{R}_+ \times D)]. \tag{5.4}$$

We note that $r > 0$. Otherwise, if $r = 0$, in this case, (5.3) can be written as

$$\psi(G(x)) \geq 0, \ \forall x \in A, \tag{5.5}$$

and (5.4) becomes

$$\psi(\operatorname{icr}[\operatorname{cone}(G(A)) + \operatorname{icr}(D)]) > 0. \tag{5.6}$$

It follows from assumption (iv) and (5.6) that

$$\psi(\operatorname{icr}(D)) > 0. \tag{5.7}$$

By assumption (iii), there exists $x_0 \in A$, such that $G(x_0) \in -\operatorname{icr}(D)$, by (5.7), $\psi(G(x_0)) < 0$. Which contradicts (5.5). So $r > 0$.

Since $\bar{x} \in S$ and $\psi \in D^+$, there exists $\bar{z} \in G(\bar{x}) \cap -D$ such that $\psi(\bar{z}) \leq 0$. Taking $x = \bar{x}$, $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ in (5.3), we obtain $\psi(\bar{z}) \geq 0$, thus, $\psi(\bar{z}) = 0$. Hence,

$$0 \in \psi(G(\bar{x}) \cap -D). \tag{5.8}$$

From $r > 0$ and $\varphi \in C^{+i}$, we can select $c \in C$ such that

$$r\varphi(c) = 1. \tag{5.9}$$

Define a linear operator $T : Z \to Y$ as

$$T(z) = \psi(z)c. \tag{5.10}$$

Obviously, $T(D) \subset C$, i.e., $T \in L^+(Z, Y)$ and by (5.8), we have

$$0 \in T(G(\bar{x}) \cap -D). \tag{5.11}$$

From (5.3), (5.9) and (5.10), we obtain

$$r\varphi(F(x) + T(G(x))) = r\varphi(F(x)) + \psi(G(x))r\varphi(c) = r\varphi(F(x)) + \psi(G(x)) \geq r\varphi(\bar{y}), \ \forall x \in A.$$

Dividing the above inequality by $r > 0$ leads to

$$\varphi(\bar{y}) \leq \varphi(F(x) + T(G(x))).$$

Since $\bar{y} \in F(\bar{x}) \subset F(\bar{x}) + T(G(\bar{x})) = L(\bar{x}, T)$, it means that $(\bar{x}, \bar{y})$ is a minimizer element of the following scalar optimization problem

$$\operatorname{Min} \varphi(L(x, T)), \ \text{s.t.} \ x \in A.$$

According to Theorem 4.1, $(\bar{x}, \bar{y})$ is a Benson proper minimizer efficient of $(\mathrm{UVP})_{\mathrm{T}}$.

Remark 5.1. *Theorem 5.1 extends Theorem 4.3 of [22] and Theorem 5.5 of [24]. In Theorem 5.1, the condition (iv)* $\operatorname{icr}(D) \subset \operatorname{icr}[\operatorname{cone}(G(A)) + \operatorname{icr}(D)]$ *can be replaced by the condition* $\operatorname{aff}(D) = \operatorname{aff}[\operatorname{cone}(G(A)) + \operatorname{icr}(D)]$. *Actually, if* $\operatorname{aff}(D) = \operatorname{aff}[\operatorname{cone}(G(A)) + \operatorname{icr}(D)]$, $\operatorname{icr}(D) \subset \operatorname{icr}[\operatorname{cone}(G(A)) + \operatorname{icr}(D)]$. *However, the example below indicates that the condition* $\operatorname{icr}(D) \subset \operatorname{icr}[\operatorname{cone}(G(A)) + \operatorname{icr}(D)]$ *is weaker than* $\operatorname{aff}(D) = \operatorname{aff}[\operatorname{cone}(G(A)) + \operatorname{icr}(D)]$.

Example 5.1. *Let $X = Z = \mathbb{R}^2$, $D = \{(y_1, 0) \mid y_1 \geq 0\}$ and $A = \{(x_1, x_2) \mid -2 \leq x_1 + x_2 \leq 3\}$. The set-valued map $F : A \rightrightarrows Y$ is defined as follows:*

$$F(x_1, x_2) = \{(x_1, x_2)\}, \ \forall (x_1, x_2) \in A.$$

Clearly, $\operatorname{icr}(D) \subset \operatorname{icr}(\operatorname{cone}(F(A)) + \operatorname{icr}(D))$. *However,* $\operatorname{aff}(D) = \{(y_1, 0) \mid y_1 \in \mathbb{R}\} \neq \mathbb{R}^2 = \operatorname{aff}(\operatorname{cone}(F(A)) + \operatorname{icr}(D))$.

Analogously to Theorem 5.2 in [14], we can obtain the following theorem.

Theorem 5.2. *If there exists a pair $(\bar{x}, \bar{y})$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ and a positive linear operator $\bar{T} \in L^+(Z, Y)$ such that:*

(i) $0 \in \bar{T}(G(\bar{x}) \cap -D)$*;*

(ii) $(\bar{x}, \bar{y})$ *is a Benson proper efficient element of* $(\mathrm{UVP})_{\bar{T}}$.

Then $(\bar{x}, \bar{y})$ *is a Benson proper efficient element of* (CVP).

6 Benson proper saddle points

Definition 6.1. *A pair $(\bar{x}, \bar{T}) \in A \times L^+(Z, Y)$ is said to be a Benson proper saddle point of $L(x, T)$ if*

$$L(\bar{x}, \bar{T}) \cap \operatorname{PMin}\left(L(A, \bar{T}), C\right) \cap \operatorname{PMax}\left(L(\bar{x}, L^+), C\right) \neq \emptyset,$$

where $L(A, \bar{T}) = \bigcup_{x \in A} L(x, \bar{T})$ and $L(\bar{x}, L^+) = \bigcup_{T \in L^+(Z,Y)} L(\bar{x}, T)$.

In the following, we will present an important equivalent characterization for Benson proper saddle point of the Lagrangian map $L(x, T)$ by virtue of a strong separation theorem.

Theorem 6.1. *Let D be v-closed. Then, $(\bar{x}, \bar{T}) \in A \times L^+(Z, Y)$ is a Benson proper saddle point of $L(x, T)$ if and only if there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that*

(i) $\bar{y} \in \operatorname{PMin}\left(L(A, \bar{T}), C\right)$*;*

(ii) $G(\bar{x}) \subset -D$*;*

(iii) $\bar{T}(\bar{z}) = 0$*;*

(iv) $\bar{y} \in \operatorname{PMax}(F(\bar{x}), C)$.

Proof. Necessity. Since $(\bar{x}, \bar{T})$ is a Benson proper saddle point of $L(x, T)$, there exist $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$ such that

$$\bar{y} + \bar{T}(\bar{z}) \in \operatorname{PMin}\left(L(A, \bar{T}), C\right), \tag{6.1}$$

and

$$\bar{y} + \bar{T}(\bar{z}) \in \operatorname{PMax}\left(L(\bar{x}, L^+), C\right). \tag{6.2}$$

From (6.2), it follows that

$$\operatorname{vcl}[\operatorname{cone}(L(\bar{x}, L^+) - C - (\bar{y} + \bar{T}(\bar{z})))] \cap C = \{0\}. \tag{6.3}$$

Now, for any $T \in L^+(Z, Y)$,

$$T(\bar{z}) - \bar{T}(\bar{z}) = (\bar{y} + T(\bar{z})) - (\bar{y} + \bar{T}(\bar{z})) \in F(\bar{x}) + T(G(\bar{x})) - (\bar{y} + \bar{T}(\bar{z})) = L(\bar{x}, T) - (\bar{y} + \bar{T}(\bar{z})).$$

As a result,

$$\bigcup_{T \in L^+(Z,Y)} T(\bar{z}) - C - \bar{T}(\bar{z}) \subset L(\bar{x}, L^+) - C - (\bar{y} + \bar{T}(\bar{z})).$$

Hence, by (6.3), one has

$$\operatorname{vcl}(\operatorname{cone}(\bigcup_{T \in L^+(Z,Y)} T(\bar{z}) - C - \bar{T}(\bar{z}))) \cap C = \{0\}. \ T \in L^+(Z, Y). \tag{6.4}$$

Define a map $f : L^+(Z, Y) \to Y$ as

$$f(T) = -T(\bar{z}).$$

Then, (6.4) is equivalent to

$$\mathrm{vcl}(\mathrm{cone}(f(L^+) + C - f(\bar{T})) \cap (-C) = \{0\}.$$

This implies that $\bar{T} \in L^+(Z, Y)$ is an Benson proper efficient solution of the following vector optimization problem

$$\mathrm{Min}\, f(T) \ \text{ s.t. } T \in L^+(Z, Y).$$

Since f is a linear map, clearly, f is relatively solid generalized C-subconvexlike on $L^+(Z, Y)$. Thus, by Theorem 4.2, there exists $\varphi \in C^{+i}$ such that

$$\varphi(-\bar{T}(\bar{z})) = \varphi(f(\bar{T})) \leq \varphi(f(T)) = \varphi(-T(\bar{z})), \forall T \in L^+(Z, Y),$$

that is

$$\varphi(\bar{T}(\bar{z})) \geq \varphi(T(\bar{z})), \ \forall T \in L^+(Z, Y). \tag{6.5}$$

We affirm that $-\bar{z} \in D$. Otherwise, $0 \notin D + \bar{z}$. Since D is v-closed, $\mathrm{vcl}(D + \bar{z}) = D + \bar{z}$. Thus, $0 \notin \mathrm{vcl}(D + \bar{z})$. From Lemma 2.5, $\mathrm{icr}(D + \bar{z}) = \mathrm{icr}(D) + \bar{z} \neq \emptyset$. By Lemma 2.2, there exists $z^\star \in Z^\star \backslash \{0\}$ such that

$$z^\star(\lambda d) > z^\star(-\bar{z}) \ \forall \lambda \geq 0, \ \forall d \in D. \tag{6.6}$$

Taking $\lambda = 0$ in (6.6), we obtain

$$z^\star(\bar{z}) > 0.$$

By (6.6), we have

$$z^\star(d) > \frac{1}{\lambda} z^\star(-\bar{z}), \ \ \forall \lambda > 0, \ \forall d \in D. \tag{6.7}$$

Letting $\lambda \to +\infty$ gives

$$z^\star(d) > 0, \ \forall d \in D.$$

Thus, $z^\star \in D^+ \setminus \{0\}$. Choose $c_0 \in \mathrm{icr}(C)$ be fixed, and define the map $T_0 : Z \to Y$ as

$$T_0(z) = \frac{z^\star(z)}{z^\star(\bar{z})} c_0 + T(\bar{z}), \ \forall z \in Z.$$

Obviously, $T_0 \in L^+(Z, Y)$ and

$$T_0(\bar{z}) - T(\bar{z}) = c_0, \ \forall z \in Z.$$

It follows from $\varphi \in C^{+i}$ and Remark 2.1 that

$$\varphi(T_0(\bar{z})) - \varphi(T(\bar{z})) = \varphi(c_0) > 0,$$

which contradicts (6.5) and so $-\bar{z} \in D$. Thus, $-\bar{T}(\bar{z}) \in C$. If $\bar{T}(\bar{z}) \neq 0$, then $-\bar{T}(\bar{z}) \in C \setminus \{0\}$, and so

$$\varphi(\bar{T}(\bar{z})) < 0. \tag{6.8}$$

Taking $T = 0 \in L^+(Z, Y)$ in (6.5) gives $\varphi(\bar{T}(\bar{z})) \geq 0$. Which conflicts with (6.8). So condition (iii) holds. Together with (6.1), condition (i) holds.

Now we show $G(\bar{x}) \subset -D$. If not, there exist $z_1 \in G(\bar{x})$ such that $-z_1 \notin D$. Analogously to the above proof of $-\bar{z} \in D$, there exists $z_1^\star \in D^+ \setminus \{0\}$ such that $z_1^\star(z_1) > 0$. Choose $c_1 \in \mathrm{icr}(C)$ be fixed and define the map $T_1 : Z \to Y$ as follows

$$T_1(z) = z_1^\star(z) c_1.$$

Clearly, $T_1 \in L^+(Z, Y)$ and

$$T_1(z_1) = z_1^\star(z_1)c_1 \in \mathrm{icr}(C) \subset C \setminus \{0\}. \tag{6.9}$$

On the other hand, from condition (iii) and (6.2), we have

$$\bar{y} \in \mathrm{PMax}\left(L(\bar{x}, L^+), C\right) \subset \mathrm{Max}\left(L(\bar{x}, L^+), C\right),$$

i.e.,

$$y - \bar{y} \notin C \setminus \{0\},\ \forall y \in \bigcup_{T \in L^+(Z,Y)} L(\bar{x}, T).$$

By

$$\bar{y} + T_1(z_1) \in F(\bar{x}) + T_1[G(\bar{x})] = L(\bar{x}, T_1) \subset \bigcup_{T \in L^+(Z,Y)} L(\bar{x}, T).$$

Therefore,

$$T_1(z_1) = \bar{y} + T_1(z_1) - \bar{y} \notin C \setminus \{0\}.$$

Which contradicts (6.9). So, $G(\bar{x}) \subset -D$ holds, i.e., condition (ii) holds.

Finally, from (6.3),

$$\mathrm{vcl}(\mathrm{cone}(F(\bar{x}) + T(G(\bar{x})) - C - (\bar{y} + \bar{T}(\bar{z}))) \cap C = \{0\},\ \forall T \in L^+(Z, Y). \tag{6.10}$$

Taking $T = 0 \in L^+(Z, Y)$ in (6.10)

$$\mathrm{vcl}(\mathrm{cone}(F(\bar{x}) - C - \bar{y})) \cap C = \{0\},$$

which means that condition (iv) holds.

Sufficiency. Since condition (i) and condition (iii), we only prove $\bar{y} \in L(\bar{x}, \bar{T}) \cap \mathrm{PMax}\left(L(\bar{x}, L^+), C\right)$. From $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$ and condition (iii), it follows that

$$\bar{y} = \bar{y} + \bar{T}(\bar{z}) \in F(\bar{x}) + \bar{T}(G(\bar{x})) = L(\bar{x}, \bar{T}) \tag{6.11}$$

By condition (ii),

$$T(G(\bar{x})) \subset -C,\ \forall T \in L^+(Z, Y).$$

Hence,

$$L(\bar{x}, L^+) - C - \bar{y} = F(\bar{x}) + \bigcup_{T \in L^+(Z,Y)} T(G(\bar{x})) - C - \bar{y} \subset F(\bar{x}) - C - \bar{y}. \tag{6.12}$$

From condition (iv) and (6.12), it follows that

$$\mathrm{vcl}(\mathrm{cone}(L(\bar{x}, L^+) - C - \bar{y})) \cap C = \{0\},$$

i.e., $\bar{y} \in \mathrm{PMax}\left(L(\bar{x}, L^+), C\right)$.

Now, the example below explains the sufficiency of Theorem 6.1.

Example 6.1. *Let $X = Y = Z = \mathbb{R}^2$, $C = D = \mathbb{R}^2_+$ and $A = \{0\} \times [-1, 0]$. The set-valued maps $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ are, respectively, defined as*

$$F(x_1, x_2) = \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1, y_2) \in \{4\} \times [-1 + x_2, 1 + x_2]\},\ \forall (x_1, x_2) \in A;$$

$$G(x_1, x_2) = \{(z_1, z_2) \in \mathbb{R}^2 \mid (z_1, z_2) = \lambda(x_2, x_2),\ \lambda \in [0, 1]\},\ \forall (x_1, x_2) \in A.$$

Take $\bar{x} = (0, -1)$, $\bar{y} = (4, 0)$, $\bar{z} = (0, 0)$, $\bar{T}(z_1, z_2) = (0.5z_1, 0.5z_2) \in L^+(Z, Y)$. Clearly, all conditions of Theorem 6.1 are satisfied. Consequently, $(\bar{x}, \bar{T})$ is a Benson proper saddle point of the Lagrangian set-valued map $L(x, T)$.

Remark 6.1. *Theorem 6.1 extends Proposition 6.1 in [14] which were done in the framework of topological linear spaces.*

Theorem 6.2. *Let D be v-closed. $(\bar{x}, \bar{T}) \in A \times L^+(Z, Y)$ is a Benson proper saddle point of $L(x, T)$. Then, there exist $\bar{y} \in F(\bar{x})$ such that $(\bar{x}, \bar{y})$ is a Benson proper efficient element of* (VP).

Proof. It follows directly from the necessity of Theorem 6.1 and Theorem 5.2.

Theorem 6.3. *Let C, D be v-closed and C^+ is solid. Let $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$. Assume that the following conditions hold:*

(i) $(\bar{x}, \bar{y})$ is a Benson proper efficient element of (CVP) *and $F - \bar{y}$ is relatively solid generalized C-subconvexlike on S;*

(ii) $(F - \bar{y}, G)$ is relatively solid generalized $C \times D$-subconvexlike on A;

(iii) (CVP) *satisfies the generalized Slater constraint condition;*

(iv) $\mathrm{icr}(D) \subset \mathrm{icr}[\mathrm{cone}(G(A)) + \mathrm{icr}(D)]$;

(v) $G(\bar{x}) \subset -D$, $\bar{y} \in \mathrm{PMax}(F(\bar{x}), C)$.

Then, there exists $\bar{T} \in L^+(Z, Y)$ such that $(\bar{x}, \bar{T}) \in A \times L^+(Z, Y)$ is a Benson proper saddle point of L. Furthermore, $0 \in \bar{T}(G(\bar{x} \cap -D))$.

Proof. This conclusion follows by Theorem 5.1 and the sufficiency of Theorem 6.1.

7 Duality

A set-valued map $\Phi : L^+(Z, Y) \rightrightarrows Y$ defined by

$$\Phi(T) := \mathrm{PMin}\left(L(A, T), C\right), \quad T \in L^+(Z, Y)$$

is called a proper dual map of (CVP).

The vector maximization problem with set-valued map Φ

$$\text{(VD)} \qquad \mathrm{Max} \bigcup_{T \in L^+} \Phi(T)$$

is said to be a Lagrange dual problem of (CVP).

Definition 7.1. *A point $\bar{y} \in Y$ is called a feasible point of* (VD) *if $\bar{y} \in \bigcup_{T \in L^+} \Phi(T)$. A feasible point $\bar{y}$ is called an efficient point of* (VD) *if*

$$y - \bar{y} \notin C \setminus \{0\}, \quad \forall y \in \bigcup_{T \in L^+} \Phi(T).$$

Analogously to Theorem 7.1 and Theorem 7.2 in [14], we can obtain the following two theorems.

Theorem 7.1. *Weak Duality. Let $\bar{x} \in S$ and $\bar{y}$ be any feasible point of* (VD). *Then*

$$\bar{y} \notin F(\bar{x}) + C \setminus \{0\}.$$

Theorem 7.2. *Let $\bar{x} \in S$. Then, $\bar{y} \in F(\bar{x}) \cap \bigcup_{T \in L^+} \Phi(T)$ if and only if $\bar{x}$ is a Benson proper efficient solution of* (CVP) *and $\bar{y}$ is an efficient point of* (VD).

The following strong duality results show us a necessary and sufficient condition of proper efficiency.

Theorem 7.3. *Strong duality. Let C is v-closed and C^+ is solid. Let* (CVP) *satisfies the generalized Slater constraint condition such that*

(i) F is relatively solid generalized C-subconvexlike on S;

(ii) (F, G) is relatively solid generalized $C \times D$-subconvexlike on A;

(iii) $\mathrm{icr}(D) \subset \mathrm{icr}[\mathrm{cone}(G(A)) + \mathrm{icr}(D)]$.

Then, $\bar{x}$ *is a Benson proper efficient solution of* (CVP) *if and only if* $\bar{x} \in S$ *and there exists* $\bar{y} \in F(\bar{x})$ *such that* $\bar{y}$ *is an efficient point of* (VD).

Proof. Suppose that $\bar{x}$ is a Benson proper efficient solution of (CVP). By Definition 5.1, $\bar{x} \in S$ and there exists $\bar{y} \in F(\bar{x})$ such that $(\bar{x}, \bar{y})$ is a Benson proper efficient element of (CVP). By Theorem 5.1, there exists $\bar{T} \in L^+(Z, Y)$ with $0 \in \bar{T}(G(\bar{x}) \cap -D)$ such that

$$\bar{y} \in \mathrm{PMin}(L(x, \bar{T}), C) = \Phi(\bar{T}) \subset \bigcup_{T \in L^+} \Phi(T).$$

By Theorem 7.2, $\bar{y}$ is an efficient point of (VD).

Conversely, suppose that $\bar{x} \in S$, and there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y}$ is an efficient point of (VD). Then, $\bar{y} \in \bigcup_{T \in L^+} \Phi(T)$. Again, by Theorem 7.2, we get $\bar{x}$ is a Benson proper efficient solution of (CVP).

Remark 7.1. *Theorem 7.3 extends Theorem 3.6 and Theorem 3.7 in [23] to vector optimization problems with set-valued maps.*

8 Conclusions

In this paper, we introduce the concept of relatively solid generalized cone-subconvexlike set-valued maps. With the help of the concept, we establish scalarization theorems, Lagrangian type duality theorems, saddle-point theorems and duality theorems to characterize Benson proper efficient solutions.

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